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Statistical stationary states for a two-layer quasi-geostrophic system

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Abstract. Existence of a family of locally invariant probability measures for large scale flows in enclosed temperate sea is proved. This model is extremely important for understanding the meso-scale phenomena in oceans. The techniques used are those developed by Albeverio and his collaborators.

Keywords. Quasi-geostrophic flows; infinite dimensional dynamical system; Luoville's theorem; invariant measures; rigged Hilbert spaces; white noise; Koopmann's formulation; infinite dimensional analysis; Louville operator; self-adjointness.

1. Introduction

Dynamical systems with infinite degrees of freedom arise in all field theories in physics. At a formal level they have many similarities with dynamical systems with finitely many degrees of freedom. Often they admit a Hamiltonian description. It is tempting to attempt an investigation of their ergodic properties. Any such attempt requires the construction of an invariant probability measure on the phase space of the system. As the phase space is in general infinite dimensional a natural approach would be to exploit the specific topological structure which may be present on the phase space. If the phase space is a Hilbertian space (i.e. admits a structure of a Hilbert space) one could exploit Gaussian measures on this space. The only canonical Gaussian probability measures associated with such spaces are of the white noise type. But these measures are supported on generalized vectors, the Hilbert space itself being measurable but with measure zero. As the system is non-linear the existence of dynamics would have to be established, as the vector field concerned would involve products of distributions. These non-linear terms would have to be regularized and energy renormalized. It is this approach that is successfully exploited in [1, 2] to construct locally invariant measures for two dimensional Eulerian flows on a flat torus.

Periodic two dimensional Eulerian flows which have been investigated at great length in [2, 3, 5, 7], although of great interest in understanding two dimensional flows in laboratory experiments, are not of much use in understanding the motion of temperate seas. Flows with horizontal spatial scale of a few hundred kilometers and temporal scales of a few weeks are essentially two dimensional with little or no vertical variability (the depth of the sea is about 4 kms on an average). The motion to a large extent is controlled by the variable rotation of the Earth. This motion is called quasi-geostrophic motion. The variability of Coriolis force with latitude gives rise to a restoring mechanism, which allows for the existence of propagating Rossby waves and leads to the observed westward intensification of oceanic currents. These effects stand in sharp contrast to 2-D Eulerian flows. A clear account of quasi-geostrophy can be found in [11, 12]. Quasi-geostrophic flows are very close to 2-D Eulerian flows. Formal investigations into the construction of invariant probability measures for such flows have been carried out in [9, 13, 17]. A rigorous investigation on the construction of locally invariant probability measures for such flows using the techniques of Albeverio and his group was carried out for single layer model in [10]. The analysis of the authors is strictly valid for an unstratified ocean, therefore of limited validity. Real oceans have a vertical density stratification. This stratification gives rise to baroclinic motion. The quasi-geostrophic motion in the sea is characterized by the coupling of the barotropic and baroclinic modes. In the sequel a simple two layer quasi-geostrophic model, which has been used with great effect by oceanographers, is studied. The existence of locally invariant probability measures is proved. The techniques used are essentially those of Albeverio and his collaborators.

2. Two layer quasi-geostrophic model

A two layer quasi-geostrophic (see [12]) model consists of a layer of an ideal fluid with density ρ_1 and thickness H_1 superposed on another layer with density ρ_2 and thickness H_2 with $\rho_1 < \rho_2$. The equations governing this model on the β-plane i.e. local tangent plane at a reference latitude λ_0, with the X-axis oriented eastwards and the Y-axis oriented northwards are

$$\left[\frac{\partial}{\partial t}+\frac{\partial \Psi_1}{\partial x}\frac{\partial}{\partial y}-\frac{\partial \Psi_1}{\partial y}\frac{\partial}{\partial x}\right][\nabla^2\Psi_1 - F_1(\Psi_1-\Psi_2)+\beta y]=0, \tag{2.1}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial \Psi_2}{\partial x}\frac{\partial}{\partial y}-\frac{\partial \Psi_2}{\partial y}\frac{\partial}{\partial x}\right][\nabla^2\Psi_2 - F_2(\Psi_2-\Psi_1)+\beta y]=0. \tag{2.2}$$

Change of Coriolis frequency at latitude λ_0 and $f_0 = 2\alpha\sin\lambda_0$, with $\alpha = (2\pi/24)\,hr^{-1}$. $\Psi_1(x,y)$ and $\Psi_2(x,y)$ are the stream functions for the two dimensional velocity fields in the upper and lower layers respectively.

