GENERALIZED THEORY OF DIFFERENTIAL EQUATIONS

By

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D E C L A R A T I O N

I declare that this thesis has been composed by me and has not previously formed the basis for the award of any Degree, Diploma or any other similar title.

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C E R T I F I C A T E

Certified that the thesis entitled " Generalized Theory of Differential Equations" by Jayasree K.N. has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

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CONTENTS

INTRODUCTION

Theory of differential equations is a rich area of research. Several new contributions continue to enrich it further. New analytical methods, including application of abstract mathematics to study the qualitative properties of solutions, in the absence of their explicit representations are being developed by differential equationists. In addition, the applications to physical problems yield new types of equations. In such situations it is necessary to generalize the existing theory to study the qualitative properties of new equations. The present thesis is an attempt to further this process so as to include the differential equations involving piecewise constant delays.

Consider the initial value problem studied in [8,26]

$$x'(t) = f(t,x(t)) , \quad x(t_o) = x_o , \qquad x' = \frac{dx}{dt} .$$

Here, the derivative of solution x(t) depends on t and x(t) (the present state of x). For quite a long time, it was assumed that the physical systems are determined only by present state. However, the experience has shown that this assumption is only a first approximation to the exact situation and for a better approximation, one needs to take into account the past history of the system. The continuous flow of contributions [14, 16, 22, 25] to equations involving past history during the last four decades is due to two reasons. Firstly, such equations are faithful mathematical representations of physical phenomena in several sciences and secondly, these equations are mathematically rich and therefore, need many analytical tools to exhibit their complex behaviour.

The study of differential equations involving piecewise constant arguments, the subject of the present thesis, has been initiated recently by Cooke and Wiener [9, 10, 11]. This area of research is also now extended by the contributions of Aftabizadeh and Wiener [2, 3, 4, 35] and Shah and Wiener [27, 34].

There are very few research papers published so far on differential equations with piecewise constant arguments, since the work in this area began in the present decade. However, it will be noted through subsequent discussion that this area of research is rich in content and will attract large number of applications.

It is seen that equations with piecewise constant deviating arguments (PCDA) are closely related to impulse and loaded equations and, especially, to difference equations of a discrete argument. In fact, these equations have the structure of continuous dynamical systems within an interval of certain length. Continuity of a solution at a point joining any two consecutive intervals then implies recursion relation for the values of the solution at such points.

The equations with PCDA are similar in structure to those found in certain "sequential-continuous" models of disease dynamics treated by Busenberg and Cooke [6]. Thus, the so far known results show sufficient evidence of potential application of these kind of equations.

Now we give a brief history of the present problem:

In [9], Cooke and Wiener develop some theory of equations of the form

$$x'(t) = f(t, x(t), x(h(t))) \tag{0.1}$$

which are of retarded type. They have studied the equation (0.1) for $h(t) = [t]$, $h(t) = t-n[t]$, etc. Here the notation [ ] denotes greatest integer function.

Cooke and Wiener also consider equations with two PCDA [t] and [t-1] (refer [9]), and [t-1] and [t+1] (refer [11]). In a similar manner, the results in [27] include the theory of equations with delay argument [t] and advanced argument [t+1]. For equations with two PCDA, the instrument of continued fractions plays an important role in computation of solutions and in the study of asymptotic behaviour of solutions [34].

Recently, Cooke and Wiener [10] studied an interesting differential equation alternately of retarded and advanced type

$$x'(t) = ax(t)+bx(2[(t+1)/2]), \quad x(0) = c_o \,. \tag{0.2}$$

The argument deviation $\tau(t) = t-2[(t+1)/2]$ is negative in $[2n-1, 2n)$ and $\tau(t)$ is positive in $(2n, 2n+1)$, n is an integer. So the equation (0.2) is of advanced type on $[2n-1, 2n)$ and of retarded type on $(2n, 2n+1)$. They have established that all types of equations with PCDA have similar characteristics. Some stability properties have also been obtained in [9, 10].

Aftabizadeh and Wiener have studied the oscillatory properties of scalar equations in [1, 2] and system of two first order linear differential equations in [3] with PCDA. In [33], oscillatory and periodic properties of the solutions of a linear system of differential equations with the argument $[t+\frac{1}{2}]$ are studied. Some oscillation results for the solution of linear differential equation with argument $[t-1]$ have been obtained in [5].

Existence of a second order differential equation with PCDA is proved by Ladas, Parthemiadis and Evan [20]. In [4],the authors used monotone method to prove the existence of minimal and maximal solutions for the nonlinear differential equation (0.1) with PCDA $h(t) = [t]$.

As a generalization of the results in [10] Wiener and Aftabizadeh [35] discuss the existence and uniqueness of solutions of

$$x'(t) = f(t,x(m[(t+k)/m]), \quad x(0) = c_0$$

where k, m are positive integers and $k<m$.

Based on the results known so far in this area of research, the author of the present thesis has made some new contributions.

In the following chapters, we develop some basic tools needed in the study of qualitative properties of solutions. These include iterative method of finding solutions, variation of parameters formula, Gronwall type integral inequalities, oscillation result, stability, etc.

Taking the clue from the books [14, 16] and the results incorporated in [9, 10], we study the equations with two types of delays, namely, continuous delay and piecewise constant argument.

In the present thesis, the following types of equations are considered.

(i) $x'(t) = ax(t)+bx([t])$

(ii) $x'(t) = ax(t)+bx(2[(t+1)/2])$

(iii) $x'(t) = ax(t)+bx([t])+cx(t-\tau)$

(iv) $x'(t) = ax(t)+bx([t])+L(x_t)$

(v) $x'(t) = f(t,x(t),x([t]))$

(vi) $u_t(x,t) = au_{xx}(x,t)+bu_{xx}(x,[t])$

(vii) $u_{tt}(x,t) = a^2u_{xx}(x,t)+b^2u_{xx}(x,[t])$

(viii) $u_t(x,t) = au_{xx}(x,t)+bu_{xx}(x,2[(t+1)/2])$.

We give below a brief summary of each chapter included in this thesis.

In chapter 1, we organize together, as a prerequisite, some known relevant results, which have been published earlier. The results summarised are from the papers [9, 10, 1, 23] which include equations with constant coefficients involving PCDA, system of equations with PCDA, method of finding solutions of nonlinear equations, equations alternately of retarded and advanced type, some integral inequalities and variation of parameters formula. Many illstrative examples explaining the properties of solutions have been constructed which are not available elsewhere.

In chapter 2, we present some basic tools to study the qualitative properties of differential equations. In section 2.3, we prove the existence and uniqueness of solution for equation with PCDA by using Banach fixed point theorem. We develop a variation of parameters formula for linear equations with PCDA in section 2.4. An extension of Gronwall's integral inequality is made in section 2.5. Applications of the results proved in the earlier sections are given in section 2.6. Section 2.7 generalizes an oscillatory result for the case of a system of equations with PCDA. In the last two sections, we introduce some equations with two types of delays, namely, continuous delay and piecewise constant deviating argument.

In chapter 3, section 3.1 gives closed form solution of an equation alternately of retarded and advanced type with variable coefficients. Variation of parameters formula is also developed in section 3.2. In section 3.3, an integral inequality is proved.

In chapter 4, we introduce some second order partial differential equations having PCDA. In section 4.2, we prove an existence theorem in the framework of semigroup theory. Many illustrative examples are presented giving explicit solution by using method of seperation of variables. General diffusion equation and general wave equation are discussed in section 4.3 and section 4.4, respectively.

Chapter 5 deals with the study of some nonlinear differential equations involving PCDA. In section 5.2, we extend the method of finding solution of scalar equation to the case of system of equations. An existence theorem using Schauder's fixed point theorem is proved in section 5.3. Some comparison results are also included in this section. In section 5.4, we develop a nonlinear variation of parameters formula of Alekseev type [23].

Several illustrative examples have been constructed by the author to explain the theory. The thesis ends with a complete bibliography.

CHAPTER 1

SURVEY OF EXISTING LITERATURE

1.1 INTRODUCTION

Theory of ordinary differential equations with piecewise constant arguments has been studied during the last few years. It is seen that not many research papers in this area are published so far. However, this topic of research appears to be potentially rich, since some significant applications have been already noted.

The present chapter includes some relevant results, which have been published earlier. The results summarised here forms the pre-requisite of the new work that the author intends to include in the present thesis. The contribution of the author is appearing in chapters 2 to 5. Below we quote the results from the papers [9, 10, 1, 23] which include equations with constant coefficients involving piecewise constant argument, system of equations with variable coefficients, method of finding solutions of nonlinear equations, equation alternately of retarded and advanced type, some integral inequalities and variation of parameters formula.

It has been noted that not many illustrative examples have been included in the literature published so far in this area. The author felt that this gap needs to be filled up by adding suitable examples. This has been done throughout this chapter.

1.2 EQUATION WITH CONSTANT COEFFICIENTS [9]

Consider the scalar initial value problem (IVP)

$$y'(t) = ay(t)+by([t]), \quad t \in J = [0, \infty), \tag{1.1}$$

$$y(0) = c_o, \tag{1.2}$$

where a, b, c_o are real constants. Here [t] designates the greatest integer less than or equal to t.

This equation is very closely related to impulse and loaded equations. Indeed, write equation (1.1) as

$$y'(t) = ay(t)+\sum_{i=-\infty}^{\infty} by(i)(H(t-i)-H(t-i-1))$$

where H(t) = 1 for t>0 and H(t) = 0 for t<0. If we admit distributional derivatives, then by differentiating the latter equation, we get

$$y''(t) = ay'(t)+\sum_{i=-\infty}^{\infty} by(i)(\delta(t-i)-\delta(t-i-1))$$

where δ is the delta functional. This impulse equation contains the value of the unknown solution for the integral values of t.

<u>**DEFINITION 1.1:**</u> A solution of equation (1.1) on $J = [0,\infty)$ is a function y(t) that satisfies the conditions:

(i) y(t) is continuous on J,

(ii) the derivative y'(t) exists at each point $t \in J$ except possibly at integral points, there only one-sided derivatives exist,

(iii) equation (1.1) is satisfied on each interval $[n,n+1) \subset J$ with integral end points.

The solution of the IVP (1.1), (1.2) is given in the following theorem.

THEOREM 1.1: The unique solution of IVP (1.1), (1.2) is given by

$$y(t) = \lambda(t-[t])(\lambda(1))^{[t]}c_o, \quad t\in J \tag{1.3}$$

where

$$\lambda(t) = \exp(at)(1+a^{-1}b)-a^{-1}b .$$

REMARK 1.1: (i) If a = 0 in (1.1), then the solution (1.3) becomes

$$y(t) = (1+b(t-[t]))(1+b)^{[t]}c_o, \quad t\in J,$$

(ii) if b = 0 in (1.1), then (1.3) has the form

$$y(t) = \exp(at)c_o, \quad t\in J.$$

REMARK 1.2: (i) If $b = \frac{-a\exp(a)}{\exp(a)-1}$, then in view of (1.3), $y(t)\equiv 0$, $t\in J$,

(ii) if $b< \frac{-a\exp(a)}{\exp(a)-1}$, then the solution (1.3) is oscillatory,

(iii) if $b = \frac{a(1+\exp(a))}{1-\exp(a)}$, then the zero solution of (1.1) is stable,

(iv) if $\frac{a(1+\exp(a))}{1-\exp(a)} < b < -a$, $a > 0$, then the zero solution of (1.1) is asymptotically stable.

Next theorem establishes the fact that the IVP for equation (1.1) may be posed at any point, not necessarily at integer points.

THEOREM 1.2: If $\lambda(1) \neq 0$ and $t_o \in J$ is such that $\lambda(t_o-[t_o])\neq 0$, then equation (1.1) with the initial condition $y(t_o) = y_o$ has a unique solution on $(-\infty, \infty)$ given by

$$y(t) = \frac{\lambda(t-[t])}{\lambda(t_o-[t_o])}(\lambda(1))^{[t]-[t_o]}y_o,$$

where

$$\lambda(t) = \exp(at)\ (1+a^{-1}b)-a^{-1}b.$$

Clearly, for $t_o = 0$, $y(t)$ reduces to the solution (1.3).

It is interesting to study the oscillatory behaviour of solution of (1.1) which are caused by deviating arguments and which do not appear in the corresponding ordinary differential equation. Below we give a result which is proved in [1].

THEOREM 1.3: Consider the delay differential inequality

$$y'(t)+a(t)y(t)+b(t)y([t]) \leq 0, \quad t \in J, \tag{1.4}$$

where a and b are continuous functions on J. Assume that

$$\limsup_{n \to \infty} \int_n^{n+1} b(t)\exp\{\int_n^t a(s)ds\}dt > 1, \tag{1.5}$$

then (1.4) has no eventually positive solutions.

Under the same conditions it is proved that, if the assumption (1.5) is true, then

$$y'(t)+ a(t)y(t)+b(t)y([t]) \geq 0, \quad t \in J,$$

has no eventually negative solution.

It follows from these results that the delay differential equation

$$y'(t)+a(t)y(t)+b(t)y([t]) = 0$$

has oscillatory solutions only, if the condition (1.5) holds.

EXAMPLE 1.1: In the delay differential equation

$$y'(t) = y(t)-\exp(t)y([t])$$

the coefficients satisfy the condition (1.5). Hence the solution is oscillatory.

EXAMPLE 1.2: Consider the scalar equation (1.1) with $a = 0$ and $c_0 = 1$. The solution is given by

$$y(t) = (1+b(t-\lfloor t\rfloor))(1+b)^{\lfloor t\rfloor}, \quad t \in J.$$

This solution has interesting behaviour for various values of b. For any $b < -1$, the solution has infinity of isolated zeros on $\lfloor T,\infty)$, $T > 0$ and hence it is oscillatory. For all other values of b, solution y(t) is nonoscillatory.

The solution y(t) is asymptotically stable for $-2 < b < 0$, it is stable for $b = -2$ and unstable for all other values of b.

The nature of solution y(t) for some values of b is indicated in figures given in the next page.

1.3 SYSTEM WITH VARIABLE COEFFICIENTS [9]

This section deals with study of systems of differential equations of the form

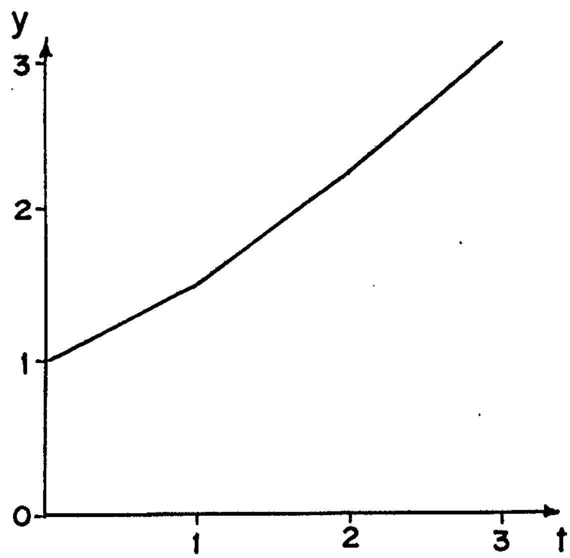
$$x'(t) = A(t)x(t), \tag{1.6}$$

$$y'(t) = A(t)y(t)+B(t)y(\lfloor t\rfloor), \tag{1.7}$$

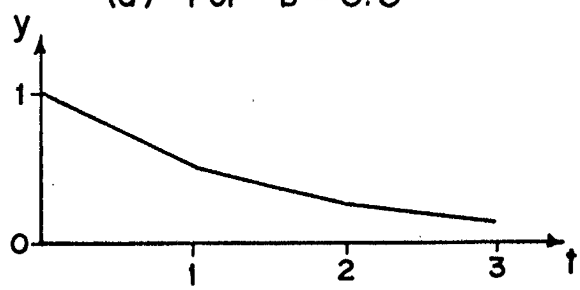
$t \in J$ with initial conditions

$$x(0) = y(0) = c_0, \tag{1.8}$$

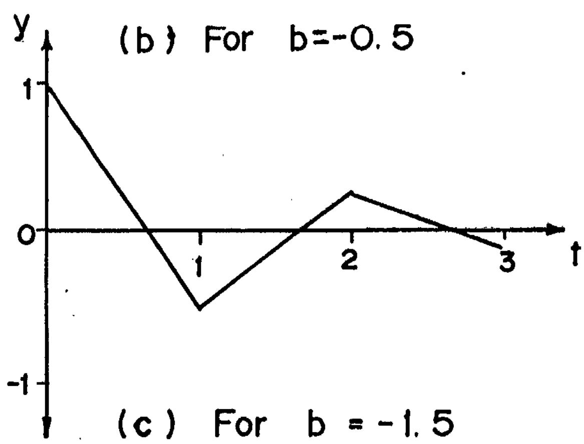
where A, B are n x n matrices with entries as real-valued continuous



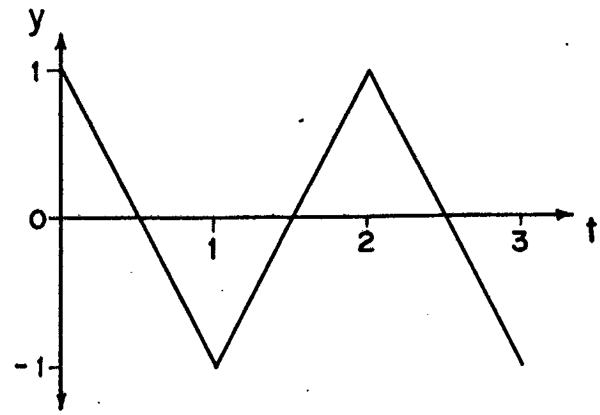
(a) For b = 0.5



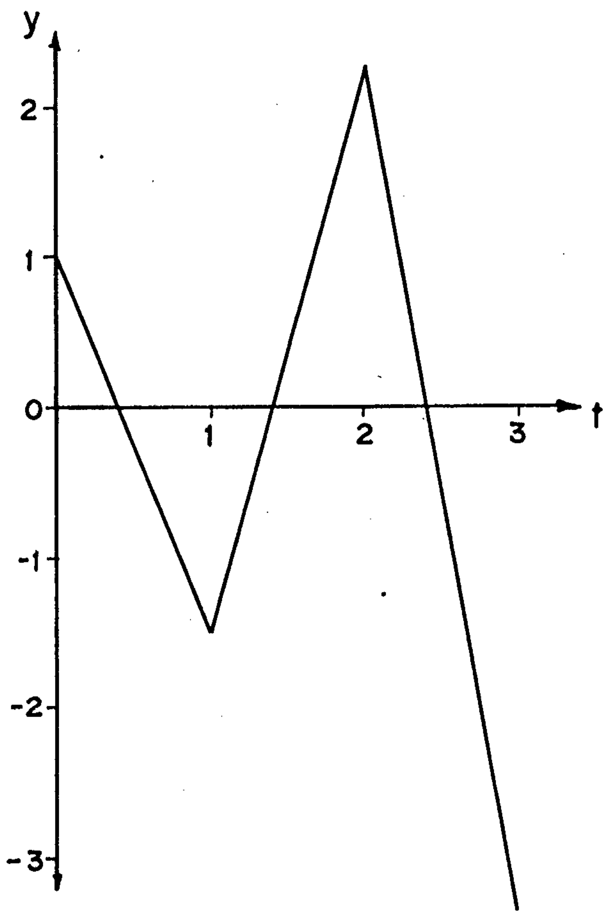
(b) For b=-0.5



(c) For b = -1.5



(d) For b = -2



(d) For b = -2.5

functions defined for $t \in J$, x, y are n-vectors and c_o is a real constant n-column vector.

The following theorem shows that the solution of the IVP (1.7), (1.8) can be expressed in terms of the fundamental matrix $\Phi(t)$ of (1.6).

THEOREM 1.4: Let A, B be n x n matrices with entries as real-valued continuous functions defined on J. Then there exists a unique solution of (1.7), (1.8) given by

$$y(t) = \left(\Phi(t,[t]) + \int_{[t]}^{t} \Phi(t,s)B(s)ds \right) c_{[t]}, \quad t \in J, \tag{1.9}$$

where

$$c_{[t]} = \prod_{k=[t]}^{1} \left(\Phi(k,k-1) + \int_{k-1}^{k} \Phi(k,s)B(s)ds \right) c_o, \tag{1.10}$$

$\Phi(t)$ is the fundamental matrix of (1.6), $\Phi(0) = E_n$, the identity matrix and $\Phi(t,s) = \Phi(t)\,\Phi^{-1}(s)$, $0 \le s \le t < \infty$.

