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CAUCHY TYPE GENERALIZATIONS OF HOLOMORPHIC MEAN VALUE THEOREMS

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Abstract. We extend the results on the Mean Value Theorem obtained by Flett, Myers, Sahoo, Cakmak and Tiryaki to holomorphic functions.

1. Introduction

In this note, we extend upon some variants of mean value theorems in the real variable case to holomorphic function in the spirit of Evard and Jaffari [3]. It is well known that the natural extension of mean value theorems to the case of holomorphic case do not hold, for example Rolle's theorem does not hold for $f(z) = e^z - 1$, for $z \in [0, 2\pi i]$ (line joining 0 and $2\pi i$). Our main inputs here are Cauchy type extensions of Flett's theorem and Myers's theorem for a pair of functions in the real variable case.

We note that Flett's theorem [4, 12] has received a lot of attention recently (see for example [9, 10, 7, 11, 5]), where as, its twin, the Myers's theorem (see [8]) did not get that much. One possible explanation could be that if a result holds in reference to one end point of an interval, it is natural to guess that an identical (or similar) result holds in reference to the other end point. However, in this note, we use these two results side by side. We note that these variants of mean value theorems have found applications in solving functional equations (see for example [12]).

This is a continuation of work in [6]. In this note, instead of a single function of a single variable, we consider a pair of functions of a single variable (in both real and complex variables).

In the second section we state and prove the basic results that form input for the main results in Section 3. In the third section we also prove a Cauchy type extension of the standard Cauchy mean value theorem. Extensions of results of both Davit et al, and Cakmak-Tiryaki [1] are also obtained there for a pair of holomorphic functions. All these extensions are in the spirit of Evarad-Jaffari.

Throughout this note we denote by $[a, b]$ the line segment joining a and b (both endpoints included) in the appropriate space ($\mathbb{R}$ or $\mathbb{C}$). In the same spirit we define (a, b), $[a, b)$, $(a, b]$. For a complex valued function f, its real and imaginary parts are denoted by $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ respectively.

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2. Basic results

We start with Flett's theorem [4, 12].

Theorem 2.1 ([4]). *If $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ and that $f'(a) = f'(b)$, then there exists a $c \in (a, b)$ such that*

$$f(c) - f(a) = (c - a)f'(c).$$

The next one is a slight modification of the above result, known as Myers's theorem [8].

Theorem 2.2 ([8]). *If $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ and that $f'(a) = f'(b)$, then there exists a $c \in (a, b)$ such that*

$$f(b) - f(c) = (b - c)f'(c).$$

However, the conditions $f'(a) = f'(b)$ can be dropped to obtain more general results: the first one in the below is a generalization of Flett's result and is due to Sahoo and Riedel and the next one is a generalisation of Myers's result, which we call Cakmak-Tiryaki theorem.

Theorem 2.3 ([12]). *Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$. Then there exists a $c \in (a, b)$ such that*

$$f(a) - f(c) = (a - c)f'(c) + \frac{1}{2}\frac{f'(b) - f'(a)}{b - a}(c - a)^2.$$

Theorem 2.4 ([1]). *Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$. Then there exists a $c \in (a, b)$ such that*

$$f(b) - f(c) = (b - c)f'(c) + \frac{1}{2}\frac{f'(b) - f'(a)}{b - a}(b - c)^2.$$

Now we are ready to prove Cauchy type generalization for a pair of functions, which is an extension of Sahoo-Riedel theorem.

Theorem 2.5. *If $f, h : [a, b] \to \mathbb{R}$ are two differentiable functions on $[a, b]$, then there exists a $c \in (a, b)$ such that*

$$\begin{aligned} &[h(b) - h(a)]h'(b)\{f(c) - f(a) - (c - a)f'(c)\} \\ &= [f'(b) - f'(a)][h(c) - h(a)]\Big\{\frac{1}{2}[h(c) - h(a)] - h'(c)(c - a)\Big\} \end{aligned} \tag{2.1}$$

Proof. Let

$$g(x) = [h(b) - h(a)]f(x)h'(b) - \frac{1}{2}[f'(b) - f'(a)][h(x) - h(a)]^2.$$

Then

$$g'(x) = [h(b) - h(a)]f'(x)h'(b) - [f'(b) - f'(a)][h(x) - h(b)]h'(x).$$

Now it is easy to check that

$$g'(a) = g'(b) = [h(b) - h(a)]f'(a)h'(b).$$

So applying Flett's theorem 2.1 for g and substituting the expression of g in terms of f and h we get the asserted result. □

The next result is an extension of Cakmak-Tiryaki Theorem for a pair of functions.