Introducing new stream functions $\Phi = \Psi_1 - \Psi_2$ the barotropic and $\Psi = (H_1H_2)^{1/2}(H_1\Psi_1 + H_2\Psi_2)/H^2$ the baroclinic stream functions, where $H = H_1 + H_2$. Equations (2.1) and (2.2) take the form

$$\frac{\partial}{\partial t}(\nabla^2\Psi)+J(\Psi,\nabla^2\Psi)+J(\Phi,\nabla^2\Phi)+\beta\frac{\partial\Psi}{\partial x}=0, \tag{2.3}$$

$$\frac{\partial}{\partial t}[(-\nabla^2\Phi+F\Phi)]+\kappa J(\Phi,\nabla^2\Phi)+J(\Psi,\nabla^2\Phi)$$
$$+J(\Phi,\nabla^2\Psi)-FJ(\Psi,\Phi)+\beta\frac{\partial\Phi}{\partial x}=0, \tag{2.4}$$

where $\kappa = (H_1-H_2)/(H_1H_2)^{1/2}, F = F_1 + F_2$ and

$$J(\Phi,\Psi)=\frac{\partial\Phi}{\partial x}\frac{\partial\Psi}{\partial y}-\frac{\partial\Phi}{\partial y}\frac{\partial\Psi}{\partial x}$$

is the Jacobian of Ψ and Φ.

This system admits two quadratic constants of motion.

$$H(\Psi,\Phi)=\int\!\!\int\{|\nabla\Psi|^2+|\nabla\Phi|^2+F\Phi^2\}\mathrm{d}x\,\mathrm{d}y, \tag{2.5}$$

$$S(\Psi,\Phi)=\frac{1}{2}\int\!\!\int\{(\nabla^2\Psi)^2+((-\nabla^2+F)\Phi)^2\}\mathrm{d}x\,\mathrm{d}y. \tag{2.6}$$

$S(\Psi,\Phi)$ is called the enstrophy, $H(\Psi,\Phi)$ is the energy of the system.

Let $\omega_1 = -\nabla^2\Psi$ be the vorticity of the baroclinic mode and $\omega_2 = (-\nabla^2 + F)\Phi$ be the modified vorticity of the barotropic mode. In terms of these modes enstrophy and energy take the form

$$S(\omega_1, \omega_2) = \frac{1}{2}\int\int\{\omega_1^2 + \omega_2^2\}\mathrm{d}x\,\mathrm{d}y, \tag{2.7}$$

$$H(\omega_1, \omega_2) = \frac{1}{2}\int\int\{(-\nabla^2)^{-1}(\omega_1)^2 + (-\nabla^2 + F)^{-1}(\omega_2)^2\}\mathrm{d}x\,\mathrm{d}y. \tag{2.8}$$

Energy and enstrophy being positive, attempts to construct Gibbs-like distributions on the space of vortices have been made in [9, 13, 14, 17]. There have been enthusiastic attempts at examining the ergodic properties of these systems in [4, 19]. As the space of vortices is infinite dimensional, such formal distribution functions do not exist. In this note we construct probability measures of the Gibbsian type for the two-layer quasi-geostrophic model following the techniques of Albeverio and his collaborators.