EXAMPLE 1.3: Consider the system

$$y'(t) = Ay(t) + By(\lfloor t \rfloor), \quad y(0) = c_o,$$

with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad c_o = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

The fundamental matrix Φ in this case is given by

$$\Phi(t) = \begin{bmatrix} \exp(t) & 0 & 0 \\ 0 & \exp(t) & 0 \\ 0 & 0 & \exp(t) \end{bmatrix}$$

Using the relations (1.9) and (1.10), we get the solution

$$y(t) = \lambda(t-[t])(\lambda(1))^{[t]}c_o,$$

where

$$\lambda(t) = \begin{bmatrix} \exp(t) & 0 & \exp(t)-1 \\ 0 & 2\exp(t)-1 & \exp(t)-1 \\ \exp(t)-1 & 0 & \exp(t) \end{bmatrix}.$$

The following theorem provides us a pointwise bound for the solution of the IVP (1.7), (1.8). The estimate obtained here illustrates the nature and growth of the solution.

THEOREM 1.5: Let A, B be n x n matrices with entries as continuous functions and let $a(t) = \max|A(s)|$, $b(t) = \max|B(s)|$, $0 \leq s \leq t$, then the solution of (1.7), (1.8) satisfies the estimate

$$|y(t)| \leq \exp\{(t+1)a(t)\}(b(t)+1)^{t+1}|c_o|, \quad t \in J.$$

1.4 METHOD OF FINDING SOLUTIONS OF NONLINEAR EQUATIONS [9]

Let

$$x'(t) = f(x(t),x([t])),x(0) = c_o, \tag{1.11}$$

where c_o is a constant and f is a piecewise continuous function on $\mathbb{R} \times \mathbb{R}$.

If the system with non zero parameter λ is such that $f(x,\lambda) \neq 0$ everywhere, then

$$F(x, \lambda) = \int \frac{dx}{f(x, \lambda)} = t+g(\lambda),$$

where $g(\lambda)$ is an arbitrary function.

For computing solution of (1.11), assume that $x_n(t)$ is a solution of (1.11) in $[n, n+1)$ with the condition $x_n(n) = c_n$, $n = 0, 1, 2, \ldots$. By putting $\lambda = c_n$, we obtain

$$F(x_n(t), c_n) = t + g(c_n), \quad t \in [n, n+1).$$

Put $t = n$, to get

$$F(c_n, c_n) = n + g(c_n).$$

The two expressions above may be treated as equations for unknown $g(c_n)$. By solving them, we get

$$F(x_n(t), c_n) = F(c_n, c_n) + t - n. \tag{1.12}$$

Clearly, by taking limit as $t \to n+1$ in (1.12), we obtain

$$F(c_{n+1}, c_n) = F(c_n, c_n) + 1.$$

Here (1.12) gives explicit solution of (1.11) in each interval $[n, n+1)$.

EXAMPLE 1.4: Choose $f(x(t), x([t])) = -2x^{\frac{1}{2}}(t)x([t])$, $t \in J$ in the IVP (1.11). Let λ be a nonzero parameter, then the integral

$$F(x, \lambda) = \int \frac{dx}{-2x^{\frac{1}{2}}\lambda} = \frac{-x^{\frac{1}{2}}}{\lambda}.$$

The solution can be obtained using (1.12) as

$$x(t) = c_{[t]}(1 - (t-[t])c_{[t]}^{\frac{1}{2}})^2, \quad t \in J,$$

where

$$c_{[t]} = c_{[t]-1}(1 - c_{[t]-1}^{\frac{1}{2}})^2.$$

REMARK 1.3: If $c_0 = 1$, $x(t)$ tends to zero as t tends to 1 and $x(t) = 0$ for $t \geq 1$.

1.5 EQUATION ALTERNATELY OF RETARDED AND ADVANCED TYPE [10]

Another interesting class of differential equations, which have been studied recently is of the form

$$y'(t) = ay(t) + by(2[(t+1)/2]), y(0) = c_0, \quad t \in J, \qquad (1.13)$$

where a, b, c_0 are constants. The argument deviation $\tau(t) = t - 2[(t+1)/2]$ is negative for $2n-1 \leq t < 2n$ and positive for $2n < t < 2n+1$ (n is an integer). Equation (1.13) is of advanced type on $[2n-1, 2n)$ and of retarded type on $(2n, 2n+1)$.

DEFINITION 1.2: A solution of equation (1.13) on J is a function y(t) that satisfies the conditions:

(i) y(t) is continuous on J,

(ii) the derivative y'(t) exists at each point $t \in J$, with possible exceptions of the points $t = 2n-1$, $n = 1, 2, 3 \ldots$ where only one-sided derivative exists,

(iii) equation (1.13) satisfies on each interval $2n-1 \leq t < 2n+1$, $n = 1, 2, 3 \ldots$.

A closed form solution for the problem (1.13) is obtained in the following result.

THEOREM 1.6: The IVP (1.13) has a unique solution given by

$$y(t) = \lambda(t-2[(t+1)/2])\left(\frac{\lambda(1)}{\lambda(-1)}\right)^{[(t+1)/2]} c_0 \quad \text{if} \quad \lambda(-1) \neq 0 \qquad (1.14)$$

where

$$\lambda(t) = \exp(at)(1+a^{-1}b) - a^{-1}b, \quad t \in J.$$

One of the qualitative properties of this solution is established in the theorem stated below. In view of the representation (1.14) the proof immediately follows.

THEOREM 1.7: The zero solution of (1.13) is asymptotically stable if and only if

$$\left| \frac{\lambda(1)}{\lambda(-1)} \right| < 1.$$

The existence and uniqueness of solutions of (1.13) with variable coefficients is the consequence of the following result.

THEOREM 1.8: The IVP

$$y'(t) = a(t)y(t) + b(t)y(2[(t+1)/2]), y(0) = c_0$$

has a unique solution on J if a and b are continuous for $t \in J$, and

$$\int_{2n-1}^{2n} u^{-1}(t)b(t)dt \neq u^{-1}(2n), \quad n = 1, 2, \ldots,$$

where u^{-1} is the reciprocal of u and $u(t) = \exp(\int_0^t a(s)ds)$.

REMARK 1.4: It can be easily shown that the solution $y \equiv 0$ of (1.13) is asymptotically stable if and only if one of the conditions is satisfied:

(i) $a < 0,\ b > -\dfrac{a(\exp(2a)+1)}{(\exp(a)-1)^2}$ or $b < -a$;

(ii) $a > 0,\ -\dfrac{a(\exp(2a)+1)}{(\exp(a)-1)^2} < b < -a$;

(iii) $a = 0,\ b < 0.$

EXAMPLE 1.5: Let us consider the IVP

$$y'(t) = y(t) + 2y(2[(t+1)/2]),\ y(0) = 1, \quad t \in J.$$

Here we have chosen $a = 1$, $b = 2$ and $c_o = 1$ in (1.13). Use the relation (1.14) to get the solution

$$y(t) = (3\exp(t-2[(t+1)/2]) - 2)\left(\frac{3\exp(1) - 2}{3\exp(-1) - 2}\right)^{[(t+1)/2]}, \quad t \in J.$$

EXAMPLE 1.6: In a similar way, as in the Example 1.2, we choose $a = 0$ and $c_o = 1$ in (1.13). The solution is given by

$$y(t) = (1 + b(t-2[(t+1)/2]))\left(\frac{1 + b}{1 - b}\right)^{[(t+1)/2]}, \quad t \in J.$$

For any b such that $-1 > b$ or $b > 1$ the above solution is oscillatory. For all $-1 \le b < 1$ it is nonoscillatory. The solution $y(t)$ is asymptotically stable for $b < 0$ and it is stable for $b = 0$. $y(t)$ is unstable for $b > 0$ and it is not defined at $b = 1$.

1.6 GRONWALL'S INTEGRAL INEQUALITY

In this section, we state Gronwall's integral inequality [23] which is useful in the study of the qualitative behaviour of solutions of differential equations.

THEOREM 1.9: Let c be a nonnegative constant and let m and v be nonnegative continuous functions on some interval $t_0 \leq t < t_0 + a$ satisfying

$$m(t) \leq c + \int_{t_0}^{t} v(s)m(s)ds, \quad t \in [t_0, t_0+a).$$

Then , the inequality

$$m(t) \leq c \exp\left(\int_{t_0}^{t} v(s)ds \right), \quad t \in [t_0, t_0+a),$$

holds.

A minor modification of the above theorem is given in the following result.

THEOREM 1.10: Let m and v be nonnegative continuous functions on $[t_0, t_0+a)$ and let n be a continuous function on $[t_0, t_0+a)$ satisfying the inequality

$$m(t) \leq n(t) + \int_{t_0}^{t} v(s)m(s)ds, \quad t \in [t_0, t_0+a).$$

Then, we have

$$m(t) \leq n(t) + \int_{t_0}^{t} v(s)n(s)\exp\left(\int_{s}^{t} v(r)dr \right)ds, \quad t \in [t_0, t_0+a).$$

If, in addition, the derivative $n'(t)$ exists for $t \in [t_o, t_o+a)$, then

$$m(t) \leq n(t_o)\exp\left(\int_{t_o}^{t} v(s)ds\right) + \int_{t_o}^{t} \exp\left(\int_{s}^{t} v(r)dr\right)n'(s)ds.$$

1.7 VARIATION OF PARAMETERS FORMULA

The method of variation of parameters is one of the most important techniques in the study of the qualitative properties of ordinary differential equations.

Variation of parameters formula for linear differential equation is given below [23].

Consider the IVP

$$x'(t) = a(t)x(t), \quad x(t_o) = x_o \tag{1.15}$$

and associated perturbed IVP

$$y'(t) = a(t)y(t) + b(t), \quad y(t_o) = x_o, \tag{1.16}$$

where a,b are continuous functions on J and x_o is a real constant.

It can be verified that

$$x(t,t_o,x_o) = \exp\left(\int_{t_o}^{t} a(s)ds\right)x_o$$

and

$$y(t,t_o,x_o) = \exp\left(\int_{t_o}^{t} a(s)ds\right)x_o + \int_{t_o}^{t} \exp\left(\int_{s}^{t} a(r)dr\right)b(s)ds$$

are solutions of (1.15) and (1.16), respectively.

If we denote $\phi(t) = \exp(\int_{t_o}^{t} a(s)ds)$ and $\Phi(t,s) = \Phi(t)\Phi^{-1}(s)$, then the solution $y(t,t_o,x_o) = y(t)$ of (1.16) can be written as

$$y(t) = \phi(t,t_o)x_o + \int_{t_o}^{t} \phi(t,s)b(s)ds,$$

which is known as variation of parameters formula for linear differential equation.

The generalization of variation of parameters technique for nonlinear case is not obvious which is also described below [23].

Consider the initial value problems

$$x'(t) = f(t,x(t)),\ x(t_o) = x_o,\quad t \in J \tag{1.17}$$

$$y'(t) = f(t,y(t)) + F(t,y(t)), y(t_o) = x_o,\ t \in J, \tag{1.18}$$

where f, F are continuous functions defined on $J \times \mathbb{R}$ and x_o is a constant. Assume that f possesses continuous partial derivatives $\frac{\partial f}{\partial x}$ on $J \times \mathbb{R}$. Let $x(t,t_o,x_o)$ and $y(t,t_o,x_o)$ be the solutions of (1.17) and (1.18), respectively. Denote $H(t,t_o,x_o) = \frac{\partial f}{\partial x_o}$. Then

$$\phi(t,t_o,x_o) = \frac{\partial x(t,t_o,x_o)}{\partial x_o} \text{ exists, } \phi(t_o,t_o,x_o) = 1$$

and is the solution of the linear equation

$$y'(t) = H(t,t_o,x_o)y(t),\quad t \in J\ . \tag{1.19}$$

Also $\frac{\partial x(t,t_o,x_o)}{\partial t_o}$ exists and is a solution of (1.19) satisfying

$$\frac{\partial x(t,t_o,x_o)}{\partial t_o} = -\phi(t,t_o,x_o)f(t_o,x_o), \quad t \in J.$$

Now we state a theorem from [23] which is due to Alekseev.

THEOREM 1.11: Let f and F be continuous functions on J x $\mathbb{R}$ to $\mathbb{R}$ and let $\frac{\partial f}{\partial x}$ exist and be continuous on J x $\mathbb{R}$. If $x(t,t_o,x_o)$ is the solution of (1.17) existing for $t \geq t_o$, any solution $y(t,t_o,x_o)$ of (1.18), satisfies the integral equation

$$y(t,t_o,x_o) = x(t,t_o,x_o) + \int_{t_o}^{t} \phi(t,s,y(s,t_o,x_o))F(s,y(s,t_o,x_o))ds,$$

for $t \geq t_o$, where $\phi(t,t_o,x_o) = \frac{\partial x(t,t_o,x_o)}{\partial x_o}$.

1.8 DELAY DIFFERENTIAL EQUATIONS

In this section, we discuss some results on delay differential equations which are needed for the present work.

Consider a linear delay differential equation of the form

$$x'(t) = ax(t) + bx(t-\tau), \quad \tau > 0, \quad t \in J, \tag{1.20}$$

with initial data

$$x(t) = \phi(t) \quad \text{in} \quad -\tau \leq t \leq 0,$$

where a, b are constants and ϕ is a real-valued continuous function defined on $[-\tau, 0]$.

We need the following definition [16].

DEFINITION 1.3: The solution $\phi_1(t), t \geq 0$, of equation (1.20) with initial data

$$\phi(t) = \begin{cases} 0, & -\tau \leq t < 0 \\ 1, & t = 0; \end{cases}$$

is called the fundamental solution.

It is known in [16] that if $h(\lambda) = \lambda - a - b\exp(-\lambda\tau) = 0$ is the characteristic equation of (1.20), then the Laplace transform of $\phi_1(t)$ is $h^{-1}(\lambda)$, that is $L(\phi_1)(\lambda) = h^{-1}(\lambda)$.

Further, the solution $y(t)$ of the perturbed equation

$$y'(t) = ay(t) + by(t-\tau) + cy([t]),$$

with the same initial condition is given by

$$y(t) = x(t) + \int_0^t \phi_1(t-s)cy([s])ds,$$

where $x(t)$ is the solution of the IVP (1.20).

The following known theorem from [16] is employed in a subsequent chapter.

THEOREM 1.12: Let $h(\lambda)$ be the characteristic equation of (1.20); if $\alpha_0 = \max\{\mathrm{Re}\,\lambda ; h(\lambda) = 0\}$, then, for any $\alpha > \alpha_0$, there is a constant $k = k(\alpha)$ such that the fundamental solution $\phi_1(t)$ of (1.20) satisfies the inequality

$$|\phi_1(t)| \leq k \exp(\alpha t), \quad t \geq 0. \tag{1.21}$$

Further, if x(t) is the solution of (1.20) with initial condition ϕ, a real-valued continuous function on $[-\tau, 0]$, then

$$|x(t)| \leq k \exp(\alpha t)|\phi|, \quad t \geq 0, \tag{1.22}$$

where

$$|\phi| = \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)|.$$

In the case of functional differential equations, a linear eqution is represented in the form

$$x'(t) = ax(t) + bL(x(t+\theta)) \tag{1.23}$$

and its perturbed equation is

$$y'(t) = ay(t) + bL(y(t+\theta)) + cy([t]) \tag{1.24}$$

with initial data

$$x(t) = y(t) = \phi(t), \quad -\tau \leq t \leq 0,$$

where L is a linear operator on $[-\tau, 0]$ to $\mathbb{R}$, $t \in J$ and ϕ is a continuous function defined on $[-\tau, 0]$, $\tau > 0$ a constant.

We need the following definition from[16].

DEFINITION 1.4: The function $\phi_2(t)$ satisfying the relation

$$\frac{\partial \phi_2(t)}{\partial t} = a\,\phi_2(t) + bL(\phi_2(t+\theta)), \quad -\tau \leq \theta \leq 0,$$

with initial data

$$\phi(t) = \begin{cases} 0, & -\tau \leq t < 0 \\ 1, & t = 0; \end{cases}$$

is called the fundamental solution of (1.23).

It is known that $\phi_2(t,s) = \phi_2(t)\phi_2^{-1}(s) = \phi_2(t-s)$. By variation of parameters formula proved in [16], the solution y(t) of (1.24) has a representation

$$y(t) = x(t) + \int_0^t \phi_2(t,s)cy([s])ds, \quad t \geq 0,$$

where x(t) is the solution of (1.23) and $\phi_2(t)$ is the fundamental solution of (1.23).

CHAPTER 2

QUALITATIVE PROPERTIES OF DIFFERENTIAL EQUATIONS INVOLVING PIECEWISE CONSTANT DEVIATING ARGUMENT

2.1 INTRODUCTION

Qualitative theory of ordinary differential equations is an interesting study because of its great practical utility in many branches of science and engineering. In this chapter, we prove an existence theorem, a variation of parameters formula, an integral inequality and an oscillation result which are some of the basic results to study further properties of differential equations.

In section 2.3, we prove the existence and uniqueness of solution of a system with PCDA by using Banach contraction mapping principle. Section 2.4 gives the variation of parameters formula for piecewise constant delay equations. Well known Gronwall's integral inequality is extended in section 2.5. Applications of variation of parameters formula and Gronwall's type integral inequality are given in section 2.6. In section 2.7, the oscillatory behaviour of solutions of the system with PCDA is discussed. Equations with two types of delays, namely, continuous and piecewise constant deviating argument are introduced in section 2.8. Section 2.9, is concerned with the study of some functional differential equations. In the same section, we introduce a new class of delay and functional differential equations.

$$x'(t) = ax(t) + bx([t]) + cx((t))$$

$$x'(t) = L(x_{[t]}),$$

where the notation (t) designates fractional part of t.

2.2 PRELIMINARIES

We consider the following systems of differential equations

$$x'(t) = A(t)x(t) \tag{2.1}$$

$$y'(t) = A(t)y(t) + B(t)y([t]) \tag{2.2}$$

$$z'(t) = A(t)z(t) + B(t)z([t]) + c(t) \tag{2.3}$$

for $t \in J = [0, \infty)$ and with initial conditions

$$x(0) = y(0) = z(0) = c_o, \tag{2.4}$$

where A, B are n x n matrices with entries as real-valued continuous functions of $t \in J$, c is an n-column vector with entries as real-valued continuous functions for $t \in J$, x, y, z are n-vectors and c_o is a real constant n-column vector. Here the notation [ ] designates the greatest integer function.

The definition of solutions of (2.2), (2.4) and (2.3), (2.4) now can be given with modification in the Definition 1.1.

The following notations will be used in our subsequent discussion: $\mathbb{R}^n$ is the space of Euclidean n-vectors and for $x \in \mathbb{R}^n$, $|x|$ is the Euclidean vector norm.

Let $C[J,\mathbb{R}^n]$ or $C(J)$ denotes the space of continuous functions mapping from $J = [0, \infty)$ into $\mathbb{R}^n$. $BC(J)$ is the space of bounded continuous functions mapping from J into $\mathbb{R}^n$ and for $x \in BC(J)$,

$$|x|^* = \sup_{t \in J} |x(t)| .$$

It is known that BC(J) is a Banach space with the norm defined above.

The norm of an n x n matrix $M = (m_{ij})$ is defined to be

$$|M| = \max_j \sum_i |m_{ij}| .$$

The notation E_n denotes the n x n identity matrix.

DEFINITION 2.1: A solution $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))$ of system (2.2) is said to be oscillatory if each of its component has arbitrary large number of zeros for $t \geq T$, $0 \leq T < \infty$. A solution of (2.2) is nonoscillatory if atleast one of its component is eventually of a constant sign.

2.3 METHOD OF ITERATION

In this section we employ the method of iteration described in [13] to obtain the solution of the IVP (2.2), (2.4).

THEOREM 2.1: Let A, B be n x n matrices with entries as real-valued continuous functions defined on J. Then there exists a unique solution to the IVP (2.2), (2.4) and the solution is given by

$$y(t) = \lim_{k}\{\Phi(t,0) + \int_0^t \Phi(t,t_1)B(t_1)\,\Phi([t_1],0)dt_1$$

$$+ \int_0^t \int_0^{[t_1]} \Phi(t,t_1)B(t_1)\,\Phi([t_1],t_2)B(t_2)\,\Phi([t_2],0)dt_2dt_1 + \dots$$

$$+ \int_0^t \int_0^{[t_1]} \dots \int_0^{[t_{k-1}]} \Phi(t,t_1)B(t_1)\,\Phi([t_1],t_2)B(t_2) \dots$$

$$\dots B(t_k)\Phi([t_k],0)dt_k \dots dt_2dt_1\}c_o \qquad (2.5)$$

$t \in J$, where $\Phi(t)$ is a fundamental matrix of (2.1), $\Phi(0) = E_n$ and

$\Phi(t,s) = \Phi(t)\,\Phi^{-1}(s)$.

