OPERATOR ALGEBRAS AND THEIR APPLICATIONS TO PHYSICS

THESIS SUBMITTED

TO

GOA UNIVERSITY

FOR THE AWARD OF THE DEGREE

OF

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

By

STEPHEN DIAS BARRETO

Department of Mathematics

Goa University

Goa-403206

1998

DECLARATION

I do hereby declare that this thesis entitled "OPERATOR ALGEBRAS AND THEIR APPLICATIONS TO PHYSICS" submitted to Goa University for the award of the degree of Doctor of Philosophy in Mathematics is a record of original and independent work done by me under the supervision and guidance of Professor Y. S. Prahalad, and it has not previously formed the basis for the award of any Degree, Diploma, Associateship, Fellowship or other similar title to any candidate of any University.

Stephen Dias Barreto

CERTIFICATE

This is to certify that this thesis entitled "OPERATOR ALGEBRAS AND THEIR APPLICATIONS TO PHYSICS" submitted to Goa university by Shri. Stephen Dias Barreto is a bonafide record of original and independent research work done by the candidate under my guidance. I further certify that this work has not formed the basis for the award of any Degree, Diploma, Associateship, Fellowship or other similar title to any candidate of any other University.

Y. S. Prahalad

Professor,

Department of mathematics,

Goa University

ACKNOWLEDGEMENT

During the course of this investigation, I had the help, support and encouragement of a number of people whom I would like to thank.

I thank Professor Prahalad for suggesting the problem, and for his encouragement and guidance. I am thankful to the faculty members and the administrative staff of the department of mathematics, both past and present, for their help and co-operation. I also acknowledge the encouragement received from a number of friends.

Special thanks are due, to Sheryl, for diligently reading some of the early drafts and pointing out a number of typographical errors, and making some useful suggestions. I also appreciate her constant support and encouragement. I am thankful to my family for having always stood by me, and for all the support and encouragement they have given me.

Finally, I thank the Council for Scientific and Industrial Research for supporting the research.

Stephen Dias Barreto

A Psalm of David

The Lord is my shepherd,
I shall not want;
he makes me lie down in green pastures.

He leads me beside still waters;
he restores my soul.
He leads me in paths of righteousness
for his name's sake.

Even though I walk through the
valley of shadow of death,
I fear no evil;
for thou art with me;
thy rod and thy staff,
they comfort me.

Thou preparest a table before me
in the presence of my enemies;
thou anointest my head with oil,
my cup overflows.

Surely goodness and mercy
shall follow me
all the days of my life;
and I shall dwell in the house of the Lord
for ever.

Abstract

In this thesis, we attempt the study of the dynamics of a quantum spin glass with the help of two models, one a quantum system on an infinite connected graph with deterministic nearest neighbour type of interactions, and the other a quantum spin system on an infinite lattice with random interactions. The problem to which we address ourselves is that of explaining the behaviour of a quantum spin glass through the dynamics of the quantum spin systems studied by us and the associated KMS states.

We construct the global dynamics for the quantum spin system on an infinite connected graph with countably infinite number of vertices. As expected, the existence of an equilibrium state at a fixed temperature T, is established. The equilibrium state satisfies the KMS condition and is invariant under the action of the time evolution group. However, all attempts to establish the maximum entropy principle for the infinite system were thwarted due to the absence of spatial homogeneity.

In the case of the quantum spin system on an infinite lattice $\mathbb{Z}^{\nu}$ with random interactions, we establish the existence of a family of strongly continuous, one–parameter groups of automorphisms $\{\tau_t(\omega)\}$, of the quasi–local algebra $\mathcal{A}$ associated with the spin system, where ω lives in a probability space $(\Omega, \mathcal{S}, P)$. These automorphism groups

$\tau_t(\omega)$ determine the evolution of the infinite spin system. The joint measurability of the map $(t, \omega) \mapsto \tau_t(\omega)(A)$ for all $A \in \mathcal{A}$ is also proved. Some interesting algebraic properties of the generators $\overline{\delta}(\omega)$ of these automorphism groups have been derived. The notion of ergodicity of a measure preserving group of automorphisms of Ω, is used to prove the almost sure independence of the Arveson spectrum $Sp(\tau(\omega))$, of the evolution group $\tau_t(\omega)$. Next, the existence of a family of states $\{\rho(\omega)\}$, which are $(\tau(\omega), \beta)$–KMS states of $\mathcal{A}$ with respect to the evolution groups $\tau_t(\omega)$, and satisfy $\rho(\omega)(A) = \rho(T_{-a}\omega)(\alpha_a(A))$, for $A \in \mathcal{A}$ and all $a \in \mathbb{Z}^\nu$, is established for all $\beta \in \mathbb{R} \setminus \{0\}$. It is assumed that there exists one such family of $(\tau(\omega), \beta)$–KMS states $\{\rho(\omega)\}$, where $\omega \mapsto \rho(\omega)(A)$ is measurable for all $A \in \mathcal{A}$. We show that the spin system on the infinite lattice with random interactions, exhibits a phase structure. In fact, we establish that there is an unique KMS state $\rho(\omega)$ with respect to the evolution group $\tau_t(\omega)$, above a certain critical temperature T_c almost surely independent of ω.

Now, the Arveson spectrum of the evolution group $\tau_t(\omega)$ is closely connected with the spectrum of the generator of the unitary group $U_t(\omega)$, which implements $\tau_t(\omega)$ in the cyclic representation π_ω associated with the $(\tau(\omega), \beta)$–KMS state $\rho(\omega)$. We exploit this fact to establish the almost sure independence of the spectrum of the generators.

Next, the cyclic representations π_ω associated with the $(\tau(\omega), \beta)$–KMS states $\rho(\omega)$, give rise to an ensemble of von Neumann algebras

$\{\pi_\omega(\mathcal{A})''\}$. Each of these von Neumann algebras acts on a separable Hilbert space $\mathcal{H}_\omega$. Thus, given this structure, we are obliged to invoke the theory of measurable fields of von Neumann algebras. Using the cyclicity of the representation π_ω, we construct a collection of measurable vector fields $\mathcal{F}$, which endows the field of separable Hilbert spaces $\omega \mapsto \mathcal{H}_\omega$, with a measurable structure. Equipped with this structure, we construct the direct integral Hilbert space $\mathcal{H} = \int_\Omega^\oplus \mathcal{H}_\omega dP(\omega)$. We also show that for each $t \in \mathbb{R}$, $\omega \mapsto U_t(\omega)$ is a measurable field of unitary operators. The joint measurability of $(t, \omega) \mapsto \langle U_t(\omega)\xi(\omega), \eta(\omega)\rangle_\omega$ for all $\xi, \eta \in \mathcal{F}$, is also established. Here $\langle .,.\rangle_\omega$ denotes the inner product on the Hilbert space $\mathcal{H}_\omega$. By using the theory of measurable fields of operators, we derive some interesting ergodic properties of the spectra of the generators $H(\omega)$, of the unitary groups $U_t(\omega)$.

In the final part of the thesis, we prove that the field of von Neumann algebras $\omega \mapsto \pi_\omega(\mathcal{A})''$, is a measurable field of von Neumann algebras and construct a direct integral von Neumann algebra

$$\mathcal{M} = \int_\Omega^\oplus \pi_\omega(\mathcal{A})'' dP(\omega).$$

The existence of a strongly continuous, one parameter group of unitaries U_t, on the direct integral Hilbert space $\mathcal{H}$, is established. Moreover, this group of unitaries in turn gives rise to a σ-weakly continuous group of automorphisms $\tilde{\tau}_t$, of the von Neumann algebra $\mathcal{M}$. We construct a faithful normal state $\tilde{\rho}$ of $\mathcal{M}$ from the measurable field $\omega \mapsto \tilde{\rho}(\omega)$, of faithful normal KMS states $\tilde{\rho}(\omega)$, which are extensions of the KMS states $\rho(\omega)$ to the von Neumann algebras $\pi_\omega(\mathcal{A})''$. Finally, this faithful normal state $\tilde{\rho}$ is shown to be a $(\tilde{\tau}, \beta)$–KMS state of the direct integral von Neumann algebra $\mathcal{M}$.

Contents

1 Introduction **4**

2 Mathematical Preliminaries **12**

2.1 Analysis in Normed Linear Spaces 12

2.1.1 Analytic Functions 12

2.1.2 Measure Theoretic Preliminaries 14

2.2 Operator Theoretic Preliminaries 18

2.3 Operator Algebraic Preliminaries 21

2.3.1 Standard Results in the Theory of C^*–Algebras and von Neumann Algebras 21

2.3.2 KMS States and Associated Representations 29

2.3.3 Arveson Spectrum 31

2.4 Standard Results in the Theory of Direct Integrals and Decompositions 31

2.4.1 Measurable Vector Fields 32

2.4.2 Square Integrable Vector Fields 34

2.4.3 Measurable Fields of Operators 36
2.4.4 Measurable Fields of von Neumann Algebras 39

3 Dynamics of a Quantum Spin Glass **41**
3.1 A Quantum Spin System on an Infinite Graph 43
3.2 Interactions 45
3.3 Time Evolution 46
3.4 Equilibrium State and the KMS Condition 52
3.5 Description of the Random Model 60
3.6 Random Interactions 63
3.7 Random Evolution 68

4 Ergodic Properties of Spectra of Evolution Groups **84**
4.1 Arveson Spectrum 84
4.2 KMS States 88
4.2.1 Construction of a Family of KMS States 89
4.2.2 Uniqueness of KMS States 93
4.2.3 Representations Associated with the KMS States . . . 106
4.3 Direct Integral von Neumann Algebra. 112
4.3.1 Measurable Field of Hilbert spaces 113
4.3.2 Measurable Field of von Neumann Algebras. 119
4.3.3 Automorphism Group of the Direct Integral von Neumann Algebra 123

4.3.4 Construction of a KMS State of the Direct Integral von Neumann Algebra 127

5 Summary **133**

Chapter 1

Introduction

Spin glasses still present something of a mystery. In fact, spin glasses are among the least understood systems in equilibrium statistical mechanics. In particular, their low temperature regime and critical behaviour are extremely complex. Spin glasses, which usually occur as dilute solutions of atoms with large magnetic moments (e.g. Fe, Co and Mn) in paramagnetic substrates (e.g. Cu and Au), have a number of interesting physical properties. The basic ingredient of a spin glass is the random distribution of the impurity in the form of magnetic atoms. At very low temperatures, there is a freezing of the magnetic moments in random directions which leads to an increase in susceptibility. The spin–glass phase may therefore be regarded as an arrangement of blocks of spins, each block with its own characteristic orientation in such a way that, there is no net magnetic moment [Bin 86]. More details concerning the physics of a spin glass can be obtained from the review article by Binder and Young [Bin 86].

Edwards and Anderson [And 75] proposed a spin Hamiltonian to account for the basic properties of a spin glass. The Hamiltonian is written as

$$H_\Lambda = \sum_{i,j \in \Lambda} J_{i,j} \phi(|i-j|) \sigma_i \sigma_j,$$

where Λ is a finite region in $\mathbb{Z}^\nu$, $\sigma_i = \pm 1$ and ϕ is a deterministic potential. The magnetic impurities are simulated by the set $\{J_{i,j}\}$ of independent, identically distributed random variables with distribution depending on the distance $|i-j|$. This model describes the important concept of frustration in a spin glass and the related problem of defining a suitable order parameter [Bin 86, Hem 83]. Analytical investigations concerning the equilibrium statistical mechanics of a spin glass have focussed attention on the mean field model of Sherrrington and Kirkpatrick [She 78]. Here, the model is defined by a Hamiltonian

$$H_n = -\sum_{i,j} J_{i,j} \sigma_i \sigma_j - H \sum_{i=1}^{n} \sigma_i,$$

where $\sigma_i = \pm 1$, H is the external magnetic field and $\{J_{i,j}\}$ is a set of independent, identically distributed random variables with probability density

$$p(J_{i,j}) = \left(\frac{2\pi J^2}{n} \right)^{-1/2} \exp \left[-\frac{n(J_{i,j} - J_0/n)^2}{2J^2} \right].$$

For a pure spin glass in zero field, we have $J_0 = 0$ and $H = 0$. Using the replica trick and the concept of replica symmetry breaking, Parisi et al [Par 80] have obtained a rather appealing picture of the low temperature phase of this model. It has been shown that there exists infinitely many extremal Gibbs states at very low temperatures. As the temperature is raised, spins with increasing distance from each other coalesce, until, above a freezing temperature, the equilibrium state is unique.

The other model of a spin glass which has been investigated is the Cayley tree model [Tho 86, Cha 86, Car 91]. Here, one usually looks for thermodynamic quantities in the innermost regions of the tree, namely, on a Bethe lattice, in order to avoid the problems caused by the presence of a large number of boundary points. Although boundary conditions do have an important influence, this model does not suffer from the problems associated with boundary effects. The Hamiltonian for this model is given by

$$\mathbf{H} = -\sum_{i,j} J_{i,j}\sigma_i\sigma_j - H\sum_i \sigma_i,$$

where $\sigma_i = \pm 1$ and H is an external magnetic field. The sum (i,j) is over nearest neighbour sites and $\{J_{i,j}\}$'s are independent, identically distributed random variables such that, $E(J_{i,j}) = J_0$ and $E(J_{i,j}^2) = J^2$. For a pure spin–glass phase, $J_0 = 0$.

Traditionally, quantum spin glasses have been studied as systems of quantum spins interacting through random interactions. These models are essentially Ising–type models with random coupling. The coupling coefficients are assumed to be independent, identically distributed random variables. Extensive investigations on the existence of the thermodynamic limit have been made by van Hemmen et al [Ent 83, Hem 83]. The almost sure existence of the free energy of an infinite spin system on a lattice with random interactions has been established. This is a generalization of the result of Khanin and Sinai [Sin 79], in the classical case. An alternate model of a quantum spin glass can be based on the realization that the magnetic ions in a spin glass are randomly distributed at lattice sites. The spins therefore, may be considered to be located at the vertices of an infinite connected graph with countably infinite number of vertices. Hence, one can caricature a quantum spin glass as a quantum spin system on an infinite connected graph with countably infinite number of vertices. This model may be regarded as the quantum analogue of systems studied by Preston and others [Pre 74]. But, unlike a classical spin glass, a quantum spin glass admits a dynamics naturally. Despite this fact, this aspect of a quantum spin glass has not been investigated.

In this thesis, we have attempted to study the dynamics of a quantum spin glass with the help of both these models, namely, a quantum spin system on an infinite graph with deterministic nearest neighbour type of interactions, and a quantum spin system on an infinite lattice with random interactions. The problem to which we address ourselves is that of explaining the behaviour of a quantum spin glass through the dynamics of these spin systems and the associated KMS states.

The C^*–algebraic approach has met with a fair amount of success in the study of quantum spin systems. Here, we study a quantum spin glass as a quantum spin system in the C^*–algebraic frame work. Usually, a quantum spin system consists of a set of points confined to a lattice and interacting with each other. In some applications it is important that the lattice has a symmetry, for example the case $L = \mathbb{Z}^\nu$. Nevertheless, in many cases, and in particular, in the construction of dynamics, it is enough to assume that L is a countably infinite set. The kinematical structure associated with the quantum spin system is a quasi-local UHF algebra constructed over the finite subsets of L. Usually, the dynamical evolution of a quantum spin system in the Heisenberg picture is given by a strongly continuous, one-parameter group of automorphisms τ_t, of the quasi-local algebra. Thus, one of the most important tasks in the study of quantum spin systems is to construct an evolution group of the spin system. In many situations, this is achieved by establishing the existence of the thermodynamic limit of the local evolution group τ_t^{Λ}, of the spin system confined to the finite region Λ. Closely connected with the strongly continuous, one-parameter group of automorphisms τ_t of the quasi-local algebra, is a set of states called the KMS states. These states are known to be invariant with respect to the automorphism group, and they satisfy certain analytic conditions in a strip, in the complex plane. A detailed account of these facts has been given in chapters (3) and (4), both in the case of a quantum spin system on an infinite graph, as well as in the case of a spin system on a lattice with random interactions.

We construct the global dynamics (3.3.0.11) for a quantum spin system on an infinite graph. As expected, the existence of an equilibrium state at a fixed temperature T, is established. The equilibrium state satisfies the KMS condition (3.4.0.15) and is invariant under the action of the time evolution group as shown in corollary 3.4.0.16. However, all attempts to establish the maximum entropy principle for the infinite system were thwarted due to the absence of spatial homogeneity. Thus, this approach did not prove to be very useful in understanding the behaviour of a quantum spin glass. As a result, we were obliged to take recourse to the more traditional line of thinking in understanding and explaining the behaviour.

To this end, we consider a quantum spin system on an infinite lattice $\mathbb{Z}^\nu$ with random interactions $\Phi(.,\omega)$, where ω lives in a probability space $(\Omega, \mathcal{S}, P)$. Here, P is the completion of a probability measure with respect to the sigma algebra $\mathcal{S}$, containing the Borel sigma algebra generated by the topology of the complete, separable metric space Ω. We establish the existence of a family of strongly continuous, one–parameter groups of automorphisms $\{\tau_t(\omega)\}$, of the quasi–local algebra $\mathcal{A}$, associated with the spin system(3.7.0.28). These automorphism groups $\tau_t(\omega)$ determine the evolution of the infinite spin system. The joint measurability (3.7.0.29) of the map $(t,\omega) \mapsto \tau_t(\omega)(A)$ for all A in $\mathcal{A}$ is also proved. Some interesting algebraic properties of the generators $\overline{\delta}(\omega)$ of these automorphism groups have been derived (3.7.0.33). The notion of ergodicity of a measure preserving group of automorphisms of Ω, is used to prove the almost sure independence of the Arveson spectrum $Sp(\tau)(\omega)$, of the evolution group $\tau_t(\omega)$ (4.1.0.35).
Next, the existence of a family of states $\{\rho(\omega)\}$, which are $(\tau(\omega), \beta)$–KMS states of the quasi–local algebra $\mathcal{A}$, with respect to the evolution groups $\tau_t(\omega)$ and satisfy $\rho(\omega)(A) = \rho(T_{-a}\omega)(\alpha_a(A))$, for $A \in \mathcal{A}$ and all $a \in \mathbb{Z}^\nu$, is established for all $\beta \in \mathbb{R} \setminus \{0\}$. We assume that there exists one such family denoted by $\{\rho(\omega)\}$, where $\omega \mapsto \rho(\omega)(A)$ is measurable for all $A \in \mathcal{A}$. We show that the spin system on the infinite lattice with random interactions exhibits a phase structure. In fact, we establish that there is an unique KMS

state $\rho(\omega)$ (4.2.2.2), above a certain critical temperature T_c almost surely independent of ω.

The Arveson spectrum of the evolution group $\tau_t(\omega)$ is closely connected with the spectrum of the generator of the unitary group $U_t(\omega)$, which implements $\tau_t(\omega)$ in the cyclic representation π_ω associated with the $(\tau(\omega),\beta)$-KMS state $\rho(\omega)$. We shall exploit this fact to establish the almost sure independence of the spectrum of the generators (4.2.3.3).

Now, the cyclic representations π_ω associated with the $(\tau(\omega),\beta)$–KMS states $\rho(\omega)$, satisfying the conditions mentioned above, give rise to an ensemble of von Neumann algebras $\{\pi_\omega(\mathcal{A})''\}$. Each of these von Neumann algebras is defined on a separable Hilbert space $\mathcal{H}_\omega$. The separability follows from the fact that [illegible] state $\rho(\omega)$ [illegible] is a locally normal state. Thus, given this structure one is obliged to invoke the theory of measurable fields of von Neumann algebras. Using the cyclicity of the representation π_ω, we construct a collection of measurable vector fields $\mathcal{F}$, which endows the field of separable Hilbert spaces $\omega \mapsto \mathcal{H}_\omega$, with a measurable structure. Equipped with this structure, we construct the direct integral Hilbert space $\mathcal{H} = \int_\Omega^\oplus \mathcal{H}_\omega dP(\omega)$. We also show in proposition 4.3.1.2 that, for each $t \in \mathbb{R}$, $\omega \mapsto U_t(\omega)$ is a measurable field of unitary operators. The joint measurability of $(t,\omega) \mapsto \langle U_t(\omega)\xi(\omega), \eta(\omega)\rangle_\omega$ for all $\xi, \eta \in \mathcal{F}$, is established in proposition 4.3.1.3. Here $\langle .,. \rangle_\omega$ denotes the inner product on the Hilbert space $\mathcal{H}_\omega$. By using the theory of measurable fields of operators, we derive some interesting ergodic properties (4.3.1.6) of the spectra of the generators $H(\omega)$, of the unitary groups $U_t(\omega)$.

In the final part of the thesis, we prove in proposition 4.3.2.2, that the field of von Neumann algebras $\omega \mapsto \pi_\omega(\mathcal{A})''$, is a measurable field of von Neumann algebras. Next, we construct a direct integral von Neumann algebra

$$\mathcal{M} = \int_\Omega^\oplus \pi_\omega(\mathcal{A})'' dP(\omega),$$

from the measurable field of von Neumann algebras $\omega \mapsto \pi_\omega(\mathcal{A})''$. The existence of a strongly continuous, one–parameter group of unitaries on the direct integral Hilbert space $\mathcal{H}$, is established in theorem 4.3.3.2. This was achieved by constructing a family of decomposable operators $\{U_t\}$, from the measurable field of unitaries $\omega \mapsto U_t(\omega)$. Moreover, this strongly continuous group of unitaries in turn gives rise to a σ–weakly continuous group of automorphisms $\tilde{\tau}_t$, of the von Neumann algebra $\mathcal{M}$. From the measurable field $\omega \mapsto \tilde{\rho}(\omega)$, of KMS states $\tilde{\rho}(\omega)$, which are extensions of the KMS states $\rho(\omega)$ to the von Neumann algebras $\pi_\omega(\mathcal{A})''$, we construct a faithful normal state $\tilde{\rho}$ of $\mathcal{M}$. This fact is established in theorem 4.3.4.1. In theorem 4.3.4.3, this faithful normal state $\tilde{\rho}$ is shown to be a $(\tilde{\tau}, \beta)$–KMS state of the direct integral von Neumann algebra $\mathcal{M}$.

The final Chapter in this thesis is devoted to a discussion on the results obtained in chapters (3) and (4) and their implications. Some of the open problems which remain unresolved are identified.

Chapter 2

Mathematical Preliminaries

This chapter is devoted to a discussion on mathematical preliminaries encompassing several areas in analysis. For the convenience of the reader we include some standard results which one may have the occasion to use in the thesis. We begin with a section on analysis in normed linear spaces. By and large, this section will feature notions of measurability of functions taking values in a Banach space and properties of the Bochner integral. Some important results involving complex valued analytic functions have also been included.

2.1 Analysis in Normed Linear Spaces

2.1.1 Analytic Functions

Theorem 2.1.1.1 (Vitali's convergence theorem). *Let $f_n(z)$ be a sequence of functions, each regular in a region D; let*

$$|f_n(z)| \leq M$$

for every n, and z in D, and let $f_n(z)$ tend to a limit as $n \to \infty$, at a set of points having a limit point inside D. Then $f_n(z)$ tends to a limit in any region bounded by a contour, interior to D, the limit being an analyitc function of z.

Proof See [Tit 91](Theorem 5.21). △

Theorem 2.1.1.2 (Phragmen–Lindelöf). *Let $\mathcal{D}$ be the open strip in $\mathbb{C}$ defined by*

$$\mathcal{D} = \{z; z \in \mathbb{C}, a < \Im z < b\}$$

and $\overline{\mathcal{D}}$ the closure of $\mathcal{D}$. Let f be a complex function which is analytic on $\mathcal{D}$, and bounded and continuous on $\overline{\mathcal{D}}$. It follows that the function

$$y \in [a, b] \mapsto g(y) = \log\left(\sup_{x \in \mathbb{R}} |f(x + iy)|\right)$$

is convex. In particular,

$$\sup_{z \in \overline{\mathcal{D}}} |f(z)| = \max\left\{\sup_{x \in \mathbb{R}} |f(x + ia)|, \sup_{x \in \mathbb{R}} |f(x + ib)|\right\}.$$

Proof Vide [Rob 81](Proposition 5.3.5). △

Theorem 2.1.1.3 *Suppose that γ is the boundary of an unbounded region Ω, $f \in H(\Omega)$, f is continuous on $\Omega \cup \Gamma$, and there are constants $B < \infty$ and $M < \infty$, such that, $|f| \leq M$ on Γ and $|f| \leq B$ in Ω. Then, we actually have $|F| \leq M$ in Ω.*

Proof See problem 11 on page 264 in [Rud 87]. △

2.1.2 Measure Theoretic Preliminaries

Let Ω be an abstract set, $\mathcal{C}$ a σ–ring of subsets of Ω, and m defined on Ω be a σ–finite measure. In this section, we study the notion of measurability for vector valued functions $f(\sigma)$ on Ω, taking values in a Banach space X, relative to the measure m. There are several notions of measurability for vector valued functions.

The following definitions have been taken from [Hil 57].

Definition 2.1.2.1 *1. $f(\sigma)$ is said to be finitely–valued if it is constant on each of a finite number of disjoint measurable sets E_j and equal to 0 on $\Omega \setminus \cup E_j$.*

2. It is a simple function if it is finitely–valued and if the set for which $\|f(\sigma)\| > 0$ is of finite measure .

3. $f(\sigma)$ is a countably–valued function if it assumes at most a countable set of values in X, assuming each value different from 0 on a measurable subset.

Definition 2.1.2.2 *$f(\sigma)$ is said to be separably–valued if its range, $f(\Omega)$ is separable. It is almost separably–valued if there exists a m–null set $E_0 \in \mathcal{C}$ such that $f(\Omega \setminus E_0)$ is separable.*

Definition 2.1.2.3 *1. $f(\sigma)$ is said to be weakly measurable in Ω if the numerical functions $x^\star(f(\sigma))$ are measurable for each $x^\star \in X^\star$.*

2. *$f(\sigma)$ is strongly measurable if there exists a sequence of countably-valued functions converging almost everywhere in Ω to $f(\sigma)$.*

Note that if $m(\Omega) < \infty$, then we may replace "countably–valued" in part (2) by "simple".

A subset $\Lambda \subseteq X^*$ is said to be determining for X if $\|x\| = sup\{|x^*(x)|; x^* \in \Lambda\}$ for all $x \in X$.

Theorem 2.1.2.4 *If $f(\sigma)$ is weakly measurable and if there exists a denumerable set Λ which is determining for X, then the numerically valued function $\|f(\sigma)\|$ is measurable.*

Proof Refer to [Hil 57](Theorem 3.5.4). $\triangle$

Theorem 2.1.2.5 *A vector valued function on Ω taking values in X is strongly measurable if and only if it is weakly measurable and almost separably valued.*

Proof Theorem 3.5.3 in [Hil 57]. $\triangle$

Corollary 2.1.2.6 *If X is separable, then strong and weak measurability are equivalent notions.*

Theorem 2.1.2.7

1. *If $f(\sigma)$ and $g(\sigma)$ are strongly measurable functions on Ω taking values in X, and γ_1, γ_2 are constants, then $\gamma_1 f(\sigma) + \gamma_2 g(\sigma)$ is strongly measurable.*

2. *If $h(\sigma)$ is a finite numerically valued function which is measurable, then $h(\sigma)f(\sigma)$ is strongly measurable if $f(\sigma)$ has this property.*

3. *If $f(\sigma)$ is the limit almost everywhere of a sequence of strongly measurable functions, then $f(\sigma)$ is strongly measurable.*

4. *The same conclusion is valid if in (3) the word "limit" (that is, strong limit) is replaced by "weak limit".*

5. *The conclusion is also valid if the "limit almost everywhere" is replaced by the "limit in measure".*

Proof See theorem 3.5.4 in [Hil 57]. △

Next, we introduce the Bochner integral. The results listed in this part of the section have been taken from [Hil 57], chapter 3.

Definition 2.1.2.8 *A countably valued function $f(\sigma)$ from Ω to X is Bochner integrable, if and only if, $\|f(\sigma)\|$ is Lebesgue integrable. By definition, the Bochner integral of $f(\sigma)$ on $E \in \mathcal{C}$, denoted by $(B)\int_E x(\sigma)dm$ is given by*

$$(B)\int_E f(\sigma)dm = \sum_{k=1}^{\infty} x_k m(E_k \cap E),$$

where $f(\sigma) = x_k$ on $E_k \in \mathcal{C}\,(k = 1, 2, \ldots)$. This integral is well defined for all $E \in \mathcal{C}$ and for Ω itself. This follows from the fact that $\|f(\sigma)\|$ is integrable.