3. Mathematical formulation of the quasi-geostrophic model

In order to facilitate analysis, we consider an ocean basin occupying the region $\Omega = [0, \pi] \times [0, \pi] \subset R^2$. Consider the Hilbert space $L^2(\Omega, \mathrm{d}x\,\mathrm{d}y)$. The Friedrichs extension of the Dirichlet Laplacian $-\nabla^2$ on $C_0^\infty(\Omega) \subset L^2(\Omega, \mathrm{d}x\,\mathrm{d}y)$ is self-adjoint and has a Hilbert–Schmidt inverse. The eigenvalues and eigenfunctions can be computed explicitly. The eigenvalues of the operator are $K^2 = k_1^2 + k_2^2$, where $K = (k_1, k_2) \in Z^2$ and $k_1 \neq 0$ for $i = 1, 2$. Denote the eigenfunction corresponding to eigenvalue K^2 by ξ_K. It is assumed that $\omega_i, i = 1, 2$ are in $L^2(\Omega, \mathrm{d}x\,\mathrm{d}y)$. Expanding $\omega_i, i = 1, 2$ in terms of eigenfunctions ξ_K, we have $\omega_1 = \sum X_K\xi_K$ and $\omega_2 = \sum Y_K\xi_K$. Denote the sequences $\{X_K\}_{K\in Z}$ and $\{Y_K\}_{K\in Z}$ by X and Y. Using these expansions both equations (2.3) and (2.4) are reduced to a countably infinite system of equations. The explicit form of these equations are

$$\begin{aligned}\frac{\partial X_N}{\partial t} =& \frac{1}{2\pi}\sum_K K \cdot N\left\{\frac{X_K X_{N-K}}{(N-K)^2} + \frac{Y_K Y_{N-K}}{N-K)^2 + F}\right\} - \frac{1}{2\pi}\\ &\times\sum_K K \cdot N\left\{\frac{X_K X_{J(N-K)}}{(N-K)^2} + \frac{Y_K Y_{J(N-K)}}{N-K)^2 + F}\right\} = A_N(X, Y),\end{aligned} \tag{3.1}$$

$$\begin{aligned}\frac{\partial Y_N}{\partial t} =& \frac{\kappa}{2\pi}\sum_K E_K^N\{Y_K Y_{N-K} - Y_K Y_{J(N-K)}\}\\ &+ \frac{1}{2\pi}\sum_K F_K^N\{Y_K X_{N-K} - Y_K X_{J(N-K)}\} = B_N(X, Y),\end{aligned} \tag{3.2}$$

where summation is taken over all $K \in Z^2, K \neq 0, J(N) = (n_1, -n_2), K.N = k_1 n_2 - k_2 n_1$,

$$E_K^N = K \cdot N/((N-K)^2 + F)$$

and

$$F_K^N = K \cdot N\{K^2 - (N-K)^2 - F\}/\{K^2((N-K)^2 + F)\}.$$

Energy and enstrophy take the form

$$H(X, Y) = \sum_K\left\{\frac{X_K^2}{K^2} + \frac{Y_K^2}{K^2 + F}\right\}, \tag{3.3}$$

and

$$S(X,Y) = \sum_K \{X_K^2 + Y_K^2\}, \tag{3.4}$$

the summation being taken over $K \in Z^2$ and $K \neq 0$.

For a fixed $K \in Z^2, K \neq 0, A_K(X,Y)$ and $B_K(X,Y)$ are independent of X_K and Y_K. Since enstrophy and energy are conserved it can be easily seen that

$$\sum_K \{X_K A_K(X,Y) + Y_K B_K(X,Y)\} = 0, \tag{3.5}$$

$$\sum_K \left\{ \frac{X_K}{K^2} A_K(X,Y) + \frac{Y_K}{K^2+F} B_K(X,Y) \right\} = 0. \tag{3.6}$$

Following Daletskii [6], consider the space $\{(X,Y)\}$ of real sequences in $\mathbb{R}^2$. For $m \in \mathbb{N}$, the subspace H_m defined by

$$H_m = \left\{ (X,Y) | \sum_K \{K_K^{2m} X_K^2 + (K^2+F)^m Y_K^2\} < \infty \right\}$$

is a Hilbert space. Set $H_\infty = \cap_{m=0}^\infty H_m$. Clearly H_∞ is a locally convex space with dual $H_{-\infty} = \cup_{m=0}^\infty H_{-m}$. Clearly $H_\infty \subset H_0 \subset H_{-\infty}$ is a Gelfand triplet (rigged Hilbert space).