Theorem 2.6. *If $f, h : [a, b] \to \mathbb{R}$ are two differentiable functions on $[a, b]$, then there exists a $c \in (a, b)$ such that*

$$\begin{aligned}&[h(b) - h(a)]h'(a)\{f(b) - f(c) - (b - c)f'(c)\}\\&= [f'(b) - f'(a)][h(c) - h(b)]\big\{\frac{1}{2}[h(b) - h(c)] - h'(c)(b - c)\big\}\end{aligned} \tag{2.2}$$

Proof. Let

$$g(x) = [h(b) - h(a)]f(x)h'(a) - \frac{1}{2}[f'(b) - f'(a)][h(x) - h(b)]^2.$$

Then

$$g'(x) = [h(b) - h(a)f'(x)h'(a) - [f'(b) - f'(a)][h(x) - h(b)]h'(x).$$

Now it is easy to check that

$$g'(a) = g'(b) = [h(b) - h(a)]f'(b)h'(a).$$

So applying Myers's theorem 2.2 for g and substituting the expression of g in terms of f and h we get the asserted result. □

Remark 2.7. By setting $h(x) = x$ in the Theorem 2.5 (Theorem 2.6 respectively), we obtain the assertions of Theorem 2.3 (Theorem 2.4 respectively).

3. Mean Value Theorem for holomorphic functions

In this section, in the spirit of Evard-Jaffari (see [3]), we will prove some mean value theorems for holomorphic functions, which are extensions of results of Davitt et al. (see [2]), and that of Cakmak-Tiryaki. We need the following Rolle's type result on holomorphic functions due to Evard and Jafari, to prove a complex version of Cauchy type mean value theorem.

Theorem 3.1 ([3, 12])**.** *Let f be holomorphic on a convex open domain D of $\mathbb{C}$. Let $a, b \in D$ with $a \neq b$ such that $f(a) = f(b) = 0$. Then there exists $z_1, z_2 \in (a, b)$ such that $\operatorname{Re} f'(z_1) = 0 = \operatorname{Im} f'(z_2)$.*

First we prove a Cauchy type of result for a pair of holomorphic functions.

Theorem 3.2. *Let f and h be holomorphic on a convex open domain $D \subset \mathbb{C}$. Let $a, b \in D$ be such that $a \neq b$. Then there exists $z_1, z_2 \in D$ such that*

$$\operatorname{Re}\{f'(z_1)[h(b) - h(a)]\} = \operatorname{Re}\{h'(z_1)[f(b) - f(a)]\} \tag{3.1}$$

and

$$\operatorname{Im}\{f'(z_2)[h(b) - h(a)]\} = \operatorname{Im}\{h'(z_2)[f(b) - f(a)]\} \tag{3.2}$$

Proof. Let

$$g(z) = [f(z) - f(a)][h(b) - f(a)] - [h(z) - h(a)][f(b) - f(a)].$$

Then $g'(z) = f'(z)[h(b)-h(a)]-h'(z)[f(b)-f(a)]$. Since $g(a) = g(b) = 0$, from the above theorem we conclude that there exists $z_1, z_2 \in (a, b)$ such that $\operatorname{Re} g'(z_1) = 0 = \operatorname{Im} g'(z_2)$, which upon expanding yields the identities (3.1) and (3.2). □

The following two theorems are holomorphic versions of Theorems 2.3 and 2.4. Our next objective is to extend these theorems for a pair of the holomorphic functions. Here we use the following notation: for any $z, \omega \in \mathbb{C}$, we denote $\langle z, \omega\rangle$ by

$$\langle z, \omega\rangle = \operatorname{Re}(z\overline{\omega}) = \operatorname{Re}(z)\operatorname{Re}(\omega) + \operatorname{Im}(z)\operatorname{Im}(\omega). \tag{3.3}$$

Theorem 3.3 ([2]). *Let f be a holomorphic function defined on an open convex subset D of $\mathbb{C}$. Let a and b be two distinct points in D. Then there exists $z_1, z_2 \in D$ such that, in accordance with* (3.3)*,*

$$\operatorname{Re}(f'(z_1)) = \frac{\langle b-a, f(z_1)-f(a)\rangle}{\langle b-a, z_1-a\rangle} + \frac{1}{2}\frac{\operatorname{Re}(f'(b)-f'(a))}{b-a}(z_1-a) \tag{3.4}$$

and

$$\operatorname{Im}(f'(z_2)) = \frac{\langle b-a, -i[f(z_2)-f(a)]\rangle}{\langle b-a, z_2-a\rangle} + \frac{1}{2}\frac{\operatorname{Im}(f'(b)-f'(a))}{b-a}(z_2-a) \tag{3.5}$$