Definition 2.1.2.9 *A function $f(\sigma)$ from Ω to X is Bochner integrable if, and only if, there exists a sequence of countably valued Bochner integrable functions $\{f_n(\sigma)\}$ converging almost everywhere to $f(\sigma)$ and such that*

$$\lim_{n\to\infty} \int_\Omega \|f(\sigma) - f_n(\sigma)\|dm = 0.$$

By definition,

$$(B)\int_E f(\sigma)dm = \lim_{n\to\infty}(B)\int_E f_n(\sigma)dm,$$

for each $E \in \mathcal{C}$ and $E = \Omega$.

Theorem 2.1.2.10 *A necessary and sufficient condition for $f(\sigma)$ from Ω to X be Bochner integrable is that, $f(\sigma)$ be strongly measurable and that $\int_\Omega \|f(\sigma)\|dm < \infty$.*

We shall denote the class of all Bochner integrable functions relative to m, by $B(\Omega, X, m)$. Some interesting properties of the Bochner integral have been listed below

Proposition 2.1.2.11 *If $f_1(\sigma)$ and $f_2(\sigma) \in B(\Omega, X, m)$ and γ_1, γ_2 are constants, then $\gamma_1 f_1(\sigma) + \gamma_2 f_2(\sigma) \in B(\Omega, X, m)$ and*

$$\int_E (\gamma_1 f_1(\sigma) + \gamma_2 f_2(\sigma))dm = \gamma_1 \int_E f_1(\sigma)dm + \gamma_2 \int_E f_2(\sigma)dm,$$

for all $E \in \mathcal{C}$ and $E = \Omega$.

Proposition 2.1.2.12 *If $f(\sigma) \in B(\Omega, X, m)$, then*

$$\left\|\int_E f(\sigma)dm\right\| \leq \int_E \|f(\sigma)\|dm,$$

for all $E \in \mathcal{C}$ and $E = \Omega$.

Proposition 2.1.2.13 *Let T be a closed linear transformation from X to Y. If $f(\sigma) \in B(\Omega, X, m)$ and $T(f(\sigma)) \in B(\Omega, Y, m)$, then*

$$T\left(\int_E f(\sigma)dm\right) = \int_E T(f(\sigma))dm$$

for all $E \in \mathcal{C}$ and $E = \Omega$.

If, in particular, T is a bounded, linear transformation from X to Y, then the theorem applies if only $f(\sigma) \in B(\Omega, X, m)$.

The last result in this section is an analogue of Fubini's theorem for Bochner integrals.

Suppose that S and T are abstract sets possessing σ–rings of subsets $\mathcal{C}$ and $\mathcal{F}$, with σ–finite measures m and n defined on $\mathcal{C}$ and $\mathcal{F}$, respectively. We denote the σ –ring of subsets of $S \times T$ generated by the class of measurable rectangles by $\mathcal{C} \times \mathcal{F}$. Finally we denote the product measure by $m \times n$.

Theorem 2.1.2.14 *If a function $f(\sigma, \tau)$ on $S \times T$ taking values in X, is Bochner integrable, then the functions $g(\sigma) = \int_T f(\sigma, \tau) dn$ and $h(\tau) = \int_S f(\sigma, \tau) dm$ are defined almost everywhere in S and T respectively, Bochner integrable on S and T respectively, and*

$$\int_{S \times T} f(\sigma, \tau) d(m \times n) = \int_S g(\sigma) dm = \int_T h(\tau) dn.$$

Proof Vide [Hil 57], theorem 3.7.13. $\triangle$

In the next section we collect some standard results pertaining to the spectral theory of self adjoint operators. We also include some important results which arise in the theory of one–parameter groups of unitaries.

2.2 Operator Theoretic Preliminaries

Theorem 2.2.0.15 (Spectral Theorem). *Let A be a self adjoint operator on a Hilbert space $\mathcal{H}$ with inner product $\langle .,. \rangle$. Then, there exists an unique* *spectral family $E(\lambda)$ on $\mathcal{H}$ such that,*

$$A\phi = \int_{\mathbb{R}} \lambda\, dE(\lambda).$$

and the domain of A,

$$D(A) = \{\phi \in \mathcal{H} : \int_{\mathbb{R}} \lambda^2 d\langle E(\lambda)\phi, \phi\rangle < \infty\}.$$

Proposition 2.2.0.16 *Let A be a self adjoint operator with spectral family $E(\lambda)$, then $s \in \sigma(A)$ if and only if, $E(s+\epsilon) - E(s-\epsilon) \neq 0$ for every $\epsilon > 0$.*

Proof See [Wei 80](Theorem 7.22). △

We denote the essential spectrum of a self adjoint operator A by $\sigma_e(A)$ and the discrete spectrum by $\sigma_d(A)$.

Proposition 2.2.0.17 *Let A be as in the above proposition, then $s \in \sigma_e(A)$ if and only if, for every $\epsilon > 0$, we have $\dim(R(E(s+\epsilon) - E(s-\epsilon))) = \infty$.*

Proof Vide [Wei 80](Theorem 7.24). △

Proposition 2.2.0.18 *Let A be a self adjoint operator on a Hilbert space $\mathcal{H}$, and $R(A, z)$ denote the resolvent $(A - zI)^{-1}$ at z. Then for ϕ, $\psi \in \mathcal{H}$, we have*

$$\begin{aligned} &\langle E(\lambda)\phi, \psi\rangle \\ = \; &\lim_{\delta \to 0+} \lim_{\epsilon \to 0+} \frac{1}{2\pi i} \int_{-\infty}^{\lambda+\delta} \langle \left((s - i\epsilon - A)^{-1} - (s + i\epsilon - A)^{-1}\right) \phi, \psi\rangle ds. \end{aligned}$$

Proof Refer to theorem 7.17 in [Wei 80]. △

Proposition 2.2.0.19 *Let A be a self adjoint operator on a Hilbert space $\mathcal{H}$ and ϕ, $\psi \in \mathcal{H}$. If $\Im z > 0$, then*

$$\langle (R(A,z))\phi, \psi\rangle = i\int_0^\infty e^{izt}\langle e^{-iAt}\phi, \psi\rangle dt$$

and if $\Im z < 0$,

$$\langle (R(A,z))\phi, \psi\rangle = -i\int_0^\infty e^{-izt}\langle e^{iAt}\phi, \psi\rangle dt,$$

where the integral is a Riemann integral.

Proof Vide [Dun 63](Chapter XII, Section VI, Theorem 1) △

Definition 2.2.0.20 *An operator function $U(t)$ on a Hilbert space $\mathcal{H}$, satisfying*

1. *For each $t \in \mathbb{R}$, $U(t)$ is a unitary operator and $U(t+s) = U(t)U(s)$ for all $t \in \mathbb{R}$.*
2. *If $\phi \in \mathcal{H}$ and $t \to t_0$, then $U(t)\phi \to U(t_0)\phi$,*

is called a strongly continuous, one–parameter group of unitary operators.

Theorem 2.2.0.21 (Stone's Theorem). *Let $U(t)$ be a strongly continuous, one–parameter unitary group on a Hilbert space $\mathcal{H}$. Then there is a self adjoint operator A on $\mathcal{H}$ such that $U(t) = e^{itA}$.*

Proof See [Sim 80](Theorem VIII.8). △

Definition 2.2.0.22 *If $U(t)$ is a strongly continuous, one–parameter unitary group, then the self adjoint operator A with $U(t) = e^{itA}$, is called the infinitesimal generator of $U(t)$.*

It is worth noting that if $U(t)$ is weakly continuous, then it is strongly continuous.

In the next section we collect a number of results in the theory of operator algebras which are relevant to the study of Quantum Statistical Mechanics.

2.3 Operator Algebraic Preliminaries

2.3.1 Standard Results in the Theory of $C^\star$–Algebras and von Neumann Algebras

Definition 2.3.1.1 *A normed algebra $\mathcal{A}$ with an involution which is complete and has the property $\|A^\star\| = \|A\|$, is called a Banach *–algebra.*

Definition 2.3.1.2 *A $C^\star$–algebra is a Banach *–algebra with the property*

$$\|A^\star A\| = \|A\|^2.$$

Definition 2.3.1.3 *A linear functional ρ over a $C^\star$–algebra $\mathcal{A}$ is defined to be positive if,*

$$\rho(A^\star A) \geq 0,$$

for all $A \in \mathcal{A}$. A positive linear functional ρ over a $C^\star$–algebra $\mathcal{A}$ with $\|\rho\| = 1$ is called a state.

Note that the set of states $E_\mathcal{A}$ of the $C^\star$–algebra is weak*–compact if, and only if, $\mathcal{A}$ contains an identity.

Definition 2.3.1.4 *A von Neumann algebra on $\mathcal{H}$ is a *–algebra $\mathcal{M}$ of $\mathcal{L}(\mathcal{H})$ such that*

$$\mathcal{M} = \mathcal{M}'',$$

where $\mathcal{M}'$ denotes the commutant of $\mathcal{M}$ and $\mathcal{M}''$ denotes the commutant of $\mathcal{M}'$. The center $\mathcal{E}(\mathcal{M})$ of a von Neumann algebra is defined by

$$\mathcal{E}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'.$$

A von Neumann algebra is called a factor, if it has a trivial center, i.e. if $\mathcal{E}(\mathcal{M}) = \mathbb{C}I$

Definition 2.3.1.5 *Let $\mathcal{M}$ be a von Neumann algebra and ρ a positive linear functional on $\mathcal{M}$. If $\rho(l.u.b._\alpha A_\alpha) = l.u.b._\alpha \rho(A_\alpha)$ for all increasing nets $\{A_\alpha\}$ in $\mathcal{M}_+$ with an upper bound, then ρ is defined to be normal.*

Definition 2.3.1.6 *A von Neumann algebra $\mathcal{M}$ is said to be σ–finite if all collections of mutually orthogonal projections have at most a countable cardinality.*

Definition 2.3.1.7 *A representation of a $C^\star$–algebra $\mathcal{A}$ is defined to be a pair $(\mathcal{H}, \pi)$, where $\mathcal{H}$ is a complex Hilbert space and π a *–morphism of $\mathcal{A}$ into $\mathcal{L}(\mathcal{H})$. The representation is said to be faithful if and only if, π is a *–isomorphism between $\mathcal{A}$ and $\pi(\mathcal{A})$.*

It is worth mentioning that if π is a representation of a $C^\star$–algebra $\mathcal{A}$, then π is continuous and $\|\pi(A)\| \leq \|A\|$ for all $A \in \mathcal{A}$. The equality holds only in the case of a faithful representation.

Definition 2.3.1.8 *A vector Θ in a Hilbert space $\mathcal{H}$ is said to be cyclic for a set of bounded operators $\mathcal{N}$, if the set $\{A\Theta | A \in \mathcal{M}\}$ is dense in $\mathcal{H}$.*

Definition 2.3.1.9 *A cyclic representation of a C^*–algebra $\mathcal{A}$, is defined to be a triple $(\mathcal{H}, \pi, \Theta)$, where $(\mathcal{H}, \pi)$ is a representation of $\mathcal{A}$, and Θ is a vector in $\mathcal{H}$ which is cyclic for π in $\mathcal{H}$.*

Theorem 2.3.1.10 *Let ρ be a state over the C^*–algebra $\mathcal{A}$. It follows that there exists a cyclic representation $(\mathcal{H}_\rho, \pi_\rho, \Theta_\rho)$ of $\mathcal{A}$ such that,*

$$\rho(A) = (\Theta, \pi_\rho(A)\Theta),$$

for all $A \in \mathcal{A}$ and consequently, $\|\Theta_\rho\|^2 = \|\rho\| = 1$. Moreover, the representation is unique upto unitary equivalence.

Proof Refer to theorem 2.3.16 in [Rob 87]. △

Definition 2.3.1.11 *A state ρ of a C^*–algebra is called a primary state, or a factor state, if $\pi_\rho(\mathcal{A})''$ is a factor, where π_ρ is the associated cyclic representation.*

Definition 2.3.1.12 *Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$. A subset $\mathcal{R} \subseteq \mathcal{H}$ is separating for $\mathcal{M}$ if for any $A \in \mathcal{M}$, $A\xi = 0$ for all $\xi \in \mathcal{R}$ implies $A = 0$.*

Definition 2.3.1.13 *A subset $\mathcal{R} \subseteq \mathcal{H}$ is cyclic for $\mathcal{M}$ if $[\mathcal{M}\mathcal{R}] = \mathcal{H}$, where $[\mathcal{M}\mathcal{R}]$ denotes the closure of the linear span of elements of the form $A\xi$, where $A \in \mathcal{M}$ and $\xi \in \mathcal{H}$.*

Proposition 2.3.1.14 *Let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$ and $\mathcal{R} \subseteq \mathcal{H}$ a subset. The following conditions are equivalent:*

1. $\mathcal{R}$ is cyclic for $\mathcal{M}$;

2. $\mathcal{R}$ is separating for $\mathcal{M}'$.

Proof Vide [Rob 87](Proposition 2.5.3). △

Next, a directed set J is said to possess an orthogonality relation if there exists a relation $\perp$, between pairs of elements of J such that,

1. if $\alpha \in J$ then there is a $\beta \in J$ with $\alpha \perp \beta$;
2. if $\alpha \leq \beta$ and $\beta \perp \gamma$ then $\alpha \perp \gamma$;
3. if $\alpha \perp \beta$ and $\alpha \perp \gamma$ then there exists a $\delta \in J$ such that, $\alpha \perp \delta$ and $\delta \geq \beta, \gamma$.

Remark If σ is an automorphism of a $C^\star$–algebra which satisfies $\sigma^2 = \imath$, then each element $A \in \mathcal{A}$, has an unique decomposition into odd and even parts with respect to σ. This decomposition is defined by

$$A = A^+ + A^-; \quad A^\pm = \frac{A \pm \sigma(A)}{2}.$$

It follows that $\sigma(A^\pm) = \pm A$, the even elements of $\mathcal{A}$ form a $C^\star$–subalgebra $\mathcal{A}^e$ of $\mathcal{A}$ and the odd elements $\mathcal{A}_o$ form a Banach space.

Definition 2.3.1.15 *A quasi-local algebra is a $C^\star$–algebra $\mathcal{A}$ and a net $\{\mathcal{A}_\alpha\}_{\alpha \in J}$ of $C^\star$–subalgebras such that, index set J has an orthogonality relation and the following properties are valid:*

1. if $\alpha \geq \beta$ then $\mathcal{A}_\alpha \supseteq \mathcal{A}_\beta$;

2. $\mathcal{A} = \overline{\cup_\alpha \mathcal{A}_\alpha}$, where the bar denotes the uniform closure;

3. *The algebras* $\mathcal{A}_\alpha$ *have a common identity* I*:*

4. *there exists an automorphism* σ *such that* $\sigma^2 = \imath$, $\sigma(\mathcal{A}_\alpha) = \mathcal{A}_\alpha$ *and* $[\mathcal{A}^e_\alpha, \mathcal{A}^e_\beta] = \{0\}$, $[\mathcal{A}^e_\alpha, \mathcal{A}^o_\beta] = \{0\}$, $\{\mathcal{A}^o_\alpha, \mathcal{A}^o_\beta\} = \{0\}$ *whenever* $\alpha \perp \beta$, *where* $\mathcal{A}^o_\alpha \subseteq \mathcal{A}_\alpha$ *and* $\mathcal{A}^e_\alpha \subseteq \mathcal{A}_\alpha$ *are odd and even elements with respect to* σ.

We have used the notation $\{A, B\} = AB + BA$. One case covered by this definiton is $\sigma = \imath$ and then $\mathcal{A}^e_\alpha = \mathcal{A}_\alpha$ and the condition (4) simplifies to

$$[\mathcal{A}_\alpha, \mathcal{A}_\beta] = \{0\}$$

whenever $\alpha \perp \beta$.

Proposition 2.3.1.16 *Let* $\mathcal{A}$, $\{\mathcal{A}_\alpha\}_{\alpha \in J}$ *be a quasi–local algebra and assume that each* $\mathcal{A}_\alpha$ *is simple. It follows that* $\mathcal{A}$ *is simple.*

Proof See corollary 2.6.19 in [Rob 87]. △

Definition 2.3.1.17 *A* $C^\star$*–algebra* $\mathcal{A}$ *with unit* I*, is said to be uniformly matricial if there is a sequence* $\{\mathcal{A}_n\}$ *of* $C^\star$*–subalgebras of* $\mathcal{A}$ *and a sequence* $\{n_j\}$ *of positive integers such that,* $\mathcal{A}_j$ *is* **–isomorphic to the algebra of all* $n_j \times n_j$ *complex matrices,*

$$I \in \mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \cdots,$$

and $\mathcal{A}$ *is the norm–closure of* $\cup \mathcal{A}_j$*. We then describe* $\mathcal{A}$ *in more detail, as uniformly matricial of type* $\{n_j\}$*, and refer to the sequence* $\{\mathcal{A}_j\}$ *as a generating nest of type* $\{n_j\}$ *for* $\mathcal{A}$*.*

Proposition 2.3.1.18 *There is a uniformly matricial C^*–algebra of type $\{n_j\}$, if and only if, the sequence $\{n_j\}$ of positive integers is strictly increasing and n_j divides $n_{j+1}(j = 1, 2, \ldots)$. When these conditions are satisfied, all uniformly matricial algebras of type $\{n_j\}$ are *–isomorphic and are simple C^*–algebras.*

Proof See proposition 10.4.18 in [Kad 86]. △

Definition 2.3.1.19 *Let $\{\mathcal{A}_j : j \in J\}$ be a family of C^*–algebras (with unit I_j in $\mathcal{A}_j$), in which the index set J is directed by a binary relation $\leq$. Suppose that, whenever $j, k \in J$ and $j \leq k$, there is specified, a *–isomorphism $\Phi_{k,j}$ from $\mathcal{A}_j$ into $\mathcal{A}_k$(with $\Phi_{k,j}(I_j) = I_k$); and finally , suppose that $\Phi_{l,k} \circ \Phi_{k,j} = \Phi_{l,j}$ whenever $j, k, l \in J$ and $j \leq k \leq l$. In these circumstances, we say that the C^*–algebras $\{\mathcal{A}_j; j \in J\}$, together with the *–isomorphisms $\{\Phi_{j,k} : j, k \in J, j \leq k\}$, constitute a directed system of C^*–algebras. Note that $\Phi_{j,j}$ is the identity mapping on $\mathcal{A}_j$.*

Proposition 2.3.1.20 *Suppose that the C^*–algebras $\{\mathcal{A}_j : j \in J\}$, and the *–isomorphisms $\Phi_{j,k} : \mathcal{A}_j \mapsto \mathcal{A}_k$ $(j, k \in J; j \leq k)$, together form a directed system.*

1. *There is a C^*–algebra $\mathcal{A}$ and for each j in J, a *–isomorphism ϕ_j, from $\mathcal{A}_j$ into $\mathcal{A}$ (carrying the unit of $\mathcal{A}_j$ into that of $\mathcal{A}$), such that $\phi_j = \phi_k \circ \Phi_{k,j}$ when $j \leq k$ and $\cup\{\phi_j(\mathcal{A}_j); j \in J\}$ is everywhere dense in $\mathcal{A}$.*

2. *The $C^\star$–algebra $\mathcal{A}$ occuring in (1) is uniquely determined, up to *–isomorphism; if $\mathcal{C}$ is a $C^\star$–algebra, $\psi_J : \mathcal{A}_j \to \mathcal{C}$ is a *–isomorphism (for each j in J) and conditions analogous to those in (1) are satisfied, then there is a *–isomorphism Ψ from $\mathcal{A}$ onto $\mathcal{C}$, such that $\psi_j = \Psi \circ \phi_j$, for each j in J.*

Proof Refer to proposition 11.4.1 in [Kad 86]. △

The $C^\star$–algebra $\mathcal{A}$ occuring in the above proposition is called the inductive limit of the directed system $\{\mathcal{A}_j; j \in J\}$.

Proposition 2.3.1.21 *If $\mathcal{A}$ is the inductive limit of a directed system of simple $C^\star$–algebras, then $\mathcal{A}$ is simple.*

Proof Vide proposition 11.4.2 on [Kad 86]. △

Definition 2.3.1.22 *A one–parameter family $t \in \mathbb{R} \mapsto \tau_t$ of automorphisms of the $C^\star$–algebra $\mathcal{A}$, is called a strongly continuous group of automorphisms of $\mathcal{A}$, if,*

1. $\tau_{t_1+t_2} = \tau_{t_1} \circ \tau_{t_2}$, $t_1, t_2 \in \mathbb{R}$, *and* $\tau_0 = \imath$;

2. $t \mapsto \tau_t(A)$ *is continuous in norm for all* $A \in \mathcal{A}$.

Definition 2.3.1.23 *A one–parameter family $t \in \mathbb{R} \mapsto \tau_t$ of automorphisms of a von Neumann algebra $\mathcal{M}$ is called a weakly continuous group of automorphisms of $\mathcal{M}$ if*

1. $\tau_{t_1+t_2} = \tau_{t_1} \circ \tau_{t_2}$, $t_1, t_2 \in \mathbb{R}$, *and* $\tau_0 = \imath$;

2. $t \mapsto \tau_t(A)$ *is weakly continuous for all* $A \in \mathcal{M}$.

Definition 2.3.1.24 *A derivation* δ *of a* $C^\star$*-algebra* $\mathcal{A}$ *is a linear operator from a* **-subalgebra* $D(\delta)$*, the domain of* δ*, into* $\mathcal{A}$ *with the following properties:*

1. $\delta(A)^\star = \delta(A^\star)$, $A \in D(\delta)$;
2. $\delta(AB) = \delta(A)B + A\delta(B)$; $A, B \in D(\delta)$.

Definition 2.3.1.25 *Let* S_n *be a sequence of operators on a Banach space* X *and let* $G(S_n) \subseteq X \times X$ *be their graphs. Define*

$$G = \lim_{n\to\infty} G(S_n)$$

as the set of pairs $(A, B) \in X \times X$ *such that there exists a sequence* $(A_n, B_n) \in X \times X$ *with* $A_n \in D(S_n)$, $B_n = S_n A_n$, *and*

$$A = \lim_{n\to\infty} A_n, \quad B = \lim_{n\to\infty} B_n.$$

Define $D(G)$ *as the set of* $A \in X$ *such that, there exists* $B \in X$ *with* $(A, B) \in G$ *and similarly,* $R(G)$ *is the set of* $B \in X$ *such that,* $(A, B) \in G$ *for some* $A \in X$. *If* G *is the graph of an operator* S*, then* S *is called the graph limit of* S_n. *Then clearly* $D(S) = D(G)$ *and* $R(S) = R(G)$.

Definition 2.3.1.26 *Let* $(\mathcal{A}, G, \tau)$ *be a* $C^\star$*-dynamical system. We say that the system is asymptotically abelian if there is a net* g_α *in* G*, such that*

$$\lim_{\alpha} \|A\tau_{g_\alpha}(B) - \tau_{g_\alpha}(B)A\| \to 0.$$

Furthermore, the states ρ for which there exists a net g_α in G such that,

$$\lim_\alpha |\rho(A\tau_{g_\alpha}(B)) - \rho(A)\rho(\tau_{g_\alpha}(B))| = 0$$

are called strongly clustering states.

2.3.2 KMS States and Associated Representations

Definition 2.3.2.1 *Let $(\mathcal{A}, \tau)$ be a $C^\star$–dynamical system, or a $W^\star$–dynamical system and ρ a state over $\mathcal{A}$ which is assumed to be normal in the $W^\star$ case. Then, ρ is said to be a (τ, β)–KMS state if, for $\beta > 0$ and any pair $A, B \in \mathcal{A}$, there exists a complex function $F_{A,B}$ which is analytic on the open strip $0 < \Im z < \beta$, uniformly bounded and continuous on the closed strip $0 \leq \Im z \leq \beta$ such that,*

$$F_{A,B}(t) = \rho(A\tau_t B) \quad and \quad F_{A,B}(t + i\beta) = \rho(\tau_t(B)A).$$

If $\beta < 0$, then ρ is said to be a (τ, β)–KMS state if there exists a complex function $F_{A,B}$ which is analytic on the open strip $\beta < \Im z < 0$, uniformly bounded and continuous for $\beta \leq \Im z \leq 0$ such that,

$$F_{A,B}(t) = \rho(A\tau_t B) \quad and \quad F_{A,B}(t + i\beta) = \rho(\tau_t(B)A).$$

Proposition 2.3.2.2 *Let ρ be a (τ, β)–KMS state of the $C^\star$–dynamical system $(\mathcal{A}, \tau)$ with $\beta \in I\!R \setminus \{0\}$ and let $\tilde{\rho}$ be the normal extension of ρ to the weak closure $\mathcal{M}_\rho = \pi(\mathcal{A})''$ of $\mathcal{A}$ in the cyclic representation $(\mathcal{H}_\rho, \pi_\rho, \Theta_\rho)$. It follows that there exists an unique σ–weakly continuous group $t \mapsto \tilde{\tau}_t$ of $\star$–automorphisms of $\mathcal{A}_\rho$ such that*

$$\tilde{\tau}_t(\pi_\rho(A)) = \pi_\rho(\tau_t(A))$$

for all $A \in \mathcal{A}$ *and* $t \in \mathbb{R}$. *Moreover,* $\tilde{\rho}$ *is a* $(\tilde{\tau}, \beta)$*–KMS state on* $\mathcal{M}_\rho$.

Proof Refer to corollary 5.3.4 in [Rob 81]. △

Proposition 2.3.2.3 *If* ρ *is the only state satisfying the* (τ, β)*–KMS condition, then* ρ *is a primary state.*

Proof Vide corollary 4.14 in [Hug 72]. △

Proposition 2.3.2.4 *An extremal invariant state* ρ*, which satisfies the* (τ, β)*–KMS condition is primary.*

Proof See corollary 4.15 in [Hug 72]. △

Let $(\mathcal{A}, \tau)$ be a $C^\star$–dynamical system. If $\mathcal{E}$ be the set of states of the $C^\star$–algebra $\mathcal{A}$, then an extremal invariant state is an extreme point of the convex set $\mathcal{E}$, which is invariant under the action of the automorphism group τ.

Some algebraic properties of a KMS state and that of its associated representation are as follows:

1. If ρ is a (τ, β)–KMS state, then $\rho(\tau_t A) = \rho(A)$.

2. The sets $I_1 = \{A \in \mathcal{A} | \, \rho(A^\star A) = 0\}$ and $I_2 = \{A \in \mathcal{A} | \, \rho(AA^\star) = 0\}$ are identical and form a two sided ideal.

3. If $(\mathcal{H}_\rho, \pi_\rho, \Theta_\rho)$ is the cyclic representation of $\mathcal{A}$ associated with the state ρ, then the von Neumann algebra $\pi_\rho(\mathcal{A})''$ has a cyclic and separating vector in Θ_ρ.

2.3.3 Arveson Spectrum

Let $\mathcal{A}$ be a $C^\star$–algebra and τ_t a strongly continuous, one–parameter group of automorphisms of $\mathcal{A}$. Now, the Bochner integral

$$\int_{-\infty}^{\infty} f(t)\tau_t(A)dt = \Gamma(f)A; \quad A \in \mathcal{A}, f \in L^1(I\!R),$$

defines a representation Γ of $L^1(I\!R)$ into the bounded operators on $\mathcal{A}$. Then the Arveson spectrum $Sp(\tau)$ of τ is given by

$$Sp(\tau) = \{s \in I\!R : \hat{f}(s) = 0, \forall f \in \ker\Gamma\}.$$

Proposition 2.3.3.1 *If τ is a strongly continuous, one–parameter group of automorphisms of a $C^\star$–algebra $\mathcal{A}$, then the following statements are equivalent:*

1. $s \in Sp(\tau)$.
2. *For every f in $L^1(I\!R)$ we have $|\hat{f}(s)| \leq \|\Gamma(f)\|$.*
3. *If $f \in L^1(I\!R)$ such that $\Gamma(f) = 0$ then $\hat{f}(s) = 0$.*

Proof Refer to proposition 8.1.9 in [Ped 79]. △

The last section deals with the theory of direct integrals and decompositions.

2.4 Standard Results in the Theory of Direct Integrals and Decompositions

All the results listed here can be found in [Dix 81].