For every $\gamma > 0$ the function $C : H_\infty \to \mathbb{R}$ defined by

$$C[X,Y] = e^{(-\gamma/2)\sum_K \{X_K^2 + Y_K^2\}}$$

is a positive definite Frechet continuous functional on H_∞, normalized to 1 at 0. By Bochner–Minlos theorem (see [8]) there exists a Borel probability measure P_γ on $H_{-\infty}$, which has $C[X,Y]$ as its characteristic functional. The properties of P_γ are well known. Its two important properties are

1. H_0 is a measurable subset of $H_{-\infty}$ and has measure 0.
2. If $\gamma_1 \neq \gamma_2, P_\gamma \perp P_\gamma$.

The enstrophy allows us to construct a one parameter family of mutually singular measures, with support on $H_{-\infty}$. Clearly these measures are supported not on regular vortices but on generalized vortices i.e. distributional vortices. In the sequel these probability measures play the role analogous to the Lebsegue measure on $\mathbb{R}^n$. Further these measures are a pure Hilbert space construction.

If one were to regard these measures as the one sought, the baroclinic and modified barotropic vorticity components are statistically independent identically distributed normal random variable with mean zero and variance γ-indicating an equipartition of enstrophy and energy.

4. Renormalization of energy

PROPOSITION 4.1

Energy is P_γ almost surely infinite.

Proof. For each $n \in \mathbb{N}$, consider the cylinder function

$$H_n(X,Y) = \sum_{K^2<n} \left\{ \frac{X_K^2}{K^2} + \frac{Y_K^2}{K^2+F} \right\}. \tag{4.1}$$

H_n is measurable and H_n converges monotonically to H, hence H is measurable. For each $n, H_n > 0$ therefore

$$\lim_{n\to\infty} \int H_n \mathrm{d}P_\gamma = \int H \mathrm{d}P_\gamma.$$

But

$$\int H_n \mathrm{d}P_\gamma = \gamma \sum_{K^2<n} \left\{ \frac{1}{K^2} + \frac{1}{K^2+F} \right\}, \tag{4.2}$$

which diverges to infinity as $n \to \infty$. Hence H is almost surely infinite. [As $\{X_k^2/K^2 + Y_k^2/(K^2+F)\}_k$ are mutually independent random variables by the Kolmogorov zero-one law it follows that $\sum_K \{X_k^2/K^2 + Y_k^2/(K^2+F)\}$ either converges or diverges almost surely.] □

The support of P_γ indicates that the enstrophy is P_γ almost surely infinite. In a system occupying a finite region of space, clearly this is unphysical. For $n \in \mathbb{N}$, define

$$H_n^R(X,Y) = \sum_{K^2<n} \left\{ \frac{1}{K^2}(X_K^2 - \gamma) + \frac{1}{K^2+F}(Y_K^2 - \gamma) \right\}. \tag{4.3}$$

Then for each n

$$\int H_n^R(X,Y)\mathrm{d}P_\gamma = 0.$$

PROPOSITION 4.2

$\{H_n^R(X,Y)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(P_\gamma)$ and converges to a limit $H^R(X,Y)$ in $L^2(P_\gamma)$.