PROOF: Let Δ be any compact interval in J, such that $0 \in \Delta$. The space $C(\Delta)$ of continuous functions $y : t \longrightarrow y(t)$ from Δ into $\mathbb{R}^n$, with norm

$$|y| = \sup_{t\in\Delta}|y(t)|$$

is complete. Let us also consider the space $C_\lambda(\Delta)$, $\lambda \geq 0$ of continuous functions $y : t \longrightarrow y(t)$ from Δ into $\mathbb{R}^n$, with norm

$$|y|_\lambda = \sup_{t\in\Delta}\{|y(t)|\exp(-\lambda\int_0^t |\Phi(t,s)B(s)|ds)\}.$$

Clearly $C_o(\Delta) = C(\Delta)$. It is seen that the norms $|y|_\lambda$ are all equivalent for $\lambda \geq 0$ so that $C_\lambda(\Delta)$ is also a complete space. Let us now consider the mapping $T : C_\lambda(\Delta) \longrightarrow C_\lambda(\Delta)$ defined by

$$(Ty)(t) = \Phi(t,0)c_o + \int_0^t \Phi(t,s)B(s)y([s])ds.$$

We show that T is a contraction mapping. Observe that

$$|(Ty_1)(t) - (Ty_2)(t)| \leq \int_0^t |\Phi(t,s)B(s)|\,|y_1([s]) - y_2([s])|ds$$

$$= \int_0^t |\Phi(t,s)B(s)|\,|y_1([s]) - y_2([s])|$$

$$.\exp(\lambda\int_0^s |\Phi(s,r)B(r)|dr - \lambda \int_0^s |\Phi(s,r)B(r)|dr)ds$$

$$\leq \sup_{t\in\Delta}|y_1(t)-y_2(t)|\exp(-\lambda \int_0^t |\Phi(t,s)B(s)|ds)$$

$$\cdot \int_0^t \frac{1}{\lambda}\frac{d}{ds}(\exp(\lambda\int_0^s |\Phi(t,r)B(r)|dr))ds$$

$$\leq |y_1 - y_2|_\lambda \; \frac{1}{\lambda} \exp(\lambda \int_0^t |\Phi(t,s)B(s)|ds).$$

Hence, it follows that

$$|Ty_1 - Ty_2|_\lambda \leq \frac{1}{\lambda}|y_1 - y_2|_\lambda .$$

For $\lambda > 1$, T is a contraction map on $C_\lambda(\Delta)$. Hence, by Banach contraction mapping theorem there exists a unique y in $C_\lambda(\Delta)$ such that

$$(Ty)(t) = y(t) = \Phi(t,0)c_o + \int_0^t \Phi(t,s)B(s)y([s])ds.$$

Defining

$$y_o(t) = c_o$$

$$y_k(t) = \Phi(t,0)c_o + \int_0^t \Phi(t,s)B(s)y_{k-1}([s])ds, \quad k = 1, 2, \ldots$$

and using the method of successive approximations, the required result (2.5) follows.

REMARK 2.1: If $A(t) \equiv 0$ in (2.2), then the solution (2.5) reduces to the form

$$y(t) = \lim_k \{E_n + \int_o^t B(t_1)dt_1 + \int_o^t \int_o^{[t_1]} B(t_1)B(t_2)dt_2dt_1 + \ldots$$

$$+ \int_o^t \int_o^{[t_1]} \ldots \int_o^{[t_{k-1}]} B(t_1)B(t_2)\ldots B(t_k)dt_k \ldots dt_2dt_1\}c_o.$$

For computing solution of (2.2),(2.4), we can use the method given in [9]. For this purpose, assume that $y_n(t)$ is a solution in the interval $n \le t < n+1$, with the initial condition $y_n(n) = c_n$, $n = 0, 1, 2 \ldots$. The unique solution of (2.2) on the given interval is

$$y_n(t) = (\Phi(t,n) + \int_n^t \Phi(t,s)B(s)ds)c_n.$$

Since $y_{n-1}(n) = y_n(n)$, we obtain the recurrence relation

$$c_n = (\Phi(n, n-1) + \int_{n-1}^n \Phi(n,s)B(s)ds)c_{n-1}.$$

Hence, the solution of (2.2), (2.4) is given by

$$y(t) = (\Phi(t,[t]) + \int_{[t]}^t \Phi(t,s)B(s)ds)c_{[t]}, \quad t \in J \tag{2.6}$$

where

$$c_{[t]} = \prod_{k=[t]}^{1} (\Phi(k,k-1) + \int_{k-1}^k \Phi(k,s)B(s)ds)c_o. \tag{2.7}$$

In order to study perturbation effects on the equation (2.2), we now treat (2.2) as our basic equation (in place of (2.1)). This approach leads us to the study of equation (2.3). For this purpose, we first define the fundamental matrix solution of (2.2).

DEFINITION 2.2: The function

$$\Psi(t) = \left(\Phi(t,[t]) + \int_{[t]}^{t} \Phi(t,s)B(s)ds \right) \cdot \prod_{k=[t]}^{1} \left(\Phi(k,k-1) + \int_{k-1}^{k} \Phi(k,s)B(s)ds \right), \quad t \in J,$$

where $\Phi(t)$, $t \in J$ is a fundamental matrix of (2.1), satisfies the matrix IVP

$$Y'(t) = A(t)Y(t) + B(t)Y([t]), \; Y(0) = E_n .$$

We call matrix Ψ a fundamental matrix for the equation (2.2). We also use the notation $\Psi(t, k) = \Psi(t)\Psi^{-1}(k)$, $k = 0,1,2, \ldots [t]$.

It is seen that $\Psi(t)C$ where C is a nonsingular constant n x n matrix is also a fundamental matrix of (2.2). In view of Definition 2.2, it is clear from (2.6), (2.7) that any solution y(t) of (2.2) is given by

$$y(t) = \Psi(t)c_o, \; t \in J. \tag{2.8}$$

From (2.8) we can prove the following properties.

(i) $\Psi(t, \theta)\Psi(\theta,s) = \Psi(t,s)$

(ii) $\Psi^{-1}(t,s) = \Psi(s,t)$

(iii) $\Psi(t,t) = E_n$

(iv) $\dfrac{\partial \Psi(t,\theta)}{\partial t} = A(t)\Psi(t,\theta) + B(t)\Psi([t],\theta)$,

where $t, \theta, s \in J$, $t \geq \theta \geq s$.

2.4 VARIATION OF PARAMETERS METHOD

Let x(t), y(t), z(t) be solutions of (2.1), (2.2), (2.3), respectively, satisfying the initial condition (2.4). The relationship between the solutions x, y, and z is established below through variation of parameters formula.

THEOREM 2.2: The unique solution of (2.3), (2.4) is given by

$$z(t) = y(t) + \sum_{k=1}^{[t]} \int_{k-1}^{k} \Psi(t,k)\,\Phi(k,s)c(s)ds + \int_{[t]}^{t} \Phi(t,s)c(s)ds, \quad t \in J \tag{2.9}$$

where Φ and Ψ are fundamental matrices of linear systems (2.1) and (2.2), respectively, and y(t) is the solution of (2.2), (2.4).

PROOF: It is enough to prove that

$$\tilde{z}(t) = \sum_{k=1}^{[t]} \int_{k-1}^{k} \Psi(t,k)\,\Phi(k,s)c(s)ds + \int_{[t]}^{t} \Phi(t,s)c(s)ds, \quad t \in J$$

is a solution of (2.3). We have for $t \in J$,

$$\tilde{z}'(t) = \sum_{k=1}^{[t]} \int_{k-1}^{k} \Psi'(t,k)\,\Phi(k,s)c(s)ds + \frac{d}{dt}\left(\int_{[t]}^{t} \Phi(t,s)c(s)ds\right)$$

$$= \sum_{k=1}^{[t]} \int_{k-1}^{k} (A(t)\Psi(t,k) + B(t)\Psi([t],k))\Phi(k,s)c(s)ds + c(t)$$

$$+ \int_{[t]}^{t} A(t)\Phi(t,s)c(s)ds$$

$$= A(t)\tilde{z}(t) + B(t)\tilde{z}([t]) + c(t).$$

The proof is complete.

REMARK 2.2: If $A \equiv 0$ in (2.3) the solution (2.9) becomes

$$z(t) = y(t) + \sum_{k=1}^{[t]} \int_{k-1}^{k} \Psi(t,k)c(s)ds + \int_{[t]}^{t} c(s)ds, \quad t \in J$$

where y(t) is the solution of (2.2), (2.4) with $A \equiv 0$.

We can arrive at the result (2.9) by another method by taking (2.1) as basic equation and using usual variation of parameters formula (refer section 1.7) to get the solution of (2.3). For this purpose, we prove the following theorem.

THEOREM 2.3: Let ϕ and ψ be fundamental matrices of (2.1) and (2.2), respectively, then

$$z(n) = y(n) + \sum_{k=1}^{n} \int_{k-1}^{k} \Psi(n,k)\Phi(k,s)c(s)ds \tag{2.10}$$

where $n \geq 1$ is an integer,

$$\left.\begin{aligned} \Psi(n,k) &= \Phi(n,k) + \sum_{r=k+1}^{n} \int_{r-1}^{r} \Psi(r-1,k)\Phi(n,s)B(s)ds, \text{ for } n > k \\ &\text{and} \\ \Psi(n,k) &= E_n \text{ for } n = k,\ n = 1, 2, \ldots[t]. \end{aligned}\right] \tag{2.11}$$

PROOF: Let $y(t)$ and $z(t)$ be solutions of (2.2), (2.4) and (2.3), (2.4), respectively, for $t \in J$. Using variation of parameters formula (refer section 1.7), we have, for $t \in J$,

$$y(t) = \Phi(t,0)c_0 + \int_0^t \Phi(t,s)B(s)y([s])\,ds \tag{2.12}$$

and

$$z(t) = \Phi(t,0)c_0 + \int_0^t \Phi(t,s)B(s)z([s])ds + \int_0^t \Phi(t,s)c(s)ds. \tag{2.13}$$

It is easy to see that, for $n = 1$,

$$z(1) = y(1) + \int_0^1 \Psi(1,1)\,\Phi(1,s)c(s)ds$$

where

$$\Psi(1,1) = E_n.$$

Assume that the result (2.10) is true for $n = 1,2,\dots m$. For $t = m+1$, we obtain from (2.13)

$$z(m+1) = \Phi(m+1,0)c_o + \sum_{k=1}^{m+1} \int_{k-1}^{k} \Phi(m+1,s)\{B(s)z(k-1) + c(s)\}ds.$$

In similar form we can write $y(m+1)$ from (2.12) in terms of $y(0), y(1), \dots y(m)$. Since (2.10) holds for $n = 1,2,\dots m$, we get

$$z(m+1) = \Phi(m+1,0)c_o + \sum_{k=1}^{m+1} \int_{k-1}^{k} \Phi(m+1,s)\{B(s)y(k-1)$$
$$+B(s) \sum_{r=1}^{k-1} \int_{r-1}^{r} \Psi(k-1,r)\,\Phi(r,p)c(p)dp\}ds$$
$$+ \sum_{k=1}^{m+1} \int_{k-1}^{k} \Phi(m+1,s)c(s)ds,$$

where

$$\sum_{r=1}^{0} \int_{r-1}^{r} \Psi(k-1,r)\,\Phi(r,p)c(p)dp = 0.$$

From the expression of $z(m+1)$, sum of the terms containing $y(0), y(1), \dots y(m)$ can be replaced by $y(m+1)$, then

$$z(m+1) = y(m+1) + \sum_{k=1}^{m+1} \int_{k-1}^{k} \Phi(m+1,s)B(s)\{\sum_{r=1}^{k-1} \int_{r-1}^{r} \Psi(k-1,r)\,\Phi(r,p)c(p)dp\}ds$$
$$+ \sum_{k=1}^{m+1} \int_{k-1}^{k} \Phi(m+1,k)\,\Phi(k,s)c(s)ds.$$

By changing the order of integration in the second term on the right side, we obtain

$$z(m+1) = y(m+1) + \sum_{k=1}^{m+1} \int_{k-1}^{k} \{\,\Phi(m+1,k)$$
$$+ \sum_{r=k+1}^{m+1} \int_{r-1}^{r} \Psi(r-1,k)\,\Phi(m+1,p)B(p)dp\}\,\Phi(k,s)c(s)ds. \quad (2.14)$$

Now use (2.11) in (2.14) to see the result

$$z(m+1) = y(m+1) + \sum_{k=1}^{m+1} \int_{k-1}^{k} \Psi(m+1,k)\, \Phi(k,s)c(s)ds.$$

Hence the theorem.

In order to establish (2.9) as a by product of the above theorem, it is necessary to get the solution z(t) for each t, t ∈ J. Hence we need to add the following steps.

Use (2.10) for n = 1, 2, ... [t] in (2.13) to get z(t) in terms of y(0), y(1), ... y([t]). From the expression of z(t) we can replace the terms of (2.12) as y(t). Using the same argument as in the case of above theorem, we get the result (2.9).

In the following example, we verify the relation (2.9) for the one dimensional case.

EXAMPLE 2.1: Consider (2.2) and (2.3) with scalar functions $A(t) \equiv a(t)$, $B(t) \equiv b(t)$. Assume that $z_n(t)$ is the solution of the perturbed equation

$$z'(t) = a(t)z(t) + b(t)z([t]) + c(t) \tag{2.15}$$

in the interval $n \le t < n+1$, with the condition $z_n(n) = d_n$, for n = 0, 1, 2, ... Hence,

$$z_n(t) = d_n \exp\left(\int_n^t a(p)dp\right) + \int_n^t \exp\left(\int_s^t a(p)dp\right)\{b(s)d_n + c(s)\}ds. \tag{2.16}$$

Since $z_{n-1}(n) = z_n(n) = d_n$, we obtain the recursion relation

$$d_n = d_{n-1}\left\{\exp\left(\int_{n-1}^n a(p)dp\right) + \int_{n-1}^n \exp\left(\int_s^n a(p)dp\right)b(s)ds\right\}$$

$$+ \int_{n-1}^n \exp\left(\int_s^n a(p)dp\right)c(s)ds, \tag{2.17}$$

n = 1, 2, 3 ... .

Since $c_o = d_o$, we can write $d_{[t]}$ in terms of c_o by using the recursion relation (2.17). Now (2.16) and (2.17) yield

$$z(t) = d_{[t]}\{\exp(\int_{[t]}^{t} a(p)dp) + \int_{[t]}^{t} \exp(\int_{s}^{t} a(p)dp)b(s)ds\}$$
$$+ \int_{[t]}^{t} \exp(\int_{s}^{t} a(p)dp)c(s)ds. \qquad (2.18)$$

In view of the relation (2.17), (2.18) can be written as

$$z(t) = \{\exp(\int_{[t]}^{t} a(p)dp) + \int_{[t]}^{t} \exp(\int_{s}^{t} a(p)dp)b(s)ds\}$$
$$\cdot \prod_{k=[t]}^{1} \{\exp(\int_{k-1}^{k} a(p)dp) + \int_{k-1}^{k} \exp(\int_{s}^{k} a(p)dp)b(s)ds\}c_o$$
$$+ \sum_{k=[t]}^{1} \int_{k-1}^{k} \{\exp(\int_{[t]}^{t} a(p)dp) + \int_{[t]}^{t} \exp(\int_{\tau}^{t} a(p)dp)b(\tau)d\tau\}$$
$$\cdot \prod_{r=k+1}^{[t]} \{\exp(\int_{r-1}^{r} a(p)dp) + \int_{r-1}^{r} \exp(\int_{\tau}^{r} a(p)dp)b(\tau)d\tau\}$$
$$\cdot \exp(\int_{s}^{k} a(p)dp)c(s)ds + \int_{[t]}^{t} \exp(\int_{s}^{t} a(p)dp)c(s)ds,$$

where

$$\prod_{r=[t]+1}^{[t]} \{\exp(\int_{r-1}^{r} a(p)dp) + \int_{r-1}^{r} \exp(\int_{\tau}^{r} a(p)dp)b(\tau)d\tau\} = 1.$$

It can be seen from (1.9), (1.10) that in the expression of $z(t)$, first term on right side can be replaced by $y(t)$. Since

$$\phi(t) = \exp(\int_{o}^{t} a(p)dp) \text{ and}$$

$$\psi(t) = \{\exp(\int_{[t]}^{t} a(p)dp) + \int_{[t]}^{t} \exp(\int_{\tau}^{t} a(p)dp)b(\tau)d\tau\}$$
$$\cdot \prod_{r=1}^{[t]} \{\exp(\int_{r-1}^{r} a(p)dp) + \int_{r-1}^{r} \exp(\int_{\tau}^{r} a(p)dp)b(\tau)d\tau$$

for $t \in J$, we obtain

$$z(t) = y(t) + \sum_{k=1}^{[t]} \int_{k-1}^{k} \psi(t,k)\, \phi(k,s)c(s)ds + \int_{[t]}^{t} \phi(t,s)c(s)ds.$$

2.5 GRONWALL TYPE INTEGRAL INEQUALITY

In this section, we extend the Theorem 1.9, the Gronwall's integral inequality, which is referred to as a fundamental inequality of differential equation. This has been done in the following theorem.

THEOREM 2.4: Let $c_o \geq 0$ be a constant and $u, a, b \in C[J, \mathbb{R}^+]$. If the inequality

$$u(t) \leq c_o + \int_o^t \{a(s)u(s) + b(s)u([s])\}ds, \quad t \in J \tag{2.19}$$

holds, then for $t \in J$,

$$u(t) \leq c_o \prod_{k=1}^{[t]} \{\exp(\int_{k-1}^{k} a(p)dp) + \int_{k-1}^{k} \exp(\int_{s}^{k} a(p)dp)b(s)ds\}$$
$$.\{\exp(\int_{[t]}^{t} a(p)dp) + \int_{[t]}^{t} \exp(\int_{s}^{t} a(p)dp)b(s)ds\}. \tag{2.20}$$

PROOF: In the interval $n \leq t < n+1$

$$u(t) \leq u(n)(1 + \int_n^t b(s)ds) + \int_n^t a(s)u(s)ds.$$

Then by using Theorem 1.10, we get

$$u(t) \leq u(n)\{\exp(\int_n^t a(p)dp) + \int_n^t \exp(\int_s^t a(p)dp)b(s)ds\}, \tag{2.21}$$

$n = 0, 1, 2, \ldots$. Applying the inequality (2.21) successively for $u(n)$, $u(n-1)$, ... $u(1)$, we get the desired conclusion (2.20).

REMARK 2.3: Observe that the right hand side of the inequality (2.20) is infact a solution of the related delay differential equation

$$y'(t) = a(t)y(t) + b(t)y([t]), \quad y(0) = c_o.$$

REMARK 2.4: When b = 0 in (2.19) the inequality (2.20) reduces to

$$u(t) \le c_o \exp\left(\int_o^t a(s)ds\right).$$

Further, when a = 0 in (2.19) we get

$$u(t) \le c_o(1+b(t-[t]))(1+b)^{[t]} \le c_o \exp([t+1]b).$$

REMARK 2.5: Assume a = 0 in (2.19). Replace b(t) by b([t]) and put t = n in (2.19), then the inequality (2.20) becomes

$$u(n) \le \prod_{k=1}^{[t]} (1+b(k)).$$

This estimate has been proved in [31] .

2.6 APPLICATIONS

Consider the system

$$y'(t) = Ay(t) + By([t]) \tag{2.22}$$

and the perturbed system

$$z'(t) = Az(t) + Bz([t]) + f(t,z(t),z([t])) \tag{2.23}$$

where A, B are n x n constant matrices, A is nonsingular and $f \in C[J \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$. Assume that

$$|f(t,z(t),z([t]))| \le \alpha(t)|z(t)| + \beta(t)|z([t])| \tag{2.24}$$

where α, β are nonnegative continuous functions on J, satisfying

$$\int_0^\infty \alpha(s)ds < \infty \quad , \quad \int_0^\infty \beta(s)ds < \infty \quad . \tag{2.25}$$

Under suitable conditions, we prove the solution of (2.23) exists and further, if (2.22) has asymptotic stability property, then (2.23) also possesses the same. Similar results for ordinary differential equations are proved in [26].

THEOREM 2.5: Assume the fundamental matrices Φ and Ψ of (2.22) and (2.23), respectively, satisfy the conditions

$$|\Psi(t)| \le N, \ |\Phi(t,s)| \le M_o, \ 0 \le s \le t < \infty ,$$

$$|\Psi(t,k)\,\Phi(k,s)| \le M_k \ , \ k = 1, 2, \ldots [t], \ 0 \le s \le k \le t < \infty \tag{2.26}$$

and $M = \max\{M_o, M_1, \ldots M_{[t]}\}$,

where N and M_k, k = 1, 2, ... [t] are constants.

Let the conditions (2.24), (2.25) hold. Then all the solutions z(t) of (2.23) exists for $t \in \bar{J}$ and there exists a constant $K > 0$ such that

$$|z(t)| \le K|c_o|, \ t \in J . \tag{2.27}$$

Further, if y(t) is the solution of (2.22) with $y(0) = c_o$ and y(t) satisfies $\lim_{t\to\infty} y(t) = 0$, then $\lim_{t\to\infty} z(t) = 0$.