2.4.1 Measurable Vector Fields

Let Ω be a Borel space and μ a finite measure on Ω. A mapping $\omega \mapsto \mathcal{H}_\omega$ on Ω, such that $\mathcal{H}_\omega$ is a Hilbert space for every $\omega \in \Omega$, with inner product $\langle.,.\rangle_\omega$, is called a field of complex Hilbert spaces. Now let $\mathcal{F}$ be the collection of all mappings $\omega \mapsto x(\omega)$ such that, $x(\omega) \in \mathcal{H}_\omega$. Such a mapping is called a vector field. It is clearly seen that $\mathcal{F}$ is a complex vector space.

Definition 2.4.1.1 *Let $\omega \mapsto \mathcal{H}_\omega$ be a field of complex Hilbert spaces over Ω and $\mathcal{F}$ the vector space of vector fields. We say that $\omega \mapsto \mathcal{H}_\omega$ is a μ–measurable field of Hilbert spaces if there is given a subspace $\mathcal{K}$ of $\mathcal{F}$ having the following properties:*

1. *For every $x \in \mathcal{K}$, the function $\omega \mapsto \|x(\omega)\|$ is μ–measurable;*
2. *If $y \in \mathcal{F}$ is such that, for every $x \in \mathcal{K}$, the function $\omega \mapsto \langle x(\omega), y(\omega)\rangle_\omega$ is μ–measurable, then, $y \in \mathcal{K}$;*
3. *There exists a sequence $\{x_1, x_2, \ldots\}$ of elements of $\mathcal{K}$, such that, for every $\omega \in \Omega$, the $x_n(\omega)$'s form a total sequence in $\mathcal{H}_\omega$.*

The vector fields belonging to $\mathcal{K}$ are then called μ–measurable vector fields. A sequence $\{x_1, x_2, \ldots\}$ of μ–measurable vector fields possessing property (3) is called a fundamental sequence of μ–measurable vector fields. In fact property (3) implies that the $\mathcal{H}_\omega$'s are separable.

Hence, it is easily seen that if x and y are measurable vector fields then, $\omega \mapsto \langle x(\omega), y(\omega)\rangle_\omega$ is a measurable function of ω. By property (2) of the above definition, the product of a measurable vector field with a complex valued measurable function is a measurable vector field. The same property also implies that the weak limit of a sequence of measurable vector fields which converges at each point of Ω is a measurable vector field.

Proposition 2.4.1.2 *Let Ω be a Borel space, μ a finite measure and $\omega \mapsto \mathcal{H}_\omega$ a measurable field of Hilbert spaces.*

1. *The set Ω_p of all $\omega \in \Omega$ such that the dimension $d(\omega)$ of $\mathcal{H}_\omega$ is equal to p is measurable.*

2. *There exists a sequence $\{y_1, y_2, \ldots\}$ of measurable vector fields possessing the following properties:*

   (a) if $d(\omega) = \aleph$, $\{y_1(\omega), y_2(\omega), \ldots\}$ is an orthonormal basis of $\mathcal{H}_\omega$;

   (b) if $d(\omega) < \aleph$, $\{y_1(\omega), y_2(\omega), \ldots, y_{d(\omega)}(\omega)\}$ is an orthonormal basis of $\mathcal{H}_\omega$, and $y_i(\omega) = 0$ for all $i > d(\omega)$.

Proof. See proposition 1 in [Dix 81] (Chapter 1 of Part II). $\triangle$

Definition 2.4.1.3 *A sequence $\{y_1, y_2, \ldots\}$ of measurable vector fields having the properties listed in (2), of the above proposition, is called a measurable field of orthonormal bases.*

Proposition 2.4.1.4 *Let $\{x_1, x_2, \ldots\}$ be a fundamental sequence of measurable fields. For a vector field x over Ω to be measurable, it is necessary and sufficient that the functions $\omega \mapsto \langle x(\omega), x_i(\omega)\rangle_\omega$ be measurable.*

Proof Vide proposition 2 in [Dix 81] (Chapter 1 of Part II). △

Proposition 2.4.1.5 *Let Ω be a Borel space, μ a finite measure on Ω, and $\omega \mapsto \mathcal{H}_\omega$ a field of Hilbert spaces over Ω. Let $\{x_1, x_2, \ldots\}$ be a sequence of vector fields having the following properties:*

1. *The functions $\omega \mapsto \langle x_i(\omega), x_j(\omega)\rangle_\omega$ are measurable;*
2. *For every $\omega \in \Omega$, the $x_i(\omega)$ form a total sequence in $\mathcal{H}_\omega$.*

Then, there exists exactly one measurable field structure on the $\mathcal{H}_\omega$'s such that the fields x_i are measurable.

Proof Vide proposition 4 in [Dix 81] (Chapter 1 of PartII). △

2.4.2 Square Integrable Vector Fields

Let $\omega \mapsto \mathcal{H}_\omega$ be a μ-measurable field of complex Hilbert spaces over Ω. A vector field x is said to be square integrable, if it is measurable and if,

$$\int_\Omega \|x(\omega)\|^2 d\mu(\omega) < \infty.$$

The set of square integrable fields is a complex vector space $\mathcal{N}$. For $x, y \in \mathcal{N}$, $\langle x(\omega), y(\omega)\rangle_\omega$ is an integrable function of ω. On putting

$$\langle x, y\rangle = \int_\Omega \langle x(\omega), y(\omega)\rangle_\omega d\mu(\omega),$$

the space $\mathcal{N}$ is endowed with a complex pre-Hilbert space structure. We have for $x \in \mathcal{N}$,

$$\|x\|^2 = \int_\Omega \|x(\omega)\|^2 d\mu(\omega).$$

Thus, the $x \in \mathcal{N}$ such that $\|x\| = 0$, are just those x's which vanish almost everywhere. We will identify two elements of $\mathcal{N}$ which are equal almost everywhere. In other words, we consider the pre–Hilbert space $\mathcal{H}$ associated with $\mathcal{N}$. The elements of $\mathcal{H}$ may be regarded as vector fields. For $x \in \mathcal{H}$ we may therefore speak of the values $x(\omega) \in \mathcal{H}_\omega$. It should be noted that the $x(\omega)$'s are determined to within negligible sets.

Proposition 2.4.2.1 *$\mathcal{H}$ is a Hilbert space.*

Proof Refer to proposition 5 in [Dix 81] (Chapter 1 of Part II). △

Definition 2.4.2.2 *The space $\mathcal{H}$ is called the direct integral of the $\mathcal{H}_\omega$'s and is denoted by $\int_\Omega^\oplus \mathcal{H}_\omega d\mu(\omega)$.*

Proposition 2.4.2.3 *let $\{y_1, y_2, \ldots\}$ be a measurable field of orthonormal bases. Let x be a vector field. Then $x \in \mathcal{H}$ if and only if the functions $\omega \mapsto \langle x(\omega), y_i(\omega)\rangle_\omega$ are square integrable and*

$$\sum_{i=1}^{\infty} \int_\Omega |\langle x(\omega), y_i(\omega)\rangle_\omega|^2 d\mu(\omega) < \infty.$$

Proof Refer to proposition 6 in [Dix 81] (Chapter 1 of Part II). △

Proposition 2.4.2.4 *Let $\omega \mapsto \mathcal{H}_\omega$ be a μ–measurable field of complex Hilbert spaces over Ω and $\{x_i\}$ a fundamental sequence of measurable vector fields. For every measurable vector field $\omega \mapsto x(\omega)$, there exists a sequence of vector fields of the form*

$$\omega \mapsto \sum_{i=1}^{n} f_i(\omega) x_i(\omega),$$

where the $f_i(\omega)$'s are measurable complex valued functions on Ω, which converge to $x(\omega)$ almost everywhere on Ω.

Proof Vide problem 3 in [Dix 81] (Page (176)). △

2.4.3 Measurable Fields of Operators

Let Ω be a Borel space, μ a finite measure on Ω, and $\omega \mapsto \mathcal{H}_\omega$ a μ–measurable field of complex Hilbert spaces over Ω. For every $\omega \in \Omega$, let $T(\omega)$ be an element of $\mathcal{L}(\mathcal{H}_\omega)$, i.e., a bounded linear operator on $\mathcal{H}_\omega$. Then, the mapping $\omega \mapsto T(\omega)$ is called a field of bounded linear operators over Ω.

Definition 2.4.3.1 *The field of bounded linear operators $\omega \mapsto T(\omega)$ is said to be measurable if, for every measurable vector field $\omega \mapsto x(\omega) \in \mathcal{H}_\omega$, the vector field $\omega \mapsto T(\omega)x(\omega) \in \mathcal{H}_\omega$ is measurable.*

Proposition 2.4.3.2 *Let $\{x_1, x_2, \ldots\}$ be a fundamental sequence of measurable vector fields with values in the $\mathcal{H}_\omega$'s. For the field $\omega \mapsto T(\omega)$ to be measurable, it is necessary and sufficient that the functions $\omega \mapsto \langle T(\omega)x_i(\omega), x_j(\omega)\rangle_\omega$ be measurable.*

Proof Refer to proposition 1 in [Dix 81] (Chapter 2 of Part II). △

Let $\omega \mapsto \mathcal{H}_\omega$ be a μ–measurable field of complex Hilbert spaces over Ω. Let

$$\mathcal{H} = \int_\Omega^\oplus \mathcal{H}_\omega d\mu(\omega).$$

A measurable field of bounded linear operators $\omega \mapsto T(\omega) \in \mathcal{L}(\mathcal{H}_\omega)$ is said to be essentially bounded if the essential supremum M of the function $\omega \mapsto$

$\|T(\omega)\|$ is finite. If this is the case, for every square integrable vector field x, the vector field $\omega \mapsto x'(\omega) = T(\omega)x(\omega)$ is also a square integrable vector field and we have $\|T\| \leq M$. Thus $x \mapsto x'$ establishes a correspondence $T : \mathcal{H} \to \mathcal{H}$ such that, T is a bounded linear operator on $\mathcal{H}$ with $\|T\| \leq M$.

Proposition 2.4.3.3 *We have* $\|T\| = M$.

Proof Refer to proposition 2 in [Dix 81] (Chapter 2 of PartII). △

This proposition yields the following corollary.

Corollary 2.4.3.4 *If two essentially bounded measurable fields of bounded linear operators define the same element of* $\mathcal{L}(\mathcal{H})$*, they are equal almost everywhere.*

Definition 2.4.3.5 *An operator* $T \in \mathcal{L}(\mathcal{H})$ *is said to be decomposable, if it is defined by an essentially bounded measurable field of operators* $\omega \mapsto T(\omega)$. *We then write*

$$\int_{\Omega}^{\oplus} T(\omega)d\mu(\omega).$$

It follows from the corollary that the $T(\omega)$'s may be defined upto negligible subsets of Ω. In particular, given a point $\omega \in \Omega$ of measure zero , $T(\omega)$ may be chosen arbitrarily.

Proposition 2.4.3.6 *Let* T_1, T_2 *be decomposable operators. If*

$$T_1 = \int_{\Omega}^{\oplus} T_1(\omega)d\mu(\omega) \quad and \quad T_2 = \int_{\Omega}^{\oplus} T_2(\omega)d\mu(\omega),$$

we have

$$T_1 + T_2 = \int_{\Omega}^{\oplus} (T_1(\omega) + T_2(\omega))d\mu(\omega), \quad T_1T_2 = \int_{\Omega}^{\oplus} T_1(\omega)T_2(\omega)d\mu(\omega)$$

and

$$\lambda T_1 = \int_{\Omega}^{\oplus} \lambda T_1(\omega)d\mu(\omega), \quad T_1^* = \int_{\Omega}^{\oplus} T_1^*(\omega)d\mu(\omega).$$

Proof Vide proposition 3 in [Dix 81] (Chapter 2 of Part II). △

Proposition 2.4.3.7 *Let*

$$T_i = \int_{\Omega}^{\oplus} T_i(\omega)d\mu(\omega) \quad (i = 1, 2, \ldots)$$

and

$$T = \int_{\Omega}^{\oplus} T(\omega)d\mu(\omega)$$

be decomposable operators.

1. *If T_i converges strongly to T, then there exists a subsequence T_{n_k} such that, $T_{n_k}(\omega)$ converges strongly to $T(\omega)$ almost everywhere.*

2. *If $T_i(\omega)$ converges strongly to $T(\omega)$ almost everywhere and if $\sup_i \|T_i\| < \infty$, then T_i converges strongly to T.*

Proof See proposition 4 in [Dix 81] (Chapter 2, Part II). △

Let $L^\infty(\Omega, \mu)$ be the set of essentially bounded, complex valued measurable functions on Ω, in which we identify any two functions which are equal almost everywhere. If $f \in L^\infty(\Omega, \mu)$, then the field of operators $\omega \mapsto f(\omega)I \in \mathcal{L}(\mathcal{H}_\omega)$ is measurable and essentially bounded. Let T_f be the corresponding operator of $\mathcal{H}$.

Definition 2.4.3.8 *The operators of the form T_f, where $f \in L^\infty(\Omega, \mu)$ are said to be diagonalisable.*

If $\mathcal{Z}$ denotes all such operators, then, $\mathcal{Z}$ is a *–algebra of $\mathcal{L}(\mathcal{H})$.

Proposition 2.4.3.9 *The algebra $\mathcal{Z}$ is an abelian von Neumann algebra and $\mathcal{Z}'$ is σ–finite.*

Proof Refer to proposition 7 in [Dix 81] (Chapter 2, Part II). △

2.4.4 Measurable Fields of von Neumann Algebras

In this section, Ω will continue to be a Borel space, μ a finite measure on Ω and $\omega \mapsto \mathcal{H}_\omega$ a μ–measurable field of complex Hilbert spaces. For every $\omega \in \Omega$, let $\mathcal{A}_\omega$ be a von Neumann algebra on $\mathcal{H}_\omega$. The mapping $\omega \mapsto \mathcal{A}_\omega$ is called a field of von Neumann algebras.

Definition 2.4.4.1 *A field of von Neumann algebras $\omega \mapsto \mathcal{A}_\omega$ over Ω is said to be measurable, if there exists a sequence $\omega \mapsto T_1(\omega), \omega \mapsto T_2(\omega), \ldots$ of measurable fields of operators such that, $\mathcal{A}_\omega$ is the von Neumann algebra generated by the $T_i(\omega)$'s almost everywhere.*

Proposition 2.4.4.2 *Let $\omega \mapsto \mathcal{A}_\omega$ be a measurable field of von Neumann algebras. The set $\mathcal{M}$ of decomposable operators*

$$\int_\Omega^\oplus T(\omega) d\mu(\omega),$$

such that $T(\omega) \in \mathcal{A}_\omega$ almost everywhere, is a von Neumann algebra on $\mathcal{H}$ such that, $\mathcal{Z} \subseteq \mathcal{M} \subseteq \mathcal{Z}'$. Moreover $\mathcal{M}$ is generated by $\mathcal{Z}$ and a countable family of elements $\{T_i\}$, where the $T_i(\omega)$'s generate $\mathcal{A}_\omega$ for almost every ω.

Proof See proposition 1 in [Dix 81] (Chapter 2, Part II). △

Definition 2.4.4.3 *A von Neumann algebra $\mathcal{M}$ on a Hilbert space $\mathcal{H}$ is said to be decomposable, if it is defined by a measurable field of $\omega \mapsto \mathcal{A}_\omega$ of von Neumann algebras. We then write*

$$\mathcal{M} = \int_\Omega^\oplus \mathcal{A}_\omega d\mu(\omega).$$

The $\mathcal{A}_\omega$'s are defined by $\mathcal{M}$ to within negligible sets.

Theorem 2.4.4.4 *For a von Neumann algebra $\mathcal{M}$ to be decomposable it is necessary and sufficient that it be the von Neumann algebra generated by $\mathcal{Z}$ and a countable family of decomposable operators.*

Proof Vide theorem 2 in [Dix 81] (Chapter 2, Part II). △

Chapter 3

Dynamics of a Quantum Spin Glass

In this chapter we give a detailed account of the models of a quantum spin glass investigated by us. In the sequel, we give a description of the models and establish the existence of global dynamics among other things. Traditionally, quantum spin glasses have been studied as systems of quantum spins interacting through random interactions. These models are essentially Ising–type models with random coupling. Generally, the coupling coefficients are assumed to be independent, identically distributed random variables. An alternate model of a quantum spin glass can be based on the realization that the magnetic ions are randomly distributed at lattice sites. The spins therefore, may be considered to be located at the vertices of an infinite graph in a lattice. There is no translation invariance in such a system, the lattice itself plays no significant role. Therefore, one can caricature a quantum spin glass as a quantum spin system with spins located at the vertices of an infinite connected graph with countably infinite number of vertices. In such a system, it is not necessary to consider random interactions. The study is restricted to deterministic interactions of the nearest neighbour type, with two spins defined as neighbours if an edge connects the two. This model may be regarded as a quantum analogue of the systems studied by Preston [Pre 74] and others. In the sequel, we establish the existence of the global dynamics of this infinite system of quantum spins, discuss the equilibrium state and establish the Kubo-Martin-Schwinger (KMS) condition. However, our attempts to establish the maximum entropy principle failed on account of absence of spatial homogeneity.

As expected, the thermodynamic limit of the local Gibbs states exists. Thus, an equilibrium state at a fixed inverse temperature β, exists for a quantum spin system on an infinite graph. But this state is by no means unique. It is shown that it satisfies the Kubo–Martin–Schwinger condition. We would like to point out that these equilibrium states which arise as thermodynamic limits of the the local Gibbs state are known to exist in the case of quantum spin systems, where the spins are located at each point of a countably infinite set L. In such cases, there is no additional structure imposed on the set L. However, in order to construct the dynamics for such spin systems, one has to put stringent conditions on the nature of the interactions between spins. In fact, in many cases, the interaction potentials are assumed to be of exponential nature. Whereas, in the case of a quantum spin system on an infinite graph, because of the additional structure, one does not have to be very restrictive regarding the class of interaction potentials.

3.1 A Quantum Spin System on an Infinite Graph

Definition 3.1.0.5 *A graph is said to be simple if it has no loops or multiple edges. Such a graph is said to have a finite valency if there exists an* $\alpha \in \mathbb{Z}^+$ *such that, at most* α *edges are incident on any vertex. Here* $\mathbb{Z}^+$ *denotes the set of all positive integers.*

Definition 3.1.0.6 *A non empty finite subset* $S \subseteq V$ *is said to be a simplex of the graph G(V,E) if, for every* $v_1, v_2 \in S$, *there exists an edge connecting the two. A subset* $S \subseteq V$ *is said to be a n-simplex* ($n \geq 0$) *if S is a simplex of the graph G and* $|S| = n + 1$. *Here* $|.|$ *denotes the cardinality of the set.*

Lemma 3.1.0.7 *It is easily seen that, given a simple graph G(V,E) with finite valency* $\alpha \in \mathbb{Z}^+$ *and* $v \in V$, *there is no n-simplex for* $n > \alpha$ *and there are at most only a finite number of simplexes containing v.*

Consider a quantum spin system on an infinite connected graph G(V,E), where V is the set of countably infinite mumber of vertices and E the collection of edges. The graph is assumed to be simple and has finite valency, say, $\alpha \in \mathbb{Z}^+$. By a connected graph we mean that there is a path connecting any two vertices of the graph. A quantum spin is assumed to be located at each of these vertices. Two spins interact if they are connected by an edge. A quasi–local UHF algebra constructed over finite subsets of the vertices of the graph is associated with this spin system. Explicitly, one can order the collection of all finite subsets of vertices by inclusion. With each vertex v

of the graph $G(V, E)$, one can associate a two dimensional complex Hilbert space $\mathcal{H}_v$. Then, with each finite [1] $\Lambda \subseteq V$, we associate the tensor product space

$$\mathcal{H}_\Lambda = \bigotimes_{v \in \Lambda} \mathcal{H}_v.$$

We then define the local $C^\star$–algebra $\mathcal{A}_\Lambda$ for each finite subset $\Lambda \subseteq V$ by $\mathcal{A}_\Lambda = \mathcal{L}(\mathcal{H}_\Lambda)$, where $\mathcal{L}(\mathcal{H}_\Lambda)$ denotes the space of all bounded linear operators on $\mathcal{H}_\Lambda$. Now, if $\Lambda_1 \cap \Lambda_2 = \emptyset$ for $\Lambda_1, \Lambda_2 \subseteq V$, then $\mathcal{H}_{\Lambda_1 \cup \Lambda_2} = \mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda_2}$ and $\mathcal{A}_{\Lambda_1}$ is isomorphic to the $C^\star$–subalgebra $\mathcal{A}_{\Lambda_1} \otimes I_{\Lambda_2}$, where I_{Λ_2} is the identity operator on $\mathcal{H}_{\Lambda_2}$. Further, if $\Lambda_1 \subseteq \Lambda_2$, one can identify $\mathcal{A}_{\Lambda_1}$ with the subalgebra $\mathcal{A}_{\Lambda_1} \otimes I_{\Lambda_2 \setminus \Lambda_1}$ of $\mathcal{A}_{\Lambda_2}$. Let the identification map be given by $i_{\Lambda_2, \Lambda_1} : A \in \mathcal{A}_{\Lambda_1} \to A \otimes I_{\Lambda_2 \setminus \Lambda_1} \in \mathcal{A}_{\Lambda_2}$. The collection $\{\mathcal{A}_\Lambda | \Lambda \subseteq V\}$ along with the collection of maps $\{i_{\Lambda_2, \Lambda_1}\}$ has the structure of a directed system of $C^\star$–algebras. Therefore, there exists a $C^\star$–algebra $\mathcal{A}$ with an identity I, which is the inductive limit of the collection $\{\mathcal{A}_\Lambda | \Lambda \subseteq V\}$ of $C^\star$-algebras with identity I_Λ. i.e., there exists a $C^\star$–algebra $\mathcal{A}$ and injective *–homomorphisms $i_\Lambda : \mathcal{A}_\Lambda \to \mathcal{A}$ such that,

$$\Lambda_1 \subseteq \Lambda_2 \Longrightarrow i_{\Lambda_1}(\mathcal{A}_{\Lambda_1}) \subseteq i_{\Lambda_2}(\mathcal{A}_{\Lambda_2}),$$

$$\overline{\bigcup_{\Lambda \subseteq V} i_\Lambda(\mathcal{A}_\Lambda)} = \mathcal{A}$$

and

$$i_\Lambda(I_\Lambda) = I; \quad \forall \Lambda \subseteq V.$$

[1]Throughout this chapter and the next, all Λ's, X's and Y's which feature as subsets of either V or $\mathbb{Z}^\nu$, with or without subscripts, should be taken to be finite unless stated otherwise.

Also, for

$$\Lambda_1 \cap \Lambda_2 = \emptyset; \quad [i_{\Lambda_1}(\mathcal{A}_{\Lambda_1}), i_{\Lambda_2}(\mathcal{A}_{\Lambda_2})] = 0,$$

where $[.,.]$ is the commutator. We will hence forth leave out the i_{Λ_2,Λ_1}'s and i_Λ's whenever no confusion can arise and regard $\mathcal{A}_\Lambda$'s as subalgebras of $\mathcal{A}$. This object $\mathcal{A}$, along with the net of local C^*–algebras $\{\mathcal{A}_\Lambda\}_{\Lambda \subseteq V}$ is a quasi–local algebra (The orthogonality relation $\perp$ between Λ's is defined by $\Lambda_1 \perp \Lambda_2$ if $\Lambda_1 \cap \Lambda_2 = \emptyset$). It is worth noting that $\mathcal{A}$ is an uniformly matricial algebra (UHF), and hence a separable C^*–algebra which is simple [Rob 81]. The local algebra $\mathcal{A}_\Lambda$ represents the physical observables associated with the spins located in a finite region Λ, where as the quasi–local algebra $\mathcal{A}$, corresponds to the observables of the infinite spin system.

3.2 Interactions

Definition 3.2.0.8 *An interaction Φ is a function from the collection $\mathcal{F}$ of finite subsets X of V into the Hermitian (self adjoint) elements in $\mathcal{A}$ such that, for every finite $X \subseteq V$, $\Phi(X) \in \mathcal{A}_X$.*

Definition 3.2.0.9 *An interaction Φ is said to be of the nearest neighbour type if, $\Phi(X) = 0$ whenever X is not a simplex of the graph G.*

Now for a finite X, $\Phi(X)$ represents the interaction energy of the spins confined to $X \subseteq V$. Hence, the total interaction energy for a finite $\Lambda \subseteq V$, consists of the interaction energy of all finite subsystems $X \subseteq \Lambda$. Thus, we define this total energy as the Hamiltonian $H(\Lambda)$ associated with $\Lambda \subseteq V$, i.e

$$H(\Lambda) = \sum_{X \subseteq \Lambda} \Phi(X).$$

$H(\Lambda)$ is a Hermitian (self adjoint) element of $\mathcal{A}_\Lambda$.

3.3 Time Evolution

In order to study the evolution of the infinite spin system, we write down the following equation of motion:

$$\frac{dA_t^\Lambda}{dt} = i[H(\Lambda), A_t^\Lambda], \quad A_t^\Lambda \in \mathcal{A}_\Lambda.$$

Here, $t \mapsto A_t^\Lambda$ describes the evolution of the observable $A \in \mathcal{A}_\Lambda$. This equation of motion defines a rule by which the observables associated with a finite $\Lambda \subseteq V$, evolve. With every $A \in \mathcal{A}_\Lambda$, it associates the observable $\tau_t^\Lambda(A) = A_t^\Lambda = e^{iH(\Lambda)t} A e^{-iH(\Lambda)t}$, which yields the quantum evolution of the finite spin system. Clearly, $\tau_t^\Lambda(A)$ is an element of $\mathcal{A}_\Lambda$ and τ_t^Λ is a one–parameter group of *–automorphisms of $\mathcal{A}_\Lambda$, which defines the time evolution of the finite subsystem confined to $\Lambda \subseteq V$. As the system consists of infinite number of spins, computing the time evolution of a fixed observable $A \in \mathcal{A}_{\Lambda_0}$, where $\Lambda_0 \subseteq V$, entails calculating the limit of $\tau_t{}^\Lambda(A)$ as $\Lambda \to \infty$. Here we adopt the convention that, $\Lambda \to \infty$ indicates Λ eventually contains all finite subsets of V. It is our endeavour to show that for a certain class of potentials this limit exists for all $A \in \mathcal{A}_{\Lambda_0}$. In order to make this notion of convergence precise, we observe that the collection $\mathcal{F}$ of all finite subsets Λ of V which is partially ordered by inclusion, is an increasing directed set. Hence, when we say that a net S_Λ converges to S in $\mathcal{A}$, as $\Lambda \to \infty$ (Λ eventually contains all finite subsets of V), we mean that for a given $\epsilon > 0$, there exists a finite subset Λ' of V such that, $\|S_\Lambda - S\| < \epsilon$, whenever $\Lambda \supseteq \Lambda'$. This is equivalent to showing that for a given $\epsilon > 0$, there exists a finite subset Λ' of V such that, $\|S_{\Lambda_1} - S_{\Lambda_2}\| < \epsilon$ whenever $\Lambda_1 \supseteq \Lambda'$ and $\Lambda_2 \supseteq \Lambda'$.

Next, note that the time evolution ${\tau_t}^\Lambda(A)$ of a finite system can be expanded in terms of commutators as

$$ {\tau_t}^\Lambda(A) = e^{iH(\Lambda)t} A e^{-iH(\Lambda)t} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} [H(\Lambda), A]^{(n)}, \tag{3.3.1} $$

where

$$ [B, A]^{(0)} = A, \quad [B, A]^{(1)} = [B, A] = BA - AB, $$

and

$$ [B, A]^{(n+1)} = [B, [B, A]^{(n)}]. $$

This formula is easily verified by taking derivatives of the expression in the middle and that of the expression on the right hand-side of (3.3.1).

In order to establish the dynamics of the spin system, we prove the following proposition.