Theorem 3.4 ([1]). *Let f be a holomorphic function defined on an open convex subset D of $\mathbb{C}$. Let a and b be distinct points of D. Then there exists $z_1, z_2 \in D$ such that, in accordance with* (3.3)*,*

$$\operatorname{Re}(f'(z_1)) = \frac{\langle b-a, f(z_1)-f(b)\rangle}{\langle b-a, z_1-b\rangle} + \frac{1}{2}\frac{\operatorname{Re}(f'(b)-f'(a))}{b-a}(z_1-b) \tag{3.6}$$

and

$$\operatorname{Im}(f'(z_2)) = \frac{\langle b-a, -i[f(z_2)-f(b)]\rangle}{\langle b-a, z_2-b\rangle} + \frac{1}{2}\frac{\operatorname{Im}(f'(b)-f'(a))}{b-a}(z_2-b) \tag{3.7}$$

Theorem 3.5. *Let f and h be holomorphic on a convex open domain $D \subset \mathbb{C}$. Then there exist $z_1, z_2 \in (a,b)$ such that (i)*

$$\begin{aligned}
&[\operatorname{Re}(h'(a))]\langle b-a, h(b)-h(a)\rangle\Big\{\frac{\langle b-a, f(b)-f(z_1)\rangle}{\langle b-a, b-z_1\rangle} - \operatorname{Re}(f'(z_1))\Big\}\\
&= \{\operatorname{Re}[f'(b)-f'(a)]\}\langle b-a, h(z_1)-h(b)\rangle\\
&\quad\times\Big\{\frac{1}{2}\frac{\langle b-a, h(b)-h(z_1)\rangle}{\langle b-a, b-z_1\rangle} - \operatorname{Re}(h'(z_1))\Big\}
\end{aligned} \tag{3.8}$$

and (ii)

$$\begin{aligned}
&[\operatorname{Re}(h'(b))\langle b-a, h(b)-h(a)\rangle]\Big\{\frac{\langle b-a, f(a)-f(z_2)\rangle}{\langle b-a, a-z_2\rangle} - \operatorname{Re} f'(z_2)\Big\}\\
&= \{\operatorname{Re}[f'(b)-f'(a)]\}\langle b-a, h(z_2)-h(a)\rangle\\
&\quad\times\Big\{\frac{1}{2}\frac{\langle b-a, h(a)-h(z_2)\rangle}{\langle b-a, a-z_2\rangle} - \operatorname{Re}(h'(z_2))\Big\}
\end{aligned} \tag{3.9}$$

Proof. Let $\operatorname{Re}(a) = a_1$, $\operatorname{Im}(a) = a_2$, $\operatorname{Re}(b) = b_1$, $\operatorname{Im}(b) = b_2$, $\operatorname{Re}(f) = u$, $\operatorname{Im}(f) = v$, $\operatorname{Re}(h) = u_1$, $\operatorname{Im}(h) = v_1$. For $t \in [0,1]$, define

$$\begin{gathered}
\phi(t) = (b_1-a_1)u(a+t(b-a)) + (b_2-a_2)v(a+t(b-a)),\\
\psi(t) = (b_1-a_1)u_1(a+t(b-a)) + (b_2-a_2)v_1(a+t(b-a)).
\end{gathered}$$

Then by Theorem 2.6 there exists a $c \in (0,1)$ such that

$$\begin{aligned}
&[\psi(1)-\psi(0)]\psi'(0)[\phi(1)-\phi(c)-(1-c)\phi'(c)]\\
&= [\phi'(1)-\phi'(0)][\psi(c)-\psi(1)]\Big\{\frac{1}{2}(\psi(1)-\psi(c))-(1-c)\psi'(c)\Big\}
\end{aligned} \tag{3.10}$$