PROOF: Using Theorem 2.2, we have

$$z(t) = y(t) + \sum_{k=1}^{[t]} \int_{k-1}^{k} \Psi(t,k)\, \Phi(k,s) f(s,z(s),z([s]))\, ds$$

$$+ \int_{[t]}^{t} \Phi(t,s) f(s,z(s),z([s]))\, ds, \quad t \in J.$$

The conditions (2.24) and (2.26) yield for $t \in J$,

$$|z(t)| \leq N|c_o| + M \int_0^t (\alpha(s)|z(s)| + \beta(s)|z([s])|)\, ds .$$

Applying the inequality (2.20), we obtain

$$|z(t)| \leq K_1 N \; |c_o|$$

where

$$K_1 = \prod_{k=1}^{[t]} \{\exp(M \int_{k-1}^{k} \alpha(r)dr) + \int_{k-1}^{k} \exp(M \int_s^k \alpha(r)dr) M \beta(s)ds\}$$

$$.\{\exp(M \int_{[t]}^{t} \alpha(r)dr) + \int_{[t]}^{t} \exp(M \int_s^t \alpha(r)dr) M \beta(s)ds\}.$$

Take

$$K_2 = \exp(\int_o^\infty M \alpha(r)dr) \prod_{k=1}^{\infty} (1 + \int_{k-1}^{k} M \beta(s)ds) , \qquad (2.28)$$

Clearly, in view of (2.25) K_2 is a constant and is an upper bound of K_1 ,taking $K = K_2 N$, we get the required result (2.27) . The condition $\lim_{t \to \infty} y(t) = 0$ implies that, given any $\varepsilon > 0$, there exist a $T(\varepsilon) > 0$ such that $|y(t)| < \varepsilon$ for all $t \geq T(\varepsilon)$.Proceeding as before, for $t \geq T(\varepsilon)$,we get

$$|z(t)| \leq \varepsilon + M \int_o^t (\alpha(s)|z(s)| + \beta(s)|z([s])|)ds.$$

Hence $|z(t)| \leq K_2 \varepsilon$ where K_2 is given by (2.28) . Observe that K_2 is finite and is independent of ε and T. This implies $\lim_{t \to \infty} z(t) = 0.$

EXAMPLE 2.2: Consider the IVP

$$z'(t) = -z(t)-2z([t]) + \frac{z(t) + z([t])}{(1+z^2(t))(1+t)^2} \; ; \; z(0) = 1.$$

Clearly, the condition (2.26) is true. Since

$$\frac{z(t) + z([t])}{(1+z^2(t))(1+t)^2} \leq \frac{z(t)}{(1+t)^2} + \frac{z([t])}{(1+t)^2}$$

and

$$\int_0^\infty \frac{1}{(1+t)^2} \, dt = 1 < \infty$$

the conditions (2.24) and (2.25) are true. Hence by using Theorem 2.5, the zero solution of the given IVP is asymptotically stable.

Consider the system (2.2) and the perturbed system

$$z'(t) = A(t)z(t) + B(t)z([t]) + f(t,z(t)) + g(t,z(t)), \; t \in J, \tag{2.29}$$

where $f, g \in C[J \times \mathbb{R}^n, \mathbb{R}^n]$.

Under suitable conditions, we prove (2.29) possesses a solution and further, if (2.2) has a stability property, then (2.29) also possesses a stability property. Some similar results for measure differential equations are proved in [24].

THEOREM 2.6: Let Φ and Ψ be fundamental matrices of (2.1) and (2.2), respectively, such that

$$\sup_{t \in J} \Big\{ \sum_{k=1}^{[t]} \int_{k-1}^{k} | \Psi(t,k) \Phi(k,s) | ds + \int_{[t]}^{t} | \Phi(t,s) | ds \Big\} < a_0,$$

where a_0 is a positive constant.

Let (i) $f,\ g \in C[J \times R^n,\ R^n]$ are integrable and $g(t,0) = 0$.

(ii) For each $\gamma > 0$, there is a $\delta > 0$ such that $|f(t,z)| \leq \gamma\ |z|$, uniformly in $t \in J$, whenever $|z| \leq \delta$.

(iii) For each $\xi > 0$, there is an $\eta > 0$ such that $|g(t,z_1) - g(t,z_2)| \leq \xi\ |z_1 - z_2|$ uniformily in $t \in J$, whenever $|z_1| \leq \eta$, $|z_2| \leq \eta$.

Suppose $y \in BC(J)$ is a solution of (2.2), then there is a number $\varepsilon_0 > 0$ with the following property. For any ε, $0 < \varepsilon \leq \varepsilon_0$, there corresponds a $\delta_0 > 0$ such that whenever $|y|^* = \sup_{t \in J} |y(t)| \leq \delta_0$, there exists at least one solution $z(t)$ of (2.29) such that $z \in BC(J)$ and $|z|^* \leq \varepsilon$.

<u>**PROOF:**</u> Fix a $\xi > 0$ such that $\xi a_0 < 1$ and choose an $\eta > 0$ such that condition (iii) holds. Let $\gamma = \frac{1 - \xi' a_0}{2a_0}$, for this γ, select a $\delta > 0$ such that condition (ii) holds. Let $\varepsilon_0 = \min(\eta, \delta)$. For any $\varepsilon \in (0, \varepsilon_0)$, define $S(\varepsilon) = \{z \in BC(J),\ |z|^* \leq \varepsilon\}$. Clearly $S(\varepsilon)$ is a closed convex subset of $BC(J)$ under the norm $|\ |^*$.

Define the operators P and Q on $S(\varepsilon)$ by

$$(Pz)(t) = y(t) + \sum_{k=1}^{[t]} \int_{k-1}^{k} \Psi(t,k)\, \Phi(k,s) f(s,z(s))ds + \int_{[t]}^{t} \Phi(t,s) f(s,z(s))ds,$$

and

$$(Qz)(t) = \sum_{k=1}^{[t]} \int_{k-1}^{k} \Psi(t,k)\, \Phi(k,s) g(s,z(s))ds + \int_{[t]}^{t} \Phi(t,s) g(s,z(s))ds.$$

For $z_1, z_2 \in S(\varepsilon)$, we have

$$|Pz_1 + Qz_2|^* \leq \sup_{t \in J} |(Pz_1)(t) + (Qz_2)(t)|$$

$$\leq |y|^* + \sup_{t\in J} \sum_{k=1}^{[t]} \int_{k-1}^{k} |\Psi(t,k)\, \Phi(k,s)|\gamma\, |z_1(s)|ds$$

$$+ \sup_{t\in J} \int_{[t]}^{t} |\Phi(t,s)|\gamma\, |z_1(s)|ds$$

$$+ \sup_{t\in J} \sum_{k=1}^{[t]} \int_{k-1}^{k} |\Psi(t,k)\, \Phi(k,s)|\gamma\, |z_2(s)|ds$$

$$+ \sup_{t\in J} \int_{[t]}^{t} |\Phi(t,s)|\, \gamma |z_2(s)|ds$$

$$\leq \delta_o + \gamma\varepsilon a_o + \xi\varepsilon a_o$$

$$\leq \varepsilon \text{ provided } \delta_o < (1-\xi a_o)(\varepsilon/2).$$

This shows that $Pz_1 + Qz_2 \in S(\varepsilon)$, for every $z_1, z_2 \in S(\varepsilon)$. Next, $|Qz_1 - Qz_2|^* \leq \xi a_o |z_1 - z_2|^*$ together with the fact $\xi a_o < 1$ implies that Q is a contration on $S(\varepsilon)$. We show that P is completely continuous. It is enough to prove that any bounded sequence $\{z_k\}$ in $S(\varepsilon)$ has a convergent subsequence. Since $\{Pz_k\}$ is uniformily bounded and equicontinuous set of functions by Ascoli's theorem, there is a subsequence $\{z_{k_j}\}$ which converges to some $z \in S(\varepsilon)$. Thus P maps bounded subset of $S(\varepsilon)$ into relatively compact subsets and so P is completely continuous. Therefore, by Krasnoselskii's fixed point theorem [28], there exists $z \in S(\varepsilon)$ such that

$$Pz + Qz = z.$$

That is

$$z(t) = y(t) + \sum_{k=1}^{[t]} \int_{k-1}^{k} \psi(t,k)\,\phi(k,s)\{f(s,z(s)) + g(s,z(s))\}ds$$

$$+ \int_{[t]}^{t} \phi(t,s)\{f(s,z(s)) + g(s,z(s))\}ds,$$

From definition of $S(\varepsilon)$, $z \in S(\varepsilon)$ and implies that $|z|^* \leq \varepsilon$.

2.7 OSCILLATORY BEHAVIOUR

In [1], Aftabizadeh and Wiener have obtained a result on oscillatory behaviour of solutions of scalar equations involving piecewise constant delays. In this section, we extend this result for a system of equation (2.2).

First we need the following definition [12].

DEFINITION 2.3: The measure $\mu(B)$ of the matrix B is defined by

$$\mu(B) = \lim_{\theta \to 0^+} \frac{|E_n + \theta B| - 1}{\theta}$$

where $|\ |$ is the norm of the matrix.

It is well known, if we adopt the following norm of matrix

$$|B| = \max_j \sum_{i=1}^{n} |b_{ij}|$$

then the corresponding measure is

$$\mu(B) = \max_j (b_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^{n} |b_{ij}|) .$$

In view of Theorem 1.3 of chapter 1 and Definition 2.3, we prove the following theorem.

THEOREM 2.7: Let $\mu(.)$ denote the matrix measure. Assume that the matrices A(t) and B(t) in (2.2) are such that

$$\limsup_{m \to \infty} \int_m^{m+1} -\mu(B(s))\exp\left(\int_m^s -\mu(A(r))dr\right)ds > 1. \tag{2.30}$$

Then every solution of (2.2) is oscillatory.

PROOF: Suppose that there exists a solution, say, $\tilde{y}(t) = (y_1(t), \ldots y_n(t))$, of (2.2) which is nonoscillatory. Then there exists a constant $m > 0$ such that for $t > m$, no component of $\tilde{y}$ has a zero. In such a case, we have

$$\frac{d}{dt}\left(\sum_{i=1}^{n} |y_i(t)|\right) \leq \mu(A(t))\sum_{i=1}^{n} |y_i(t)| + \mu(B(t))\sum_{i=1}^{n} |y_i([t])|$$

for sufficiently large $t \geq T > m$. Let $u(t) = \sum_{i=1}^{n} |y_i(t)|$, then u(t) satisfies the inequality

$$\frac{d}{dt}(u(t)) \leq \mu(A(t))u(t) + \mu(B(t))u([t]). \tag{2.31}$$

Since the solution $\tilde{y}(t)$ of (2.2) is nonoscillatory, $u(t) > 0$ for $t \geq T$. That is, the inequality (2.31) has a positive solution for $t \geq T$ which contradicts Theorem 1.3 in view of the condition (2.30). Hence the theorem.

EXAMPLE 2.3: Consider the system

$$y'(t) = Ay(t) + By([t]) \tag{2.32}$$

with

$$A = \begin{bmatrix} -10 & 3 & 3 \\ 1 & -8 & -5 \\ 2 & 0 & -15 \end{bmatrix} \text{ and } B = \begin{bmatrix} -14 & -3 & 4 \\ -3 & -17 & 2 \\ 1 & 1 & -18 \end{bmatrix}.$$

Here $\mu(A) = -5$ and $\mu(B) = -10$. The system (2.32) satisfies the contition (2.30) and hence every solution of (2.32) is oscillatory.

2.8 RETARDED EQUATIONS INVOLVING TWO TYPES OF DELAYS

This section deals with the study of differential equations involving piecewise constant dealy and continuous delay. Consider the following scalar retarded differential equations

$$x'(t) = ax(t) + bx(t-\tau) \tag{2.33}$$

$$y'(t) = ay(t) + by(t-\tau) + cy([t]) \tag{2.34}$$

$$z'(t) = az(t) + bz(t-\tau) + cz([t]) + f(t) \tag{2.35}$$

where $\tau > 0$, $t \in J$, with initial functions

$$x(t) = y(t) = z(t) = \phi(t) \text{ for } -\tau \leq t \leq 0, \tag{2.36}$$

where ϕ is a real-valued continuous function and a,b,c are real constants.

<u>**DEFINITION 2.4:**</u> A solution of IVP (2.34), (2.36) on J is a function $y : J \longrightarrow \mathbb{R}$ satisfying the following conditions:

(i) $y(t)$ is continuous on J,

(ii) the derivative $y'(t)$ exists at each point $t \in J$ except possibly at $t = 0, 1, 2, \ldots$ where $y'(0)$ represents right hand side derivative and at all other points only the left hand side derivative exists; and

(iii) $y(t)$ satisfies equation (2.34), for each interval $[n, n+1)$, it coincides with $\phi(t)$ on $-\tau \leq t \leq 0$.

By using the iterative method, we can get the existence and uniqueness of solution of (2.34). We state the following theorem.

THEOREM 2.8: Let x(t) be the solution of (2.33), (2.36) and $\phi_1(t)$ be the fundamental solution (see Definition 1.3) of (2.33). Then, there exists unique solution to the IVP (2.34), (2.36) and it is given by

$$y(t) = \begin{cases} \phi(t), & -\tau \le t \le 0, \\ \lim_k \{x(t) + c\int_0^t \phi_1(t-t_1)x([t_1])dt_1 \\ \quad + c^2\int_0^t\int_0^{[t_1]} \phi_1(t-t_1)\ \phi_1([t_1]-t_2)x([t_2])dt_2dt_1 + \dots \\ \quad + c^k\int_0^t\int_0^{[t_1]} \cdots \int_0^{[t_{k-1}]} \phi_1(t-t_1)\ \phi_1([t_1]-t_2) \dots \\ \quad \dots \phi_1([t_{k-1}]-t_k)x([t_k])dt_k \dots dt_2dt_1\}, & t \ge 0. \end{cases}$$

PROOF: The proof can be formulated by following the argument of Theorem 2.1. The details are omitted.

The following examples illustrates the method of steps to solve equation (2.34).

EXAMPLE 2.4: Consider the equation (2.34), with $\tau = 1$ and initial function

$$\phi(t) = \begin{cases} 0, & -1 \le t < 0, \\ 1 & t = 0. \end{cases}$$

The solution y(t) in [0,1) is given by

$$y(t) = \exp(at)(1+a^{-1}c) - a^{-1}c$$

and hence

$$y(1) = \exp(a)(1+a^{-1}c)-a^{-1}c.$$

In [1,2), we obtain the solution y(t) as

$$y(t) = y(1)\{\exp(a(t-1))(1+a^{-1}c)-a^{-1}c\}$$

$$+ b\int_1^t \exp(a(t-s))\{\exp(a(s-1))(1+a^{-1}c)-a^{-1}c\}ds$$

$$= y(1)\{\exp(a(t-1))(1+a^{-1}c)-a^{-1}c\} +$$

$$b\exp(a(t-1))(1+a^{-1}c)(t-1)-b\,a^{-2}c\{1-\exp(a(t-1))\}.$$

Continuing the same procedure, we get the solution for any $t \in [n,n+1)$, n is a positive integer.

<u>**EXAMPLE 2.5:**</u> Consider the delay differential equation

$$x'(t) = bx(t-1) + cx([t]), \quad t \geq 0$$

where b, c are constants, with initial function x(t) = 1 in $-1 \leq t \leq 0$.

In [0,1), the above equation becomes $x'(t) = b+c$

Hence $x(t) = x(0) + (b+c)t$.

Clearly $x(1) = 1 + b + c$.

In [1,2),

$x'(t) = b(x(0) + (b+c)(t-1)) + cx(1)$.

Hence

$$x(t) = x(1) + bx(0)(t-1) + (b+c)\frac{(t-1)^2}{2} + cx(1)(t-1).$$

Following the same way, that is by using 'method of steps' we obtain the solution

$$x(t) = \sum_{j=0}^{[t]} b^j x([t]-j)\left\{ \frac{(t-[t])^j}{j!} + c\frac{(t-[t])^{j+1}}{(j+1)!} \right\} + b^{[t]+1}\frac{(t-[t])^{[t]+1}}{([t]+1)!}, \quad t \geq 0,$$

where

$$x([t]-j) = \sum_{k=0}^{[t]-j-1} b^k x([t]-j-k-1)\left\{ \frac{1}{k!} + \frac{c}{(k+1)!} \right\} + \frac{b^{[t]-j}}{([t]-j)!}$$

for $j = 0, 1, 2, \ldots [t]-1$, and $x(0) = 1$.

REMARK 2.6: When $b = 0$, we obtain

$$x(t) = (1+c(t-[t]))(1+c)^{[t]}.$$

REMARK 2.7: When $c = 0$, we have

$$x(t) = \sum_{j=0}^{[t]} b^j x([t]-j)\frac{(t-[t])^j}{j!} + b^{[t]+1}\frac{(t-[t])^{[t]+1}}{([t]+1)!}, \qquad t \geq 0$$

where

$$x([t]-j) = \sum_{k=0}^{[t]-j} \frac{b^k x([t]-j-k-1)}{k!} \text{ for } j = 0, 1, 2, \ldots [t]-1$$

and $x(-1) = 1$.

EXAMPLE 2.6: Consider the scalar delay differential equation

$$x'(t) = bx(t-2) + cx([t])$$

where b, c are constants and $t \geq 0$, with initial function

$$x(t) = 1 \text{ in } -2 \leq t \leq 0.$$

Using the method of steps, we obtain the solution

$$x(t) = \begin{cases} \displaystyle\sum_{k=0}^{[t]/2} x([t]-2k)b^k\left\{ \frac{(t-[t])^k}{k!} + \frac{c(t-[t])^{k+1}}{(k+1)!} \right\} \\ \qquad + \dfrac{b^{([t]/2)+1}(t-[t])^{([t]/2)+1}}{(([t]/2)+1)!} \\ \quad \text{if } [t] \text{ is even} \\ \displaystyle\sum_{k=0}^{([t]-1)/2} x([t]-2k)b^k\left\{ \frac{(t-[t])^k}{k!} + \frac{c(t-[t])^{k+1}}{(k+1)!} \right\} \\ \qquad + \dfrac{b^{([t]+1)/2}(t-[t])^{([t]+1)/2}}{(([t]+1)/2)!} \\ \quad \text{if } [t] \text{ is odd.} \end{cases}$$

In order to calculate $x([t]-2k) = x(n)$ (say), we use

$$x(n) = \begin{cases} \displaystyle\sum_{j=0}^{(n/2)-1} x(n-2j-1)b^j\left\{ \frac{1}{j!} + \frac{c}{(j+1)!} \right\} + \frac{b^{n/2}}{(n/2)!} \\ \quad \text{if } n \text{ is even} \\ \displaystyle\sum_{j=0}^{(n-1)/2} x(n-2j-1)b^j\left\{ \frac{1}{j!} + \frac{c}{(j+1)!} \right\} + \frac{b^{(n+1)/2}}{((n+1)/2)!} \\ \quad \text{if } n \text{ is odd.} \end{cases}$$

The following result gives an estimate on the growth of the solution of (2.34), (2.36). In the proof, we use the inequalities (1.21) and (1.22) given in chapter 1.

THEOREM 2.9: Let k be a constant given in (1.21) and (1.22). Then the solution $y(t)$ of (2.34),(2.36) satisfies the estimate

$$|y(t)| \leq k\exp(\alpha t)|\phi|(1-\frac{|c|k}{\alpha}\{\exp(\alpha)-1\})^{-1}$$

$$.(1-\frac{|c|k}{\alpha}\{\exp(\alpha)-\exp(\alpha(t-[t]))\}).$$

PROOF: By using relation (6.1) of [16, page 21], we can write the solution y(t) of (2.34) in the interval [n,n+1)

$$y(t) = x(t) + |c| \int_n^t \phi_1(t-s)y(n)ds,$$

where x(t) is the solution of (2.33) and $\phi_1(t)$ is its fundamental solution with $\phi_1(0) = 1$.

Hence,

$$|y(t)| \leq |x(t)| + |c| \int_n^t |\phi_1(t-s)||y(n)|ds \tag{2.37}$$

From (2.37), it follows that

$$|y(r)| \leq |x(r)| + |c| \int_{r-1}^r |\phi_1(r-s)||y(r-1)|ds \tag{2.38}$$

for r = 1, 2, ... n. Use the recurrence relation (2.38) successively to get a bound for $|y(n)|$ interms of $|x(n)|$, $|\phi_1|$ and $|\phi|$. In view of the estimates (1.21) and (1.22), we get

$$|y(n)| \leq k\exp(\alpha n)|\phi|(1-\frac{|c|M}{\alpha}\{\exp(\alpha)-1\})^{-1}.$$

Substitute $|y(n)|$ in (2.37) and use the estimates for x(t) from (1.22) and for $\phi_1(t)$ from (1.21) to obtain the required result.

REMARK 2.8: When c = 0, in the above estimate of y(t), we get estimate (1.22).