Proof. We prove that $\{H_n^R(X,Y)\}_{n\in\mathbb{N}}$ is Cauchy sequence in $L^2(P_\gamma)$. For $n, m \in \mathbb{N}$ and $n > m$,

$$\int |H_n^R(X,Y) - H_m^R(X,Y)|^2 \mathrm{d}P_\gamma = 2\gamma^2 \sum_{(m<K^2<n)} \left\{ \frac{1}{K^4} + \frac{1}{(K^2+F)^2} \right\}, \tag{4.4}$$

which tends to zero as $n, m \to \infty$. Hence the result. □

We now use the fact that $H^R(X,Y) \in L^2(P_\gamma)$ to construct the measure which corresponds to the formal distribution introduced by Rhines, Salmon and others.

Theorem 4.3. *For $\alpha > 0$, the formal object*

$$\mathrm{d}P_{\alpha,\gamma} = \mathrm{e}^{-\alpha H^R(X,Y)} \mathrm{d}P_\gamma$$

is a well-defined probability measure on $H_{-\infty}$, which is absolutely continuous with respect to P_γ and for $\gamma_1 \neq \gamma_2, P_{\alpha,\gamma}$ and $P_{\alpha,\gamma}$ are mutually singular. □

5. Regularization of $A_N(X,Y)$ and $B_N(X,Y)$

The right hand side of equations (3.1) and (3.2) involve products of distributions i.e. generalized vectors and are as such ill-defined. These can be given acceptable meaning only by a process of regularization. In order to regularize A_N and B_N we resort to a high wave number cut-off and a subsequent limit process. Note that for $N, M \in Z^2$ and $N \neq M, X_N$ and X_M, Y_N and Y_M are statistically independent, normally distributed random variables. Also note that X_N and Y_M are independent. The random variable X_N is normally distributed with mean 0 and variance $\gamma N^2/(\alpha\gamma + N^2)$. Also Y_N is normally distributed with mean 0 and variance $\gamma(N^2 + F)/[\alpha\gamma + (N^2 + F)]$. For $m \in \mathbb{N}$ define A_N^m and B_N^m by

$$A_N^m(X,Y) = \frac{1}{2\pi}\sum_{K^2<m} K\cdot N\left\{\frac{X_K X_{(N-K)}}{(N-K)^2} + \frac{Y_K Y_{(N-K)}}{(N-K)^2+F} - \frac{X_K X_{J(N-K)}}{(N-K)^2} - \frac{Y_K Y_{J(N-K)}}{(N-K)^2+F}\right\} \tag{5.1}$$

and

$$B_N^m(X,Y) = \frac{\kappa}{2\pi}\sum_{K^2<m} E_K^N\{Y_K Y_{N-K} - Y_K Y_{J(N-K)}\} + \frac{1}{2\pi}\sum_{K^2<m} F_K^N\{Y_K X_{N-K} - Y_K X_{J(N-K)}\}. \tag{5.2}$$

Theorem 5.1 *For each $m \in \mathbb{N}, A_N^m(X,Y)$ and $B_N^m(X,Y) \in L^2(P_{\alpha,\gamma})$ and converge to limit $A_N^R(X,Y)$ and $B_N^R(X,Y) \in L^2(P_{\alpha,\gamma})$ respectively, as $m \to \infty$.*

Proof. $A_N^m(X,Y)$ and $B_N^m(X,Y)$ are cylinder functions and hence measurable. It is clear that $A_N^m(X,Y)$ and $B_N^m(X,Y) \in L^2(P_{\alpha,\gamma})$. We now prove that the sequence $\{A_N^m(X,Y)\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^2(P_{\alpha,\gamma})$. It is enough to show that the sequence $\{D_N^m(X,Y)\}_{m\in\mathbb{N}}$ defined by

$$D_N^m(X,Y) = \sum_{K^2<m} K\cdot N \frac{X_K X_{N-K}}{(N-K)^2} \tag{5.3}$$

is a Cauchy sequence. Since KN is antisymmetric in K and N, equation (5.3) takes the form

$$D_N^m(X,Y) = \frac{1}{2}\sum_{K^2<m} K\cdot N\left\{\frac{1}{(N-K)^2} - \frac{1}{K^2}\right\} X_N X_{N-K}. \tag{5.4}$$