Proposition 3.3.0.10 *Let Φ be a nearest neighbour type of interaction for the quantum spin system on the infinite graph $G(V, E)$ with valency α such that,*

$$ \sup_{v \in V} \left(\sum_{X \ni v} \|\Phi(X)\| \right) < \infty. $$

Then, for $A \in \mathcal{A}_{\Lambda_0}$ with $\Lambda_0 \subseteq V$, we have

$$\|[H(\Lambda), A]^{(n)}\| \leq \|A\| e^{|\Lambda_0|} n! \left(2 \left(\sup_{v \in V} \sum_{X \ni v} \|\Phi(X)\| \right) e^{(\alpha+1)} \right)^n . \tag{3.3.2}$$

Proof Take $A \in \mathcal{A}_{\Lambda_0}$, where $\Lambda_0 \subseteq V$. One has, $\Phi(X) \in \mathcal{A}_X$ for $X \subseteq V$. Now the local algebras $\mathcal{A}_{\Lambda_1}$, $\mathcal{A}_{\Lambda_2}$ commute whenever $\Lambda_1 \cap \Lambda_2 = \emptyset$. Therefore,

$$\begin{aligned}
\|[H(\Lambda), A]^{(n)}\| &= \| \sum_{X_1 \subseteq \Lambda} \cdots \sum_{X_n \subseteq \Lambda} [\Phi(X_n), [\ldots [\Phi(X_1), A]]] \| \\
&= \sum_{X_1 \subseteq \Lambda} \cdots \sum_{X_n \subseteq \Lambda} \|[\Phi(X_n), [\ldots [\Phi(X_1), A]]]\| \\
&\leq \sum_{X_1 \cap S_0 \neq \emptyset} \cdots \sum_{X_n \cap S_{n-1} \neq \emptyset} \|[\Phi(X_n), [\ldots [\Phi(X_1), A]]]\|
\end{aligned}$$

where

$$S_0 = \Lambda_0$$

and

$$S_j = X_j \cup X_{j-1} \cup \ldots \cup X_1 \cup \Lambda_0, \quad \text{for } j \geq 1.$$

Since Φ is a nearest neighbour type of interaction potential, on applying lemma 3.1.0.7, we notice that if

$$[\Phi(X_j), [\ldots [\Phi(X_1), A]]] \neq 0,$$

where

$$[\Phi(X_j), [\ldots [\Phi(X_1), A]]] \in \mathcal{A}_{S_j},$$

then,

$$|X_i| \leq \alpha + 1, \quad \forall i = 1, 2, \ldots, j,$$

where α is the valency of the graph $G(V, E)$. Therefore,

$$\begin{aligned} |S_j| &\leq |X_j| + |X_{j-1}| + \cdots + |X_1| + |\Lambda_0| \\ &\leq j(\alpha+1) + |\Lambda_0|, \end{aligned}$$

Thus, we get

$$\begin{aligned} \|[H(\Lambda), A]^{(n)}\| &\leq 2^n \|A\| \sum_{X_1 \cap S_0 \neq \emptyset} \cdots \sum_{X_n \cap S_{n-1} \neq \emptyset} \|\Phi(X_n)\| \ldots \|\Phi(X_1)\| \\ &\leq 2^n \|A\| \sum_{v_1 \in S_0} \sum_{X_1 \ni v_1} \cdots \sum_{v_n \in S_{n-1}} \sum_{X_n \ni v_n} \|\Phi(X_n)\| \ldots \|\Phi(X_1)\| \\ &\leq 2^n \|A\| \prod_{i=1}^{n} ((i-1)(\alpha+1) + |\Lambda_0|) \left(\sup_{v_i \in V} \sum_{X_i \ni v_i} \|\Phi(X_i)\| \right) \\ &\leq 2^n \|A\| \prod_{i=1}^{n} ((i-1)(\alpha+1) + |\Lambda_0|) \left(\sup_{v \in V} \sum_{X \ni v} \|\Phi(X)\| \right)^n \\ &\leq 2^n \|A\| (n(\alpha+1) + |\Lambda_0|)^n \left(\sup_{v \in V} \sum_{X \ni v} \|\Phi(X)\| \right)^n . \end{aligned}$$

Now $a^n \leq n! e^a$, for $a > 0$ hence,

$$\begin{aligned} \|[H(\Lambda), A]^{(n)}\| &\leq \|A\| e^{|\Lambda_0|} 2^n n! \left(\sup_{v \in V} \sum_{X \ni v} \|\Phi(X)\| \right)^n e^{n(\alpha+1)} \\ &\leq \|A\| e^{|\Lambda_0|} n! \left(2 \left(\sup_{v \in V} \sum_{X \ni v} \|\Phi(X)\| \right) e^{(\alpha+1)} \right)^n \end{aligned}$$

Notice that this estimate is independent of Λ and hence, we have

$$[H(\Lambda), A]^{(n)} \to \sum_{X_1 \subseteq V} \cdots \sum_{X_n \subseteq V} [\Phi(X_n), [\ldots [\Phi(X_1), A]]]$$

as $\Lambda \to \infty$. $\triangle$

Theorem 3.3.0.11 *Let Φ be a nearest neighbour type of interaction potential for the quantum spin system on the infinite graph $G(V, E)$ with valency α such* *that,*

$$\sup_{v\in V}\left(\sum_{X\ni v}\|\Phi(X)\|\right)<\infty.$$

Then, there exists a strongly continuous, one–parameter group of **–automorphisms* τ_t *of* $\mathcal{A}$ *such that, for all* $A\in\mathcal{A}$ *we have*

$$\tau_t(A)=\lim_{\Lambda\to\infty}\tau_t^{\Lambda}(A),$$

where

$$\tau_t^{\Lambda}(A)=e^{iH(\Lambda)t}Ae^{-iH(\Lambda)t},$$

and the limit is uniform for t on compact sets.

Proof We shall use the fact that

$$[H(\Lambda),A]^{(n)}\to\sum_{X_1\subseteq V}\cdots\sum_{X_n\subseteq V}[\Phi(X_n),[\ldots[\Phi(X_1),A]]],$$

as $\Lambda\to\infty$ and inequality 3.3.2 to demonstrate that for $A\in\mathcal{A}_{\Lambda_0}$, the limit of ${\tau_t}^{\Lambda}(A)$ exists as $\Lambda\to\infty$.

Put

$$T=\left(2\left(\sup_{v\in V}\sum_{X\ni v}\|\Phi(X)\|\right)e^{(\alpha+1)}\right)^{-1}.$$

It follows from equation 3.3.1 that for $A\in\mathcal{A}_{\Lambda_0}$.

$$\begin{aligned}&\|\tau_t^{\Lambda_1}(A)-\tau_t^{\Lambda_2}(A)\|\\ \le\;&\|\sum_{n=0}^{N}\frac{i^n}{n!}\left([H(\Lambda_1),A]^{(n)}-[H(\Lambda_2),A]^{(n)}\right)t^n\|+\|\sum_{n=N+1}^{\infty}\frac{i^n}{n!}[H(\Lambda_1),A]^{(n)}t^n\|\\ +\;&\|\sum_{n=N+1}^{\infty}\frac{i^n}{n!}[H(\Lambda_2),A]^{(n)}t^n\|.\end{aligned}$$

Let $\epsilon > 0$ be given, and choose t such that $|t| \leq t_1 \leq T$. It follows from inequality 3.3.2 in proposition 3.3.0.10 that, one can find $N \in \mathbb{Z}^+$ such that for $l = 1, 2$

$$\begin{aligned} \|\sum_{n=N+1}^{\infty} \frac{i^n}{n!}[H(\Lambda_l), A]^{(n)} t^n\| &\leq \sum_{n=N+1}^{\infty} \|\frac{i^n}{n!}[H(\Lambda_l), A]^{(n)}\| |t^n| \\ &\leq \|A\| e^{|\Lambda_0|} \sum_{n=N+1}^{\infty} \left(\frac{t_1}{T}\right)^n \\ &< \frac{\epsilon}{4}. \end{aligned}$$

Now, using the fact that

$$[H(\Lambda), A]^{(n)} \to \sum_{X_1 \subseteq V} \cdots \sum_{X_n \subseteq V} [\Phi(X_n), [\ldots [\Phi(X_1), A]]],$$

as $\Lambda \to \infty$, we can find a finite subset Λ' of V such that,

$$\|\frac{i^n}{n!}\left([H(\Lambda_1), A]^{(n)} - [H(\Lambda_2), A]^{(n)}\right)\| \;<\; t_1^{-n} \frac{\epsilon}{2N+2},$$

for all $n \leq N$ whenever $\Lambda_1 \supseteq \Lambda'$ and $\Lambda_2 \supseteq \Lambda'$. Thus, given $\epsilon > 0$, there exists a finite $\Lambda' \subseteq V$ such that,

$$\|\tau_t^{\Lambda_1}(A) - \tau_t^{\Lambda_2}(A)\| < \epsilon,$$

whenever $\Lambda_1 \supseteq \Lambda'$ and $\Lambda_2 \supseteq \Lambda'$. Hence, it follows that the convergence is uniform in t on any closed subinterval of $(-T, T)$ and in a ball around zero. Since for $t \in (-T, T)$, the mapping $A \mapsto \tau_t{}^{\Lambda}(A)$ is a *–automorphism and

$$\bigcup_{\Lambda \subseteq V} \mathcal{A}_{\Lambda}$$

is dense in $\mathcal{A}$, we conclude that

$$\lim_{\Lambda \to \infty} \tau_t^{\Lambda}(A),$$

exists for all $A \in \mathcal{A}$ and $t \in (-T, T)$. Therefore,

$$\tau_t(A) = \lim_{\Lambda \to \infty} \tau_t^{\Lambda}(A)$$

exists for all $A \in \mathcal{A}$ and $t \in (-T, T)$, and thus, defines a *–automorphism of $\mathcal{A}$ for each $t \in (-T, T)$. If we take t, s and $t + s$ in the interval $[-T, T]$ and use the group property of $\tau_t{}^{\Lambda}$, then on taking the limit as $\Lambda \to \infty$, we get

$$\tau_s \circ \tau_t(A) = \tau_{t+s}(A).$$

This group property of τ_t for $|t| < T$ allows us to define τ_t for all values of t. The strong continuity of τ_t follows from the series expansion. $\triangle$

3.4 Equilibrium State and the KMS Condition

We now focus our attention on the study of equilibrium states of the quantum spin system on an infinite graph. It is known that the equilibrium states of infinite systems are stationary. The analytic properties of these states are going to be the object of our study. In the sequel, we establish the existence of the thermodynamic limit of the local Gibbs states, and derive some interesting properties connected with these states.

As discussed earlier, there is a Hamiltonian $H(\Lambda) \in \mathcal{A}_{\Lambda}$ associated with each finite $\Lambda \subseteq V$. We are interested in the thermodynamic limit of the local Gibbs states ρ_{Λ}. Let us start by defining a local Gibbs state ρ_{Λ} for a finite $\Lambda \subseteq V$ as,

$$\rho_{\Lambda}(A) = \frac{Tr(e^{-\beta H(\Lambda)}A)}{Tr(e^{-\beta H(\Lambda)})},$$

where $A \in \mathcal{A}_\Lambda$. Here $\beta = KT^{-1}$, where K is the Boltzmann's constant and T the temperature.

Definition 3.4.0.12 *Let $\{\rho_\Lambda\}$ be the collection of the local Gibbs states defined on the local algebras $\mathcal{A}_\Lambda$. If there is a state ρ on $\mathcal{A}$ such that, ρ is the weak*-limit of a net of extensions of ρ_Λ to $\mathcal{A}$, then we call ρ the thermodynamic limit of the local Gibbs states. If $\hat{\rho}_{\Lambda_\alpha}$ is one such net of extensions, then for arbitrary $A \in \mathcal{A}_{\Lambda_0}$ and $\Lambda_\alpha \supseteq \Lambda_0$,*

$$\lim_{\Lambda_\alpha \to \infty} \rho_{\Lambda_\alpha}(A) = \rho(A).$$

Notice that the thermodynamic limit need not be unique, as different weak–limit points of the extensions of ρ_Λ to $\mathcal{A}$ give rise to different thermodynamic limits of ρ_Λ.*

A state obtained as the thermodynamic limit of the local Gibbs states $\{\rho_\Lambda\}$ will be called the equilibrium state of the infinite quantum spin system.

Now, the thermodynamic limit of the local Gibbs states $\{\rho_\Lambda\}$ exists by virtue of the fact that each ρ_Λ can be extended to the whole of $\mathcal{A}$, and if $\hat{\rho}_\Lambda$ is one such extension, then the collection $\{\hat{\rho}_\Lambda\}$ being weak*–compact, one can always find an accumulation point ρ. Since $\mathcal{A}$ is separable, we can extract a sequence $\hat{\rho}_{\Lambda_n}$ from the net $\hat{\rho}_\Lambda$ such that,

$$\rho(A) = \lim_{n \to \infty} \rho_{\Lambda_n}(A),$$

for all $A \in \mathcal{A}_\Lambda$ and all Λ. Thus, the thermodynamic limit of the local Gibbs states ρ_Λ exists. In the discussion that follows, we derive an interesting property of the local Gibbs states ρ_Λ and establish the Kubo–Martin–Schwinger (KMS) condition for the equilibrium state ρ of the infinite system.

Proposition 3.4.0.13 *Let $A, B \in \mathcal{A}_\Lambda$ and $\beta > 0$. There exists a complex valued function $F^\Lambda_{A,B}$, which is analytic everywhere and uniformly bounded in the strip $0 \leq \Im z \leq \beta$ such that, for real t,*

$$F^\Lambda_{A,B}(t) = \rho_\Lambda(A\tau_t^\Lambda(B))$$

and

$$F^\Lambda_{A,B}(t + i\beta) = \rho_\Lambda(\tau_t{}^\Lambda(B)A).$$

If $\beta < 0$, then there exists a complex valued function $F^\Lambda_{A,B}$, which is analytic everywhere and uniformly bounded in the strip $\beta \leq \Im z \leq 0$ such that, for real t,

$$F^\Lambda_{A,B}(t) = \rho_\Lambda(A\tau_t^\Lambda(B))$$

and

$$F^\Lambda_{A,B}(t + i\beta) = \rho_\Lambda(\tau_t{}^\Lambda(B)A).$$

Proof Let A, $B \in \mathcal{A}_\Lambda$ and $\beta > 0$. Since $H(\Lambda) \in \mathcal{A}_\Lambda$, where $\mathcal{A}_\Lambda$ is a matrix algebra, $\tau_t^\Lambda(B) = e^{iH(\Lambda)t}Be^{-iH(\Lambda)t}$ makes sense for all complex t and hence, has an analytic extension to the entire complex plane. Therefore, it follows that $\rho_\Lambda(A\tau_t^\Lambda(B))$ can be extended to an entire function $F^\Lambda_{A,B}(z)$ on $\mathbb{C}$. Now, for real t, consider

$$\begin{aligned}
\rho_\Lambda(A\tau^\Lambda_{t+i\beta}(B)) &= \frac{Tr(e^{-\beta H(\Lambda)}Ae^{iH(\Lambda)(t+i\beta)}Be^{-iH(\Lambda)(t+i\beta)})}{Tr(e^{-\beta H(\Lambda)})} \\
&= \frac{Tr(e^{-\beta H(\Lambda)}e^{iH(\Lambda)t}Be^{-iH(\Lambda)t}A)}{Tr(e^{-\beta H(\Lambda)})} \\
&= \rho_\Lambda(\tau_t{}^\Lambda(B)A).
\end{aligned}$$

The last equality follows from the cyclicity of the trace. Further, $|F^{\Lambda}_{A,B}(z)|$ is bounded in the open strip and $|F^{\Lambda}_{A,B}(z)| \leq \|A\|\|B\|$ on the boundary of the strip. Therefore, it follows from a version of the Phragmen–Lindelöf theorem ([Rob 81], Prop 5.3.5) that, the maximum of the function $|F^{\Lambda}_{A,B}(z)|$ is attained on the boundary. Hence the theorem holds for $\beta > 0$. Similarly, the theorem can be proved for $\beta < 0$. $\triangle$

In order to study the analytic property of the thermodynamic limit ρ of the local Gibbs states ρ_Λ, one needs to prove the following proposition.

Proposition 3.4.0.14 *Let $\{\Lambda_n\}$ be a sequence of finite subsets of V such that, $lim_{n\to\infty}\rho_{\Lambda_n}(A) = \rho(A)$, $\forall A \in \mathcal{A}_{\Lambda_0}$ and all $\Lambda_0 \subseteq V$. Then, for $A, B \in \mathcal{A}_{\Lambda_0}$, we have*

$$\lim_{n\to\infty} \rho_{\Lambda_n}(A{\tau_t}^{\Lambda_n}(B)) = \rho(A\tau_t(B)),$$

where the limit exists for all real t and uniformly for t in a small ball around zero.

Proof Let $A, B \in \mathcal{A}_{\Lambda_0}$ where $\Lambda_0 \subseteq V$. Now we have from theorem 3.3.0.11 that,

$$\lim_{n\to\infty} {\tau_t}^{\Lambda_n}(B) = \tau_t(B),$$

for $B \in \mathcal{A}_{\Lambda_0}$, where the limit is uniform in t, in some ball around zero. Therefore, given $\epsilon > 0$ and a fixed t, there exists $n_0 \in \mathbb{Z}^+$, which can be chosen independent of t in a ball around zero such that, for $n, m > n_0$,

$$\|{\tau_t}^{\Lambda_n}(B) - {\tau_t}^{\Lambda_m}(B)\| < \frac{\epsilon}{4\|A\|} \quad \textit{and} \quad \|{\tau_t}^{\Lambda_m}(B) - \tau_t(B)\| < \frac{\epsilon}{4\|A\|}.$$

Further, since

$$\lim_{n\to\infty} \rho_{\Lambda_n}(A\tau_t^{\Lambda_m}(B)) = \rho(A\tau_t^{\Lambda_m}(B)),$$

we have for given m, and $n > n_0$

$$|\rho_{\Lambda_n}(A\tau_t^{\Lambda_m}(B)) - \rho(A\tau_t^{\Lambda_m}(B))| < \frac{\epsilon}{2}.$$

These estimates allow us to arrive at the following inequalities:

$$\begin{aligned}
|\rho(A\tau_t(B)) - \rho_{\Lambda_n}(A\tau_t^{\Lambda_n}(B))| &\leq |\rho(A\tau_t(B)) - \rho(A\tau_t^{\Lambda_m}(B))| \\
&+ |\rho(A\tau_t^{\Lambda_m}(B)) - \rho_{\Lambda_n}(A\tau_t^{\Lambda_m}(B))| \\
&+ |\rho_{\Lambda_n}(A\tau_t^{\Lambda_m}(B)) - \rho_{\Lambda_n}(A\tau_t^{\Lambda_n}(B))| \\
&\leq \|\tau_t^{\Lambda_n}(B) - \tau_t^{\Lambda_m}(B)\|\|A\| \\
&+ \|\tau_t^{\Lambda_m}(B) - \tau_t(B)\|\|A\| + \frac{\epsilon}{2} \\
&< \epsilon.
\end{aligned}$$

This proves the proposition for real t, and t in a small ball around zero. △

It is evident that the time evolution bears some relation with the equilibrium state of the infinite system. One such relation is the Kubo–Martin–Schwinger (KMS) condition. This condition may be formulated as follows for the equilibrium state ρ.

Theorem 3.4.0.15 *Let ρ be the equilibrium state of the quantum spin system on the infinite graph $G(V, E)$ and $A, B \in \mathcal{A}$. Then, for $\beta > 0$, there exists a function $F_{A,B}$, which is analytic in the open strip $0 < \Im z < \beta$, continuous and uniformly bounded in the closed strip $0 \leq \Im z \leq \beta$ such that,*

$$F_{A,B}(t) = \rho(A\tau_t(B)) \quad \textit{and} \quad F_{A,B}(t + i\beta) = \rho(\tau_t(B)A).$$

If $\beta < 0$, then there exists a function $F_{A,B}$, which is analytic in the open strip $\beta < \Im z < 0$, continuous and uniformly bounded in the closed strip $\beta \leq \Im z \leq 0$ such that,

$$F_{A,B}(t) = \rho(A\tau_t(B)) \quad \textit{and} \quad F_{A,B}(t + i\beta) = \rho(\tau_t(B)A).$$

Proof We shall prove the theorem for the case $\beta > 0$. Let $\{\Lambda_n\}$ be a sequence of finite subsets of $\mathbb{Z}^\nu$ such that, $\lim_{n\to\infty} \rho_{\Lambda_n}(B) = \rho(B)$, for all $B \in \mathcal{A}_{\Lambda_0}$ and all $\Lambda_0 \subseteq V$. It follows from proposition 3.4.0.13, that, for $\beta > 0$ and $A, B \in \mathcal{A}_{\Lambda_0}$, there exists a sequence of entire functions $F^{\Lambda_n}_{A,B}(z)$, which is uniformly bounded in the closed strip $0 \leq \Im z \leq \beta$ such that, for real t,

$$F^{\Lambda_n}_{A,B}(t) = \rho_\Lambda(A\tau_t^{\Lambda_n}(B)) \quad \textit{and} \quad F^{\Lambda_n}_{A,B}(t + i\beta) = \rho_{\Lambda_n}(\tau_t{}^{\Lambda_n}(B)A).$$

Therefore, it follows from proposition 3.4.0.14, that this sequence converges pointwise on the real axis and in a neighbourhood of zero. Hence, as a consequence of Vitali's theorem, see [Tit 91], the sequence $F^{\Lambda_n}_{A,B}$ of analytic functions converges uniformly on every compact subset of the strip to a function $F_{A,B}$, which is analytic in the open strip $0 < \Im z < \beta$, continuous and uniformly bounded in the closed strip $0 \leq \Im z \leq \beta$ such that,

$$F_{A,B}(t) = \rho(A\tau_t(B)) \quad \textit{and} \quad F_{A,B}(t + i\beta) = \rho(\tau_t(B)A)$$

The general case can be handled by approximating arbitrary $A \in \mathcal{A}$ by local elements and using a version of the Phragmen–Lindelöf theorem ([Rob 81], Prop 5.3.5).

For $\beta < 0$, the theorem can be proved along the same lines by considering the closed strip $\beta \leq \Im z \leq 0$. $\triangle$

Corollary 3.4.0.16 *The equilibrium state ρ of the infinite spin system is invariant under time evolution given by the automorphism group τ_t.*

Proof Take $B = I$ in proposition (3.4.0.14). $\triangle$

Thus, having established the existence of an equilibrium state of the spin system on an infinite graph, we set our sights on proving the maximum entropy principle for the infinite spin system. In view of this, we attempted to establish the existence of thermodynamic quantities such as mean entropy and the free energy of this infinite system. To this end we constructed a nested sequence $\{G_n(V_n, E_n)\}$ of finite subgraphs of the infinite connected graph $G(V, E)$, with set of vertices V_n and collection of edges E_n. Each of these subgraphs $G_n(V_n, E_n)$ is constructed from the preceding subgraph by simply adding those vertices of the graph $G(V, E)$, which are connected to it by an edge. The choice of the initial subgraph can be arbitrary. The investigation concerning the existence of mean entropy and the free energy of the infinite system in the state ρ, entails computing the limit of the entropy per site $\frac{S_\rho(V_n)}{|V_n|}$ and the free energy per site $-\beta^{-1}\frac{\log\left(Tr\left(e^{-\beta H(V_n)}\right)\right)}{|V_n|}$, as $n \to \infty$. Here $|V_n|$ denotes the cardinality of the set of vertices of the subgraph $G_n(V_n, E_n)$. The entropy $S_\rho(V_n) = -Tr(\rho_{V_n} \log(\rho_{V_n}))$, where ρ_{V_n} is the density matrix corresponding to the restriction of the state ρ to the local algebra $\mathcal{A}_{V_n}$, associated with the subgraph $G_n(V_n, E_n)$. But, all attempts at proving the existence of these limits failed, primarily because of the absence of spatial homogeneity. For, unlike in the case of a quantum spin system on a lattice, the local entropy is not translation invariant. Besides, the absence of translation invariance also hindered the investigation pertaining to the existence of free energy of the spin system on the infinite graph. Despite the fact that the local entropy satisfies the strong subadditivity property, none of the results pertaining to the existence of the limit of objects such as $\frac{f(x)}{x}$ as $x \to \infty$, where f is a real valued subadditive function defined on $\mathbb{R}$ ($\mathbb{R}^+, \mathbb{Z}^+$), could be applied in this case. Such results are known to have a role to play, in demonstrating the existence of mean entropy for quantum spin systems on a lattice with deterministic interaction potentials [Rue 69]. Thus, the question of existence of these quantities remains unresolved. Therefore, one conjectures that the maximum entropy principle may not hold for a quantum spin system on an infinite connected graph with deterministic interaction potential of the nearest neighbour type. However, in the case of some random models of a spin glass, subadditivity along with the appropriate ergodic theorem have been employed to establish the existence of some thermodynamic quantities under fairly stringent conditions on the random interaction potential. For, in the study of equilibrium spin glass theory through random models on a lattice, van Hemmen et al [Hem 83, Ent 83] have shown that the thermodynamic limit of the local free energy $F(\Lambda)$ exists. In fact, it has been established that the free energy of the infinite system exists as a non–random limit of $\frac{F(\Lambda)}{|\Lambda|}$, with probability one. Thus, one is obliged to conclude that the attempt at understanding the behaviour of a quantum spin glass through quantum spin systems on an infinite graph, has not proved to be very useful. Therefore, one is obliged to take recourse to the more traditional approach.

As mentioned earlier, in the more traditional line of thinking, quantum spin glasses have been studied as systems of quantum spins interacting through random interactions. These models are essentially Ising–type models with random coupling. Extensive investigations on the existence of the thermodynamic limit have been made e. g. van Hemmen et al [Hem 83, Ent 83], and the equilibrium statistical mechanics of such systems has been studied. Although quantum spin glasses admit a natural dynamics, no attempt has been made to study the dynamics of a quantum spin glass. Hence, we study the dynamics of a quantum spin glass, as a quantum spin system on an infinite lattice with random interactions. We establish the existence of a family of one–parameter groups of *–automorphisms $\{\tau_t(\omega)\}$, of the quasi–local algebra $\mathcal{A}$ associated with the infinite system. Here ω lives in a probability space $(\Omega, \mathcal{S}, P)$, where Ω is a set, $\mathcal{S}$ a sigma algebra and P a complete probability measure. The strong measurability of $(t, \omega) \mapsto \tau_t(\omega)(A)$, for all $A \in \mathcal{A}$ is established. Some interesting algebraic properties of the automorphism groups $\tau_t(\omega)$ as well as those of their generators $\overline{\delta}(\omega)$ have been derived.

3.5 Description of the Random Model

Consider a quantum spin system with spins located at the vertices of an infinite lattice $\mathbb{Z}^\nu$. The interaction between spins of course taken to be random. A quasi–local UHF algebra similar to the one in 3.1, constructed over the finite subsets of $\mathbb{Z}^\nu$, is associated with this spin system. One can order the collection of all finite subsets of $\mathbb{Z}^\nu$ by inclusion. With each point in $\mathbb{Z}^\nu$, one associates a two dimensional Hilbert space $\mathcal{H}_x$. Then with each finite $\Lambda \subseteq \mathbb{Z}^\nu$, we associate the tensor product space

$$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x,$$

where $\Lambda \subseteq \mathbb{Z}^\nu$. We define a local $C^\star$–algebra for each finite $\Lambda \subseteq \mathbb{Z}^\nu$ by $\mathcal{A}_\Lambda = \mathcal{L}(\mathcal{H}_\Lambda)$, where $\mathcal{L}(\mathcal{H}_\Lambda)$ denotes the space of all bounded linear operators on $\mathcal{H}_\Lambda$. Now if $\Lambda_1 \cap \Lambda_2 = \emptyset$ for $\Lambda_1, \Lambda_2 \subseteq \mathbb{Z}^\nu$, then $\mathcal{H}_{\Lambda_1 \cup \Lambda_2} = \mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda_2}$ and $\mathcal{A}_{\Lambda_1}$ is isomorphic to the $C^\star$–subalgebra $\mathcal{A}_{\Lambda_1} \otimes I_{\Lambda_2}$, where I_{Λ_2} is the identity operator on $\mathcal{H}_{\Lambda_2}$. Further, if $\Lambda_1 \subseteq \Lambda_2$, one can identify $\mathcal{A}_{\Lambda_1}$ with the subalgebra $\mathcal{A}_{\Lambda_1} \otimes I_{\Lambda_2 \setminus \Lambda_1}$ of $\mathcal{A}_{\Lambda_2}$. Let the identification map be given by $i_{\Lambda_2,\ \Lambda_1} : A \in \mathcal{A}_{\Lambda_1} \rightarrow A \otimes I_{\Lambda_2 \setminus \Lambda_1} \in \mathcal{A}_{\Lambda_2}$. The collection $\{\mathcal{A}_\Lambda | \Lambda \subseteq \mathbb{Z}^\nu\}$, with the collection of maps $\{i_{\Lambda_2,\Lambda_1}\}$ has the structure of a directed system of $C^\star$–algebras. Therefore, there exists a $C^\star$–algebra $\mathcal{A}$ with an identity, which is the inductive limit of the collection $\{\mathcal{A}_\Lambda | \Lambda \subseteq \mathbb{Z}^\nu\}$ of $C^\star$–algebras with identity I_Λ. i.e., there exists a $C^\star$–algebra $\mathcal{A}$ and injective *–homomorphisms $i_\Lambda : \mathcal{A}_\Lambda \rightarrow \mathcal{A}$ such that,

$$\Lambda_1 \subseteq \Lambda_2 \Longrightarrow i_{\Lambda_1}(\mathcal{A}_{\Lambda_1}) \subseteq i_{\Lambda_2}(\mathcal{A}_{\Lambda_2}),$$

$$\overline{\bigcup_{\Lambda \subseteq \mathbb{Z}^\nu} i_\Lambda(\mathcal{A}_\Lambda)} = \mathcal{A}$$

and

$$i_\Lambda(I_\Lambda) = I \quad \forall \Lambda \subseteq \mathbb{Z}^\nu.$$

Also, for

$$\Lambda_1 \cap \Lambda_2 = \emptyset; \quad [i_{\Lambda_1}(\mathcal{A}_{\Lambda_1}), i_{\Lambda_2}(\mathcal{A}_{\Lambda_2})] = 0,$$

where $[.,.]$ is the commutator.