Next we state a theorem which gives the solution of the perturbed equation (2.35).

THEOREM 2.10: If $y(t)$ is the solution of (2.34), (2.36), then the unique solution $z(t)$ of (2.35), (2.36) is given by

$$z(t) = y(t) + \sum_{k=1}^{[t]} \int_{k-1}^{k} \psi_1(t,k)\, \phi_1(k,s) f(s)ds + \int_{[t]}^{t} \phi_1(t,s) f(s)ds,$$

where $\phi_1(t)$ is the fundamental solution of (2.33),

$$\phi_1(t,s) = \phi_1(t-s), \quad 0 \le s \le t.$$

$$\psi_1(t,k) = \phi_1(t,k) + c\int_{k}^{t} \psi_1([s],k)\, \phi_1(t,s)ds \quad \text{for } t > k$$

and $\psi_1(t,k) = 1$ for $t = k$.

2.9. FUNCTIONAL DIFFERENTIAL EQUATIONS

In this section, we have obtained some similar results as in the previous section for the following equations

$$x'(t) = ax(t) + bL(x(t+\theta)) \tag{2.39}$$

$$y'(t) = ay(t) + bL(y(t+\theta)) + cy([t]) \tag{2.40}$$

$$z'(t) = az(t) + bL(z(t+\theta)) + cz([t]) + f(t) \tag{2.41}$$

with initial conditions

$$x(t) = y(t) = z(t) = \phi(t), \quad -\tau \le t \le 0, \tag{2.42}$$

where L is a linear operator mapping $C[[-\tau,0],\mathbb{R}] \longrightarrow \mathbb{R}$ for each $t \ge 0$, a, b, c are real constants, ϕ is a continuous real-valued function defined on $[-\tau,0]$, τ being a constant and f is a continuous function on J.

By iterative method we can get the solution of (2.40)

THEOREM 2.11: Let $x(t)$ be solution of (2.39), (2.42) and $\phi_2(t,s)$ is its fundamental solution (See Definition 1.4). Then there exists a unique solution to the IVP (2.40), (2.42) given by

$$y(t) = \lim_k \{x(t) + c \int_0^t \phi_2(t,t_1)x([t_1])dt_1 + c^2 \int_0^t \int_0^{[t_1]} \phi_2(t,t_1)\phi_2([t_1],t_2)x([t_2])\, dt_2 dt_1 + \cdots + c^k \int_0^t \int_0^{[t_1]} \cdots \int_0^{[t_{k-1}]} \phi_2(t,t_1)\,\phi_2([t_1],t_2) \cdots \phi_2([t_{k-1}],t_k)x([t_k])dt_k \ldots dt_2 dt_1\}, \quad t \geq 0.$$

The details of proof are omitted.

The next theorem provides a variation of parameters formula for the equation (2.41), (2.42). The proof is not given.

THEOREM 2.12: Let $y(t)$ be the solution of (2.40),(2.42). Then the unique solution of (2.41) with (2.42) is given by

$$z(t) = y(t) + \sum_{k=1}^{[t]} \int_{k-1}^{k} \psi_2(t,k)\,\phi_2(k,s)f(s)ds + \int_{[t]}^{t} \phi_2(t,s)f(s)ds,$$

$t \geq 0$, where $\phi_2(t,s) = \phi_2(t-s)$ is the fundamental solution of (2.39),

$$\psi_2(t,k) = \phi_2(t,k) + c \int_k^t \psi_2([s],k)\,\phi_2(t,s)ds \text{ for } t > k$$

and

$$\psi_2(t,k) = 1 \quad \text{for } t = k.$$

Here we introduce a new class of delay and functional differential equations of the form

$$x'(t) = ax(t) + bx([t]) + cx((t)) \tag{2.43}$$

$$x'(t) = L(x_{[t]}) \tag{2.44}$$

where a, b, c are real constants and the notation (t) denotes the fractional part of t. L is a linear operator on a Banach space $C[[-1,0],\mathbb{R}]$ with supnorm. As in [16], we define

$$x_{[t]}(\theta) = x([t]+\theta), t \in [0,\infty) = J, \ \theta \in [-1,0].$$

Thus for each $t \in J$, $x_{[t]}$ defines an element of the space $C[[-1,0],\mathbb{R}]$.

To avoid repeatation, we omit existence of solution, variation of parameters formula for these equations. The following two examples are illustrative.

EXAMPLE 2.7: Consider (2.43) with initial condition $x(0) = 1$. For computing solution of (2.43), we use method of steps in each interval. In [0,1), we have

$$x'(t) = (a+c)x(t) + b.$$

Hence

$$x(t) = \exp((a+c)t)(1+(a+c)^{-1}b) - (a+c)^{-1}b.$$

In the limit,

$$x(1) = \exp(a+c)(1+(a+c)^{-1}b) - (a+c)^{-1}b.$$

in the interval [1,2),

$$x'(t) = ax(t) + bx(1) + cx(t-1).$$

Hence

$$x(t) = x(1)\exp(a(t-1)) + \int_1^t (bx(1) + cx(s-1))\exp(a(t-s))ds$$

$$= x(1)\exp(a(t-1)) + \int_1^t bx(1)\exp(a(t-s))ds$$

$$+ \int_1^t c\exp(at+cs-a-c)(1+(a+c)^{-1}b)ds - \int_1^t c(a+c)^{-1}b\exp(a(t-s))ds$$

$$= x(1)\exp(a(t-1)) + a^{-1}bx(1)\{1-\exp(a(t-1))\}$$

$$+ \{\exp((a+c)(t-1)) - \exp(at-a)\}(1+(a+c)^{-1}b)$$

$$- a^{-1}(a+c)^{-1}cb\{1-\exp(a(t-1))\}.$$

In the same way, we can get the solution in the interval [n,n+1), n = 2,3,... .

EXAMPLE 2.8: Let the operator L in (2.44) be defind by

$$L(x_{[t]}) = \int_{-1}^0 x_{[t]}(u)du.$$

Suppose x(t) satisfies the initial function $\phi(t) = \exp(t), -1 \leq t \leq 0$. The solution of (2.44) in [0,1) is the solution of

$$x'(t) = \int_{-1}^0 x(u)du = \int_{-1}^0 \exp(u)du.$$

Hence

$$x(t) = 1 + (1-\exp(-1))t.$$

In the limit

$$x(1) = 2-\exp(-1).$$

In [1,2), the equation becomes

$$x'(t) = \int_{-1}^{0} x(u+1)du$$

$$= \int_{-1}^{0} \{1+(1-\exp(-1))(u+1)\}du.$$

Hence

$$x(t) = (1/2)\{1-\exp(-1)\} + (1/2)\{3-\exp(-1)\}t.$$

Following the same method of steps we get the solution in each interval [n, n+1), n = 2, 3, ...

CHAPTER 3

ON DIFFERENTIAL EQUATION ALTERNATELY OF RETARDED AND ADVANCED TYPE

3.1 INTRODUCTION

In [10], Cooke and Wiener studied an interesting differential equation alternately of retarded and advanced type. They have shown that all types of equations with piecewise constant arguments have similar characteristics.

The method of variation of parameters is one of the most important technique in the study of the qualitative properties of ordinary differential equations. In particular, perturbation theory heavily depends on this method. Integral inequalities also play a useful role in the qualitative behaviour of solutions of differential equations.

In section 3.2, we establish variation of parameters formula for the equation (3.3), which is given below. In section 3.3, the well known Gronwall's integral inequality is extended for the equation alternately of retarded and advanced type. Equations with two types of delays are studied in section 3.4.

Consider the following equations

$$x'(t) = a(t)x(t) \tag{3.1}$$

$$y'(t) = a(t)y(t) + b(t)y(2[(t+1)/2]) \tag{3.2}$$

$$z'(t) = a(t)z(t) + b(t)z(2[(t+1)/2]) + f(t) \tag{3.3}$$

for $t \in J$, with the initial conditions

$$x(0) = y(0) = z(0) = c_o \tag{3.4}$$

where a, b, f are real-valued continuous functions of t defined on J and c_o is a real constant.

We use the following notations.

$$\lambda(t) = (1+a^{-1}b)\exp(at)-a^{-1}b \tag{3.5}$$

$$\lambda(t,2n) = \exp(\int_{2n}^{t} a(p)dp) + \int_{2n}^{t} \exp(\int_{s}^{t} a(p)dp)b(s)ds. \tag{3.6}$$

The existence of solution of (3.2), (3.4) is already mentioned in section 1.5 of chapter 1. To obtain closed form solution of (3.2), (3.4), we prove the following theorem.

THEOREM 3.1: The IVP (3.2), (3.4) has a unique solution

$$y(t) = c_o \frac{\prod_{k=0}^{[(t+1)/2]-1} \lambda(2k+1,2k)}{\prod_{k=1}^{[(t+1)/2]} \lambda(2k-1,2k)} \lambda(t,2[(t+1)/2]), \quad t \in J,$$

if $\lambda(2k-1,2k) \neq 0$ for $k = 1, 2, \ldots [(t+1)/2]$.

PROOF: Assume that $y_n(t)$ is a solution of (3.2), (3.4) in $[2n-1,2n+1)$. Further, let $y_n(2n) = c_{2n}$ for $n = 0, 1, 2, \ldots$. It can be verified that the solution of (3.2) in $[2n-1, 2n+1)$ is

$$y_n(t) = c_{2n} \lambda(t,2n). \tag{3.7}$$

Put $t = 2n-1$, to get

$$y_n(2n-1) = c_{2n-1} = c_{2n} \ \lambda(2n-1,2n) . \tag{3.8}$$

In the limit as $t \longrightarrow 2n+1$,

$$y_n(2n+1) = c_{2n+1} = c_{2n} \ \lambda(2n+1,2n) . \tag{3.9}$$

Eliminate c_{2n} from (3.8) and (3.9), we obtain

$$c_{2n+1} = c_{2n-1} \frac{\lambda(2n+1,2n)}{\lambda(2n-1,2n)} . \tag{3.10}$$

Application of (3.10) repeatedly for c_{2n-1}, c_{2n-3}, ... c_3, yields

$$c_{2n+1} = c_1 \prod_{k=1}^{n} \frac{\lambda(2k+1,2k)}{\lambda(2k-1,2k)} , \quad n = 1, 2, \ldots . \tag{3.11}$$

Observe that

$$c_1 = y_o(1) = c_o \ \lambda(1,0). \tag{3.12}$$

Use (3.11) to obtain the value of c_{2n-1} and then use (3.8) and (3.12) to find

$$c_{2n} = c_o \prod_{k=1}^{n-1} \frac{\lambda(2k+1,2k)}{\lambda(2k-1,2k)} \ \frac{\lambda(1,0)}{\lambda(2n-1,2n)} .$$

Substitute for c_{2n} in (3.7), we obtain

$$y_n(t) = c_o \frac{\prod_{k=0}^{n-1} \lambda(2k+1,2k)}{\prod_{k=1}^{n} \lambda(2k-1,2k)} \qquad \lambda(t,2n), \tag{3.13}$$

$2n-1 \leq t < 2n+1$.

If we take $n = [(t+1)/2]$, then (3.13) is true for any $t \in J$ and hence, write $y_n(t) = y(t)$ for $t \in J$. Hence the theorem.

3.2 VARIATION OF PARAMETERS METHOD

It is of interest to determine the explicit form that the variation of parameters formula takes for the equation (3.3).

Let ϕ denote fundamental solution of (3.1) such that $\phi(0) = 1$. Next we define the fundamental solution of (3.2).

DEFINITION 3.1: A solution $\psi(t)$ of (3.2) is said to be a fundamental solution if it satisfies (3.2) with the initial condition $\psi(0) = 1$.

We use below the notation $\psi(t,k) = \psi(t) \cdot \psi^{-1}(k)$, $k = 1, 2, \ldots$. In the following theorem, we develop the variation of parameters formula.

THEOREM 3.2: The unique solution of (3.3), (3.4) for $t \in J$ is given by

$$z(t) = y(t) + \sum_{k=0}^{[(t+1)/2]-1} \lambda^{-1}(1) \int_{2k}^{2k+1} \psi(t,2k)\, \phi(2k+1,s) f(s)\,ds$$

$$- \sum_{k=0}^{[(t+1)/2]} \lambda^{-1}(-1) \int_{2k}^{2k-1} \psi(t,2k)\, \phi(2k-1,s) f(s)\,ds$$

$$+ \int_{2[(t+1)/2]}^{t} \phi(t,s) f(s)\,ds \qquad (3.14)$$

where ϕ and ψ are fundamental solutions of (3.1) and (3.2), respectively, $y(t)$ is the solution of (3.2), (3.4) and $\lambda(t)$ is given by (3.5).

PROOF: It is enough to prove that

$$\tilde{z}(t) = \sum_{k=0}^{[(t+1)/2]-1} \lambda^{-1}(1) \int_{2k}^{2k+1} \psi(t,2k)\,\phi(2k+1,s)f(s)ds$$

$$- \sum_{k=1}^{[(t+1)/2]} \lambda^{-1}(-1) \int_{2k}^{2k-1} \psi(t,2k)\,\phi(2k-1,s)f(s)ds$$

$$+ \int_{2[(t+1)/2]}^{t} \phi(t,s)f(s)ds,$$

is a solution of (3.3). Differentiate $\tilde{z}(t)$ and use (3.1) and (3.2), to find

$$\tilde{z}'(t) = \sum_{k=0}^{[(t+1)/2]-1} \lambda^{-1}(1) \int_{2k}^{2k+1} \{a(t)\psi(t,2k)+b(t)\psi(2[(t+1)/2],2k)\}$$

$$\cdot\ \phi(2k+1,s)f(s)ds$$

$$- \sum_{k=1}^{[(t+1)/2]} \lambda^{-1}(-1) \int_{2k}^{2k-1} \{a(t)\psi(t,2k)+b(t)\psi(2[(t+1)/2],2k)\}$$

$$\cdot\ \phi(2k-1,s)f(s)ds$$

$$+ \int_{2[(t+1)/2]}^{t} a(t)\phi(t,s)f(s)ds + f(t)$$

$$= a(t)\tilde{z}(t) + b(t)\tilde{z}(2[(t+1)/2]) + f(t).$$

The proof is complete.

For the purpose of simplicity, we prove the next theorem, which verifies the relation (3.14) for equation (3.3) with constant coefficients. The result can be generalized to equations of the type (3.3) with minor modifications.

THEOREM 3.3: The unique solution z(t) of (3.3), (3.4) with constant functions a(t) = a, c(t) = c, on J is given by the relation (3.14), where $\phi(t) = \exp(at)$ and $\psi(t)$ is given by (1.14) with $c_o = 1$ and y(t) is the corresponding solution of (3.2), (3.4).

PROOF: Assume that $y_n(t)$ and $z_n(t)$ are solutions of (3.2) and (3.3) in the interval [2n-1, 2n+1), respectively. Further, let $z_n(2n) = d_{2n}$, for n = 0, 1, 2 ... .

It is easy to verify that the solution of (3.3) in [2n-1, 2n+1) is

$$z_n(t) = d_{2n}\ \lambda(t-2n) + \int_{2n}^{t} \exp\{a(t-s)\}f(s)ds \tag{3.15}$$

where λ is given by (3.5). From (3.15), we obtain

$$z_n(2n-1) = d_{2n-1} = d_{2n}\ \lambda(-1) + \int_{2n}^{2n-1} \exp\{a(2n-1-s)\}f(s)ds \tag{3.16}$$

and in the limit

$$z_n(2n+1) = d_{2n+1} = d_{2n}\ \lambda(1) + \int_{2n}^{2n+1} \exp\{a(2n+1-s)\}f(s)ds. \tag{3.17}$$

Eliminating d_{2n} from (3.16) and (3.17), we get

$$d_{2n+1} = \frac{\lambda(1)}{\lambda(-1)}\ \{d_{2n-1} - \int_{2n}^{2n-1} \exp\{a(2n-1-s)\}f(s)ds\}$$

$$+ \int_{2n}^{2n+1} \exp\{a(2n+1-s)\}f(s)ds. \tag{3.18}$$

Application of (3.18) repeatedly for d_{2n-1}, d_{2n-3}, ... d_3, yields

$$d_{2n+1} = \left(\frac{\lambda(1)}{\lambda(-1)} \right)^n d_1 + \sum_{k=1}^{n} \left(\frac{\lambda(1)}{\lambda(-1)} \right)^{n-k}$$

$$\cdot \left\{ -\frac{\lambda(1)}{\lambda(-1)} \int_{2k}^{2k-1} \exp\{a(2k-1-s)\}f(s)ds \right.$$

$$\left. + \int_{2k}^{2k+1} \exp\{a(2k+1-s)\}f(s)ds \right\} \qquad (3.19)$$

for $n = 0, 1, 2, \ldots$.

Observe that from (3.15) for $n = 0$ and $t = 1$, one gets

$$d_1 = z_o(1) = \lambda(1)d_o + \int_o^1 \exp\{a(1-s)\}f(s)ds. \qquad (3.20)$$

Now we obtain the value of d_{2n-1} from (3.19) and then use (3.16) and (3.20), to obtain

$$d_{2n} = \left(\frac{\lambda(1)}{\lambda(-1)} \right)^n d_o$$

$$- \lambda^{-1}(-1) \sum_{k=1}^{n} \left(\frac{\lambda(1)}{\lambda(-1)} \right)^{n-k} \int_{2k}^{2k-1} \exp\{a(2k-1-s)\}f(s)ds$$

$$+ \lambda^{-1}(-1) \sum_{k=0}^{n-1} \left(\frac{\lambda(1)}{\lambda(-1)} \right)^{n-k-1} \int_{2k}^{2k+1} \exp\{a(2k+1-s)\}f(s)ds. \qquad (3.21)$$

Substitute (3.21) in (3.15) and use the fact $d_o = c_o$, to find

$$z_n(t) = y_n(t) + \sum_{k=0}^{n-1} \lambda^{-1}(1) \int_{2k}^{2k+1} \lambda(t-2n)\left(\frac{\lambda(1)}{\lambda(-1)}\right)^{n-k} \exp\{a(2k+1-s)\} f(s)\,ds$$

$$- \sum_{k=1}^{n} \lambda^{-1}(-1) \int_{2k}^{2k-1} \lambda(t-2n)\left(\frac{\lambda(1)}{\lambda(-1)}\right)^{n-k} \exp\{a(2k-1-s)\} f(s)\,ds$$

$$+ \int_{2n}^{t} \exp\{a(t-s)\} f(s)\,ds, \quad 2n-1 \le t < 2n+1. \qquad (3.22)$$

If we take $n = [(t+1)/2]$, then (3.22) is true for any t and hence, write $z_n(t) = z(t)$, $y_n(t) = y(t)$, for $t \in J$. Observe that

$$\lambda(t-2n)\left(\frac{\lambda(1)}{\lambda(-1)}\right)^{n-k} = \psi(t,2k)$$

and hence, we get (3.22) in the form (3.14).

3.3 GRONWALL TYPE INTEGRAL INEQUALITY

In this section, we extend the well known Gronwall's integral inequality, which has got many applications in the theory of differential equations.

THEOREM 3.4: Let c_o, a, b be nonnegative constants and $u \in C[J,R^+]$. If the inequality

$$u(t) \le c_o + \int_0^t (au(s) + bu(2[(s+1)/2]))\,ds, \quad t \in J, \qquad (3.23)$$

holds and $\lambda(-1) \neq 0$, then for $t \in J$

$$u(t) \leq c_0 \lambda(t-2[(t+1)/2])\left(\frac{\lambda(1)}{\lambda(-1)}\right)^{[(t+1)/2]}, \tag{3.24}$$

where λ is defined in (3.5).

PROOF: From (3.23), we have in $[2n, 2n+1)$

$$u(t) \leq u(2n) + \int_{2n}^{t} au(s)ds + \int_{2n}^{t} bu(2n)ds.$$

Using the Theorem 1.10, we obtain

$$u(t) \leq u(2n)\exp\left(\int_{2n}^{t} ads\right) + \int_{2n}^{t} bu(2n)\exp\left(\int_{s}^{t} adp\right)ds \tag{3.25}$$

and hence

$$u(2n+1) \leq u(2n)\ \{\exp(a)(1+a^{-1}b)-a^{-1}b\}. \tag{3.26}$$

Similarly, in the interval $[2n-1,2n]$, we obtain

$$u(2n) \leq u(2n-1)\exp(a) + a^{-1}bu(2n)(\exp(a)-1)$$

which leads to

$$u(2n) \leq u(2n-1)(\exp(-a)(1+a^{-1}b)-a^{-1}b)^{-1}. \tag{3.27}$$

Applying inequalities (3.26) and (3.27) repeatedly and using (3.5) yields

$$u(2n) \leq u(0)\left(\frac{\lambda(1)}{\lambda(-1)}\right)^{n}. \tag{3.28}$$

Use (3.28) in (3.25) and put $n = [(t+1)/2]$, we get the desired conclusion (3.24) for any $t \in J$.

