

Note on a Partition Function Which Assumes All Integral Values

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Abstract Let $G(n)$ denote the number of partitions of n into distinct parts which are of the form $2m, 3m, 5m, 6m-3, 8m-3, 9m-3$ or $11m-3$ with parts of the form $2m, 3m, 6m-3$, or $11m-3$ being even in number minus the number of them with parts of the form $2m, 3m, 6m-3$, or $11m-3$ being odd in number. In this paper, we prove that $G(n)$ assumes all integral values and does so infinitely often.

Keywords: partition functions, Gaussian integers, Jacobi's triple product identity

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1. Introduction

Let $S(n)$ denote the number of partitions of n into distinct parts with even rank minus the number with odd rank (see [2]). Andrew, Dyson and Hickerson [3] used the arithmetic of $\mathbb{Q}(\sqrt{6})$ to show that $S(n)$ takes on every integral value infinitely often. This is the first time the interaction between the theory of partitions and algebraic number theory was exhibited. It was remarked in [3] that they know of no other partition function in the literature which assumes all integral values as $S(n)$ does.

Let $H(n)$ denote the number of partitions of n into parts which are repeated exactly 1, 3, 4, 6, 7, 9, or 10 times with the parts repeated exactly 1, 4, 6, or 9 times even in number minus the number of them with parts repeated exactly 1, 4, 6, or 9 times odd in number. In [5], using the arithmetic of Gaussian integers $\mathbb{Z}[i]$, it was shown that $H(n)$ assumes all integral values and does so infinitely often.

Let $G(n)$ denote the number of partitions of n into distinct parts which are of the form $2m, 3m, 5m, 6m-3, 8m-3, 9m-3$, or $11m-3$ with parts of the form $2m, 3m, 6m-3$, or $11m-3$ being even in number minus the number of them with parts of the form $2m, 3m, 6m-3$, or $11m-3$ being odd in number. For example, $G(7)$ is zero because $(2(2)) + (3(1))$, $(2(2)) + (6(1)-3)$ have even number of parts of the form $2m, 3m, 6m-3$, or $11m-3$. while $(2(1)) + (5(1))$ and $(2(1)) + (8(1)-3)$ are odd number of parts of the form $2m, 3m, 6m-3$, or $11m-3$ (here m -values are shown in bold). In this paper, we show that $G(n)$ assumes all integral values and does so infinitely often.

2. Main Results

A For (positive) integer n , consider the equation

$$u^2 + v^2 = 24n + 2 \quad (2.1)$$

We call a solution (u, v) of (2.1) admissible if $u \equiv 1 \pmod{6}$ and $v \equiv 1 \pmod{6}$. For a (positive) integer $n \equiv 2 \pmod{24}$, let $J(n)$ be the excess of the number of admissible solutions of $u^2 + v^2 = n$ with $v \equiv 1 \pmod{12}$ over the number of them with v not congruent to 1 modulo 12.

In subsequent sections, we shall be proving the following:

Theorem 1. For $n \geq 0$, $G(n) = J(24n + 2)$.

Theorem 2. $G(n)$ takes on every integer value infinitely often.

3. Proof of Theorem 1

First we note that the generating function of $G(n)$ is

$$\sum_{n \geq 0} G(n)q^n = \prod_{n \geq 1} \left(\frac{1 - q^{2n} - q^{3n} + q^{5n} - q^{6n-3}}{+q^{8n-3} + q^{9n-3} - q^{11n-3}} \right)$$

Lemma 1. For $|q| < 1$,

$$\prod_{n \geq 1} \left(\frac{1 - q^{2n} - q^{3n} + q^{5n} - q^{6n-3}}{+q^{8n-3} + q^{9n-3} - q^{11n-3}} \right) = \sum_{n, m \in \mathbb{Z}} (-1)^m q^{\frac{3}{2}(n^2 + m^2) + \frac{1}{2}(n+m)}$$

Proof. Using Jacobi's triple product identities (see [1], p. 21) we get

$$\begin{aligned} & \sum_{n,m \in \mathbb{Z}} (-1)^m q^{\frac{3}{2}(n^2+m^2)} + \frac{1}{2}(n+m) \\ &= \left(\sum_{n \in \mathbb{Z}} q^{\frac{3}{2}n^2 + \frac{1}{2}n} \right) \left(\sum_{m \in \mathbb{Z}} (-1)^m q^{\frac{3}{2}m^2 + \frac{1}{2}m} \right) \\ &= \prod_{n \geq 1} \left((1-q^{3n}) (1+q^{3n-1}) (1+q^{3n-2}) \right) \\ &\times \prod_{n \geq 1} \left((1-q^{3n}) (1-q^{3n-1}) (1-q^{3n-2}) \right) \\ &= \prod_{n \geq 1} \left((1-q^{3n})^2 (1+q^{6n-2}) (1+q^{6n-4}) \right) \\ &= \prod_{n \geq 1} \left((1-q^{3n}) (1-q^{2n}) (1-q^{6n-3}) \right) \\ &= \prod_{n \geq 1} \left(\begin{matrix} 1-q^{2n} - q^{3n} + q^{5n} - q^{6n-3} \\ +q^{8n-3} + q^{9n-3} - q^{11n-3} \end{matrix} \right) \\ &= \sum_{n \geq 0} G(n)q^n. \end{aligned}$$

This proves the Lemma.