We will hence forth leave out the i_{Λ_2,Λ_1}'s and i_Λ's whenever no confusion can arise and regard $\mathcal{A}_\Lambda$'s as subalgebras of $\mathcal{A}$. This object $\mathcal{A}$ along with the net of local $C^\star$–algebras $\{\mathcal{A}_\Lambda\}_{\Lambda \subseteq \mathbb{Z}^\nu}$ is a quasi–local algebra (The orthogonality relation $\perp$ between Λ's is defined by $\Lambda_1 \perp \Lambda_2$ if $\Lambda_1 \cap \Lambda_2 = \emptyset$). It is easily seen that the quasi–local UHF algebra $\mathcal{A}$, is a separable $C^\star$–algebra with no non–trivial closed ideals. Hence, it is a simple $C^\star$–algebra [Rob 81]. The local algebras $\mathcal{A}_\Lambda$ represent the physical observables associated with the spins located in Λ, whereas the quasi–local algebra $\mathcal{A}$ corresponds to the observables associated with the infinite spin system.

Having described the kinematical structure of the quantum spin system on the lattice $\mathbb{Z}^\nu$, we now turn our attention to the action of the symmetry group associated with the lattice $\mathbb{Z}^\nu$, on the observable algebra $\mathcal{A}$. To this end, for each $x \in \mathbb{Z}^\nu$, choose an unitary mapping $V(x) : \mathcal{H}_o \to \mathcal{H}_x$, where $\mathcal{H}_x$ is the underlying Hilbert space at x. Now for $x_1, x_2 \in \mathbb{Z}^\nu$, define $V(x_2, x_1) : \mathcal{H}_{x_1} \to \mathcal{H}_{x_2}$, by $V(x_2, x_1) = V(x_2)V(x_1)^{-1}$. It is clear that for $x_1, x_2, x_3 \in \mathbb{Z}^\nu$, $V(x_3, x_1) = V(x_3, x_2)V(x_2, x_1)$. Furthermore, for each $a \in \mathbb{Z}^\nu$, define $V_x(a) : \mathcal{H}_x \to \mathcal{H}_{x+a}$ as $V_x(a) = V(x+a, x)$. Thus, if for each $\Lambda \subseteq \mathbb{Z}^\nu$, one were to define $V_\Lambda(a) : \mathcal{H}_\Lambda \to \mathcal{H}_{\Lambda+a}$ as

$$V_\Lambda(a) = \bigotimes_{x \in \Lambda} V_x(a),$$

then $V_\Lambda(a)$ is an isomorphism and one has [2] $V_\Lambda(a)^{-1} = V_{\Lambda+a}(-a)$. We can now introduce an action α of $\mathbb{Z}^\nu$ as *–automorphisms of $\mathcal{A}$ as follows. For each $a \in \mathbb{Z}^\nu$, define

$$\alpha_a(A) = V_\Lambda(a) A V_\Lambda(a)^{-1}; \quad \forall A \in \mathcal{A}_\Lambda.$$

Thus, α is consistently defined on the union of local C^*–algebras $\bigcup \mathcal{A}_\Lambda$, as an isometric *–isomorphism and hence, can be extended by continuity to an automorprhism of $\mathcal{A}$, as

$$\alpha_a(\mathcal{A}_\Lambda) = \mathcal{A}_{\Lambda+a}.$$

Therefore, it follows from the quasi–local structure of $\mathcal{A}$ that

$$\lim_{a\to\infty} \|[\alpha_a(A), B]\| = 0, \quad \forall A, B \in \mathcal{A}$$

i.e., $\mathcal{A}$ is asymptotically abelian.

3.6 Random Interactions

Definition 3.6.0.17 *An interaction Ψ of the quantum spin system on the infinite lattice $\mathbb{Z}^\nu$, is a mapping from the collection of finite subsets X of $\mathbb{Z}^\nu$ into the Hermitian (self adjoint) elements of $\mathcal{A}$ such that, for every finite $X \subseteq \mathbb{Z}^\nu$, $\Psi(X) \in \mathcal{A}_X$.*

Before we introduce random interactions, one has to define the notion of measurability of Banach space valued functions on a measure space $(\Omega, \mathcal{S}, m)$, where Ω is a set, $\mathcal{S}$ a sigma algebra and m a sigma–finite measure on Ω.

[2] Here $V_\Lambda(a)^{-1}$ denotes the inverse of $V_\Lambda(a)$.

Definition 3.6.0.18 *Let $(\Omega, \mathcal{S}, m)$ be a measure space. A function $f : \Omega \rightarrow B$ where B is a Banach space, is said to be weakly measurable if, for every $\phi \in B^{\star}$, the map $\omega \mapsto \phi(f(\omega))$ is $\mathcal{S}$-measurable. f is said to be strongly measurable if, there exists a sequence of countably valued functions strongly convergent to f almost everywhere on Ω [Hil 57].*

In case m is a finite measure, then we may replace "countably valued" in the above definition by "simple". It can be shown that the notions of strong and weak measurability are equivalent if B is separable.

Definition 3.6.0.19 *Let $(\Omega, \mathcal{S}, P)$ be a probability space and J some index set. If T_j is a measure preserving automorphism of Ω, for each $j \in J$, then the action of T_j's is said to be ergodic if, for $A \in \mathcal{S}$, $P(A) = 0$ or 1 whenever $T_j A = A$, for all $j \in J$.*

From now on, let $(\Omega, \mathcal{S}, P)$ be a complete probability space, where Ω is a complete separable metric space. $\mathcal{S}$ is the sigma algebra of subsets of Ω, containing the Borel sigma algebra $\mathcal{B}$ generated by open sets in Ω. P is the completion of a probability measure defined on $\mathcal{B}$.

Definition 3.6.0.20 *Let $\mathcal{F}$ be the collection of all finite subsets of $\mathbb{Z}^{\nu}$. A random interaction is a map $\Phi : \mathcal{F} \times \Omega \rightarrow \mathcal{A}$ such that, for each $\omega \in \Omega$, $\Phi(., \omega)$ is an interaction of the quantum spin system on $\mathbb{Z}^{\nu}$ and $\omega \mapsto \Phi(X, \omega)$ is strongly measurable for every $X \in \mathcal{F}$.*

Now, for finite $\Lambda \subseteq \mathbb{Z}^{\nu}$, the Hamiltonian associated with the spins confined to the region Λ is given by a Hermitian (self adjoint) element

$$H(\Lambda,\omega) = \sum_{X \subseteq \Lambda} \Phi(X,\omega),$$

for $\omega \in \Omega$. Clearly, $H(\Lambda,\omega)$ is strongly measurable since each $\Phi(X,\omega)$ is strongly measurable on Ω.

In order to construct the dynamics of the quantum spin system with random interactions, we have to restrict the class of random interactions Φ. To this end, we introduce a measure preserving group of automorphisms $\{T_a\}_{a \in \mathbb{Z}^\nu}$ with an ergodic action on the probability space Ω, and thereby restrict the class of interactions to those Φ which satisfy the following condition:

$$\Phi(X + a, T_{-a}\omega) = \alpha_a(\Phi(X,\omega)).$$

From now on, we shall consider only those random interactions Φ which satisfy the above condition. Therefore, $H(\Lambda + a, T_{-a}\omega) = \alpha_a(H(\Lambda,\omega))$.

Definition 3.6.0.21 *Let Φ be a random interaction. The interaction $\Phi(.,\omega)$ is said to have a finite range if, the set*

$$\Delta_\omega = \{x \in \mathbb{Z}^\nu \,|\, \exists X \ni x;\ such\,that\, 0 \in X,\ and\, \Phi(X, T_a\omega) \neq 0,\ for\ some\ a \in \mathbb{Z}^\nu\}$$

is a finite subset of $\mathbb{Z}^\nu$. We may call Δ_ω the range of $\Phi(.,\omega)$.

Remark Clearly, for such $\Phi(.,\omega)$'s, whenever [3] $X - X \not\subseteq \Delta_\omega$, $\Phi(X,\omega) = 0$. For, if $X - X \not\subseteq \Delta_\omega$, then there exists x, $y \in X$ such that, $x - y \notin \Delta_\omega$. But, $x - y \in X - y$, therefore $X - y \not\subseteq \Delta_\omega$. Now, since $0 \in X - y$, it

[3] For $X \subseteq \mathbb{Z}^\nu$, $X - X = \{x - y \,|\, x, y \in X\}$.

follows from the above definition that $\Phi(X-y,T_a\omega)=0$ for all $a\in\mathbb{Z}^\nu$. In particular, on putting $a=y$, we get $\Phi(X-y,T_y\omega)=0$. Therefore, $\Phi(X,\omega)=\alpha_y(\Phi(X-y,T_y\omega))=0$.

Definition 3.6.0.22 *The random interaction Φ is said to be a finite range random interaction if, $\Phi(.,\omega)$ has a finite range Δ_ω for almost every $\omega\in\Omega$, and $\omega\mapsto|\Delta_\omega|$ is a measurable function of ω. Here $|.|$ denotes the cardinality of a set.*

It is clear from the above remark that if Φ is a finite range random interaction, then for almost every $\omega\in\Omega$, $\Phi(X,\omega)=0$, whenever $|X|>|\Delta_\omega|$.

We use the ergodicity of the measure preserving group of automorphisms to establish the following fact.

Lemma 3.6.0.23 *Let Φ be a finite range random interaction. Since the action of the measure preserving group of automorphisms $\{T_a\}$ is ergodic, the function $\omega\mapsto|\Delta_\omega|$ is almost surely constant.*

Proof We show that $\Delta_\omega=\Delta_{T_b\omega}$, for all $b\in\mathbb{Z}^\nu$. Fix $b\in\mathbb{Z}^\nu$. Let $x\in\Delta_\omega$. Then there exists a finite $X\ni x$ such that, $0\in X$ and $\Phi(X,T_a\omega)\neq0$ for some $a\in\mathbb{Z}^\nu$. i.e. there exists $X\ni x$ such that, $0\in X$ and $\Phi(X,T_{a-b}(T_b\omega))\neq0$, for some $a\in\mathbb{Z}^\nu$. Therefore, $x\in\Delta_{T_b\omega}$. Conversely, let $x\in\Delta_{T_b\omega}$. Then there exists $X\ni x$ such that, $0\in X$ and $\Phi(X,T_a(T_b\omega))\neq0$, for some $a\in\mathbb{Z}^\nu$. This implies that there exists $X\ni x$ such that, $0\in X$ and $\Phi(X,T_{a+b}\omega)\neq0$, for some $a\in\mathbb{Z}^\nu$. Hence, $x\in\Delta_\omega$. Thus, $\Delta_\omega=\Delta_{T_b\omega}$. Since b is arbitrary, this holds for all $b\in\mathbb{Z}^\nu$. Now, since $\Phi(.,\omega)$ has a finite range Δ_ω for almost every $\omega \in \Omega$, we have $|\Delta_\omega| = |\Delta_{T_b\omega}|$, for almost every $\omega \in \Omega$. Therefore, it is readily concluded that the measurable function $\omega \mapsto |\Delta_\omega|$ is invariant almost everywhere with respect to the measure preserving group of automorphisms $\{T_a\}$. Since the action of the group is ergodic, the lemma follows. $\triangle$

Lemma 3.6.0.24 *Let Φ be a finite range random interaction of the quantum spin system on an infinite lattice $\mathbb{Z}^\nu$, satisfying*

$$\sup_{a \in \mathbb{Z}^\nu} \left(\sum_{X \ni 0} \|\Phi(X, T_a\omega)\| \right) < \infty$$

almost everywhere, then the function

$$\omega \mapsto \sup_{a \in \mathbb{Z}^\nu} (\sum_{X \ni 0} \|\Phi(X, T_a\omega)\|)$$

is almost surely constant.

Proof The function $\omega \mapsto \Phi(X, \omega)$ is strongly measurable for all finite $X \subseteq \mathbb{Z}^\nu$. Therefore, it follows easily that $\omega \mapsto \|\Phi(X, \omega)\|$ is a numerically valued measurable function on Ω. Since for $a \in \mathbb{Z}^\nu$, T_a is a measure preserving automorphism of Ω, clearly, $\omega \mapsto \|\Phi(X, T_a\omega)\|$ is a measurable function of ω. Next, let $X_1, X_2, \ldots$ be the countable collection of all finite subsets of $\mathbb{Z}^\nu$ containing 0. Since $\Phi(., \omega)$ has a finite range for almost every $\omega \in \Omega$, the sum of non-negative terms

$$\sum_{X \ni 0} \|\Phi(X, T_a\omega)\|,$$

is finite almost everywhere. Therefore, the series $\sum_{n=1}^\infty \|\Phi(X_n, T_a\omega)\|$ converges to $\sum_{X \ni 0} \|\Phi(X, T_a\omega)\|$, almost everywhere. i.e.,

$$\sum_{n=1}^\infty \|\Phi(X_n, T_a\omega)\| = \sum_{X \ni 0} \|\Phi(X, T_a\omega)\|,$$

almost everywhere. Now, each of the terms of the series is a measurable function of ω. Hence, the measurability of $\omega \mapsto \sum_{X \ni 0} \|\Phi(X, T_a\omega)\|$ follows, for $a \in \mathbb{Z}^\nu$. Thus,

$$\omega \mapsto \sup_{a \in \mathbb{Z}^\nu} (\sum_{X \ni 0} \|\Phi(X, T_a\omega)\|)$$

is a measurable function of ω. Also, for almost every $\omega \in \Omega$,

$$\sup_{a \in \mathbb{Z}^\nu} \left(\sum_{X \ni 0} \|\Phi(X, T_a(T_b\omega))\| \right) = \sup_{a \in \mathbb{Z}^\nu} \left(\sum_{X \ni 0} \|\Phi(X, T_a\omega)\| \right),$$

for all $b \in \mathbb{Z}^\nu$. Thus,

$$\omega \mapsto \sup_{a \in \mathbb{Z}^\nu} (\sum_{X \ni 0} \|\Phi(X, T_a\omega)\|)$$

is a measurable function which is invariant under the action of the measure preserving group of automorphisms almost everywhere. Since the action of the automorphism group is ergodic, it follows that the function

$$\omega \mapsto \sup_{a \in \mathbb{Z}^\nu} (\sum_{X \ni 0} \|\Phi(X, T_a\omega)\|)$$

is almost surely constant. △

3.7 Random Evolution

For a finite spin system confined to a region $\Lambda \subseteq \mathbb{Z}^\nu$, and for $\omega \in \Omega$, the equation of motion is given by

$$\frac{dA_t^\Lambda(\omega)}{dt} = i[H(\Lambda, \omega), A_t^\Lambda(\omega)], \quad A_t^\Lambda(\omega) \in \mathcal{A}_\Lambda \quad .$$

Here $t \mapsto A_t^\Lambda(\omega)$ describes the evolution of the observable $A \in \mathcal{A}_\Lambda$. This yields the time evolution given by $\tau_t^\Lambda(\omega)(A) = A_t^\Lambda(\omega) = e^{iH(\Lambda,\omega)t} A e^{-iH(\Lambda,\omega)t}$, for $\omega \in \Omega$ and for all $A \in \mathcal{A}_\Lambda$. Clearly, $\tau_t^\Lambda(\omega)(A)$ is an element of $\mathcal{A}_\Lambda$. In fact, for all $\omega \in \Omega$, $\tau_t^\Lambda(\omega)$ is a one-parameter group of *-automorphisms of $\mathcal{A}_\Lambda$. Since the spin system consists of infinite number of spins, the construction of the time evolution of a fixed observable $A \in \mathcal{A}_{\Lambda_o}$, where $\Lambda_o \subseteq \mathbb{Z}^\nu$ involves taking the limit of $\tau_t^\Lambda(\omega)(A)$ as $\Lambda \to \infty$. Here, we adopt the convention that $\Lambda \to \infty$ indicates, Λ eventually contains all finite subsets of $\mathbb{Z}^\nu$. This notion of convergence has been made precise in subsection 3.3, in chapter 3. We shall show that for a certain class of random interaction potentials, this limit exists for almost every $\omega \in \Omega$ and for all $A \in \mathcal{A}_\Lambda$, where $\Lambda \subseteq \mathbb{Z}^\nu$.

Definition 3.7.0.25 *Let S be an operator on the Banach space X. An element $x \in X$ is defined to be analytic for S if $x \in D(S^n)$, for all $n = 1, 2, \ldots$, and if the series*

$$\sum_{n=0}^{\infty} \frac{(it)^n}{n!} \|S^n x\|$$

has a positive radius of convergence.

Definition 3.7.0.26 *Let $t \mapsto \tau_t$ be a strongly continuous group of automorphisms of a C^*-algebra $\mathcal{A}$. An element $A \in \mathcal{A}$ is called analytic for τ_t, if there exists a strip $I_\lambda = \{z \mid |\Im z| < \lambda\}$ in $\mathbb{C}$, a function $f : I_\lambda \to \mathcal{A}$ such that,*

1. *$f(t) = \tau_t(A)$, $\forall t \in \mathbb{R}$,*

2. *$z \mapsto f(z)$ is strongly analytic.*

An element $A \in \mathcal{A}$, is said to be entire analytic for τ_t if, there exists a function, $f : \mathbb{C} \to \mathcal{A}$, which is strongly analytic in the entire complex plane and $f(t) = \tau_t(A)$, $\forall t \in \mathbb{R}$.

In order to construct a family of one–parameter groups of *–automorphisms which determine the evolution of the spin system, we shall invoke the theory of derivations of C^*–algebras which usually arise as generators of automorphism groups. To this end, we have the following proposition.

Proposition 3.7.0.27 *Let Φ be an interaction of a quantum spin system satisfying*

$$P_\phi(x) = \sum_{x \in X} \|\Phi(X)\| < \infty,$$

for all $x \in L$, where L is a countable set. It follows that there exists a derivation δ of the quantum spin algebra $\mathcal{A}$ such that the domain of δ,

$$D(\delta) = \bigcup_{\Lambda \subseteq \boldsymbol{Z}^\nu} \mathcal{A}_\Lambda,$$

and for $A \in \mathcal{A}_\Lambda$,

$$\delta(A) = i \sum_{X \cap \Lambda \neq \emptyset} [\Phi(X), A].$$

The derivation δ is norm–closable and its closure $\overline{\delta}$ is the generator of a strongly continuous one–parameter group of *–automorphisms τ of $\mathcal{A}$ if, and only if, one of the following conditions is satisfied: either $\overline{\delta}$ possesses a dense set of analytic elements or $(I + \alpha\overline{\delta})(D(\overline{\delta})) = \mathcal{A}, \alpha \in \mathbb{R} \setminus \{o\}$. Finally, if $\overline{\delta}$ generates the group τ and if $\tau_t^\Lambda(A) = e^{iH(\Lambda)t} A e^{-iH(\Lambda)t}$, then

$$\lim_{\Lambda \to \infty} \|\tau_t(A) - \tau_t^\Lambda(A)\| = 0$$

for all $A \in \mathcal{A}$, uniformly, for t in compacts.

Proof See [Rob 81], vol 2, prop 6.2.3, pg 248. △

Theorem 3.7.0.28 *Let Φ be a finite range random interaction of the quantum spin system on a lattice $\mathbb{Z}^\nu$, satisfying*

$$\sup_{a\in\mathbb{Z}^\nu}\left(\sum_{X\ni 0}\|\Phi(X,T_a\omega)\|\right)<\infty$$

almost everywhere. Then, for almost every $\omega\in\Omega$, there exists a strongly continuous, one–parameter group of *–automorphisms $\tau_t(\omega)$ of $\mathcal{A}$ such that,

$$\lim_{\Lambda\to\infty}\tau_t^\Lambda(\omega)(A)=\tau_t(\omega)(A),\quad \forall A\in\mathcal{A}$$

and uniformly, for t in compacts, where $\tau_t^\Lambda(\omega)(A)=e^{iH(\Lambda,\omega)t}Ae^{-iH(\Lambda,\omega)t}$. $\tau_t(\omega)$ is called the evolution group of the spin system whenever the limit exists.

Proof Now, whenever $\Phi(.,\omega)$ has a finite range Δ_ω for $\omega\in\Omega$, we have for $x\in\mathbb{Z}^\nu$,

$$\begin{aligned}P_\phi(\omega)(x) &= \sum_{X\ni x}\|\Phi(X,\omega)\|\\ &= \sum_{X-x\ni 0}\|\alpha_x(\Phi(X-x,T_x\omega))\|\\ &= \sum_{X-x\ni 0}\|\Phi(X-x,T_x\omega)\|\\ &\leq \sum_{Y\ni 0}\|\Phi(Y,T_x\omega)\|<\infty,\end{aligned}$$

On appealing to the above proposition, there exists a derivation $\delta(\omega)$ of $\mathcal{A}$ such that, the domain of $\delta(\omega)$,

$$D(\delta(\omega))=\bigcup_{\Lambda\subseteq\mathbb{Z}^\nu}\mathcal{A}_\Lambda$$

and for $A\in\mathcal{A}_\Lambda$,

$$\delta(\omega)(A)=i\sum_{X\cap\Lambda\neq\emptyset}[\Phi(X,\omega),A].$$

Next, we shall show that $D(\delta(\omega))$ is a dense set of analytic elements for $\delta(\omega)$ and hence establish that the derivation $\delta(\omega)$ is norm–closable by the above proposition.

Take $A \in \mathcal{A}_{\Lambda_0}$, where $\Lambda_0 \subseteq \mathbb{Z}^\nu$. One has $\Phi(X,\omega) \in \mathcal{A}_X$, for finite $X \subseteq \mathbb{Z}^\nu$. Now the local algebras $\mathcal{A}_{\Lambda_1}$, $\mathcal{A}_{\Lambda_2}$ commute whenever $\Lambda_1 \cap \Lambda_2 = \emptyset$.

Therefore, we have

$$\begin{aligned} \|(\delta(\omega))^n(A)\| &= \|i^n \sum_{X_1\cap S_0\neq\emptyset} \cdots \sum_{X_n\cap S_{n-1}\neq\emptyset} [\Phi(X_n,\omega),[\ldots[\Phi(X_1,\omega),A]]]\| \\ &\leq \sum_{X_1\cap S_0\neq\emptyset} \cdots \sum_{X_n\cap S_{n-1}\neq\emptyset} \|[\Phi(X_n,\omega),[\ldots[\Phi(X_1,\omega),A]]]\|, \end{aligned}$$

where $S_0 = \Lambda_0$ and

$$S_j = \Lambda_0 \cup \bigcup_{i=1}^{j} X_i, \quad for\ j \geq 1.$$

Since $\Phi(.,\omega)$ has a finite range Δ_ω, it follows that $\Phi(X,\omega) = 0$, whenever $|X| > |\Delta_\omega|$.

Therefore, if

$$[\Phi(X_j,\omega),[\ldots[\Phi(X_1,\omega),A]]] \neq 0,$$

where

$$[\Phi(X_j,\omega),[\ldots[\Phi(X_1,\omega),A]]] \in \mathcal{A}_{S_j},$$

then

$$|X_i| \leq |\Delta_\omega|, \quad \forall i = 1,2,\ldots,j.$$

Therefore,

$$\begin{aligned} |S_j| &\leq \sum_{i=1}^{j} |X_i| + |\Lambda_0| \\ &\leq j|\Delta_\omega| + |\Lambda_0| \end{aligned}$$

Thus, whenever

$$\sup_{a\in \mathbf{Z}^\nu}\left(\sum_{X\ni 0}\|\Phi(X,T_a\omega)\|\right)<\infty,$$

we get

$$\begin{aligned}
\|(\delta(\omega))^n(A)\| &\leq 2^n\|A\|\sum_{x_1\in S_0}\sum_{X_1\ni x_1}\cdots\sum_{x_n\in S_{n-1}}\sum_{X_n\ni x_n}\|\Phi(X_n,\omega)\|\ldots\|\Phi(X_1,\omega)\| \\
&\leq 2^n\|A\|\sum_{x_1\in S_0}\sum_{X_1-x_1\ni 0}\cdots\sum_{x_n\in S_{n-1}}\sum_{X_n-x_n\ni 0}\|\Phi(X_n-x_n,T_{x_n}\omega)\| \\
&\quad \ldots\|\Phi(X_1-x_1,T_{x_1}\omega)\| \\
&\leq 2^n\|A\|\prod_{i=1}^{n}((i-1)|\Delta_\omega|+|\Lambda_0|)\left(\sup_{x_i\in \mathbf{Z}^\nu}\left(\sum_{Y_i\ni 0}\|\Phi(Y_i,T_{x_i}\omega)\|\right)\right) \\
&\leq 2^n\|A\|\prod_{i=1}^{n}((i-1)|\Delta_\omega|+|\Lambda_0|)\left(\sup_{a\in \mathbf{Z}^\nu}\left(\sum_{X\ni 0}\|\Phi(X,T_a\omega\|\right)\right)^n \\
&\leq 2^n\|A\|(n|\Delta_\omega|+|\Lambda_0|)^n\left(\sup_{a\in \mathbf{Z}^\nu}\left(\sum_{X\ni 0}\|\Phi(X,T_a\omega)\|\right)\right)^n.
\end{aligned}$$

Now, $a^n\leq n!$ for $a>0$ hence,

$$\|(\delta(\omega))^n(A)\|\leq\|A\|e^{|\Lambda_0|}2^n n!\left(\sup_{a\in \mathbf{Z}^\nu}\left(\sum_{X\ni 0}\|\Phi(X,T_a\omega)\|\right)\right)^n e^{n|\Delta_\omega|}.$$

This establishes that A is an analytic element for $\delta(\omega)$, with radius of analyticity

$$r_\omega\geq\left(2\left(\sup_{a\in \mathbf{Z}^\nu}\left(\sum_{X\ni 0}\|\Phi(X,T_a\omega)\|\right)\right)e^{|\Delta_\omega|}\right)^{-1}, \qquad (3.7.3)$$

where the radius of analyticity r_ω is independent of A. i.e.,

$$\sum_{n=0}^{\infty}\frac{|t|^n}{n!}\|(\delta(\omega))^n(A)\|<\infty$$

for

$$|t|<\left(2\left(\sup_{a\in \mathbf{Z}^\nu}\left(\sum_{X\ni 0}\|\Phi(X,T_a\omega)\|\right)\right)e^{|\Delta_\omega|}\right)^{-1}.$$

Therefore, it follows from the above proposition that, $\delta(\omega)$ is norm–closable and the norm–closure $\overline{\delta}(\omega)$ is the generator of an automorphism group $\tau_t(\omega)$ of $\mathcal{A}$ such that,

$$\tau_t^{\Lambda}(\omega)(A) \to \tau_t(\omega)(A), \quad \forall A \in \mathcal{A}.$$

The convergence of course being uniform in t. We also have

$$\delta^{\Lambda}(\omega)(A) \to \overline{\delta}(\omega)(A), \quad \forall A \in \bigcup_{\Lambda \subseteq \mathbf{Z}^{\nu}} \mathcal{A}_{\Lambda},$$

where ,

$$\delta^{\Lambda}(\omega)(A) = i[H(\Lambda,\omega), A], \quad \forall A \in \mathcal{A}.$$

Since the local elements

$$A \in \bigcup_{\Lambda \subseteq \mathbf{Z}^{\nu}} \mathcal{A}_{\Lambda},$$

are analytic for $\overline{\delta}(\omega)$, the convergence of $\tau_t^{\Lambda}(\omega)(A)$ as $\Lambda \to \infty$, is uniform in a ball around zero. It is also worth noting that,

$$D = \bigcup_{\Lambda \subseteq \mathbf{Z}^{\nu}} \mathcal{A}_{\Lambda},$$

is a core for $\overline{\delta}(\omega)$. Thus, whenever $\Phi(.,\omega)$ has a finite range and

$$\sup_{a \in \mathbf{Z}^{\nu}} \left(\sum_{X \ni 0} \|\Phi(X, T_a\omega)\| \right) < \infty,$$

there exists a strongly continuous, one–parameter group of automorphisms $\tau_t(\omega)$ of $\mathcal{A}$ such that,

$$\tau_t(\omega)(A) = \lim_{\Lambda \to \infty} \tau_t^{\Lambda}(\omega)(A), \quad \forall A \in \mathcal{A}.$$

$\tau_t(\omega)$ is called the evolution group associated with the infinite spin system. Since Φ is a finite range random interaction, $\Phi(.,\omega)$ has a finite range Δ_ω for almost every $\omega \in \Omega$. Moreover,

$$\sup_{a\in \boldsymbol{Z}^\nu}\left(\sum_{X\ni 0}\|\Phi(X,T_a\omega)\|\right)<\infty$$

almost everywhere. Therefore, we conclude that for almost every $\omega \in \Omega$, there exists a strongly continuous, one–parameter group of *–automorphisms $\tau_t(\omega)$ of $\mathcal{A}$ such that,

$$\lim_{\Lambda\to\infty}\tau_t^\Lambda(\omega)(A)=\tau_t(\omega)(A), \quad \forall A\in\mathcal{A}.$$

The convergence being uniform in t on compact subsets. Thus for almost every $\omega \in \Omega$, $\lim_{\Lambda\to\infty}\tau_t^\Lambda(\omega)(A)$ exists for all $A \in \mathcal{A}$ and determines an evolution group $\tau_t(\omega)$ associated with the spin system. Besides, for almost every $\omega \in \Omega$,

$$\delta^\Lambda(\omega)(A)\to\overline{\delta}(\omega)(A), \quad \forall A\in\bigcup_{\Lambda\subseteq \boldsymbol{Z}^\nu}\mathcal{A}_\Lambda,$$

where

$$D=\bigcup_{\Lambda\subseteq \boldsymbol{Z}^\nu}\mathcal{A}_\Lambda$$

is a core for $\overline{\delta}(\omega)$. △

Remark 1 Note that the radius of analyticity r_ω of $A \in D$, for $\overline{\delta}(\omega)$ is such that,

$$r_\omega\geq\left(2\left(\sup_{a\in \boldsymbol{Z}^\nu}\left(\sum_{X\ni 0}\|\Phi(X,T_a\omega)\|\right)\right)e^{|\Delta_\omega|}\right)^{-1},$$

where in view of lemmas 3.6.0.23 and 3.6.0.24,

$$\omega\mapsto\left(2\left(\sup_{a\in \boldsymbol{Z}^\nu}\left(\sum_{X\ni 0}\|\Phi(X,T_a\omega)\|\right)\right)e^{|\Delta_\omega|}\right)^{-1}$$

is almost surely constant.