REMARK 3.1: Observe that the right hand side of the inequality (3.24) is infact the solution of

$$y'(t) = ay(t) + by(2[(t+1)/2]), y(0) = c_0.$$

In this sense, (3.24) is the best estimate. When b = 0 in (3.23), (3.24) reduces to

$$u(t) \leq c_0 \exp(at), \quad t \in J.$$

REMARK 3.2: For equation (3.2) and its corresponding perturbed equation, results similar to Theorem 2.5 of chapter 2 can be proved by modifying the condition (2.26).

3.4 EQUATIONS INVOLVING TWO TYPES OF DELAYS

In this section, we consider the following more general equation involving two types of delays, namely, continuous and piecewise constant argument.

$$x'(t) = ax(t) + bx(t-\tau) \tag{3.29}$$

$$y'(t) = ay(t) + by(t-\tau) + cy(2[(t+1)/2]) \tag{3.30}$$

$$z'(t) = az(t) + bz(t-\tau) + cz(2[(t+1)/2]) + f(t) \tag{3.31}$$

$t \in J$, $\tau > 0$, with initial condition

$$x(t) = y(t) = z(t) = \bar{\Phi}(t) \text{ for } -\tau \leq t < 0 \tag{3.32}$$

where $\bar{\Phi}$ is a real-valued continuous function defined on $[-\tau, 0]$, a, b, c are real constants and f is a real-valued continuous function defined on J.

By using iterative method, we can get the existence and uniqueness of solution of (3.30). For solving (3.30), we use method of steps.

EXAMPLE 3.1: Consider

$$y'(t) = by(t-1) + cy(2[t+1)/2])$$

with initial function

$$\phi(t) = 1 \text{ in } -1 \leq t \leq 0.$$

For computing solution we use method of steps in each interval. Clearly, in [0,1) solution of the above equation is

$$y(t) = 1 + (b+c)t.$$

Taking limit at t tends to 1

$$y(1) = 1 + (b+c).$$

In the interval [1,2),

$$y(t) = y(1) + b\int_1^t (1+(b+c)(s-1))ds + \int_1^t cy(2)ds.$$

Taking limit as t tends to 2

$$y(2)(1-c) = 1 + b + c + b + b(b+c)/2$$

i.e. $y(2) = (1+2b+c+b(b+c)/2)/(1-c)$.

Hence, for $t \in [1,2)$

$$y(t) = 1 + b + c + b(t-1) + b(b+c)\frac{(t-1)^2}{2} + c\frac{(t-1)}{1-c}\{1+2b+c+b\frac{(b+c)}{2}\}.$$

In the same way, we can obtain the solution for any $t \in [n,n+1)$, $n = 2, 3, \ldots$.

Theorem 5.1 given in [16, page 19] provides a method of constructing the fundamental solution ϕ_1 of the equation (3.29). Also using the relation (6.1) of [16, page 21], we can construct the fundamental solution ψ_1 . Once the functions ϕ_1 and ψ_1 are available, the variation of parameters formula can be stated as follows.

THEOREM 3.4: The unique solution of (3.31), (3.32) is given by

$$z(t) = y(t) + \sum_{k=0}^{[(t+1)/2]-1} \lambda^{-1}(1) \int_{2k}^{2k+1} \psi_1(t,2k)\, \phi_1(2k+1,s)f(s)ds$$

$$+ \sum_{k=0}^{[(t+1)/2]} \lambda^{-1}(-1) \int_{2k}^{2k-1} \psi_1(t,2k)\phi_1(2k-1,s)f(s)ds$$

$$+ \int_{2[(t+1)/2]}^{t} \phi_1(t,s)f(s)ds \qquad t \in J ,$$

where ϕ_1 and ψ_1 are fundamental solutions of (3.29) and (3.30) , respectively, and $y(t)$ is the solution of (3.30) , (3.32). Here

$\lambda(t) = \exp(at)(1+a^{-1}b) - a^{-1}b$, $t \in J$.

The proof is omitted.

The procedure given above is applicable in respect of the functional differential equation of the form

$$y'(t) = ay(t) + bL(y(t+\theta)) + cy(2[(t+1)/2]) \text{ ,} t \in J,$$

$$y(t) = \bar{\phi}(t), \quad -\tau \leq t \leq 0$$

where L is a linear operator defined in [16, page 42]. We omit these details.

CHAPTER 4

SOME EQUATIONS OF MATHEMATICAL PHYSICS INVOLVING PIECEWISE CONSTANT DELAY

4.1. INTRODUCTION

We have seen that, in the papers [9,10,11] authors developed some theory of ordinary differential equations with piecewise constant delays. But it is known that large number of physical phenomena in applied sciences is best described in terms of partial differential equations [7,29,32]. Recently, existence theory of hyperbolic partial differential equations involving PCDA have been studied in [36].

In this chapter, we aim to study some partial differential equations of second order with PCDA. It is natural to expect that these equations will play a vital role in many applications. In section 4.2, we prove an existence theorem and generalize some problems in [32] by considering equations with piecewise constant delays. Section 4.3, deals with the study of general diffusion equation, which we solve by the method of separation of variables. Explicit solution is obtained for general wave equation by using the same method.

4.2 EXISTENCE THEORY

In this section, we consider some partial differential equations in the framework of semigroup theory. For this purpose, we introduce the following definitions [19].

DEFINITION 4.1: A family $\{T(t),\ 0 \leq t < \infty\}$ of linear operators from a Banach space X to X is called a strongly continuous semigroup on X if

(i) $T(t)$ is continuous for each $t \in J = [0, \infty)$,

(ii) $T(t+s) = T(t)T(s)$ for $t, s \in J$,

(iii) $T(0) = I$, the identity operator,

(iv) $t \to T(t)x$ is continuous for each $x \in X$.

DEFINITION 4.2: The infinitesimal generator A of $\{T(t), t \in J\}$ is the function from X to X defined by

$$Ax = \lim_{t \to 0} t^{-1}(T(t)x - x)$$

where x is in the domain D(A) of A if and only if this limit exists.

We say an operator $A : D(A) \to X$, $D(A) \subset X$, 'generates a semigroup' $\{T(t), t \in J\}$ if A is the infinitesimal generator of $\{T(t), t \in J\}$.

Now consider the nonlinear IVP

$$\frac{du(t)}{dt} = Au(t) + f(t,u([t])) \tag{4.1}$$

$$u(0) = u_o \tag{4.2}$$

where A is an operator from X to X with domain D(A) and for each $u \in X$, $t \to f(t,u([t]))$ is a piecewise continuous function and that $u_o \in D(A)$.

DEFINITION 4.3: A solution of IVP (4.1), (4.2) on J is a function $u : J \longrightarrow X$ satisfying the following conditions

(i) $u(t)$ is continuous on J,

(ii) the derivative $\frac{du(t)}{dt}$ exists at each point $t \in J$, except possibly at integral points where at $t = 0$ the right hand side derivative exists and at $t = 1, 2, 3, \ldots$ the left hand side derivative exists,

(iii) $u(t)$ satisfies (4.1), (4.2) for each interval $[n,n+1) \subset J$, n is a positive integer.

For each positive integer n and for fixed $u \in X$ define

$g_n : [n,n+1) \longrightarrow X$ by $g_n(t) = f(t,u(n))$.

The following theorem is purely an existential result.

THEOREM 4.1: Suppose that the operator A in (4.1) generates a strongly continuous semigroup on X and $u_o \in D(A)$. Assume that

$g_n \in C^1[[n,n+1) ; X]$, $n = 0, 1, \ldots$ and g_n is continuous in $[n,n+1]$. Then the IVP (4.1), (4.2) has a unique solution.

PROOF: In the interval $[0,1)$, $f(t,u([t])) = f(t,u_o) = g_o(t)$. By a theorem in [15, page 84] the problem (4.1), (4.2) has a unique solution of the form

$$u_o(t) = T(t)u_o + \int_o^t T(t-s)g_o(s)ds, \quad t \in [0,1).$$

By taking the limit $t \longrightarrow 1$, we obtain

$$u_1 = u_o(1) = T(1)u_o + \int_o^1 T(1-s)g_o(s)ds.$$

Now consider the equation (4.1) with the initial condition

$$u(1) = u_1. \qquad (4.2)'$$

By the same theorem [15, page 84], it has a unique solution

$$u_1(t) = T(t-1)u_1 + \int_1^t T(t-s)g_1(s)ds, \quad t \in [1,2),$$

where $g_1(s) = f(s,u_1)$. Clearly $u_o(1) = u_1(1) = u_1$. Thus, we have obtained a solution of (4.1), (4.2)' in [1,2). Repeating the procedure, we get solution $u_n(t)$ in $[n,n+1)$. Now, let

$$u(t) = \sum_{n=0}^{[t]} u_n(t) \quad \chi_{[n,n+1)}(t), \quad t \in J;$$

where χ is the characteristic function. Then it is seen that the function u(t) is the unique solution of (4.1), (4.2).

In the following theorem, we would like to give an iterative procedure to 'approximate' the solution of (4.1), (4.2) in the case when f is a linear function with respect to the second variable.

<u>THEOREM 4.2:</u> If $f(t,u([t])) = bu([t])$ where b is a constant and $u_o \in D(A)$, then the unique solution of (4.1), (4.2) for $t \in J$ is given by

$$u(t) = \lim_{n\to\infty} \{T(t) + b\int_o^t T(t-t_1)T([t_1])dt_1 +$$

$$b^2\int_o^t \int_o^{[t_1]} T(t-t_1)T([t_1]-t_2)T([t_2])dt_2dt_1 + \ldots +$$

$$b^n\int_o^t \int_o^{[t_1]} \ldots \int_o^{[t_{n-1}]} T(t-t_1)T([t_1]-t_2) \ldots$$

$$\ldots T([t_n])dt_n \ldots dt_2dt_1\}u_o \,. \qquad (4.3)$$

PROOF: It is known that $C[[0,a],X]$, the space of continuous functions, is complete with supnorm. Let us consider the space $C_\lambda[[0,a],X] = C_\lambda$, $a > 0$, $\lambda \geq 0$ of continuous functions with norm

$$|x|_\lambda = \sup_{t\in[0,a]} \{|x(t)|\exp(-\lambda \int_0^t |bT(t-s)|ds)\}.$$

So that $C_0 = C$. Observe that the norms $|x|_\lambda$ are all equivalent for $\lambda \geq 0$, so that C_λ is also complete. Now consider the mapping $P : C_\lambda \longrightarrow C_\lambda$ defined by

$$(Pu)(t) = T(t)u_0 + b\int_0^t T(t-s)u([s])ds.$$

It is easy to show that

$$|Pu_1-Pu_2|_\lambda \leq \lambda^{-1}|u_1-u_2|_\lambda \quad \text{for } \lambda > 0, \text{ so that P is a contraction}$$

for $\lambda > 1$. Hence by Banach fixed point theorem, there exists a unique u in C_λ such that

$$(Pu)(t) = u(t) = T(t)u_0 + b\int_0^t T(t-s)u([s])ds.$$

Defining

$$u^0(t) = u_0$$

$$u^n(t) = T(t)u_0 + b\int_0^t T(t-s)u^{n-1}([s])ds, \; n = 1, 2, \ldots$$

and by using successive approximations the required result (4.3) now follows.

EXAMPLE 4.1: Consider the functional differential equation

$$u_t(x,t) = a^2 u_{xx}(x,t) + f(t,u(x,[t])),\ t \in J \tag{4.4}$$

satisfying the conditions

$$u(0,t) = u(L,t) = 0,\ u(x,0) = u_o(x),\ L > 0, \tag{4.5}$$

where f satisfies the condition of Theorem 4.1 and a is a nonzero real number. If we write $u(.,t) = \tilde{u}(t)$, (4.4), (4.5) can be viewed as the problem (4.1), (4.2).

Let $X = L_2[0,L]$, $A = a^2 \frac{d^2}{dx^2}$ with domain

$$D(A) = \{\ \phi \in X,\ \phi \text{ and } \frac{d\phi}{dx} \text{ are absolutely continuous,}$$

$$\frac{d^2\phi}{dx^2} \in X,\quad \phi(0) = \phi(L) = 0\}.$$

A generates a strongly continuous semigroup $T(t)$ such that $|T(t)| \leq 1$ on X. Here $T(t)$ is given by

$$(T(t)g)(x) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \exp(-y^2/4a^2t)g(x-y)dy.$$

Using Theorem 4.1, we conclude that (4.4),(4.5) has a unique solution.

In [32, page 280-288], the author solves two problems

(i) diffusion of moisture through porous solid, and

(ii) diffusion theory used for qualitative understanding of the physical characteristics of nuclear reators. We generalize these results below by considering equations with PCDA. Definition 4.1 can be suitably modified for equations considered below.

EXAMPLE 4.2: The diffusion of moisture through porous solid can be studied by means of diffusion equation.

Assume the following

(i) the rate of diffusion across an element of surface is proportional to the area and to the space rate of change of concentration normal to the area,

(ii) the diffusion takes place from regions of higher concentration to those of lower concentration,

(iii) the medium is homogenous and isotropic,

(iv) the diffusion coefficient is constant.

Then, in cylindrical co-ordinates, the diffusion equation is

$$\frac{\partial \sigma}{\partial t} = K \left\{ \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \sigma}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 \sigma}{\partial \theta^2} + \frac{\partial^2 \sigma}{\partial z^2} \right\}$$

where K is diffusion coefficient and $\sigma = \sigma(t,\theta,z,r)$. The corresponding perturbed equation can be written as

$$\frac{\partial \sigma}{\partial t} = K \left\{ \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \sigma}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 \sigma}{\partial \theta^2} + \frac{\partial^2 \sigma}{\partial z^2} \right\} + \lambda \sigma([t],r,\theta,z), \quad (4.6)$$

where λ is a constant. Since there is no physical factor which can introduce an asymmetry in the θ direction, the required solution will be independent of θ, we assume

$$\sigma = T(t)R(r)Z(z). \quad (4.7)$$

By putting (4.7) in (4.6), we obtain

$$\frac{1}{T(t)}\left\{\frac{dT(t)}{dt} - \lambda T([t])\right\} = K\left\{\frac{1}{rR(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) + \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}\right\} \tag{4.8}$$

By making the assumptions

$$\frac{1}{rR(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) = -\alpha^2 \tag{4.9}$$

$$\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = -\beta^2 \tag{4.10}$$

(4.8) becomes

$$\frac{dT(t)}{dt} = -K(\alpha^2 + \beta^2)T(t) + \lambda T([t]). \tag{4.11}$$

It is known that solutions of (4.9) are Bessel's functions and solutions of (4.10) are sine and cosine functions. Theorem 1.1 (proved for a scalar case) gives solution of (4.11). Hence, the product solution is

$$\sigma(r,t,z) = \theta(t-[t])(\theta(1))^{[t]}\{c_1J_o(\alpha r) + c_2Y_o(\alpha r)\}$$

$$\cdot\ \{c_3\cos\beta z + c_4\sin\beta z\}$$

where

$$\theta(t) = \exp(-K(\alpha^2+\beta^2)t)(1-(K(\alpha^2+\beta^2))^{-1}\lambda) + (K(\alpha^2+\beta^2))^{-1}\lambda$$

and c_1, c_2, c_3, c_4 are arbitrary constants.

<u>**REMARK 4.1:**</u> (i) For fixed r and z, the solution σ is zero if

$$\lambda = \frac{-K(\alpha^2+\beta^2)\exp\{-K(\alpha^2+\beta^2)\}}{1-\exp\{-K(\alpha^2+\beta^2)\}}$$

(ii) the zero solution of (4.4) is stable, if

$$\lambda = \frac{-K(\alpha^2+\beta^2)(1+\exp\{-K(\alpha^2+\beta^2)\})}{1-\exp\{-K(\alpha^2+\beta^2)\}}$$

(iii) further, when $\lambda = 0$

$$\sigma(t,r,z) = \exp\{-K(\alpha^2+\beta^2)t\}\{c_1J_o(\alpha r) + c_2Y_o(\alpha r)\}$$

$$\cdot \{c_3\cos \beta z + c_4\sin \beta z\},$$

which is established in [32].

EXAMPLE 4.3: Linear diffusion theory can be used to study qualitative understanding of physical characteristics of nuclear reactors. Let $u(P,t)$ be neutron density at a point P and time t, $K(P)$ be diffusion coefficient at point P, ν be neutron speed, λ_c total neutron capture mean free path and k_c be average number of neutrons produced per capture by fission.

If we make the following assumptions

(i) Neutrons diffuse from regions of higher concentration to those of lower concentration.

(ii) The rate of diffusion across an element of surface will be proportional to the area of that surface element and to the space rate of change of neutron density normal to the surface element.

(iii) The diffusion coefficient depends only on the position within the body.

(iv) Neutron absorption is given by $u\nu/\lambda_c$.

(v) Neutron production due to fission is $k_c u\nu / \lambda_c$.

(vi) There is no distribution of neutron sources;

then the diffusion equation representing the physical situation taking K as a constant is given by

$$\frac{1}{K}\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + b^2 u(x,t),$$

where $b^2 = (k_c - 1)\nu / K\lambda_c$ is a constant. Let the perturbed equation be

$$\frac{1}{K}\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + b^2 u(x,t) + \lambda\; u(x,[t]) \tag{4.12}$$

where λ is a constant. We use the method of separation of variables to obtain its solution. Assume that $u(x,t) = X(x)T(t)$. It then follows that

$$\frac{d^2 X(x)}{dx^2} = (\alpha - b^2)X(x)$$

$$\frac{dT(t)}{dt} = K(\alpha T(t) + \lambda\; T([t])). \tag{4.13}$$

Theorem 1.1 gives solution T(t) of the equation (4.13). Hence the product solution is given by

$$u(x,t) = \theta(t-[t])(\theta(1))^{[t]}\{c_1 \cos\sqrt{b^2-\alpha}\; x + c_2 \sin\sqrt{b^2-\alpha}\, x\} \tag{4.14}$$

where c_1, c_2 are arbitrary constants and

$$\theta(t) = \exp(\alpha K t)(1+\alpha^{-1}\lambda) - \alpha^{-1}\lambda .$$

REMARK 4.2: (i) For a fixed x, solution (4.14) is zero if

$$\lambda = \frac{\alpha \exp(\alpha K)}{1-\exp(\alpha K)} ;$$

(ii) the zero solution of (4.14) is stable if

$$\lambda = \frac{\alpha (1+\exp(\alpha K))}{1-\exp(\alpha K)} ;$$

(iii) further, if $\lambda = 0$

$$u(x,t) = \exp(\alpha K t)\{c_1 \cos\sqrt{b^2-\alpha}\, x + c_2 \sin\sqrt{b^2-\alpha}\, x\}$$

which is given in [32].

4.3 GENERAL DIFFUSION EQUATION

In this section, we consider the following equations

$$u_t(x,t) = au_{xx}(x,t) + bu_{xx}(x,[t]) \quad (4.15)$$

$$u_t(x,t) = au_{xx}(x,t) + bu_{xx}(x,[t]) + h(x,t) \quad (4.16)$$

$$u_t(x,t) = au_{xx}(x,t) + bu_{xx}(x,2[(t+1)/2]) \quad (4.17)$$

with boundary conditions

$$u(0,t) = u(\pi,t) = 0, \; u(x,0) = f(x), \; t \in J, \quad (4.18)$$

where a,b are constants, f is a real-valued continuous function defined on $[0, \pi]$ such that f' is piecewise continuous and

$h : [0, \pi] \times J \longrightarrow R$ is a given continuous function.

Below, we use the method of separation of variables to get explicit solution of (4.15), (4.18).

<u>**THEOREM 4.3:**</u> The unique solution of (4.15) together with condition (4.18) is given by

$$u(x,t) = \sum_{n=1}^{\infty} c_n \theta(t-[t])(\theta(1))^{[t]} \sin nx \tag{4.19}$$

where

$$\left.\begin{aligned} \theta(t) &= \exp(-an^2 t)(1+a^{-1}b)-a^{-1}b \\ c_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx . \end{aligned}\right] \tag{4.20}$$

<u>**PROOF:**</u> Since $u(x,t) = X(x)T(t)$ satisfies the equation (4.15), we get the separated equations

$$\frac{dT(t)}{dt} = -\lambda(aT(t) + bT([t]))$$

$$\frac{d^2X(x)}{dx^2} = -\lambda X(x)$$

where λ is a constant. If $X(x)T(t)$ is to satisfy the conditions of (4.18), then it follows that $X(0) = 0$, $X(\pi) = 0$. However, if we consider the above equation in $X(x)$ with the same boundary conditions then it has nontrivial solutions $X(x) = c_1 \sin nx$, when $\lambda = n^2$, where n is a positive integer. Application of Theorem 1.1 (a scalar case) gives the solution of above equation in $T(t)$ with $T(0) = 1$. Hence, the product solution is given by

$$u_n(x,t) = \theta(t-[t])(\theta(1))^{[t]}\sin nx, \quad n = 1, 2, \ldots .$$

Clearly a linear combination of $u_n(x,t)$, $n = 1, 2, \ldots$ also satisfies (4.15). In order to satisfy the condition $u(x,0) = f(x)$, $f(x)$ has to be in the form

$$f(x) = \sum_{n=1}^{\infty} c_n \sin nx$$

and the coefficients c_n should have the values given in (4.20). Hence the theorem.