Using this lemma, it follows that:

$$\begin{aligned} & \sum_{n \geq 0} G(n)q^{24n+2} \\ &= q^2 \prod_{n \geq 1} \left(\begin{matrix} 1-q^{48n} - q^{72n} + q^{120n} - q^{144n-72} \\ +q^{192n-72} + q^{216n-72} - q^{264n-72} \end{matrix} \right) \\ &= q^2 \sum_{n,m \in \mathbb{Z}} (-1)^m q^{36(n^2+m^2)+12(n+m)} \\ &= \sum_{n,m \in \mathbb{Z}} (-1)^m q^{(6n+1)^2 + (6m+1)^2} \\ &= \sum_{n \geq 0} J(24n+2)q^{24n+2}. \end{aligned}$$

This proves Theorem 1.

4. Arithmetic of J(n)

In this section we study $J(n)$ using Gaussian integers $\mathbb{Z}[i]$, where $i = \sqrt{-1}$. For $\alpha = u + iv \in \mathbb{Z}[i]$, let $N(u + iv) = u^2 + v^2$. We define $c_4(\alpha)$ in terms of $u \pmod{4}$ and $v \pmod{4}$ by

Table 1. Values of $c_4(\alpha)$

		$v \pmod{4}$			
		0	1	2	4
$u \pmod{4}$	0	0	i	0	-i
	1	1	0	-1	0
	2	0	-i	0	i
	4	-1	0	1	0

Let $c_3(\alpha)$ be defined in terms of $u \pmod{3}$ and $v \pmod{3}$ by the following table, where $\omega = (1+i) / \sqrt{2}$:

Table 2. Values of $c_3(\alpha)$

		$v \pmod{3}$		
		0	1	2
$u \pmod{3}$	0	0	-i	i
	1	1	ω^5	ω^7
	2	-1	ω^3	ω

Let $C(\alpha) = c_3(\alpha)c_4(\alpha)$ and let

$$G(n) = \sum_{N(\alpha)=n} c(\alpha),$$

where the sum is over a complex set non-associate Gaussian integers with norm n.

Let $P = \{\alpha = u + iv \in \mathbb{Z}[i] : u \equiv 1 \pmod{6} \text{ and } v \equiv 1 \pmod{6}\}$ and let $Q = \{\beta = r + is \in \mathbb{Z}[i] : (1+i)\beta \in P \text{ or } (1-i)\beta \in P\}$. Then, for $n \equiv 1 \pmod{12}$,

$$J(2n) = \sum_{N(\alpha)=2n} \alpha \in P (-1)^{(v-1)/2},$$

(where v is the imaginary part of α)

$$= \sum_{N(\beta)=n} \beta \in Q (c(\beta) + c(\bar{\beta})),$$

(where $\bar{\beta}$ is the conjugate of β)

$$= C(n).$$

Note that this together with Theorem 1 proves the assertion made in the Remark 2 of [5].

Thus we have shown that:

Lemma 2. For $n \equiv 1 \pmod{12}$, $J(2n) = C(n)$.

Next we recall the properties of $C(n)$ from [5].

Lemma 3. (a) The function $C(n)$ is multiplicative.

(b) $C(n) = 0$ unless $n \equiv 1$ or $5 \pmod{12}$.

Lemma 4. Let p be a prime $\equiv 1 \pmod{12}$ and $n \geq 1$.

Then:

- a. $C(n)$ is either 0, 2 or -2.
- b. If $C(p) = 0$, then

$$C(p^n) = \begin{cases} (-1)^{n/2} & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

c. If $C(p) = \pm 2$, then

$$C(p^n) = \begin{cases} (n+1) & \text{if } C(p) = 2 \\ (-1)^n(n+1) & \text{if } C(p) = -2 \end{cases}$$

Lemma 5. Let p be a prime $\equiv 5 \pmod{12}$ and $n \geq 1$ be even. Then:

$$C(p^n) = \begin{cases} (-1)^{n/4} & \text{if } n \equiv 0 \pmod{4} \\ (-1)^{(n+2)/4} & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

5. Proof of Theorem 2

As in [5],

$$\begin{aligned}
G\left(\frac{61^{k-1}13^{2m}-1}{12}\right) &= C\left(61^{k-1} \cdot 13^{2m}\right) \\
&\text{(by Theorem 1 and Lemma 2)} \\
&= C\left(61^{k-1}\right)C\left(13^{2m}\right) \text{ (by Lemma 3(a))} \\
&= k(-1)^m \\
&\text{(by Lemma 4 (b) and (c))}
\end{aligned}$$

and [[5], Table 1])

$$= \begin{cases} k & \text{if } m \text{ is even} \\ -k & \text{if } m \text{ is odd.} \end{cases}$$

This proves Theorem 2.

6. Conclusion

An arithmetical function $f(n)$ is called *lacunary* if it is almost always 0 (see[4]). In [3] it is shown that $S(n)$ is lacunary. In [5] it is shown that $H(n)$ is lacunary. So is natural to ask whether $G(n)$ is so. We make the following conjecture:

Conjecture. $G(n)$ is lacunary.

References

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