Clearly $D_N^m(X,Y)$ is a cylinder function and hence measurable. Moreover

$$\int |D_N^m(X,Y)|^2 \mathrm{d}P_{\alpha,\gamma} = \frac{1}{4}\sum_{K^2,L^2<m} (K\cdot N)(L\cdot N)\left\{\frac{[K^2-(N-K)^2]}{K^2(N-K)^2}\right\} \times\left\{\frac{[L^2-(N-L)^2]}{L^2(N-L)^2}\right\}\int X_K X_{N-K} X_L X_{N-L} \mathrm{d}P_{\alpha,\gamma}. \tag{5.5}$$

But

$$\int X_K X_{N-K} X_L X_{N-L} \mathrm{d}P_{\alpha,\gamma} = \left\{\frac{\gamma K^2}{\alpha\gamma + K^2}\right\}\left\{\frac{\gamma(N-K)^2}{\alpha\gamma + (N-K)^2}\right\}\{\delta_{K,L} + \delta_{K,N-L}\}. \tag{5.6}$$

Note that $K^2 - (N - K)^2 = -N^2 + 2(N \cdot K)$. Hence

$$\int |D_N^m(X,Y)|^2 \mathrm{d}P_{\alpha,\gamma} \leq \frac{\gamma^2}{2} \sum_{K^2<m} \frac{[-N^2 + 2(N \cdot K)]^2 (K \cdot N)^2}{K^4 (N-K)^4} \tag{5.7}$$

and

$$\int |D_N^m(X,Y)|^2 \mathrm{d}P_{\alpha,\gamma} \leq 72 N^6 \gamma^2 \sum_{K^2<m} \frac{1}{(N-K)^4}. \tag{5.8}$$

In the same way we get for $p, q \in \mathbb{N}$ and $p < q$

$$\int\int |D_N^q(X,Y) - D_N^p(X,Y)|^2 \mathrm{d}P_{\alpha,\gamma} \leq 72 N^6 \gamma^2 \sum_{p<K^2<q} \frac{1}{(N-K)^4}. \tag{5.9}$$

This proves that $D_N^m(X,Y)$ converges in $L^2(P_{\alpha,\gamma})$ as $m \to \infty$. Similarly the other terms can be handled. This implies that $A_N^m(X,Y)$ converges to a limit $A_N^R(X,Y) \in L^2(P_{\alpha,\gamma})$. Similarly it can also be proved that $B_N^m(X,Y)$ converges to a limit $B_N^R(X,Y) \in L^2(P_{\alpha,\gamma})$ as $m \to \infty$. □

With $A_N^R(X,Y)$ and $B_N^R(X,Y)$ as defined above, in place of equations (3.1) and (3.2) we consider their regularized version, namely

$$\frac{\partial X_N}{\partial t} = A_N^R(X,Y), \tag{5.10}$$

$$\frac{\partial Y_N}{\partial t} = B_N^R(X,Y). \tag{5.11}$$

6. Dynamics

The dynamics of the model can be conveniently described in the Liouville–Koopman framework. Consider the Hilbert space $L^2(P_{\alpha,\gamma})$. Define the Liouville operator L as follows:

$$iL(X,Y) = \sum_K \left\{ A_K^R(X,Y) \frac{\partial}{\partial X_K} + B_K^R(X,Y) \frac{\partial}{\partial Y_K} \right\}. \tag{6.1}$$

With domain $\Im = \{F \in L^2(P_{\alpha,\gamma})/F$ depends on finitely many X_K's and Y_K's, is once continuously differentiable and vanishes at infinity$\}$.

Clearly $\Im$ is a dense subspace of $L^2(P_{\alpha,\gamma})$. L is a differential operator in countably infinite number of variables and is well-defined on $\Im$.

PROPOSITION 6.1

The operator L is a symmetric operator in $L^2(P_{\alpha,\gamma})$.