REMARK 4.3: For each $x \in [0,\pi]$,

(i) the zero solution of (4.15) is stable if

$$b = -a\frac{(1+\exp(-an^2))}{1-\exp(-an^2)} ;$$

(ii) the zero solution of (4.15) is asymptotically stable if

$$\frac{-a(1+\exp(-an^2))}{1-\exp(-an^2)} < b < a, \quad a > 0.$$

In the following theorem, we use the same method, separation of variables, to find the solution of the perturbed equation (4.16).

THEOREM 4.4: Let $x_n(t)$ be the solution of the IVP

$$\frac{dx_n(t)}{dt} = -an^2x_n(t) - bn^2x_n([t]) \qquad (4.21)$$

$$x_n(0) = f(x) \qquad (4.22)$$

$\psi_n(t)$ be the fundamental solution of (4.21) with $\psi_n(0) = 1$ and $\phi_n(t)$ be the fundamental solution of

$$\frac{dx_n(t)}{dt} = -an^2 x_n(t)$$

with $\phi_n(o) = 1$. Then the unique solution of (4.16), (4.18) is given by

$$u(x,t) = \sum_{n=1}^{\infty} y_n(t)\sin nx, \quad t \in J,\ x \in [0,\pi],$$

where

$$y_n(t) = x_n(t) + \sum_{k=1}^{[t]} \int_{k-1}^{k} \psi_n(t,k)\, \phi_n(k,s) h_n(s)ds$$

$$+ \int_{[t]}^{t} \phi_n(t,s)h_n(s)ds, \qquad (4.23)$$

$\psi_n(t,k) = \psi_n(t)\ \psi_n^{-1}(k)$, $\phi_n(k,t) = \phi_n(k)\phi_n^{-1}(t)$, $k = 1, 2, \ldots [t]$ and h_n is defined in (4.26) below.

PROOF: Solution of (4.15) is given by (4.19). Therefore, we assume the solution of (4.16) in the form

$$u(x,t) = \sum_{n=1}^{\infty} y_n(t)\sin nx. \qquad (4.24)$$

Our aim is now to determine $y_n(t)$. Further, assume that

$$h(x,t) = \sum_{n=1}^{\infty} h_n(t)\sin nx \qquad (4.25)$$

where

$$h_n(t) = \frac{2}{\pi} \int_0^{\pi} h(x,t)\sin nx\, dx. \tag{4.26}$$

Substitute (4.24) and (4.25) in (4.16), to find that the functions $y_n(t)$ must satisfy the ordinary differential equation

$$\frac{dy_n(t)}{dt} = -an^2 y_n(t) - bn^2 y_n([t]) + h_n(t). \tag{4.27}$$

Note that $u(x,0) = y_n(0) = f(x)$. Apply Theorem 2.2 for a scalar case to get the solution of (4.27) which is given by the expression (4.23) and hence, the theorem.

In the following theorem, we apply finite sine transform method to get the solution of (4.17).

THEOREM 4.5: The unique solution of (4.17), (4.18) is given by

$$u(x,t) = \sum_{n=1}^{\infty} c_n \theta(t-2[(t+1)/2])\left(\frac{\theta(1)}{\theta(-1)}\right)^{[(t+1)/2]} \sin nx, \quad \theta(-1) \neq 0 \tag{4.28}$$

$t \in J$, $x \in [0,\pi]$, where $\theta(t)$ and c_n are given in (4.20).

PROOF: Finite sine transform of $u(x,t)$ is

$\bar{u}(n,t) = \int_0^{\pi} u(s,t)\sin ns\, ds$, $n = 1, 2, \ldots$ is a parameter.

Its inversion is

$$u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \bar{u}(n,t)\sin nx.$$

Using finite sine transform on both sides of (4.17) and applying the initial condition, we get

$$\frac{d\bar{u}(n,t)}{dt} = an^2\bar{u}(n,t) + bn^2\bar{u}(n,2[(t+1)/2]).$$

Solution of this equation is given by

$$\bar{u}(n,t) = \bar{f}(n)\theta(t-2[(t+1)/2])\left(\frac{\theta(1)}{\theta(-1)}\right)^{[(t+1)/2]}, \quad \theta(-1) \neq 0$$

where

$$\bar{f}(n) = \int_0^\pi f(s)\sin ns\, ds .$$

Using the inversion of sine transform, we get the required result (4.28).

4.4 GENERAL WAVE EQUATION

Consider a more general wave equation of the form

$$u_{tt}(x,t) = a^2u_{xx}(x,t) + b^2u_{xx}(x,[t]), \; x \in [0,\pi], \; t \in J, \qquad (4.29)$$

with boundary conditions

$$u(0,t) = u(\pi,t) = 0, \; u(x,0) = f(x), \; u_t(x,0) = 0, \qquad (4.30)$$

where $f(x)$ is a real-valued continuous function on $[0,\pi]$ such that f' is continuous and a, b are nonnegative constants.

<u>**THEOREM 4.3:**</u> The problem (4.29), (4.30) has a solution for $t \in J$ and $x \in [0,\pi]$, and is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n \left[(1+\frac{b^2}{a^2})\cos na(t-[t]) - \frac{b^2}{a^2} \quad \frac{\sin na(t-[t])}{na} \right] \begin{bmatrix} c_{[t]} \\ d_{[t]} \end{bmatrix}$$

$$. \sin nx \tag{4.31}$$

where

$$\begin{bmatrix} c_{[t]} \\ d_{[t]} \end{bmatrix} = \begin{bmatrix} (1+\frac{b^2}{a^2})\cos na - \frac{b^2}{a^2} & \frac{\sin na}{na} \\ -na(1+\frac{b^2}{a^2})\sin na & \cos na \end{bmatrix}^{[t]} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx .$$

PROOF: As before, we obtain, the following two related second order ordinary differential equations

$$\frac{d^2T(t)}{dt^2} = -\lambda(a^2T(t) + b^2T([t])) \tag{4.32}$$

$$\frac{d^2X(x)}{dx^2} = -\lambda X(x).$$

Express (4.32) in a matrix form as follows:

$$\begin{bmatrix} T(t) \\ T'(t) \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\lambda a^2 & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ T'(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\lambda b^2 & 0 \end{bmatrix} \begin{bmatrix} T([t]) \\ T'([t]) \end{bmatrix}$$

Theorem 1.4 yields the product solution in the form (4.31).

Note: We can also consider the equations

$$u_t(x,t) = au(x,t) + bu_{xx}(x,[t]) + cu_{xx}(x,t-\tau), \quad \tau > 0$$

$$u_t(x,t) = au_{xx}(x,t) + bu_{xx}(x,2[(t+1)/2]) + cu_{xx}(x,t-\tau) \quad \tau > 0$$

$$u_{tt}(x,t) = au_{xx}(x,t) + bu_{xx}(x,[t]) + cu_{xx}(x,t-\tau), \quad \tau > 0$$

or several other similar modified equations with suitable initial and boundary conditions which take care of two types of delays.

After using the method of separation of variables we get an IVP involving t-variable only. At this stage we recall the method employed in solving Examples 2.4 to 2.6. Thus it is seen that explicit solution is available for partial differential equations of the above type. The details are omitted.

CHAPTER 5

NONLINEAR DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT DELAYS

5.1 INTRODUCTION

The purpose of the present chapter is to study one of the important problems in the theory of differential equations, namely, existence of solutions. The fixed point method is known as a handy tool to settle the problems of existence. The comparison principle is another area of investigation which we do here for differential equations with PCDA. The variation of parameters formula is useful in solving the perturbation problems in differential equations. Much work [23] has been done in nonlinear variation of parameters formula for ordinary differential equations. We extend some results of this kind to differential equations with PCDA.

A method to solve a scalar nonlinear equation is included in the first chapter. In section 5.2, we extend the method to solve a system of nonlinear equations. Existence of solutions of nonlinear equations with PCDA using Schauder's fixed point theorem is given in section 5.3. Some comparison results and its applications also form a part of this section. The last section is devoted to establish a nonlinear variation of parameters formula of Alakseev type [23] for differential equations with PCDA.

5.2 METHOD OF FINDING SOLUTIONS OF A SYSTEM OF NONLINEAR EQUATIONS

Consider a nonlinear coupled system of differential equations involving piecewise constant delays

$$\left.\begin{aligned} x'(t) &= f_1(x(t),x([t]),y([t])) \\ y'(t) &= f_2(y(t),y([t]),x([t])) \end{aligned}\right\} \qquad (5.1)$$

$t \in J = [0, \infty)$, with initial condition

$$(x(0),y(0)) = (c_o,d_o) \qquad (5.2)$$

where c_o, d_o are real constants and f_1, f_2 are piecewise continuous functions on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

If the system with nonzero parameters λ_1 and λ_2 is such that $f_1(x, \lambda_1, \lambda_2) \neq 0$ and $f_2(y, \lambda_1, \lambda_2) \neq 0$ everywhere, then there exists general integrals of f_1 and f_2 denoted by F_1 and F_2 given by

$$F_1(x, \lambda_1, \lambda_2) = \int \frac{dx}{f_1(x, \lambda_1, \lambda_2)} = \int \frac{dx}{x'}$$

$$= t+g_1(\lambda_1, \lambda_2)$$

and

$$F_2(y, \lambda_2, \lambda_1) = \int \frac{dy}{f_2(y, \lambda_2, \lambda_1)} = \int \frac{dy}{y'}$$

$$= t+g_2(\lambda_2, \lambda_1).$$

with arbitrary functions $g_1(\lambda_1, \lambda_2)$ and $g_2(\lambda_2, \lambda_1)$.

For computing a solution of (5.1),(5.2), assume that

$(x_n(t), y_n(t))$ is a solution of (5.1) in the interval $[n, n+1)$, with condition $(x_n(n), y_n(n)) = (c_n, d_n)$ for $n = 0, 1, 2, \ldots$. If we put $\lambda_1 = c_n$, $\lambda_2 = d_n$, then in $[n, n+1)$, we have

$$\left.\begin{aligned} F_1(x_n(t), c_n, d_n) &= t + g_1(c_n, d_n) \\ F_2(y_n(t), d_n, c_n) &= t + g_2(d_n, c_n) \,. \end{aligned}\right\} \qquad (5.3)$$

To get the solution in $[n, n+1)$, first, we have to find the values of unknown functions g_1 and g_2. For this purpose, put $t = n$ in (5.3) to obtain

$$\left.\begin{aligned} F_1(c_n, c_n, d_n) &= n + g_1(c_n, d_n) \\ F_2(d_n, d_n, c_n) &= n + g_2(d_n, c_n) \,. \end{aligned}\right\} \qquad (5.4)$$

After solving for g_1 and g_2, using (5.3) and (5.4), we obtain for $t \in [n, n+1)$

$$\left.\begin{aligned} F_1(x_n(t), c_n, d_n) &= t + F_1(c_n, c_n, d_n) - n \\ F_2(y_n(t), d_n, c_n) &= t + F_2(d_n, d_n, c_n) - n. \end{aligned}\right\} \qquad (5.5)$$

Here (5.5) gives x_n and y_n in terms of c_n and d_n. In view of our assumptions, we have $(x_{n-1}(n), y_{n-1}(n)) = (c_{n-1}, d_{n-1})$. Further, note that $x_{n-1}(n) = x_n(n)$ and $y_{n-1}(n) = y_n(n)$. These considerations yield the recursion relation

$$\left.\begin{aligned} F_1(c_n, c_{n-1}, d_{n-1}) &= F_1(c_{n-1}, c_{n-1}, d_{n-1}) + 1 \\ F_2(d_n, d_{n-1}, c_{n-1}) &= F_2(d_{n-1}, d_{n-1}, c_{n-1}) + 1 \end{aligned}\right\} \quad (5.6)$$

From (5.6) we can calculate (c_n, d_n) for $n \geq 1$, if the initial value (c_o, d_o) is given. The expression (5.5) gives solution $(x_n(t), y_n(t))$ explicitly in each interval $[n, n+1)$, $n = 0, 1, 2, \ldots$.

Below we give an illustrative example.

EXAMPLE 5.1: Consider the system

$$x'(t) = (1/2)x(t)y^2([t]) ,$$

$$y'(t) = (1/3)y^2(t)x([t]), \quad t \geq 0$$

with initial condition $(x(0), y(0)) = (c_o, d_o)$.

Clearly, the system with nonzero parameters λ_1 and λ_2 is such that $(1/2)x \ \lambda_2^2 \neq 0$ and $(1/3)y^2 \ \lambda_1 \neq 0$ everywhere. The integrals denoted by F_1 and F_2 are now as follows.

$$F_1(x, \lambda_1, \lambda_2) = \int \frac{2\,dx}{x\,\lambda_2^2} = \frac{1}{x^2\,\lambda_2^2}$$

$$F_2(y, \lambda_2, \lambda_1) = \int \frac{3\,dy}{y^2\,\lambda_1} = \frac{1}{y^3\,\lambda_1} .$$

The expression (5.5) now becomes

$$\frac{1}{x_n^2 d_n^2} = t-n + \frac{1}{c_n^2 d_n^2}$$

$$\frac{1}{y_n^3 c_n} = t-n + \frac{1}{d_n^3 c_n}$$

which leads to a solution (x_n, y_n) in $[n, n+1)$ and is given by

$$x_n(t) = \frac{c_n}{(1-d_n^2 c_n^2 (t-n))^{\frac{1}{2}}}$$

$$y_n(t) = \frac{d_n}{(1-d_n^3 c_n (t-n))^{\frac{1}{3}}}$$

where

$$c_1 = \frac{c_o}{(1-d_o^2 c_o^2)^{\frac{1}{2}}}, \qquad d_1 = \frac{d_o}{(1-d_o^3 c_o)^{\frac{1}{3}}}$$

$$c_2 = \frac{c_1}{(1-d_1^2 c_1^2)^{\frac{1}{2}}}, \qquad d_2 = \frac{d_1}{(1-d_1^3 c_1)^{\frac{1}{3}}}$$

....

and

$$c_n = \frac{c_{n-1}}{(1-d_{n-1}^2 c_{n-1}^2)^{\frac{1}{2}}}, \qquad d_n = \frac{d_{n-1}}{(1-d_{n-1}^3 c_{n-1})^{\frac{1}{3}}}$$

5.3 EXISTENCE OF SOLUTION

In this section, we consider the nonlinear differential equation

$$x'(t) = f(t.x(t),x([t])), \quad x(0) = x_o, \quad t \in J, \tag{5.7}$$

where x_o is a given real constant and f is a piecewise continuous function defined on $J \times \mathbb{R} \times \mathbb{R}$.

The following theorem gives the existence of solutions of (5.7) using Schauder's fixed point theorem.

THEOREM 5.1: Let the function $f(t,x(t),x([t]))$ be piecewise continuous and bounded in a cube $D : 0 \leq t \leq a,\ a > 0,\ |x-x_o| \leq b,$ $|x-x_o| \leq b,\ b > 0$, then there exists atleast one solution $x(t)$ of (5.7) on $0 \leq t \leq \beta \leq a$ for some $\beta > 0$.

PROOF: Assume a to be sufficiently large. Let C(I) be the space of all continuous functions on $I = [0,a)$ taking values in $\mathbb{R}$. Define the norm $|\ |_o$ of a function $x \in C(I)$ by

$$|x|_o = \sup_{t \in I} |x(t) - x_o|.$$

Clearly, C(I) is a Banach space with the above norm. Since f is bounded in D there exists a positive number M such that

$$|f(t,x(t),x([t]))| \leq M \text{ for } (t,x(t),x([t])) \in D.$$

Choose β and ρ such that $0 \leq \beta \leq a$, $0 \leq \rho \leq b$ and $M\beta \leq \rho$.

Define

$s(\rho) = \{x \in C(I), \ |x|_0 \leq \rho\}$.

It is clear that $s(\rho)$ is a closed, convex, bounded subset of the Banach space $C(I)$. Let T be an operator defined by

$$(Tx)(t) = x(m) + \int_m^t f(s,x(s),x(m))ds$$

for $x \in s(\rho)$, $t \in [m,m+1)$, $m = 0,1,2 \ldots n$, $n \leq a < n+1$

where

$$x(m) = x(m-1) + \int_{m-1}^m f(s,x(s),x(m-1))ds, \quad m = 1, 2, \ldots .$$

Observe that $\lim_{t \to m+1} Tx(t) = Tx(m+1)$, $m = 0, 1, \ldots n$.

Clearly, T is a continuous operator from $s(\rho)$ to $C(I)$. Further, we have

$$|(Tx)(t) - x_0| \leq M\beta \leq \rho .$$

This implies that T maps $s(\rho)$ into itself. Therefore by Schauder's fixed point theorem, there exists atleast one $x \in C(I)$ such that

$$(Tx)(t) = x(t) .$$

Hence the theorem.

Next we give definitions of upper and lower solutions of (5.7) which is needed in our subsequent discussions.

<u>DEFINITION 5.1:</u> A function $v \in C[I,\mathbb{R}]$ is said to be an upper solution of (5.7) if $v'(t)$ exists except possibly at integral points

where only one-sided derivatives exist; and if

$v'(t) \geq f(t,v(t),v([t]))$ on I.

A lower solution $w(t)$ may be defined similarly by reversing the inequality.

EXAMPLE 5.1: Consider the differential equation

$x'(t) = x(t) \sin x([t]),\ x(0) = c_o.$

Clearly, $x(t) = c_o \exp(t)$ is an upper solution while

$x(t) = c_o \exp(-t)$ is a lower solution.

The theory of existence of maximal and minimal solutions heavily depends on the use of upper and lower solutions. The methods are well established and are given in [23]. Below, we extend this method to prove the existence of extremal solutions of the equation (5.7). For this purpose, we need the following definition.

DEFINITION 5.2: Let $r(t)$ be a solution of the scalar differential equation (5.7) on $I = [0,a)$. Then $r(t)$ is said to be a maximal solution of (5.7) if, for every solution $x(t)$ of (5.7) existing on I, the inequality $x(t) \leq r(t)$, $t \in I$ holds.

A minimal solution $\rho(t)$ may be defined similarly by reversing the above inequality $\rho(t) \leq x(t)$.

The following example illustrates it.

EXAMPLE 5.2: Consider the nonlinear IVP

$$x'(t) = x^{\frac{1}{2}}(t), \; x(0) = 0, \; t \in J = [0, \infty).$$

It is known that it has infinite number of solutions

$$x(t) = \begin{cases} 0 & , \quad 0 \leq t \leq c, \\ \frac{1}{4}(t-c)^2 & , \quad c \leq t < \infty \; ; \end{cases}$$

where c is a real number.

Now consider the related IVP

$$x'(t) = x^{\frac{1}{2}}(t) + kx([t]), \; x(0) = 0.$$

Observe that $x(t) \equiv 0$, $t \in J$ is a solution on the IVP. For $t \in [0,1)$, the equation coincides with the above equation $x'(t) = x^{\frac{1}{2}}(t)$. Clearly, it has infinite number of solutions. In the interval $[n,n+1)$, we use the method given in [9] and show that solution $x_n(t)$ satisfies the relation

$$2(x_n^{\frac{1}{2}}(t) + c_n) - 2c_n k\log(x_n^{\frac{1}{2}}(t) + c_n) = 2(c_n^{\frac{1}{2}} + c_n) - 2c_n k\log(c_n^{\frac{1}{2}} + c_n) + t - n$$

$t \in [n,n+1)$, $n = 0, 1, 2, \ldots$.

Intervalwise, we can show that this implicit relation gives a nonzero solution.

For $k = 0$,

$x(t) = t^2/4$.

The following result on differential inequalities is profitably used subsequently. It is an extension of the corresponding result proved in [23].

THEOREM 5.2: Let $v, w \in C[I, \mathbb{R}]$, $I = [0,a)$ with property that $v(0) < w(0)$ and let $f(t,x(t),\mu)$ be nondecreasing in μ. Further,

$$v'(t) \leq f(t,v(t),v([t])), \; t \in I \tag{5.8}$$

$$w'(t) > f(t,w(t),w([t])), \; t \in I. \tag{5.9}$$

Then

$$v(t) < w(t) \text{ for all } t \in I. \tag{5.10}$$

PROOF: Suppose (5.10) does not hold. Then there exists a $t_1 > 0$ (first t_1 away from zero) such that $v(t_1) = w(t_1)$ and $v(t) < w(t)$, $t \in [0,t_1)$. For sufficiently small $h < 0$, it then follows that

$v'(t_1) > w'(t_1)$. Using (5.8), (5.9), we get

$$f(t_1,v(t_1), v([t_1])) > f(t_1,w(t_1),w([t_1]))$$

which is a contradiction in view of the fact that $f(t,x,\mu)$ is nondecreasing in μ. Hence $v(t) < w(t)$ on I.