Remark 2 Now for $b \in \mathbb{Z}^\nu$, $\Phi(\omega)$ has a finite range if, and only if, $\Phi(T_b\omega)$ has a finite range. Also,

$$\sup_{a\in\mathbb{Z}^\nu}\left(\sum_{X\ni 0}\|\Phi(X,T_a\omega)\|\right) = \sup_{a\in\mathbb{Z}^\nu}\left(\sum_{X\ni 0}\|\Phi(X,T_a(T_b\omega))\|\right).$$

Therefore, $\delta(\omega)$ is norm–closable if, and only if, $\delta(T_b\omega)$ is norm–closable and $\delta^\Lambda(\omega)(A)$ converges to $\overline{\delta}(\omega)(A)$, if and only if, $\delta^\Lambda(T_b\omega)(A)$ converges to $\overline{\delta}(T_b\omega)(A)$, for all $A \in D$. Also, note that D is a core for $\overline{\delta}(\omega)$ if, and only if, it is a core for $\overline{\delta}(T_b\omega)$. Hence,

$$\tau_t(\omega)(A) = \lim_{\Lambda\to\infty}\tau_t^\Lambda(\omega)(A)$$

defines a strongly continuous group of automorphisms of $\mathcal{A}$ if, and only if,

$$\tau_t(T_b\omega)(A) = \lim_{\Lambda\to\infty}\tau_t^\Lambda(T_b\omega)(A)$$

defines a strongly continuous group of automorphisms of $\mathcal{A}$. i.e. $\tau_t(\omega)$ is an evolution group if, and only if, $\tau_t(T_b\omega)$ is an evolution group.

Let $\mathcal{E}$ denote the sigma algebra of all Lebesgue measurable subsets of $\mathbb{R}$, with Lebesgue measure μ. Let $\mu \times P$ be the Caratheodory extension of the product measure defined on the smallest σ–algebra $\mathcal{E} \times \mathcal{S}$, containing all measurable rectangles in $\mathbb{R} \times \Omega$. Since $\mu \times P$ is obtained using the Caratheodory extension process, it is complete. Moreover, both μ and P being σ–finite, so is $\mu \times P$. Therefore, we have a measurable structure on the product space given by the triple $(\mathbb{R}\times\Omega, \overline{\mathcal{E}\times\mathcal{S}}, \mu\times P)$, where $\overline{\mathcal{E}\times\mathcal{S}}$ denotes the smallest sigma algebra containing $\mathcal{E} \times \mathcal{S}$, on which $\mu \times P$ is complete.

Proposition 3.7.0.29 *Let Φ satisfy the assumptions of theorem 3.7.0.28 and $\tau_t(\omega)$ be the strongly continuous, one–parameter group of automorphisms of $\mathcal{A}$, which determine the evolution of the spin system. Then, $\omega \mapsto \tau_t(\omega)(A)$ is strongly, jointly measurable in both t and ω, for all $A \in \mathcal{A}$.*

Proof It is sufficient to prove the strong measurability of the map $\omega \mapsto \tau_t(\omega)(A)$, for $A \in \mathcal{A}_{\Lambda_0}$ and all $\Lambda_0 \subseteq \mathbb{Z}^\nu$. Measurability in the case of an arbitrary $A \in \mathcal{A}$ can be established by approximating A in the norm by local elements. Let $A \in \mathcal{A}_{\Lambda_0}$ where $\Lambda_0 \subseteq \mathbb{Z}^\nu$. It follows from theorem 3.7.0.28 that, $\tau_t(\omega)(A) = \lim_{\Lambda \to \infty} \tau_t^\Lambda(\omega)(A)$, for almost every $\omega \in \Omega$. Now, let $\{\Lambda_n\}$ be a sequence [3] of finite subsets increasing to $\mathbb{Z}^\nu$. i.e.,

$$\Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset \cdots, \quad \textit{and} \quad \bigcup_{n=1}^{\infty} \Lambda_n = \mathbb{Z}^\nu.$$

Then for almost every $\omega \in \Omega$,

$$\tau_t(\omega)(A) = \lim_{n \to \infty} \tau_t^{\Lambda_n}(\omega)(A),$$

where $\tau_t^{\Lambda_n}(\omega)$ can be expressed in terms of commutators as

$$\tau_t^{\Lambda_n}(\omega)(A) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!}[H(\Lambda_n, \omega), A]^{(k)}.$$

Therefore, one has

$$\tau_t(\omega)(A) = \lim_{n \to \infty} \tau_t^{\Lambda_n}(\omega)(A),$$

for almost every $(t, \omega) \in \mathbb{R} \times \Omega$. Since $\omega \mapsto H(\Lambda, \omega)$ is strongly measurable, and strong measurability is preserved under products of functions, it follows

[3] Λ_n's can be taken to be cubic regions symmetric about the origin, with faces perpendicular to the co–ordinate axes and edges of length $2n$.

that $\omega \mapsto [H(\Lambda,\omega), A]^{(k)}$ is strongly measurable for all $k \in \mathbb{Z}^+$. Besides, for $t \in \mathbb{R}$, t^k is the limit almost everywhere, of numerically valued simple functions, for all $k \in \mathbb{Z}^+$. Therefore, the product of t^k with $[H(\Lambda,\omega), A]^{(k)}$ is the limit almost everywhere of countably valued functions on $\mathbb{R} \times \Omega$. Hence, each of the terms of the series is a strongly, jointly measurable function of t and ω with respect to the product measure $\mu \times P$. We know from [Hil 57] (Theorem 3.5.4, Page 74) that, strong measurability is preserved rather, well under taking limits. Therefore, the above series is strongly, jointly measurable in t and ω. Since

$$\tau_t(\omega)(A) = \lim_{n\to\infty} \tau_t^{\Lambda_n}(\omega)(A)$$

for almost every (t,ω) in $\mathbb{R} \times \Omega$, the strong, joint measurability of $(t,\omega) \mapsto \tau_t(\omega)(A)$ follows readily. Hence the proposition follows. △

It is seen in the case of quantum spin systems on a lattice $\mathbb{Z}^\nu$ with translation invariant interactions, that whenever the dynamics exists, the evolution group of *–automorphisms of the quasi–local algebra, commutes with the symmetry group of automorphisms associated with the lattice $\mathbb{Z}^\nu$. Here we prove a variant of this property. Before we set about establishing this result, the following fact is worth noting.

Lemma 3.7.0.30 *Let $\tau_t^\Lambda(\omega)$ be the strongly continuous, one–parameter group of local automorphisms associated with a finite $\Lambda \subseteq \mathbb{Z}^\nu$, where*

$$\tau_t^\Lambda(\omega)(A) = e^{iH(\Lambda,\omega)t} A e^{-iH(\Lambda,\omega)t}.$$

Then for all $a \in \mathbb{Z}^\nu$, we have

$$\alpha_a(\tau_t^\Lambda(\omega)(A)) = \tau_t^{\Lambda+a}(T_{-a}\omega)(\alpha_a(A)); \forall A \in \mathcal{A}_\Lambda.$$

Proof We have

$$\begin{aligned}\alpha_a(\tau_t^\Lambda(\omega)(A)) &= \alpha_a(e^{iH(\Lambda,\omega)t}Ae^{-iH(\Lambda,\omega)t}) \\ &= \alpha_a(e^{iH(\Lambda,\omega)t})\alpha_a(A)\alpha_a(e^{-iH(\Lambda,\omega)t}).\end{aligned}$$

Therefore, it follows from function calculus for $H(\Lambda, \omega)$ and the identity

$$H(\Lambda + a, T_{-a}\omega) = \alpha_a(H(\Lambda, \omega)) \tag{3.7.4}$$

that,

$$\alpha_a(\tau_t^\Lambda(\omega)(A)) = e^{iH(\Lambda+a,T_{-a}\omega)t}\alpha_a(A)e^{-iH(\Lambda+a,T_{-a}\omega)t}.$$

Hence, the lemma follows from this equality. △

We will have the occasion to use the above lemma in the proof of the following proposition.

Proposition 3.7.0.31 *Let $\tau_t(\omega)$ be the evolution group of the spin system on an infinite lattice $\mathbb{Z}^\nu$. Then for all $a \in \mathbb{Z}^\nu$, we have*

$$\tau_t(T_{-a}\omega)(\alpha_a(A)) = \alpha_a(\tau_t(\omega)(A)), \quad \forall A \in \mathcal{A}.$$

Proof It is sufficient to prove the above identity for $A \in \mathcal{A}_{\Lambda_0}$ and all $\Lambda_0 \subseteq \mathbb{Z}^\nu$. The general case follows easily from the fact that an arbitrary $A \in \mathcal{A}$ can be approximated in the norm by local elements. It follows from theorem 3.7.0.28, and lemma 3.7.0.30 established prior to this proposition that, for $A \in \mathcal{A}_{\Lambda_0}$, where $\Lambda_0 \subseteq \mathbb{Z}^\nu$, and all $a \in \mathbb{Z}^\nu$,

$$
\begin{aligned}
\alpha_a(\tau_t(\omega)(A)) &= \alpha_a(\lim_{\Lambda\to\infty}(\tau_t^\Lambda(\omega)(A))) \\
&= \lim_{\Lambda\to\infty} \tau_t^{\Lambda+a}(T_{-a}\omega)(\alpha_a(A)) \\
&= \lim_{\Lambda'\to\infty} (\tau_t^{\Lambda'}(T_{-a}\omega)\alpha_a(A)) \\
&= \tau_t(T_{-a}\omega)(\alpha_a(A)),
\end{aligned}
$$

where $\Lambda' = \Lambda + a$. Thus we have established the identity for all local elements. Therefore, this identity can be extended to the whole of $\mathcal{A}$ using the fact that the local elements are norm–dense in $\mathcal{A}$. △

Remark If $\tau_t(\omega)$ is the evolution group of the spin system on the infinite lattice $\mathbb{Z}^\nu$, then it follows from proposition 3.7.0.31 that, if A is entire–analytic with respect to $\tau_t(\omega)$, then $\alpha_a(A)$ is entire–analytic with respect to $\tau_t(T_{-a}\omega)$, for all $a \in \mathbb{Z}^\nu$.

In the discussion that follows, we establish some interesting algebraic properties of the generators $\overline{\delta}(\omega)$ of the evolution groups $\tau_t(\omega)$. To this end, we have the following theorem.

Theorem 3.7.0.32 *Let U_n be a sequence of C_0–semigroups of contractions on the Banach space X. with generators S_n and define the graph G_α by*

$$G_\alpha = \lim_{n\to\infty} G(I - \alpha S_n).$$

The following conditions are equivalent:

1. *there exists a C_0–semigroup U such that,*

$$\lim_{n\to\infty} \|(U_{n,t} - U_t)A\| = 0,$$

for all $A \in X$, $t \in \mathbb{R}_+$, uniformly for t in any finite interval of $\mathbb{R}_+$;

2. *the sets $D(G_\alpha)$ and $R(G_\alpha)$ are norm–dense in X for some $\alpha > 0$.*

If these conditions are satisfied, then G_α is the graph of $I - \alpha S$, where S is the generator of U.

Proof Refer to theorem 3.1.28 in [Rob 87]. △

Remark One of the situations in which the above theorem can be applied is the following: Let S_n and S be the generators of C_0–contraction semigroups and suppose there exists a core D of S such that,

$$D \subseteq \bigcup_m \left(\bigcap_{n \geq m} D(S_n) \right)$$

and

$$\lim_{n \to \infty} \|(S_n - S)A\| = 0,$$

for all $A \in D$. It then follows that S is the graph limit of the S_n's.

This theorem yields the following proposition.

Proposition 3.7.0.33 *Let $\tau_t(\omega)$ be the evolution group of the spin system and $D(\overline{\delta}(\omega))$ be the domain of the generator of the automorphism group $\tau_t(\omega)$. Then for all $a \in \mathbb{Z}^\nu$, we have $\alpha_a(D(\overline{\delta}(\omega))) = D(\overline{\delta}(T_{-a}\omega))$ and $\alpha_a(\overline{\delta}(\omega))(A) = \overline{\delta}(T_{-a}\omega)(\alpha_a(A))$, for all $A \in D(\overline{\delta}(\omega))$.*

Proof It is seen from the proof of theorem 3.7.0.28 that,

$$D = \bigcup_{\Lambda \subseteq \mathbb{Z}^\nu} \mathcal{A}_\Lambda,$$

is a core for $\overline{\delta}(\omega)$ and

$$\delta^{\Lambda}(\omega)(B) \to \overline{\delta}(\omega)(B); \quad \forall B \in D,$$

where $\delta^{\Lambda}(\omega)$ is the generator of the local automorphism group $\tau_t^{\Lambda}(\omega)$. We have $\tau_t^{\Lambda}(\omega)(B) = e^{iH(\Lambda,\omega)t}Be^{-iH(\Lambda,\omega)t}$ and $\delta^{\Lambda}(\omega)(B) = i[H(\Lambda,\omega), B]$, for all $B \in \mathcal{A}$. Let $\{\Lambda_n\}$ be a sequence of finite subsets increasing to $\mathbb{Z}^{\nu}$, then we have

$$\delta^{\Lambda_n}(\omega)(B) \to \overline{\delta}(\omega)(B); \quad \forall B \in D.$$

Therefore, we conclude from the remark made after the statement of the above theorem that, $\overline{\delta}(\omega)$ is the graph limit of $\delta^{\Lambda_n}(\omega)$. Hence, for $A \in D(\overline{\delta}(\omega))$, there exists a sequence $\{A_n\}$, where $A_n \in D(\delta^{\Lambda_n}(\omega))$ such that, $A_n \to A$ and $\delta^{\Lambda_n}(\omega)(A_n) \to \overline{\delta}(\omega)(A)$. This implies that $\alpha_a(A_n) \to \alpha_a(A)$ and $\alpha_a(\delta^{\Lambda_n}(\omega)(A_n)) \to \alpha_a(\overline{\delta}(\omega)(A))$. Now, it follows from the the identity 3.7.4 in lemma 3.7.0.30 that,

$$\alpha_a(\delta^{\Lambda_n}(\omega)(A_n)) = \delta^{\Lambda_n+a}(T_{-a}\omega)(\alpha_a(A_n)).$$

Hence, we have $\alpha_a(A_n) \to \alpha_a(A)$ and $\delta^{\Lambda_n+a}(T_{-a}\omega)(\alpha_a(A_n)) \to \alpha_a(\overline{\delta}(\omega)(A))$. Clearly, from remark 2 at the end of theorem 3.7.0.28, $\delta^{\Lambda}(T_{-a}\omega)(B)$ converges to $\overline{\delta}(T_{-a}\omega)(B)$, for all $B \in D$, and D is a core for $\overline{\delta}(T_{-a}\omega)$, where $\delta^{\Lambda}(T_{-a}\omega)$ is the generator of the local automorphism group $\tau_t^{\Lambda}(T_{-a}\omega)$. We have $\tau_t^{\Lambda}(T_{-a}\omega)(B) = e^{iH(\Lambda,T_{-a}\omega)t}Be^{-iH(\Lambda,T_{-a}\omega)t}$ and $\delta^{\Lambda}(T_{-a}\omega)(B) = i[H(\Lambda, T_{-a}\omega), B]$, for all $B \in \mathcal{A}$. Since $\{\Lambda_{n+a}\}$ is a sequence of finite subsets increasing to $\mathbb{Z}^{\nu}$, it follows that $\delta^{\Lambda_n+a}(T_{-a}\omega)(B)$ converges to $\overline{\delta}(T_{-a}\omega)(B)$, for all $B \in D$. Hence, the remark following theorem 3.7.0.32 implies that $\overline{\delta}(T_{-a}\omega)$ is the graph limit of $\delta^{\Lambda_n+a}(T_{-a}\omega)$. Therefore, as $\alpha_a(A_n) \to \alpha_a(A)$ and $\delta^{\Lambda_n+a}(T_{-a}\omega)(\alpha_a(A_n)) \to \alpha_a(\overline{\delta}(\omega)(A))$, where $\alpha_a(A_n) \in D(\delta^{\Lambda_n+a}(T_{-a}\omega))$, one concludes that $\alpha_a(A) \in D(\overline{\delta}(T_{-a}\omega))$ and $\alpha_a(\overline{\delta}(\omega))(A) = \overline{\delta}(T_{-a}\omega)(\alpha_a(A))$. Conversely, it can be shown that if $A \in D(\overline{\delta}(T_{-a}\omega))$ then $\alpha_{-a}(A) \in D(\overline{\delta}(\omega)))$. This completes the proof of the proposition. $\triangle$

In the next chapter, we aim to study the Arveson spectrum of the strongly continuous, one–parameter group of automorphisms $\tau_t(\omega)$, which determines the evolution of the spin system. We report an interesting ergodic property of the Arveson spectrum of the evolution group $\tau_t(\omega)$.

Chapter 4

Ergodic Properties of Spectra of Evolution Groups

4.1 Arveson Spectrum

Here we introduce the notion of Arveson spectrum.

Let X be a Banach space and $X_\star$ a linear subspace of the dual $X^\star$ of X such that, $\|x\| = \sup\{|\rho(x)| : \rho \in X_\star, \|\rho\| \leq 1\}$ for every $x \in X$. Let $B(X)$, $(B_w(X))$ denote the algebra of all bounded ($\sigma(X, X_\star)$–continuous) linear operators on X. As usual, denote the convolution group algebra of the additive group of real numbers $\mathbb{R}$, by $L^1(\mathbb{R})$. A representation of $\mathbb{R}$ on X is a homomorphism $t \mapsto V_t$ of $\mathbb{R}$ into the group of all invertible elements of $B_w(X)$ such that, $\sup_t \|V_t\| < \infty$ and for each $x \in X$, the map $t \mapsto V_t x$ is $\sigma(X, X_\star)$–continuous. Now, if for every $x \in X$, there is an unique vector y defined by

$$\int_{-\infty}^{\infty} f(t)\rho(V_t x)dt = \rho(y); \quad \rho \in X_\star, f \in L^1(\mathbb{R}),$$

then we obtain an operator $\Gamma(f)$ defined by $\Gamma(f)x = y$. Therefore, we have a representation Γ, of $L^1(\mathbb{R})$ in $B(X)$, associated with V.

Definition 4.1.0.34 *The Arveson spectrum $Sp V$ of V is a subset of the dual group $\hat{I\!R}$ of $I\!R$ defined as*

$$SpV = \{\sigma \in I\!R | \hat{f}(\sigma) = 0, \forall f \in \ker \Gamma\},$$

where $\hat{f}$ is the fourier transform of f.

If $\mathcal{A}$ is a C^*–algebra and τ_t a strongly continuous, one–parameter group of automorphisms of the C^*–algebra, then the Bochner integral

$$\int_{-\infty}^{\infty} f(t)\tau_t(A)dt = \Gamma(f)A; \quad A \in \mathcal{A}, \ f \in L^1(I\!R),$$

defines a representation of $L^1(I\!R)$ into the bounded operators on $\mathcal{A}$. Now, on applying the foregoing definition in this case, the Arveson spectrum $Sp(\tau)$ of τ is given by

$$Sp(\tau) = \{s \in I\!R : \hat{f}(s) = 0, \forall f \in \ker \Gamma\}.$$

It can be shown that $s \in Sp(\tau)$, if and only if, $|\hat{f}(s)| \leq \|\Gamma(f)\|$, for all $f \in L^1(I\!R)$ (Proposition 8.1.9 in [Ped 79]).

Our aim is to show that the Arveson spectrum of the evolution group $\tau_t(\omega)$ is almost surely constant. To this end, we have the following theorem.

Theorem 4.1.0.35 *Let $\tau_t(\omega)$ be the strongly continuous, one–parameter group of automorphisms of $\mathcal{A}$, which determines the evolution of the spin system. Then, the Arveson spectrum $Sp(\tau(\omega))$ of $\tau_t(\omega)$ is almost surely constant.*

Proof For $s \in I\!R$, let $T_s = \{\omega : \|\Gamma(\omega)(f)\| \geq |\hat{f}(s)| \, \forall f \in L^1(I\!R)\}$, where

$$\Gamma(\omega)(f)(A) = \int_{-\infty}^{\infty} f(t)\tau_t(\omega)(A)dt, \quad \forall A \in \mathcal{A}.$$

We show that T_s is a measurable subset of Ω. Since $L^1(\mathbb{R})$ is separable, there exists a countable dense set $F = \{f_n | n \in \mathbb{Z}^+\}$ in $L^1(\mathbb{R})$. Hence, for each $f \in L^1(\mathbb{R})$, there exists a sequence f_{n_k} in F, converging to f in the L^1-norm. Therefore,

$$
\begin{aligned}
| \|\Gamma(\omega)(f_{n_k})\| - \|\Gamma(\omega)(f)\| | &\leq \|\Gamma(\omega)(f_{n_k}) - \Gamma(\omega)(f)\| \\
&\leq \|\Gamma(\omega)(f_{n_k} - f)\| \\
&\leq \sup_{\|A\|=1} \|\Gamma(\omega)(f_{n_k} - f)(A)\| \\
&\leq \sup_{\|A\|=1} \| \int_{-\infty}^{\infty} (f_{n_k} - f)(t)\tau_t(\omega)(A)dt\| \\
&\leq \sup_{\|A\|=1} \left(\int_{-\infty}^{\infty} |(f_{n_k} - f)(t)| \|\tau_t(\omega)(A)\| dt \right) \\
&\leq \sup_{\|A\|=1} \left(\|A\| \int_{-\infty}^{\infty} |(f_{n_k} - f)(t)| dt \right) \\
&\leq \int_{-\infty}^{\infty} |(f_{n_k} - f)(t)| dt \\
&\leq \|f_{n_k} - f\|_1
\end{aligned}
$$

Therefore, $\|\Gamma(\omega)(f_{n_k})\|$ converges to $\|\Gamma(\omega)(f)\|$, for f_{n_k} converging to f, in the L^1-norm. In view of this, and the fact that F is dense in T_s, we have

$$T_s = \bigcap_{n=1}^{\infty} T_s^n,$$

where $T_s^n = \{\omega | \|\Gamma(\omega)(f_n)\| \geq |\hat{f}_n(s)|\}$. In order to show that each of these T_s^n's is a measurable subset of Ω, it is sufficient to establish the measurability of the function $\omega \mapsto \|\Gamma(\omega)(f_n))\|$, for all $n \in Z^+$. On appealing to proposition 3.7.0.29, we conclude that for $f \in L^1(\mathbb{R})$ and $A \in \mathcal{A}$, $(t, \omega) \mapsto f(t)\tau_t(\omega)(A)$

is strongly, jointly measurable in t and ω. Moreover,

$$\begin{aligned}\int_{\mathbb{R}\times\Omega}\|f(t)\tau_t(\omega)(A)\|d(\mu\times P)(t,\omega) &= \int_{\mathbb{R}\times\Omega}|f(t)|\|\tau_t(\omega)(A)\|d(\mu\times P)(t,\omega)\\ &= \int_{\mathbb{R}}\int_{\Omega}\|A\||f(t)|d\mu(t)dP(\omega)<\infty.\end{aligned}$$

Hence, it follows from theorem 3.7.4 in [Hil 57] that, $(t,\omega)\mapsto f(t)\tau_t(\omega)(A)$ is Bochner integrable on $\mathbb{R}\times\Omega$. Therefore, as a consequence of the analogue of Fubini's theorem for vector valued functions (Proposition 3.7.13, [Hil 57]), the map $\omega\mapsto\Gamma(\omega)(f)(A)$ is strongly measurable in ω. Hence, $\omega\mapsto\|\Gamma(\omega)(f)(A)\|$ is a measurable, real valued function on Ω. Thus it readily follows that for $f\in L^1(\mathbb{R})$, $\omega\mapsto\|\Gamma(\omega)(f)(A)\|$ is measurable for all $A\in\mathcal{A}$. Now, $\mathcal{A}$ being a separable C^*–algebra, we have for $c\in\mathbb{R}$ and $f\in L^1(\mathbb{R})$,

$$\{\omega|\,\|\Gamma(\omega)(f)\|\leq c\}=\bigcap_{n\in\mathbb{Z}^+}\{\omega\in\Omega|\,\|\Gamma(\omega)(f)(A_n)\|\leq c;\|A_n\|\leq 1\},$$

where $\mathcal{U}_0=\{A_n\in\mathcal{A}|\,n\in\mathbb{Z}^+\}$ is a dense subset of the closed unit ball in $\mathcal{A}$. This identity, coupled with the fact that $\omega\mapsto\|\Gamma(\omega)(f)(A_n)\|$ is a measurable function of ω for all $n\in\mathbb{Z}^+$, permits us to conclude that the set $\{\omega|\,\|\Gamma(\omega)(f)\|\leq c\}$, is a measurable subset of Ω. Since c is arbitrary, the function $\omega\mapsto\|\Gamma(\omega)(f)\|$ is a measurable function of ω. Thus, $\omega\mapsto\|\Gamma(\omega)(f)\|$ is measurable for all $f\in L^1(\mathbb{R})$. Therefore, $\omega\mapsto\|\Gamma(\omega)f_n\|$ is measurable $\forall n\in\mathbb{Z}^+$. Hence, each of these T_s^n's is a measurable subset of Ω. This proves conclusively that the set T_s is a measurable subset of Ω. Now, using the fact that the action of the measure preserving group of automorphisms is ergodic, we show that T_s has a measure either zero or one. It follows from the properties of the Bochner integral [Hil 57] (Chapter 3) and the fact that α_a is a *–automorphism of the $C^\star$–algebra $\mathcal{A}$ that, for $f \in L^1(\mathbb{R})$,

$$\begin{aligned}
\|\Gamma(\omega)(f)\| &= \sup_{\|A\|=1} \|\Gamma(\omega)(f)(A)\| \\
&= \sup_{\|A\|=1} \|\int_{-\infty}^{\infty} f(t)\tau_t(\omega)(A)dt\| \\
&= \sup_{\|A\|=1} \|\alpha_a\left(\int_{-\infty}^{\infty} f(t)\tau_t(\omega)(A)dt\right)\|. \\
&= \sup_{\|A\|=1} \|\int_{-\infty}^{\infty} f(t)\tau_t(T_{-a}\omega)(\alpha_a(A))dt\|,
\end{aligned}$$

for all $a \in \mathbb{Z}^\nu$. The last equality follows from proposition 3.7.0.31. Consequently, we have

$$\begin{aligned}
\|\Gamma(\omega)(f)\| &= \sup_{\|A\|=1} \|\Gamma(T_{-a}\omega)(f)(\alpha_a(A))\| \\
&= \|\Gamma(T_{-a}\omega)(f)\|,
\end{aligned}$$

for all $a \in \mathbb{Z}^\nu$ and $f \in L^1(\mathbb{R})$. Therefore, for all $a \in \mathbb{Z}^\nu$, $\|\Gamma(\omega)(f)\| = \|\Gamma(T_{-a}\omega)(f)\|$, for $f \in L^1(\mathbb{R})$. Hence, as the action of the measure preserving group of automorphisms is assumed to be ergodic, it is clear from the above equality that T_s is an invariant measurable subset of Ω and therefore, the set T_s has measure either zero or one. Hence, s lies in the Arveson spectrum of $\tau_t(\omega)$ with probability either zero or one. Thus, one concludes that the Arveson spectrum $Sp(\tau(\omega))$ of $\tau_t(\omega)$ is almost surely constant. $\triangle$

4.2 KMS States

In this section we analyse the KMS states of the spin system on a lattice with random interactions. The following definition of a KMS state has been taken from [Rob 81].