Proof. Consider $F, G \in \Im$ then

$$\int F \frac{\partial G}{\partial X_K} \mathrm{d}P_{\alpha,\gamma} = -\int G \left\{ \frac{\partial}{\partial X_K} - X_K \left(\frac{1}{\gamma} + \frac{\alpha}{K^2} \right) \right\} F \mathrm{d}P_{\alpha,\gamma} \tag{6.2}$$

and

$$\int F\frac{\partial G}{\partial Y_K}\mathrm{d}P_{\alpha,\gamma} = -\int G\left\{\frac{\partial}{\partial Y_K} - Y_K\left(\frac{1}{\gamma}+\frac{\alpha}{K^2+F}\right)\right\}F\mathrm{d}P_{\alpha,\gamma}. \tag{6.3}$$

From equations (3.5) and (3.6) it follows that

$$\sum_K\left\{X_K\left(\frac{1}{\gamma}+\frac{\alpha}{K^2}\right)A_K(X,Y) + Y_K\left(\frac{1}{\gamma}+\frac{\alpha}{K^2+F}\right)B_K(X,Y)\right\} = 0. \tag{6.4}$$

For each $K \in Z^2, A_K(X,Y)$ and $B_K(X,Y)$ are independent of X_K and Y_K. Hence from equations (6.2), (6.3) and (6.4) it follows that L is a symmetric operator. □

L is a well-defined symmetric operator, with $L^*1 = 0$. This implies that $P_{\alpha,\gamma}$ is locally invariant. If L is either self-adjoint or at least essentially self-adjoint, one could conclude that the probability measures $\{P_{\alpha,\gamma}\}$ for $\alpha, \gamma > 0$ are globally invariant. However all that can be proved is that it has self-adjoint extensions.

PROPOSITION 6.2

L is a symmetric operator with equal deficiency indices and therefore has self-adjoint extensions.

Proof. In order to prove that L has equal deficiency indices, it is enough to prove that there exists a conjugation operator J on the Hilbert space commutes with the operator L.

Define the conjugation $J : L^2(P_{\alpha,\gamma}) \to L^2(P_{\alpha,\gamma})$ by $J(f) = \bar{f}$ – the complex conjugate of the function. Then clearly $J(\Im) = \Im$ and $AJ = JA$. This implies that A has equal deficiency indices and therefore self adjoint extensions. □

The main result of this investigation can be stated as

Theorem 6.1 *There exists a two-parameter family of probability measures $\{P_{\alpha,\gamma}\}\alpha, \gamma > 0$ on $H_{-\infty}$, the extended phase space of the quasi-geostrophic two-layer model, which are locally invariant under evolution and if $\gamma_1 \neq \gamma_2$ then $P_{\alpha,\gamma} \perp P_{\alpha,\gamma}$.* □

7. Discussion

While we have presented a construction of a class of locally invariant probability measures for a two-layer quasi-geostrophic fluid, it should be noted that in a sense these measures are trivial and cannot be expected to reveal any interesting features that may be observed in real flows. Clearly the barotropic and baroclinic modes are statistically independent. Further the unrenormalized energy is almost surely infinite. This is to some extent quenched by renormalizing it. The renormalization of energy depends on the pseudo-temperature γ. What has not been noticed is that the total enstrophy is also almost surely infinite. Again note the fact that the enstrophy is equipartitioned among the modes. Except for mathematical difficulties involved, there is really no good ground for not considering other functions of vortices. All quantities of the form $\int \omega^k \mathrm{d}x, k \in \mathbb{N}$ are also conserved. It is conceivable that non-quadratic conserved quantities may reveal more interesting phenomena, as in the case of quantum fields.

There have been various attempts to link these states with geostrophic turbulence (see [16]). This link is extremely tenuous as the system considered has no dissipation at all.

These locally invariant statistical states are distinct from the statistical stationary solutions considered in [20].

After completing this work we became aware of the work carried out in [15] on 2-D Euler flows, where the entire emphasis is based on obtaining a descritization.

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