REMARK 5.1: Theorem 5.2 follows if we replace the inequalities (5.8),(5.9) by

$$v'(t) < f(t,v(t),v([t]))$$

$$w'(t) \geq f(t,w(t),w([t])),$$

respectively.

REMARK 5.2: The results of Theorem 5.2 have also been established by Aftabizadeh and Wiener in [4]. The present author proved the inequality independently however, got this paper only when this chapter was on the typewriter and hence it is included here.

Next we state a result giving the existence of maximal solution of (5.7). The case of minimal solution is similar.

THEOREM 5.3: Let $I = [0,a)$. Under the hypothesis of Theorem 5.1 there exists a maximal solution $r(t)$ for the IVP (5.7).

PROOF: The proof can be formulated by following the argument of Theorem 1.3.1 in [23] and the conclusion of Theorem 1.3.1 in each interval $[n,n+1)$, $n = 0,1,2 \ldots$. The details are omitted.

One of the result widely used is the following comparison theorem.

THEOREM 5.4: Let $r(t)$ be the maximal solution of (5.7) on I. Let $m \in C[I,\mathbb{R}]$, $m(0) \leq r(0)$ and if

$$m'(t) \leq f(t,m(t),m([t])), \quad t \in I, \tag{5.11}$$

then

$$m(t) \leq r(t), \quad t \in I.$$

PROOF: In view of Theorem 5.3, it can be proved that the maximal solutions $r(t, \varepsilon)$ of

$$x'(t) = f(t,x(t),x([t])) + \varepsilon \quad , \quad x(0) = x_0 + \varepsilon \tag{5.12}$$

exist on $[0, \tau)$ for all $\varepsilon > 0$ sufficiently small and

$r(t) = \lim_{\varepsilon \to 0} r(t, \varepsilon)$ uniformly on $[0, \tau)$. Using (5.11) and (5.12) and applying Theorem 5.2, we derive that $m(t) < r(t, \varepsilon)$, $t \in [0, \tau)$.

The last inequality together with $r(t) = \lim_{\varepsilon \to 0} r(t, \varepsilon)$, proves the assertion of the theorem.

This section ends with a result illustrating the usefulness of the inequalities proved above.

Consider a differential equation

$$y'(t) = a(t)y(t) + b(t)y([t]) \tag{5.13}$$

and corresponding perturbed equation

$$z'(t) = a(t)z(t) + b(t)z([t]) + f(t,z(t)) \tag{5.14}$$

with initial condition $z(0) = x_0$ where a, b are continuous functions of $t \in J$, x_0 is a real constant and $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying some conditions stated in the next theorem.

We prove the following theorem which is useful to discuss the stability and boundedness for cases when the function $f(t,z(t))$ in (5.14) is not necessarily small.

THEOREM 5.5: Let $\phi(t)$ be the fundamental solution of (5.13) with b=0 and $\psi(t)$ be the fundamental solution of (5.13) satisfying

$\psi(0) = 1$. Let $|\psi(t)| \le \alpha(t)$, where $\alpha(t)$ is a positive real-valued function defined on $J = [0, \infty)$, and let $\alpha(0) = \alpha_o$. Suppose also that the function $f(t,z(t)) : J \times R \to R$ satisfies the inequalities

$$\left| \psi^{-1}(k)\, \phi(k,s) f(s,z(s)) \right| \le W\left(s, \frac{|z(s)|}{\alpha(s)}, \frac{|z(k-1)|}{\alpha(k-1)}\right), \quad k-1 \le s < k,\ k=1,2,\ldots$$

and

$$\left| \phi(t,s) f(s,z(s)) \right| \le \alpha(t) W\left(s, \frac{|z(s)|}{\alpha(s)}, \frac{|z([t])|}{\alpha([t])}\right), \quad k=[t] \le s < t,$$

where $W(t,r(t),r([t]))$ is monotone increasing function in second and third variable. Let $r(t,0,r_o)$ be a solution of

$$r'(t) = W(t,r(t),r([t])), \quad r(0) = r_o. \tag{5.15}$$

Then the solution $z(t,0,x_o)$ of (5.14) satisfies

$$|z(t,0,x_o)| \le \alpha(t) r(t), \quad t \in J,$$

if $z(t,0,x_o)$ is such that $|x_o| \le \alpha_o\, r_o$.

__PROOF:__ Using Theorem 2.1, the solution $z(t,0,x_o)$ of (5.14) is given by

$$z(t,0,x_o) = y(t,0,x_o) + \sum_{k=1}^{[t]} \int_{k-1}^{k} \psi(t,k)\, \phi(k,s) f(s,z(s))\,ds + \int_{[t]}^{t} \phi(t,s) f(s,z(s))\,ds,$$

where $y(t,0,x_o)$ is the solution of (5.13) and ϕ is the fundamental solution of

$$x'(t) = a(t)x(t).$$

Then

$$|z(t,0,x_o)| \leq \alpha(t)|x_o| + \alpha(t)\sum_{k=1}^{[t]} \int_{k-1}^{k} W\left(s, \frac{|z(s)|}{\alpha(s)}, \frac{|z(k-1)|}{\alpha(k-1)}\right)ds$$

$$+ \alpha(t)\int_{[t]}^{t} W\left(s, \frac{|z(s)|}{\alpha(s)}, \frac{|z([t])|}{\alpha([t])}\right) ds$$

$$= \alpha(t)\left\{|x_o| + \int_0^t W\left(s, \frac{|z(s)|}{\alpha(s)}, \frac{|z([s])|}{\alpha([s])}\right)ds\right\} .$$

Put

$$u(t) = |x_o| + \int_0^t W\left(s, \frac{|z(s)|}{\alpha(s)}, \frac{|z([s])|}{\alpha([s])}\right)ds,$$

then we obtain

$$u'(t) = W\left(t, \frac{|z(t)|}{\alpha(t)}, \frac{|z([t])|}{\alpha([t])}\right).$$

Since

$$\frac{|z(t,0,x_o)|}{\alpha(t)} \leq u(t)$$

and W is increasing in second and third variables, we can see that

$$u'(t) \leq W(t,u(t),u([t])), \quad t \in J.$$

Hence by Theorem 5.4, the function u(t) satisfying the above inequality is dominated by a solution of (5.15). Then we have

$$\frac{|z(t,0,x_o)|}{\alpha(t)} \leq u(t) \leq r(t).$$

That is
$|z(t,0,x_o)| \leq \alpha(t)r(t)$, $t \in J$. The proof is complete.

Here we note that the above inequality implies the boundedness of $|z(t,0,x_o)|$ if $\alpha(t)r(t)$ is bounded.

<u>**REMARK 5.3:**</u> If we replace b(t) by b([t]) and take t = n, then this result reduces to the work done by Sugiyamma [31].

5.4 NONLINEAR VARIATION OF PARAMETERS FORMULA

Consider the nonlinear equations

$$x'(t) = f(t,x(t)) \tag{5.16}$$

$$y'(t) = f(t,y(t)) + g(y([t])) \tag{5.17}$$

$$z'(t) = f(t,z(t)) + g(z([t])) + c(t) \tag{5.18}$$

$t \in [t_o, \infty)$, $t_o \geq 0$, where f and c are continuous functions and g is a piecewise continuous function. Let the initial conditions at $t = t_o$ be given by

$$x(t_o) = y(t_o) = z(t_o) = x_o, \quad t \geq t_o \geq 0.$$

Further, we assume that (5.19)

$$x([t_o]) = y([t_o]) = z([t_o]) = c ,$$

where c is a real constant.

We state and prove the following two results.

LEMMA 5.1: Let $g(y([t]))$ be a piecewise continuous function on $\mathbb{R}$ and let $\frac{\partial g(y([t]))}{\partial y([t])}$ exist and be piecewise continuous on $\mathbb{R}$. Then

$$g(y_2([t])) - g(y_1([t])) = \left(\int_0^1 \frac{\partial g(sy_2([t]) + (1-s)y_1([t]))}{\partial y([t])} ds \right) . (y_2([t]) - y_1([t]))$$

PROOF: Setting

$$G(s) = g(sy_2([t]) + (1-s)y_1([t])), \quad 0 \leq s \leq 1,$$

the convexity of $\mathbb{R}$ implies that $G(s)$ is defined. Hence,

$$\frac{dG}{ds} = \frac{\partial g(sy_2([t]) + (1-s)y_1([t]))}{\partial y([t])} (y_2([t]) - y_1([t])). \tag{5.20}$$

Since $G(1) = g(y_2([t]))$, $G(0) = g(y_1([t]))$, the result follows by integrating (5.20) from0 to 1.

LEMMA 5.2: Assume that $f \in C[J \times \mathbb{R}, \mathbb{R}]$ and g is a piecewise continuous function on $\mathbb{R}$ and possesses partial derivatives

$$\frac{\partial f(t,y(t))}{\partial y} \quad \text{and} \quad \frac{\partial g(y([t]))}{\partial y([t])}.$$

Denote

$$H_1(t, t_o, x_o) = \frac{\partial f(t,y(t))}{\partial y}$$

$$H_2([t],t_o, x_o) = \frac{\partial g(y([t]))}{\partial y([t])} .$$

Then

$\psi(t, t_o, x_o) = \dfrac{\partial y(t, t_o, x_o)}{\partial x_o}$ exists and is the solution of

$$z'(t) = H_1(t, t_o, x_o)z(t) + H_2([t], t_o, x_o)z([t]) \tag{5.21}$$

such that $\psi(t_o, t_o, x_o) = 1$.

PROOF: Let h be scalar and for small h, let $y(t,h) = y(t, t_o, x_o + h)$, $t \in J$. It is known that $\lim_{h\to 0} y(t,h) = y(t, t_o, x_o) = y_o(t)$ (say), uniformly on J. In view of the Lemma 2.5.2 in [23] and Lemma 5.1, we get

$$(y(t,h) - y_o(t))' = \int_0^1 \frac{\partial f(t, sy(t,h) + (1-s)y_o(t))}{\partial y} ds(y(t,h)-y_o(t))$$

$$+ \int_0^1 \frac{\partial g(sy([t],h) + (1-s)y_o([t]))}{\partial y([t])} ds(y([t],h)-y_o([t])).$$

If we write

$$y_h(t) = \frac{y(t,h) - y_o(t)}{h}, \quad h \neq 0$$

the existence of $\dfrac{\partial y}{\partial x_o}(t, t_o, x_o)$ is equivalent to the existence of $\lim_{h\to 0} y_h(t)$. Thus, $y_h(t)$ is the solution of IVP

$$z'(t) = H_1(t,t_o,x_o,h)z(t) + H_2([t],t_o,x_o,h)z([t]), z(t_o) = 1 \quad (5.22)$$

where

$$H_1(t,t_o,x_o,h) = \int_o^1 \frac{\partial f(t, sy(t,h) + (1-s)y_o(t))}{\partial y} ds$$

and

$$H_2([t],t_o,x_o,h) = \int_o^1 \frac{\partial g(sy([t],h) + (1-s)y_o([t])}{\partial y([t])} ds.$$

$$y(t,h) \longrightarrow y_o(t) \text{ as } h \longrightarrow 0$$

$$\lim_{h \to 0} H_1(t,t_o,x_o,h) = H_1(t,t_o,x_o)$$

and

$$\lim_{h\to 0} H_2([t],t_o,x_o,h) = H_2([t],t_o,x_o)$$

uniformily on J. Considering (5.22) as a family of initial value problems depending on a parameter h, and observing that solution of (5.22) is unique [23], it is clear that the solution of (5.22) is a continuous function of h. In particular,

$$\lim_{h\to 0} y_h(t) = y(t)$$

exists and is the solution of (5.21). Hence the lemma.

We now prove the variation of parameters formula.

THEOREM 5.6: Let $y(t,t_o,x_o)$ be a solution of (5.17), (5.19) and $\phi(t,t_o,x_o)$ and $\psi(t,t_o,x_o)$ be solutions of the variational equations (1.19) and (5.22), respectively, for $t \geq t_o$. Then there exists a unique solution z(t) for (5.18), (5.19) given by

$$z(t,t_o,x_o) = \begin{cases} y(t,t_o,x_o) + \int_{t_o}^{t} \phi(t,s,z(s))c(s)ds, \quad t \in [t_o,1), \ 0 \leq t_o < 1, \\ y(t,t_o,x_o) + \int_{t_o}^{1} \psi(t,1,y(1))\phi(1,s,z(s))c(s)ds \\ \quad + \sum_{k=2}^{[t]} \int_{k-1}^{k} \psi(t,k,y(k))\phi(k,s,z(s))c(s)ds \\ \quad + \int_{[t]}^{t} \phi(t,s,z(s))c(s)ds, \quad t \geq 1. \end{cases} \tag{5.23}$$

PROOF: We prove the expression (5.23) in each interval. First consider the interval [0,1) and let $0 \leq t_o < 1$. By using the Theorem 1.11 of chapter 1

$$y(t,t_o,x_o) = x(t,t_o,x_o) + \int_{t_o}^{t} \phi(t,u,y(u,t_o,x_o))g(y([u],t_o,x_o))du. \tag{5.24}$$

Write $z(t,t_o,x_o) = z(t)$, then from (5.24), we note that

$$\frac{dy(t,\dot{s},z(s))}{ds} = \frac{dx(t,s,z(s))}{ds} - \phi(t,s,y(s,s,z(s)))g(z([s]))$$

$$= -\phi(t,s,z(s))f(s,z(s)) + \phi(t,s,z(s))z'(s)$$

$$- \phi(t,s,z(s))g(z([s]))$$

$$= \phi(t,s,z(s))c(s).$$

Integrate between t_o and t to get (5.23) in [0,1). Now consider the second interval [1,2), $0 \leq t_o < 1$, we have

$$\int_{t_o}^{1} \frac{dy}{ds}(t,s,z(s,t_o,x_o))\, ds = \int_{t_o}^{1} \frac{dy}{ds}(t,1,y(1,s,z(s)))\, ds$$

since $y(t,t_o,x_o) = y(t,1,y(1,t_o,x_o))$.

$$\int_{t_o}^{1} \frac{dy}{ds}(t,1,y(1,s,z(s)))\, ds = \int_{t_o}^{1} \frac{\partial y(t,1,y(1))}{\partial y(1)} \frac{dy}{ds}(1,s,z(s))\, ds$$

$$= \int_{t_o}^{1} \psi(t,1,y(1))\, \phi(1,s,z(s))c(s)ds. \qquad (5.25)$$

And

$$\int_{1}^{t} \frac{dy}{ds}(t,s,z(s,t_o,x_o))\, ds = \int_{1}^{t} \{\frac{dx}{ds}(t,s,z(s)) - \phi(t,s,z(s))g(z([s]))\}ds$$

$$= \int_{1}^{t} \phi(t,s,z(s))c(s)ds. \qquad (5.26)$$

Adding (5.25) and (5.26) we get (5.23) in [1,2). That is, for $t \in [1,2)$

$$z(t,t_o,x_o) = y(t,t_o,x_o) + \int_{t_o}^{1} \psi(t,1,y(1))\, \phi(1,s,z(s))c(s)ds$$

$$+ \int_{1}^{t} \phi(t,s,z(s))c(s)ds.$$

For getting (5.23) for $t \geq 2$, in each interval $[n,n+1)$ use the fact

$$y(t,n,z(n)) = y(t,n+1,y(n+1,n,z(n))), \quad n = 1, 2, \ldots$$

to obtain

$$\int_n^{n+1} \frac{dy}{ds}(t,s,z(s,t_o,x_o))\,ds = \int_n^{n+1} \frac{dy}{ds}(t,n,y(n,s,z(s)))\,ds,$$

$$= \int_n^{n+1} \psi(t,n,y(n))\ \phi(n,s,z(s))c(s)ds,\ n=1,\ 2,\ \ldots$$

Now following same steps as in the case of first two intervals, we prove (5.23) for any $t \geq t_o$.

REMARK 5.4: If $t_o = 0$, then (5.23) becomes

$$z(t,0,x_o) = y(t,0,x_o) + \sum_{k=1}^{[t]} \int_{k-1}^{k} \psi(t,k,y(k))\ \phi(k,s,z(s))c(s)ds$$

$$+ \int_{[t]}^{t} \phi(t,s,z(s))c(s)ds.$$

REMARK 5.5: Replace $f(t,y(t)) = a(t)y(t)$ and $g(y([t])) = b(t)y([t])$ in (5.16), (5.17) and (5.18), further, take $t_o = 0$, then (5.23) takes the form

$$z(t,0,x_o) = y(t,0,x_o) + \sum_{k=1}^{[t]} \int_{k-1}^{k} \psi(t,k)\ \phi(k,s)c(s)ds + \int_{[t]}^{t} \phi(t,s)c(s)ds,$$

where ϕ and ψ are the fundamental solutions of (5.16) and (5.17), respectively.

We present below a further generalization of variation of parameters formula proved above. For ordinary differential equations it has been proved by Ladde [21].

THEOREM 5.7: Suppose that $V \in C[J \times \mathbb{R}, \mathbb{R}]$ and the function $V(t,y)$ is such that $V_y(t,y)$ exists and is continuous for (t,y) in $J \times \mathbb{R}$. Let $y(t) = y(t,t_o,x_o)$ denote the solution of (5.17), (5.19) and

$z(t) = z(t,t_o,x_o)$ is a solution of (5.18), (5.19). Then

$$V(t,z(t)) = \begin{cases} V(t,y(t)) + \int_{t_o}^{t} V_y(t,y(t,s,z(s))\,\phi(t,s,z(s))c(s)ds & 0 \le t_o \le t < 1, \\ V(t,y(t)) + \int_{t_o}^{1} V_y(t,y(t,s,z(s)))\,\psi(t,1,y(1)) \cdot \phi(1,s,z(s))c(s)ds & \end{cases} \quad (5.27)$$

$$+ \sum_{k=2}^{[t]} \int_{k-1}^{k} V_y(t,y(t,s,z(s)))\,\psi(t,k,y(k)) \cdot \phi(k,s,z(s))c(s)ds$$

$$+ \int_{[t]}^{t} V_y(t,y(t,s,z(s)))\,\phi(t,s,z(s))c(s)ds, \quad t \ge 1.$$

<u>**PROOF:**</u> Let $t \in [0,1)$ and $0 \le t_o < 1$. Since we know

$$\frac{dy}{ds}(t,s,z(s)) = \phi(t,s,z(s))c(s), \quad t \in [0,1).\ 0 \le t_o < 1,$$

the integration of

$$\frac{dV}{ds}(t,y(t,s,z(s))) = V_y(t,y(t,s,z(s))) \quad \frac{dy}{ds}(t,s,z(s))$$

between t_o to t yields the result (5.27) in $[0,1)$. For $t \in [1,2)$ split the integral t_o to t into two parts from t_o to 1 and 1 to t. Note the facts

$$\int_{t_o}^{1} V_y(t,y(t,s,z(s))) \frac{dy}{ds}(t,s,z(s))ds = \int_{t_o}^{1} V_y(t,y(t,s,z(s)))\,\psi(t,1,y(1)) \cdot \phi(1,s,z(s))c(s)ds \quad (5.28)$$

and

$$\int_1^t V_y(t,y(t,s,z(s)))\frac{dy}{ds}(t,s,z(s))ds = \int_1^t V_y(t,y(t,s,z(s)))$$

$$\cdot\ \phi(t,s,z(s))c(s)ds. \qquad (5.29)$$

Add (5.28) and (5.29) to get the required result (5.27) in [1,2). Continuing the same way the result (5.27) can be proved for any t.

REMARK 5.6: If $V(t,y) = y$, then (5.27) becomes (5.23).

EXAMPLE 5.3: Consider the differential equations

$$x'(t) = -x^2(t) \qquad (5.30)$$

$$y'(t) = -(y^2(t) + y^2([t])) \qquad (5.31)$$

$$z'(t) = -(z^2(t) + z^2([t])) + c(t) \qquad (5.32)$$

with initial conditions

$$x(t_o) = y(t_o) = z(t_o) = x_o = x([t_o]) = y([t_o]) = z([t_o]). \qquad (5.33)$$

Solution $x(t,t_o,x_o)$ of (5.30), (5.33) is given by

$$x(t,t_o,x_o) = x_o(x_o(t-t_o) + 1)^{-1}.$$

Hence

$$\phi(t,t_o,x_o) = \frac{\partial x}{\partial x_o} = (x_o(t-t_o) + 1)^{-2}.$$

Solution $y(t,n,c_n) = y_n(t)$ of (5.31) with $y(n) = c_n$ in each interval $[n,n+1)$, $n = 1,2, \ldots$ is given by

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