By taking $z = a+t(b-a)$, $z_1 = a+c(b-a)$ and using (3.3), we note that the functions ϕ and ψ satisfy the following properties:

$$\begin{gathered}
\phi(1) = \langle b-a, f(b)\rangle, \quad \psi(1) = \langle b-a, h(b)\rangle,\\
\phi(0) = \langle b-a, f(a)\rangle, \quad \psi(0) = \langle b-a, h(a)\rangle,
\end{gathered}$$

$$
\begin{gathered}
\phi'(t)=|b-a|^2\operatorname{Re}(f'(z)),\quad \psi'(t)=|b-a|^2\operatorname{Re}(h'(z)),\\
\psi'(0)=|b-a|^2\operatorname{Re}(h'(a)),\quad \phi'(1)=|b-a|^2\operatorname{Re}(f'(b)),\\
\phi'(0)=|b-a|^2\operatorname{Re}(f'(a)),\quad \psi(1)-\psi(0)=\langle b-a,h(b)-h(a)\rangle,\\
\phi'(1)-\phi'(0)=|b-a|^2\operatorname{Re}[f'(b)-f'(a)],\\
(1-c)\phi'(c)=\langle b-a,b-z_1\rangle\operatorname{Re}(f'(z_1)),\\
(1-c)\psi'(c)=\langle b-a,b-z_1\rangle\operatorname{Re}(h'(z_1)),\quad c\phi'(c)=\langle b-a,z_1-a\rangle\operatorname{Re}(f'(z_1)),\\
c\psi'(c)=\langle b-a,z_1-a\rangle\operatorname{Re}(h'(z_1)),\quad \psi(1)-\psi(c)=\langle b-a,h(b)-h(z_1)\rangle.
\end{gathered}
$$

Upon substituting these expressions in (3.10) yields (3.8).

Similarly, applying Theorem 2.5 for the pair of functions ϕ and ψ there exists a $c_1\in(0,1)$ such that

$$
\begin{aligned}
&[\psi(1)-\psi(0)]\psi'(1)[\phi(c_1)-\phi(0)-c_1\phi'(c_1)]\\
&=[\phi'(1)-\phi'(0)][\psi(c_1)-\psi(0)]\big\{\frac{1}{2}(\psi(c_1)-\psi(0))-c_1\psi'(c_1)\big\},
\end{aligned}
$$

and then upon utilizing the above listed properties of ϕ and ψ and setting $z_2=a+c_1(b-a)$, we obtain (3.9). □

Corollary 3.6. *Let f and h be holomorphic on a convex open domain $D\subset\mathbb{C}$. Then there exist $z_1,z_2\in(a,b)$ such that: (i)*

$$
\begin{aligned}
&[\operatorname{Im}(h'(a))]\langle b-a,-i[h(b)-h(a)]\big\{\frac{\langle b-a,-i[f(b)-f(z_1)]\rangle}{\langle b-a,b-z_1\rangle}-\operatorname{Im}(f'(z_1))\big\}\\
&=\{\operatorname{Im}[f'(b)-f'(a)]\}\langle b-a,-i[h(z_1)-h(b)]\rangle \qquad (3.11)\\
&\quad\times\big\{\frac{1}{2}\frac{\langle b-a,-i[h(b)-h(z_1)]\rangle}{\langle b-a,b-z_1\rangle}-\operatorname{Im}(h'(z_1))\big\}
\end{aligned}
$$

and (ii)

$$
\begin{aligned}
&[\operatorname{Im}(h'(b))\langle b-a,-i[h(b)-h(a)]\rangle]\big\{\frac{\langle b-a,-i[f(a)-f(z_2)]\rangle}{\langle b-a,a-z_2\rangle}-\operatorname{Im}(f'(z_2))\big\}\\
&=\{\operatorname{Im}[f'(b)-f'(a)]\}\langle b-a,-i[h(z_2)-h(a)]\rangle \qquad (3.12)\\
&\quad\times\big\{\frac{1}{2}\frac{\langle b-a,-i[h(a)-h(z_2)]\rangle}{\langle b-a,a-z_2\rangle}-\operatorname{Im}(h'(z_2))\big\}
\end{aligned}
$$

Proof. Define $f_1=-if$ and $h_1=-ih$ and note that $\operatorname{Re}f_1'(z)=\operatorname{Im}f'(z)$ and $\operatorname{Re}h_1'(z)=\operatorname{Im}h'(z)$. Now the required results follow at once by applying the Theorem 3.5 to the pair f_1 and h_1 and rewriting them in terms of f and h. □

Remark 3.7. By setting $h(z)=z$ in the above theorem and corollary, and noting that $\frac{b-z}{b-a}=\frac{\langle b-a,b-z\rangle}{\langle b-a,b-a\rangle}$ and $\frac{a-z}{b-a}=\frac{\langle b-a,a-z\rangle}{\langle b-a,b-a\rangle}$ for all $z\in(a,b)$, one gets the results of the theorems 3.3 and 3.4.

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