Definition 4.2.0.36 *Let $(\mathcal{A}, \tau)$ be a $C^\star$–dynamical system, or a $W^\star$–dynamical system and ρ a state over $\mathcal{A}$ which is assumed to be normal in the $W^\star$ case. Then, ρ is said to be a (τ, β)–KMS state if, for $\beta > 0$ and any pair $A, B \in \mathcal{A}$, there exists a complex function $F_{A,B}$ which is analytic on the open strip $0 < \Im z < \beta$, uniformly bounded and continuous on the closed strip $0 \leq \Im z \leq \beta$ such that,*

$$F_{A,B}(t) = \rho(A\tau_t(B)) \quad \text{and} \quad F_{A,B}(t + i\beta) = \rho(\tau_t(B)A).$$

If $\beta < 0$, then ρ is a (τ, β)–KMS state if, there exists a complex function $F_{A,B}$ which is analytic on the open strip $\beta < \Im z < 0$, uniformly bounded and continuous for $\beta \leq \Im z \leq 0$ such that,

$$F_{A,B}(t) = \rho(A\tau_t(B)) \quad \text{and} \quad F_{A,B}(t + i\beta) = \rho(\tau_t(B)A).$$

4.2.1 Construction of a Family of KMS States

We know from theorem 3.7.0.28 that. for almost every $\omega \in \Omega$, there exists a strongly continuous one–parameter group of *–automorphisms $\tau_t(\omega)$, which determines the evolution of the spin system. Now, for $\omega \in \Omega$, and $\beta \in I\!R\backslash\{0\}$, the local Gibbs state associated with the interaction $\Phi(., \omega)$ is given by

$$\rho_\Lambda(\omega)(A) = \frac{Tr(e^{-\beta H(\Lambda,\omega)}A)}{Tr(e^{-\beta H(\Lambda,\omega)})}, \quad \forall A \in \mathcal{A}_\Lambda.$$

Although $\rho_\Lambda(\omega)$ is defined on $\mathcal{A}_\Lambda$, it has an extension as a state to the whole of $\mathcal{A}$. The extension is by no means unique. It follows from proposition

3.4.0.13 in chapter 3, which can be adopted to $\rho_\Lambda(\omega)$ with $\tau_t^\Lambda(\omega)$ as the local automorphism group that, these states are $(\tau^\Lambda(\omega), \beta)$–KMS states of the finite spin system confined to the region Λ, where $\tau^\Lambda(\omega)$ are the local automorphism groups. Next, for $\omega \in \Omega$, let

$$O_\omega = \{T_{-a}\omega | a \in \mathbb{Z}^\nu\}.$$

Clearly, any two O_ω's corresponding to distinct ω's are either disjoint or identical and the O_ω's form a partition of Ω. Therefore, using the axiom of choice, we pick a subset $\Omega' \subseteq \Omega$, and write the space Ω as

$$\Omega = \bigcup_{\omega \in \Omega'} O_\omega,$$

where the O_ω's in the union are pairwise disjoint. Next, for each $\omega \in \Omega'$, we establish the existence of the thermodynamic limit $\rho(T_{-a}\omega)$ of the local Gibbs states $\rho_\Lambda(T_{-a}\omega)$, for all $a \in \mathbb{Z}^\nu$. To this end, we argue as follows. Since the quasi–local algebra $\mathcal{A}$ is a separable $C^\star$–algebra, the collection of states $E_\mathcal{A}$ of $\mathcal{A}$ is weak*–compact. Therefore, for each $\omega \in \Omega'$ there exists a state $\rho(\omega)$, and a sequence $\{\Lambda_n\}$ of finite subsets of $\mathbb{Z}^\nu$ depending on ω such that, $\rho(\omega)$ is the weak*–limit of a sequence of extensions $\hat{\rho}_{\Lambda_n}(\omega)$ of $\rho_{\Lambda_n}(\omega)$. That is, for each $\omega \in \Omega'$, there exists a sequence $\{\Lambda_n\}$ of finite subsets of $\mathbb{Z}^\nu$ such that,

$$\lim_{n\to\infty} \hat{\rho}_{\Lambda_n}(\omega)(A) = \rho(\omega)(A); \quad \forall A \in \mathcal{A}.$$

In particular,

$$\lim_{n\to\infty} \rho_{\Lambda_n}(\omega)(A) = \rho(\omega)(A),$$

for all $A \in \mathcal{A}_{\Lambda_0}$ and all finite $\Lambda_0 \subseteq \mathbb{Z}^\nu$. Therefore for each $\omega \in \Omega'$, $\rho(\omega)$ is a weak*-limit point of the net of extensions of $\rho_\Lambda(\omega)$'s to $\mathcal{A}$. Hence, it follows from definition 3.4.0.12, which can be easily adopted to $\rho(\omega)$ with $\rho_\Lambda(\omega)$ as the local Gibbs sates, that for each $\omega \in \Omega'$, the state $\rho(\omega)$ is the thermodynamic limit of the local Gibbs states $\{\rho_\Lambda(\omega)\}$. Next, for each $\omega \in \Omega'$ and all $a \in \mathbb{Z}^\nu$, define

$$\rho(T_{-a}\omega)(A) = \rho(\omega)(\alpha_{-a}(A))$$

Now, keeping in mind the identity

$$H(\Lambda_n, \omega) = \alpha_{-a}(H(\Lambda_n + a, T_{-a}\omega)) = V_{\Lambda_n+a}(-a)H(\Lambda_n + a, T_{-a}\omega)V_{\Lambda_n+a}(-a)^{-1},$$

it follows from function calculus and the invariance property of the trace, that, for each $\omega \in \Omega'$ and all $a \in \mathbb{Z}^\nu$,

$$\begin{aligned}
\rho_{\Lambda_n+a}(T_{-a}\omega)(A) &= \frac{Tr(e^{-\beta H(\Lambda_n+a,T_{-a}\omega)}A)}{Tr(e^{-\beta H(\Lambda_n+a,T_{-a}\omega)})} \\
&= \frac{Tr(V_{\Lambda_n+a}(-a)e^{-\beta H(\Lambda_n+a,T_{-a}\omega)}AV_{\Lambda_n+a}(-a)^{-1})}{Tr(V_{\Lambda_n+a}(-a)e^{-\beta H(\Lambda_n+a,T_{-a}\omega)}V_{\Lambda_n+a}(-a)^{-1})} \\
&= \frac{Tr(\alpha_{-a}(e^{-\beta H(\Lambda_n+a,T_{-a}\omega)})\alpha_{-a}(A))}{Tr(\alpha_{-a}(e^{-\beta H(\Lambda_n+a,T_{-a}\omega)}))} \\
&= \frac{Tr(e^{-\beta H(\Lambda_n,\omega)}\alpha_{-a}(A))}{Tr(e^{-\beta H(\Lambda_n,\omega)})} \\
&= \rho_{\Lambda_n}(\omega)(\alpha_{-a}(A)),
\end{aligned}$$

for all $A \in \mathcal{A}_{\Lambda_0}$ and $\Lambda_n \supseteq \Lambda_0$. Hence, for each $\omega \in \Omega'$ and all $a \in \mathbb{Z}^\nu$,

$$\begin{aligned}
\rho(T_{-a}\omega)(A) &= \rho(\omega)(\alpha_{-a}(A)) \\
&= \lim_{n\to\infty} \rho_{\Lambda_n}(\omega)(\alpha_{-a}(A)) \\
&= \lim_{n\to\infty} \rho_{\Lambda_n+a}(T_{-a}\omega)(A),
\end{aligned}$$

for all $A \in \mathcal{A}_{\Lambda_0}$ and all finite $\Lambda_0 \subseteq \mathbb{Z}^\nu$. If for each $\omega \in \Omega'$ and all $a \in \mathbb{Z}^\nu$, we define

$$\hat{\rho}_{\Lambda_{n+a}}(T_{-a}\omega)(A) = \hat{\rho}_{\Lambda_n}(\omega)(\alpha_{-a}(A)); \quad \forall A \in \mathcal{A},$$

then the states $\rho(T_{-a}\omega)$ are the weak*–limits of the sequence of extensions $\{\hat{\rho}_{\Lambda_n+a}(T_{-a}\omega)\}$ of the local Gibbs states $\rho_{\Lambda_n+a}(T_{-a}\omega)$, for each $\omega \in \Omega'$ and all $a \in \mathbb{Z}^\nu$ such that, $\rho(T_{-a}\omega)(A) = \rho(\omega)(\alpha_{-a}(A))$, for all $A \in \mathcal{A}$. Thus, for each $\omega \in \Omega'$ and all $a \in \mathbb{Z}^\nu$, $\rho(T_{-a}\omega)$ is a weak*–limit point of the net of extensions of $\rho_\Lambda(T_{-a}\omega)$'s to $\mathcal{A}$. Hence, for each $\omega \in \Omega'$ and all $a \in \mathbb{Z}^\nu$, $\rho(T_{-a}\omega)$ is the thermodynamic limit of the local Gibbs states $\rho_\Lambda(T_{-a}\omega)$. Since the union of O_ω's, where $\omega \in \Omega'$, exhausts all the points in Ω i.e., $\Omega = \cup_{\omega \in \Omega'} O_\omega$, we have succeeded in establishing the existence of the thermodynamic limit $\rho(\omega)$ of the local Gibbs states $\rho_\Lambda(\omega)$, for all $\omega \in \Omega$. It is clear from the above construction that, for all $\omega \in \Omega$, these states satisfy, $\rho(\omega)(A) = \rho(T_{-a}\omega)(\alpha_a(A))$, for all $a \in \mathbb{Z}^\nu$ and $A \in \mathcal{A}$.

Now, each of these states $\rho(\omega)$, is a thermodynamic limit of the local Gibbs states $\rho_\Lambda(\omega)$. Hence, each $\rho(\omega)$ is a weak*–limit point of the extensions of the local Gibbs states $\rho_\Lambda(\omega)$, with $\tau_t(\omega)$ as the evolution group. Therefore, it follows that $\rho(\omega)$ is a $(\tau(\omega), \beta)$–KMS state [Rob 81] (Proposition 5.3.25). Thus, we have succeeded in establishing the existence of a family of $(\tau(\omega), \beta)$–KMS states $\{\rho(\omega)\}$, for each $\beta \in \mathbb{R} \setminus \{0\}$, obtained as the thermodynamic limits of the local Gibbs states $\rho_\Lambda(\omega)$, and satisfying, $\rho(\omega)(A) = \rho(T_{-a}\omega)(\alpha_a(A))$ for all $a \in \mathbb{Z}^\nu$.

4.2.2 Uniqueness of KMS States

Next, we shall demonstrate that a quantum spin system on an infinite lattice with random interactions exhibits a phase structure. To this end, we have the following theorem which establishes that there is an unique KMS state $\rho(\omega)$, associated with the evolution group $\tau_t(\omega)$, above a certain critical temperature T_c almost surely independent of ω. In order to demonstrate this, we use the fact that the function $\omega \mapsto \sup_{a\in\mathbb{Z}^\nu}(\sum_{X\ni 0}\|\Phi(X,T_a\omega)\|)$ is almost surely constant.

Definition 4.2.2.1 *Let $\mathcal{N}$ be the matrix algebra of all $n \times n$ matrices over $\mathbb{C}$ and $\{E_{p,q}\}$ be the finite collection of matrices in $\mathcal{N}$ such that, $E_{p,q}$ is the matrix with all entries zero except in position (p,q)–where the entry is 1. The $E_{p,q}$'s are such that, $E^{\star}_{p,q} = E_{p,q}$, $E_{p,q}E_{r,s} = 0$ if $q \neq r$, $E_{p,q}E_{q,r} = E_{p,r}$ and $\sum_p E_{p,p} = I$. These $E_{p,q}$'s are called matrix units in $\mathcal{N}$.*

The following theorem establishes that there is an unique KMS state $\rho(\omega)$, above a certain critical temperature T_c almost surely independent of ω.

Theorem 4.2.2.2 *If Φ is a finite range random interaction of the quantum spin system on an infinite lattice $\mathbb{Z}^\nu$, satisfying the assumptions of theorem 3.7.0.28, then there is an unique KMS state $\rho(\omega)$, associated with the evolution group $\tau_t(\omega)$, above a certain critical temperature T_c almost surely independent of ω.*

Proof Since we have etablished the existence of $(\tau(\omega),\beta)$–KMS states for all $\beta \in \mathbb{R} \setminus \{0\}$, the aim of this theorem is to show that there is an unique

KMS state $\rho(\omega)$, above a critical temperature T_c almost surely independent of ω. The proof of the theorem goes along the lines of the discussion preceding proposition 6.2.45, in [Rob 81]. The $(\tau(\omega),\beta)$–KMS condition will play a crucial role in establishing the above fact. For $x \in \mathbb{Z}^\nu$, let $e(i_x, j_x)$; $i_x, j_x = 0, 1$, be a set of matrix units for $\mathcal{A}_x$. Let $A \in \mathcal{A}_\Lambda$, where $x \notin \Lambda$. Now, the $(\tau(\omega),\beta)$–KMS condition and the identity

$$e(i_x, j_x) = \frac{1}{2}\sum_{k_x=0}^{1} e(i_x, k_x)e(k_x, j_x),$$

yield

$$\begin{aligned}
&\rho(\omega)(e(i_x, j_x)A) \\
= \;& \frac{1}{2}\sum_{k_x=0}^{1}\{\rho(\omega)(e(k_x, j_x)Ae(i_x, k_x)) \\
&+\rho(\omega)(e(k_x, j_x)A(\tau_t(\omega) - I)(e(i_x, k_x)))\}|_{t=i\beta} \\
= \;& \frac{1}{2}\delta_{i_x,j_x}\rho(\omega)(A) + \frac{1}{2}\sum_{k_x=0}^{1}\{\rho(\omega)(e(k_x, j_x)A(\tau_t(\omega) - I)(e(i_x, k_x)))\}|_{t=i\beta}.
\end{aligned}$$

In view of the fact that the local elements are dense in the quasi–local algebra $\mathcal{A}$, it is enough to establish that $\rho(\omega)(A)$ can be uniquely determined for all $A \in \mathcal{A}_\Lambda$, and all $\Lambda \subseteq \mathbb{Z}^\nu$. It is seen from the proof of theorem 3.7.0.28, and the remark following the proof, that the local elements of $\mathcal{A}$ are analytic with respect to the generator $\overline{\delta}(\omega)$ of $\tau_t(\omega)$, with radius of analyticity

$$r_\omega \geq \left(2\left(\sup_{a\in\mathbb{Z}^\nu}\left(\sum_{X\ni 0}\|\Phi(X, T_a\omega)\|\right)\right)e^{|\Delta_\omega|}\right)^{-1},$$

where

$$\omega \mapsto \left(2\left(\sup_{a\in\mathbb{Z}^\nu}\left(\sum_{X\ni 0}\|\Phi(X, T_a\omega)\|\right)\right)e^{|\Delta_\omega|}\right)^{-1}.$$

is almost surely constant. Therefore, the second term on the right hand side of the last equation can be expressed as a power series in β without a constant term, for sufficiently small β almost surely independent of ω. Besides, $A \in \mathcal{A}_\Lambda$ can be expressed as a linear combination of matrix units

$$e(I_\Lambda, J_\Lambda) = \prod_{l=1}^{n} e(i_{x_l}, j_{x_l}),$$

where $\Lambda = \{x_1, \ldots, x_n\}$; $I_\Lambda = \{i_{x_1}, \ldots, i_{x_n}\}$, and $J_\Lambda = \{j_{x_1}, \ldots, j_{x_n}\}$. Thus, it suffices to consider only the special choices $A = e(I_\Lambda, J_\Lambda)$; $I_\Lambda \in \{0,1\}^\Lambda$ *and* $\Lambda \subseteq \mathbb{Z}^\nu$. On adopting the assumptions of theorem 3.7.0.28, one has

$$\begin{aligned}&(\tau_{i\beta}(\omega) - I)(e(i_x, k_x)) \\ = &\sum_{n=1}^{\infty} \frac{(-\beta)^n}{n!} \sum_{X_1 \cap S_0 \neq \emptyset} \cdots \sum_{X_n \cap S_{n-1} \neq \emptyset} [\Phi(X_n, \omega)[\cdots[\Phi(X_1, \omega), e(i_x, k_x)]]],\end{aligned}$$

where $S_\circ = x$ and $S_j = X_j \cup X_{j-1} \cup \cdots \cup X_1 \cup x$. Now, if

$$B = \sum_{I_\Lambda, J_\Lambda} B(I_\Lambda, J_\Lambda) e(I_\Lambda, J_\Lambda)$$

is the decomposition of $B \in \mathcal{A}_\Lambda$, for some $\Lambda \subseteq \mathbb{Z}^\nu$, then the complex coefficients $B(I_\Lambda, J_\Lambda)$ satisfy, $\|B(I_\Lambda, J_\Lambda)\| \leq \|B\|$.

Hence,

$$\begin{aligned}&e(k_x, j_x) e(I_\Lambda, J_\Lambda)[\Phi(X_n, \omega), [\cdots[\Phi(X_1, \omega), e(i_x, k_x)]]] \\ = &\sum_{I_{S_n^x}, J_{S_n^x}} \gamma_{n,\omega}(I_{S_n^x}, J_{S_n^x}) e(I_{S_n}^x, J_{S_n}^x),\end{aligned}$$

where $S_n^x = x \cup \Lambda \cup S_n$, and there are atmost $2^{2|S_n|}$ nonzero coefficients $\gamma_{n,\omega}$, which satisfy

$$|\gamma_{n,\omega}(I_{S_n^x}, J_{S_n^x})| \leq 2^n \prod_{i=1}^{n} \|\Phi(X_i, \omega)\|.$$

This perturbation expansion can be combined with the previous identity for $\rho(\omega)$, evaluated with $A = e(I_\Lambda, J_\Lambda)$, to obtain a linear equation involving the family $\{\rho(\omega)(e(I_\Lambda, J_\Lambda)); \Lambda \subseteq \mathbb{Z}^\nu\}$. To this end, let $\mathcal{X}$ be the Banach space of bounded complex functions f, on the pairs $\{I_\Lambda, J_\Lambda\}$, where $I_\Lambda, J_\Lambda \subseteq \{0,1\}$, $\Lambda \subseteq \mathbb{Z}^\nu$, and $f(I_\emptyset, J_\emptyset) \in \mathbb{C}$. The space $\mathcal{X}$ is equipped with the usual operations of addition and scalar multiplication, together with the supremum norm. If $\underline{\rho}(\omega)$ denotes the family $\{\rho(\omega)(e(I_\Lambda, J_\Lambda));\ \Lambda \subseteq \mathbb{Z}^\nu\}$, where we take $e(I_\emptyset, J_\emptyset) = I$, it follows that $\underline{\rho}(\omega) \in \mathcal{X}$ and $\|\rho(\omega)\| = 1$. The foregoing identity and perturbation expansion yield the equation

$$\underline{\rho}(\omega) = \underline{\eta} + L_\beta(\omega)\underline{\rho}(\omega),$$

where $\underline{\eta}$, K, and $L_\beta(\omega)$ are defined as follows: $\underline{\eta} \in \mathcal{X}$ and

$$\underline{\eta}(I_\Lambda, J_\Lambda) = \begin{cases} 1 & if\, \Lambda = \emptyset \\ \frac{1}{2}\delta_{i_x, j_x} & if\, \Lambda = \{x\} \\ 0 & otherwise, \end{cases}$$

K is a linear operator with action $Kf(I_\Lambda, J_\Lambda) = \frac{1}{2}\delta_{i_{x_1}, j_{x_1}} f(I_{\Lambda'}, J_{\Lambda'})$, if $\Lambda = \{x_1, x_2, \ldots, x_n\}$, $\Lambda' = \{x_2, \ldots, x_n\}$ and $n \geq 2$, and $(Kf)(I_\Lambda, J_\Lambda) = 0$, if $|\Lambda| < 2$. $L_\beta(\omega)$ is a linear operator such that,

$$\begin{aligned} & (L_\beta(\omega)f)(I_\Lambda, J_\Lambda) \\ = \;& \frac{1}{2}\sum_{K_x=0}^{1}\sum_{n=1}^{\infty}\frac{(-\beta)^n}{n!}\sum_{X_1\cap S_0\neq\emptyset}\cdots\sum_{X_n\cap S_{n-1}\neq\emptyset}\sum_{I_{S_n^x}, J_{S_n^x}}\gamma_{n,\omega}(I_{S_n^x}, J_{S_n^x})f(I_{S_n^x}, J_{S_n^x}), \end{aligned}$$

where $\gamma_{n,\omega}$'s arising from the perturbation expansion are associated with a fixed splitting $\Lambda = \{x\} \cup \Lambda'$, where $\Lambda' = \Lambda \setminus \{x\}$. Thus, the above equation has the form $(I - K - L_\beta(\omega))\underline{\rho}(\omega) = \underline{\eta}$. Hence, $\underline{\rho}(\omega)$ is uniquely determined if, $\|K + L_\beta(\omega)\| < 1$. But $\|K\| = \frac{1}{2}$, and so uniqueness will follow if $\|L_\beta(\omega)\| <$

$\frac{1}{2}$. This involves estimating the norm of $L_\beta(\omega)$. To this end, we establish the following.

$$\|L_\beta(\omega)\| \leq 2^2 e \sum_{n=1}^{\infty}\left(2^{2|\Delta_\omega|+1}|\beta| e^{|\Delta_\omega|}\left(\sup_{a\in\mathbf{Z}^\nu}\left(\sum_{X\ni 0}\|\Phi(X,T_a\omega)\|\right)\right)\right)^n < \infty,$$

whenever

$$2^{2|\Delta_\omega|+1} e^{|\Delta_\omega|}|\beta|\left(\sup_{a\in\mathbf{Z}^\nu}\left(\sum_{X\ni 0}\|\Phi(X,T_a\omega)\|\right)\right) < 1.$$

The estimation procedure will be much like the one employed in the construction of global dynamics. We have

$$\begin{aligned}
&|(L_\beta(\omega)f)(I_\Lambda,J_\Lambda)| \\
\leq\ & \frac{1}{2}\sum_{k_x=0}^{1}\sum_{n=1}^{\infty}\frac{|\beta|^n}{n!}\sum_{X_1\cap S_0\neq\emptyset}\cdots\sum_{X_n\cap S_{n-1}\neq\emptyset}\sum_{I_{S_n^x},J_{S_n^x}}|\gamma_{n,\omega}(I_{S_n^x},J_{S_n^x})|\,|f(I_{S_n^x},J_{S_n^x})| \\
\leq\ & 2.\frac{1}{2}\sum_{n=1}^{\infty}\frac{|\beta|^n}{n!}\sum_{X_1\cap S_0\neq\emptyset}\cdots\sum_{X_n\cap S_{n-1}\neq\emptyset}2^{2|S_n|}(2^n\|\Phi(X_n,\omega)\|\cdots\|\Phi(X_1,\omega)\|)\|f\|_\infty
\end{aligned}$$

The last inequality follows from the remark made earlier, regarding the norms of the complex coefficients $\gamma_{n,\omega}$. Since $\Phi(.,\omega)$ has a finite range Δ_ω, $\Phi(X,\omega)=0$ whenever $|X|>|\Delta_\omega|$. Therefore,

$$\begin{aligned}
|S_j| &= |X_j\cup X_{j-1}\cup\cdots\cup X_1\cup x| \\
&\leq |X_j|+|X_{j-1}|+\cdots+|X_1|+|x| \\
&\leq (|\Delta_\omega|+|\Delta_\omega|\cdots+|\Delta_\omega|+1) \\
&\leq (j|\Delta_\omega|+1).
\end{aligned}$$

We also have

$$\sup_{x\in\mathbf{Z}^\nu}\left(\sum_{X\ni 0}\|\Phi(X,T_x\omega)\|\right) < \infty.$$

Thus,

$$
\begin{aligned}
&\|(L_\beta(\omega)f)\| \\
\leq\ & \sum_{n=1}^{\infty} 2^{2(n|\Delta_\omega|+1)} 2^n \frac{|\beta|^n}{n!} \sum_{x_1\in S_0} \sum_{X_1\ni x_1} \cdots \sum_{x_n\in S_{n-1}} \sum_{X_n\ni x_n} \|\alpha_{-x_n}(\Phi(X_n,\omega))\| \\
& \cdots \|\alpha_{-x_1}((\Phi(X_1,\omega))\|\|f\|_\infty \\
\leq\ & 2^2 \sum_{n=1}^{\infty} 2^{n(2|\Delta_\omega|+1)} \frac{|\beta|^n}{n!} \sum_{x_1\in S_0} \sum_{X_1-x_1\ni 0} \cdots \sum_{x_n\in S_{n-1}} \sum_{X_n-x_n\ni 0} \|\Phi(X_n-x_n, T_{x_n}\omega)\| \\
& \cdots \|\Phi(X_1-x_1, T_{x_1}\omega)\|\|f\|_\infty \\
\leq\ & 2^2 \sum_{n=1}^{\infty} 2^{n(2|\Delta_\omega|+1)} \frac{|\beta|^n}{n!} \prod_{i=1}^{n} (1+(i-1)|\Delta_\omega|) \left(\sup_{x_i\in \mathbf{Z}^\nu} \left(\sum_{Y_i\ni 0} \|\Phi(Y_i, T_{x_i}\omega)\| \right)\right) \|f\|_\infty \\
\leq\ & 2^2 \sum_{n=1}^{\infty} 2^{n(2|\Delta_\omega|+1)} \frac{|\beta|^n}{n!} \prod_{i=1}^{n} (1+(i-1)|\Delta_\omega|) \left(\sup_{a\in \mathbf{Z}^\nu} \left(\sum_{X\ni 0} \|\Phi(X, T_a\omega)\| \right)\right)^n \|f\|_\infty \\
\leq\ & \sum_{n=1}^{\infty} 2^{n(|\Delta_\omega|+1)} \frac{|\beta|^n}{n!} (1+n|\Delta_\omega|)^n \left(\sup_{a\in \mathbf{Z}^\nu} \left(\sum_{X\ni 0} \|\Phi(X, T_a\omega)\| \right)\right)^n \|f\|_\infty \\
\leq\ & 2^2 \sum_{n=1}^{\infty} 2^{n(2|\Delta_\omega|+1)} \frac{|\beta|^n}{n!} e^1 e^{n|\Delta_\omega|} n! \left(\sup_{a\in \mathbf{Z}^\nu} \left(\sum_{X\ni 0} \|\Phi(X, T_a\omega)\| \right)\right)^n \|f\|_\infty \\
\leq\ & 2^2 e^1 \sum_{n=1}^{\infty} (2^{2|\Delta_\omega|+1})^n |\beta|^n e^{n|\Delta_\omega|} \left(\sup_{a\in \mathbf{Z}^\nu} \left(\sum_{X\ni 0} \|\Phi(X, T_a\omega)\| \right)\right)^n \|f\|_\infty \\
\leq\ & 2^2 e \sum_{n=1}^{\infty} \left(2^{2|\Delta_\omega|+1} |\beta| e^{|\Delta_\omega|} \left(\sup_{a\in \mathbf{Z}^\nu} \left(\sum_{X\ni 0} \|\Phi(X, T_a\omega)\| \right)\right)\right)^n \|f\|_\infty .
\end{aligned}
$$

All these inequalities have been obtained by employing the estimation procedure used in the construction of dynamics. Hence, from the above estimate we have

$$
\|L_\beta(\omega)\| \leq 2^2 e \sum_{n=1}^{\infty} \left(2^{2|\Delta_\omega|+1} |\beta| e^{|\Delta_\omega|} \left(\sup_{a\in \mathbf{Z}^\nu} \left(\sum_{X\ni 0} \|\Phi(X, T_a\omega)\| \right)\right)\right)^n < \infty,
$$

whenever

$$
2^{2|\Delta_\omega|+1} e^{|\Delta_\omega|} |\beta| \left(\sup_{a\in \mathbf{Z}^\nu} \left(\sum_{X\ni 0} \|\Phi(X, T_a\omega)\| \right)\right) < 1.
$$

Now, we know that the KMS state $\rho(\omega)$ is unique whenever $\|L_\beta(\omega)\| < \frac{1}{2}$. Therefore, $\rho(\omega)$ is unique whenever

$$2^2 e \frac{2^{2|\Delta_\omega|+1} e^{|\Delta_\omega|} |\beta| \left(\sup_{a \in \mathbb{Z}^\nu} \left(\sum_{X \ni 0} \|\Phi(X, T_a\omega)\|\right)\right)}{1 - 2^{2|\Delta_\omega|+1} e^{|\Delta_\omega|} |\beta| \left(\sup_{a \in \mathbb{Z}^\nu} \left(\sum_{X \ni 0} \|\Phi(X, T_a\omega)\|\right)\right)} < \frac{1}{2}.$$

i.e., whenever

$$|\beta| < \left(2^{2|\Delta_\omega|+1} e^{|\Delta_\omega|} (1 + 2^3 e)\right)^{-1} \left(\sup_{a \in \mathbb{Z}^\nu} \left(\sum_{X \ni 0} \|\Phi(X, T_a\omega)\|\right)\right)^{-1}.$$

Next, by lemma 3.6.0.23, we have $\omega \mapsto |\Delta_\omega|$ is almost surely constant. Moreover, by lemma 3.6.0.24 $\omega \mapsto \left(\sup_{a \in \mathbb{Z}^\nu} \left(\sum_{X \ni 0} \|\Phi(X, T_a\omega)\|\right)\right)$ is also almost surely constant. Hence,

$$\omega \mapsto \left(2^{2|\Delta_\omega|+1} e^{|\Delta_\omega|} (1 + 2^3 e)\right)^{-1} \left(\sup_{a \in \mathbb{Z}^\nu} \left(\sum_{X \ni 0} \|\Phi(X, T_a\omega)\|\right)\right)^{-1}$$

is almost surely constant. Therefore, there exists a critical temperature T_c almost surely independent of ω such that, for temperatures $T > T_c$, there exists an unique KMS state $\rho(\omega)$ associated with $\tau_t(\omega)$. $\triangle$

It is worth noting that the estimate on β can be improved upon in several ways. Since $\rho(\omega)$ is an unique KMS state with respect to $\tau_t(\omega)$, above a certain critical temperature T_c almost surely independent of ω, it follows from theorem 5.3.30 in [Rob 81] that, $\rho(\omega)$ is an extremal KMS state and hence, a factor state. As the quasi–local algebra is norm asymptotically abelian, it also follows that $\rho(\omega)$ is strongly clustering with respect to the group $\mathbb{Z}^\nu$ of lattice translations.

Next, as $\rho(\omega)$ is an unique KMS state associated with the evolution group $\tau_t(\omega)$, one can easily conclude that the net of local Gibbs states $\rho_\Lambda(\omega)$ must converge in the weak*-topology to $\rho(\omega)$, as $\Lambda \to \infty$. This is a trivial consequence of the fact that each weak*-limit point of the local Gibbs states $\rho_\Lambda(\omega)$ is a $(\tau(\omega), \beta)$-KMS state, and hence, by uniqueness of the KMS state $\rho(\omega)$, it must be equal to $\rho(\omega)$. Next, since $\rho_\Lambda(\omega)(A) = \rho_{\Lambda+a}(T_{-a}\omega)(\alpha_a(A))$, for $A \in \mathcal{A}_\Lambda$, we have $\rho(\omega)(A) = \rho(T_{-a}\omega)(\alpha_a(A))$, for all $A \in \mathcal{A}_{\Lambda_0}$ and all $\Lambda_0 \subseteq \mathbb{Z}^\nu$. Since the local elements are norm dense in $\mathcal{A}$, we have $\rho(\omega)(A) = \rho(T_{-a}\omega)(\alpha_a(A))$, for all $A \in \mathcal{A}$. Let $\{\Lambda_n\}$ be a sequence of finite subsets increasing to $\mathbb{Z}^\nu$. Since $\rho(\omega)$ is an unique KMS state, above a critical temperature T_c almost surely independent of ω, we have for all $A \in \mathcal{A}_{\Lambda_0}$ and all $\Lambda_0 \subseteq \mathbb{Z}^\nu$

$$\rho(\omega)(A) = \lim_{n\to\infty} \rho_{\Lambda_n}(\omega)(A),$$

for almost every $\omega \in \Omega$, where

$$\rho_{\Lambda_n}(\omega)(A) = \frac{Tr(e^{-\beta H(\Lambda_n,\omega)}A)}{Tr(e^{-\beta H(\Lambda_n,\omega)})}.$$

However, as $\omega \mapsto H(\Lambda_n, \omega)$ is strongly measurable and $Tr(e^{-\beta H(\Lambda_n,\omega)}) \neq 0$, for all $n \in \mathbb{Z}^+$, it is clear that $\omega \mapsto \rho_{\Lambda_n}(\omega)(A)$ is a scalar valued measurable function for $A \in \mathcal{A}_{\Lambda_0}$ and all finite $\Lambda_0 \subseteq \mathbb{Z}^\nu$. Since the local elements are dense in $\mathcal{A}$, it is readily seen that $\omega \mapsto \rho(\omega)(A)$ is measurable for all $A \in \mathcal{A}$. It has been established in section 4.2.1 that for $\beta \in \mathbb{R} \setminus \{0\}$, there exists a family of states $\{\rho(\omega)\}$ on $\mathcal{A}$, satisfying $\rho(\omega)(A) = \rho(T_{-a}\omega)(\alpha_a(A))$, for all $A \in \mathcal{A}$, where the ρ_ω's are obtained as the thermodynamic limit of the local Gibbs sates $\rho_\Lambda(\omega)$. It is also seen that $\rho(\omega)$ is a $(\tau(\omega), \beta)$-KMS state with respect to the evolution group $\tau_t(\omega)$. Next, let us assume that for

$\beta \in \mathbb{R} \setminus \{0\}$, there exists one such family of $(\tau(\omega), \beta)$–KMS states $\rho(\omega)$, for which the function $\omega \mapsto \rho(\omega)(A)$ is measurable for all $A \in \mathcal{A}$. Henceforth, we shall denote this family of states satisfying the above conditions by $\{\rho(\omega)\}$. It may be noted from the discussion following the proof of theorem 4.2.2.2 that, above the critical temperature T_c almost surely independent of ω, there exists a family of unique KMS states, for which these conditions hold.

Next, we prove the following theorem.

Theorem 4.2.2.3 *If $\{\rho(\omega)\}_{\omega\in\Omega}$ be the family of $(\tau(\omega), \beta)$–KMS states on $\mathcal{A}$ satisfying the conditions mentioned above and $\beta > 0$, then for any pair A, $B \in \mathcal{A}$, we have the following:*

1. *Both $\omega \mapsto \rho(\omega)(A\tau_t(\omega)(B))$ and $\omega \mapsto \rho(\omega)(\tau_t(\omega)(B)A)$ are jointly measurable functions of t and ω.*

2. *In particular, if $\rho(\omega)$ is the unique KMS state with respect to the evolution group $\tau_t(\omega)$, at some inverse temperature $\beta > 0$ almost surely independent of ω, then both $\rho(\omega)(A\tau_t(\omega)(B))$ and $\rho(\omega)(\tau_t(\omega)(B)A)$ are strongly, jointly measurable. Moreover, there exists a function $F_{A,B}(z, \omega)$ such that, for a fixed ω, $F_{A,B}(z, \omega)$ is analytic in the strip $0 < \Im z < \beta$, continuous and uniformly bounded in the closed strip $0 \leq \Im z \leq \beta$, and*

$$F_{A,B}(t, \omega) = \rho(\omega)(A\tau_t(\omega)(B)) \quad \text{and} \quad F_{A,B}(t+i\beta, \omega) = \rho(\omega)(\tau_t(\omega)(B)A).$$

Besides, $F_{A,B}(z, \omega)$ is measurable in ω for each z in the open strip $0 < \Im z < \beta$.

Proof On appealing to theorem 3.7.0.29, we have for $A, B \in \mathcal{A}$, $\omega \mapsto A\mathcal{T}_t(\omega)(B)$ is strongly, jointly measurable in t and ω. It follows from the definition of strong measurability that, there exists a sequence of countably valued functions $g_n(t,\omega)$ on $I\!R \times \Omega$, converging almost everywhere to $A\mathcal{T}_t(\omega)(B)$. Therefore, for almost every $(t,\omega) \in I\!R \times \Omega$,

$$\rho(\omega)(A\mathcal{T}_t(\omega)(B)) = \lim_{n\to\infty} \rho(\omega)(g_n(t,\omega)).$$

In the sequel, we shall establish that for each $n \in \mathbb{Z}^+$, $\rho(\omega)(g_n(t,\omega))$ is measurable on the product space $I\!R \times \Omega$. Let $g_n(t,\omega)$ take nonzero constant values $A_{1,n}, A_{2,n}, \ldots, A_{k,n}, \ldots$, on measurable subsets $E_{1,n}, E_{2,n}, \ldots, E_{k,n}, \ldots$, of $I\!R \times \Omega$. There is no loss of generality in assuming that $\rho(\omega)(g_n(t,\omega))$ takes real values. This is because the $A_{k,n}$'s can always be written as linear combinations of self adjoint elements in $\mathcal{A}$, and $\rho(\omega)$ being a state, it takes real values on self adjoint elements of $\mathcal{A}$. Now, for $c \in I\!R$,

$$\begin{aligned}
& \{(t,\omega) \in I\!R \times \Omega | \rho(\omega)(g_n(t,\omega)) < c\} \\
= \;& \{(t,\omega) \in E_{0,n} | \rho(\omega)(g_n(t,\omega)) < c\} \bigcup \left\{ \bigcup_{k=1}^{\infty} \{(t,\omega) \in E_{k,n} | \rho(\omega)(g_n(t,\omega)) < c\} \right\} \\
= \;& \{(t,\omega) \in E_{0,n} | \rho(\omega)(0) < c\} \bigcup \left\{ \bigcup_{k=1}^{\infty} \{(t,\omega) \in E_{k,n} | \rho(\omega)(A_{k,n}) < c\} \right\},
\end{aligned}$$

where $E_{0,n}$ is the set on which g_n takes the value zero(0). Therefore, it is evident from the measurability of the function $\omega \mapsto \rho(\omega)(A)$, for all $A \in \mathcal{A}$ that, the two sets on the right hand side of the equality are measurable subsets of $I\!R \times \Omega$. Hence, as c is arbitrary, $\rho(\omega)(g_n(t,\omega))$, is a jointly measurable function of t and ω for each $n \in \mathbb{Z}^+$. Since $\rho(\omega)(A\mathcal{T}_t(\omega)(B))$ is the limit almost everywhere of $\rho(\omega)(g_n(t,\omega))$ on $I\!R \times \Omega$, we conclude that $\rho(\omega)(A\mathcal{T}_t(\omega)(B))$

is a jointly measurable function of t and ω. Similarly, it can be shown that the function $(t,\omega) \mapsto \rho(\omega)(\tau_t(\omega)(B)A)$ is jointly measurable in t and ω. This proves (1) conclusively.

Now, in order to prove (2), choose a sequence of finite subsets $\{\Lambda_n\}$ which increases to $\mathbb{Z}^\nu$. It has been shown in the discussion following the proof of theorem 4.2.2.2 that, if $\rho(\omega)$ is the unique KMS state with respect to the evolution group $\tau_t(\omega)$ at some inverse temperature β almost surely independent of ω, then for all $A \in \mathcal{A}_{\Lambda_0}$ and all $\Lambda_0 \subseteq \mathbb{Z}^\nu$,

$$\lim_{n\to\infty} \rho_{\Lambda_n}(\omega)(A) = \rho(\omega)(A)$$

almost everywhere. It was also established in this discussion that $\omega \mapsto \rho(\omega)(A)$ is measurable for all $A \in \mathcal{A}$. Since $\omega \mapsto \rho(\omega)(A)$ is measurable, the joint measurability in t and ω, of both $\rho(\omega)(A\tau_t(\omega)(B)))$ and $\rho(\omega)(\tau_t(\omega)(B)A)$ can be proved along the lines of (1). Next, for A, $B \in \mathcal{A}_{\Lambda_0}$, let

$$\tau_z^{\Lambda_n}(\omega)(B) = e^{iH(\Lambda_n,\omega)z}Be^{-iH(\Lambda_n,\omega)z},$$

where $\Lambda_n \supseteq \Lambda_0$. Also for $\Lambda_n \supseteq \Lambda_0$, define

$$F_{A,B}^{\Lambda_n}(z,\omega) = \frac{Tr(e^{-\beta H(\Lambda_n,\omega)}A\tau_z^{\Lambda_n}(\omega)(B))}{Tr(e^{-\beta H(\Lambda_n,\omega)})}.$$

Clearly, $\{F_{A,B}^{\Lambda_n}\}$ is a sequence of entire functions. which is uniformly bounded on the strip $0 \leq \Im z \leq \beta$ such that,

$$F_{A,B}^{\Lambda_n}(t,\omega) = \rho_{\Lambda_n}(\omega)(A\tau_t^{\Lambda_n}(\omega)(B)) \quad and \quad F_{A,B}^{\Lambda_n}(t+i\beta,\omega) = \rho_{\Lambda_n}(\omega)(\tau_t^{\Lambda_n}(\omega)(B)A).$$

(See the proof of theorem 3.4.0.13 which can be adopted to $\rho^\Lambda(\omega)$ with $\tau_t^\Lambda(\omega)$ as the local automorphism group). On mimicking the proof of proposition (3.4.0.14) in chapter 3, we have for $A, B \in \mathcal{A}_{\Lambda_0}$

$$\lim_{n\to\infty} \rho_{\Lambda_n}(\omega)(A\tau_t^{\Lambda_n}(\omega)(B)) = \rho(\omega)(A\tau_t(\omega)(B)),$$

where the limit exists almost everywhere in ω, for all real t and uniformly in t in a ball around zero. Hence, as a consequence of Vitali's theorem, see [Tit 91], for almost every $\omega \in \Omega$ the sequence $F_{A,B}^{\Lambda_n}(z,\omega)$ converges uniformly on every compact subset in the strip to $F_{A,B}(z,\omega)$, which for a fixed ω is analytic in the open strip $0 < \Im z < \beta$, continuous and uniformly bounded in the closed strip $0 \leq \Im z \leq \beta$ such that,

$$F_{A,B}(t) = \rho(\omega)(A\tau_t(B)) \quad and \quad F_{A,B}(t+i\beta) = \rho(\omega)(\tau_t(B)A).$$

This proves the existence of $F_{A,B}(z,\omega)$ satisfying the conditions in (2) for $A, B \in \mathcal{A}_{\Lambda_0}$, where $\Lambda_0 \subseteq \mathbb{Z}^\nu$.

Now, it follows from the strong measurability of $\omega \mapsto H(\Lambda,\omega)$ for finite $\Lambda \subseteq \mathbb{Z}^\nu$ that, both $A\tau_z^{\Lambda_n}(\omega)B$ and $e^{-\beta H(\Lambda_n,\omega)}$ are strongly measurable in ω for each z in the open strip and $n \in \mathbb{Z}^+$. Therefore, for each z in the open strip, $F_{A,B}^{\Lambda_n}(z,\omega)$ is a scalar valued measurable function of ω. This is in view of the fact that, the trace, denoted by Tr, is a continuous linear functional and hence, $F_{A,B}^{\Lambda_n}(z,\omega)$ is a ratio of two measurable functions with $Tr(e^{-\beta H(\Lambda_n,\omega)}) \neq 0$ for all $n \in \mathbb{Z}^+$. It has been seen from Vitali's theorem that, for almost every $\omega \in \Omega$ the sequence $F_{A,B}^{\Lambda_n}(z,\omega)$ converges uniformly on every compact subset in the strip to a function $F_{A,B}(z,\omega)$. Hence for each z in the open strip, $F_{A,B}^{\Lambda_n}(z,\omega)$ converges to $F_{A,B}(z,\omega)$ almost everywhere. Therefore, we conclude that for each z in the open strip, $F_{A,B}(z,\omega)$ is a measurable function of ω. This proves the measurability of $F_{A,B}(z,\omega)$ in ω for each z in the open strip, for $A, B \in \mathcal{A}_{\Lambda_0}$ and $\Lambda_0 \subseteq \mathbb{Z}^\nu$.

Next, for $A, B \in \mathcal{A}$, let $A_n \to A$ and $B_n \to B$ be sequences of local elements converging to A and B respectively. Therefore, it follows from what was established earlier that, there exists a sequence of scalar valued functions $F_{A_n,B_n}(z,\omega)$ such that, for each z in the open strip, $F_{A_n,B_n}(z,\omega)$ is measurable and for a fixed ω, $F_{A_n,B_n}(z,\omega)$ is analytic in the open strip, uniformly bounded and continuous on the closed strip. Moreover, $F_{A_n,B_n}(t,\omega) = \rho(\omega)(A_n\tau_t(\omega)(B_n))$ and $F_{A_n,B_n}(t+i\beta) = \rho(\omega)(\tau_t(\omega)(B_n)A_n)$. Now, there is a version of the Phragmen–Lindelöf theorem [Rob 81] (Vol 2, Proposition 5.3.5, Pg 81) which states that, the supremum of the modulus of a function which is bounded and analytic on the strip, is the supremum of the moduli of its boundary values. Since $A_n \to A$ and $B_n \to B$ in the norm, the sequence $F_{A_n,B_n}(t,\omega) \to \rho(\omega)(A\tau_t(\omega)(B))$ and $F_{A_n,B_n}(t+i\beta,\omega) \to \rho(\omega)(\tau_t(\omega)(B)A)$. The convergence being uniform in t. Thus, since the sequence $F_{A_n,B_n}(z,\omega)$ converges uniformly on the boundary of the strip $0 \leq \Im z \leq \beta$, it converges uniformly throughout the closed strip, to say, $F_{A,B}(z,\omega)$. $F_{A,B}(z,\omega)$ being analytic in the open strip and uniformly bounded and continuous in the closed strip, such that

$$F_{A,B}(t) = \rho(\omega)(A\tau_t(B)) \quad and \quad F_{A,B}(t+i\beta) = \rho(\omega)(\tau_t(B)A),$$

for a fixed ω. Also for each z in the open strip, $F_{A,B}(z,\omega)$ is the limit of the sequence of measurable functions $F_{A_n,B_n}(z,\omega)$ for almost every $\omega \in \Omega$. Hence, for each z in the open strip, $F_{A,B}(z,\omega)$ is a measurable function of ω.

△

The theorem can be established along the same lines in the case of $\beta < 0$, by considering the closed strip $\beta \leq \Im z \leq 0$.

4.2.3 Representations Associated with the KMS States

In this subsection, we aim to study the cyclic representations π_ω associated with the states $\rho(\omega)$. These states are thermodynamic limits of the local Gibbs states $\rho_\Lambda(\omega)$, and satisfy the following conditions: $\rho(\omega)(A) = \rho(T_{-a}\omega)(\alpha_a(A))$, for all $A \in \mathcal{A}$ and $a \in \mathbb{Z}^\nu$, and $\omega \mapsto \rho(\omega)(A)$ is measurable, for all $A \in \mathcal{A}$. It is also seen that $\rho(\omega)$ is a $(\tau(\omega), \beta)$–KMS state, where $\tau_t(\omega)$ is the evolution group. We shall exploit the quasi–local structure of the C^*–algebra to demonstrate some interesting features of the representations π_ω, and establish the separability of the Hilbert space $\mathcal{H}_\omega$. Algebraic properties of the group of unitaries $U_t(\omega)$, which implements the evolution group $\tau_t(\omega)$ of the spin system have also been derived.

Now, associated with every $\rho(\omega)$, we have a cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Theta_\omega)$ of the quasi–local algebra $\mathcal{A}$, obtained through the G.N.S construction. The idea behind this construction is to convert the C^*–algebra $\mathcal{A}$ into a pre–Hilbert space by introducing a positive semi–definite scalar product on $\mathcal{A}$. In the process, we end up with a pre–Hilbert space of equivalence classes $\psi_A(\omega)$, $\psi_B(\omega)$, defined by $\psi_A(\omega) = \hat{A}(\omega)$; $\hat{A}(\omega) = A + J$, where $J \in \mathcal{J}_\omega$, and $\mathcal{J}_\omega = \{A \in \mathcal{A} | \, \rho(\omega)(A^*A) = 0\}$, with the scalar product given by

$$\langle \psi_A(\omega), \psi_B(\omega) \rangle_\omega = \rho(\omega)(A^*B).$$

Before we complete this pre-Hilbert space to give us the Hilbert space $\mathcal{H}_\omega$, we define the representation π_ω by specifying the action of the representative $\pi_\omega(A)$ on the pre-Hilbert space as follows:

$$\pi_\omega(A)(\psi_B(\omega)) = \psi_{AB}(\omega).$$

The cyclic vector is defined by $\Theta_\omega = \psi_I(\omega)$. Note that, $\{\pi_\omega(A)\Theta_\omega; A \in \mathcal{A}\}$ is exactly the dense set of equivalence classes $\{\psi_A; A \in \mathcal{A}\}$, and hence, Θ_ω is cyclic for $(\mathcal{H}_\omega, \pi_\omega)$. Since $\mathcal{A}$ is simple, $\pi(\omega)$ is a faithful representation of $\mathcal{A}$. Moreover, $\mathcal{A}$ being a uniformly matricial $C^\star$-algebra (or UHF algebra), each of these states $\rho(\omega)$, is a locally normal state. Therefore, it follows from the remarks made on the characterization of locally normal states of an abstract $C^\star$-algebra in [Em 72] (Page 283), that, the Hilbert space $\mathcal{H}_\omega$ asssociated with the representation π_ω, is a separable Hilbert space. Since $\mathcal{A}$ is simple, and $(\mathcal{H}_\omega, \pi_\omega, \Theta_\omega)$ is a cyclic representation of $\mathcal{A}$ induced by the KMS state $\rho(\omega)$, it follows from [Win 70](Section 5, Page 253) that, the vector Θ_ω is cyclic and separating for the von Neumann algebra $\pi_\omega(\mathcal{A})''$.

Next, every element $a \in \mathbb{Z}^\nu$, induces an isomorphism $D_{-a} : \mathcal{H}_\omega \to \mathcal{H}_{T_{-a}\omega}$, as follows: Define $D_{-a}(\psi_A(\omega)) = \psi_{\alpha_a A}(T_{-a}\omega)$. Note that, D_{-a} is defined on a dense subspace of $\mathcal{H}_\omega$, and

$$\begin{aligned}
\langle D_{-a}(\psi_A(\omega)), D_{-a}(\psi_B(\omega))\rangle_{T_{-a}\omega} &= \langle \psi_{\alpha_a(A)}(T_{-a}\omega), \psi_{\alpha_a(B)}(T_{-a}\omega)\rangle_{T_{-a}\omega} \\
&= \rho(T_{-a}\omega)((\alpha_a(A))^\star(\alpha_a(B))) \\
&= \rho(T_{-a}\omega)(\alpha_a(A^\star B)) \\
&= \rho(\omega)(A^\star B)
\end{aligned}$$

$$= \langle \psi_A(\omega), \psi_B(\omega) \rangle_\omega.$$

Hence, for each $a \in \mathbb{Z}^\nu$, $D_{-a} : \mathcal{H}_\omega \to \mathcal{H}_{T_{-a}\omega}$ preserves the inner product on a dense subspace $\mathcal{V}_\omega = \{\psi_A(\omega); A \in \mathcal{A}\}$ of $\mathcal{H}_\omega$. Besides, it is clear that D_{-a} maps $\mathcal{V}_\omega$, onto a dense subspace $\mathcal{V}_{T_{-a}\omega}$ of the separable Hilbert space $\mathcal{H}_{T_{-a}\omega}$. Therefore, for each $a \in \mathbb{Z}^\nu$, D_{-a} can be extended to an isomorphism between the Hilbert spaces $\mathcal{H}_\omega$ and $\mathcal{H}_{T_{-a}\omega}$.

It is worth noting that for $\omega \in \Omega$ and $a \in \mathbb{Z}^\nu$,

$$\begin{aligned}
(D_{-a}^{-1}(\pi_{T_{-a}\omega}(\alpha_a(A)))D_{-a})(\psi_B(\omega)) &= D_{-a}^{-1}(\pi_{T_{-a}\omega}(\alpha_a(A)))(\psi_{\alpha_a(B)}(T_{-a}\omega)) \\
&= D_{-a}^{-1}(\psi_{(\alpha_a(A))(\alpha_a(B))}(T_{-a}\omega)) \\
&= D_{-a}^{-1}(\psi_{\alpha_a(AB)}(T_{-a}\omega)) \\
&= \psi_{AB}(\omega) \\
&= \pi_\omega(A)(\psi_B(\omega)),
\end{aligned}$$

for all $A \in \mathcal{A}$. Since the $\psi(\omega)$'s are dense in $\mathcal{H}_\omega$, we have

$$\pi_\omega(A) = D_{-a}^{-1}(\pi_{T_{-a}\omega}(\alpha_a(A)))D_{-a} \quad \forall A \in \mathcal{A}.$$

Thus, D_{-a} exhibits an interesting intertwinning property which establishes some sort of equivalence between the representations π_ω and $\pi_{T_{-a}\omega}$. This equivalence is reminiscent of the notion of unitary equivalence between representations. It follows readily from the identity

$$\pi_{T_{-a}\omega}(\tau_t(T_{-a}\omega)(\alpha_a(A))) = \pi_{T_{-a}\omega}(\alpha_a(\tau_t(\omega)(A))),$$

where $\tau_t(\omega)$ is the evolution group, and the intertwinning property of D_{-a} that,

$$D_{-a}^{-1}(\pi_{T_{-a}\omega}(\tau_t(T_{-a}\omega)(\alpha_a(A))))D_{-a} = \pi_\omega(\tau_t(\omega)(A)).$$

Note that for $a \in \mathbb{Z}^\nu$ and $\omega \in \Omega$,

$$\begin{aligned} D_{-a}(\Theta_\omega) &= D_{-a}(\psi_I(\omega)) \\ &= \psi_{\alpha_a(I)}(T_{-a}\omega) \\ &= \psi_I(T_{-a}\omega) \\ &= \Theta_{T_{-a}\omega}. \end{aligned}$$

In the final part of this section, we derive an interesting ergodic property of the spectrum of the generators of the unitary groups $U_t(\omega)$, which implement the evolution groups $\tau_t(\omega)$ in the representation π_ω.

Since $\rho(\omega)$ is a $(\tau(\omega), \beta)$–KMS state, we have $\rho(\omega)(\tau_t(\omega)(A)) = \rho(\omega)(A)$ for all $A \in \mathcal{A}$. It follows from the uniqueness of the cyclic representation $(\pi_\omega, \mathcal{H}_\omega, \Theta_\omega)$ that, there exists an unitary operator $U_t(\omega) : \mathcal{H}_\omega \to \mathcal{H}_\omega$ such that,

$$U_t(\omega)(\pi_\omega(A))U_t(\omega)^{-1} = \pi_\omega(\tau_t(\omega)(A)) \quad \textit{and} \quad U_t(\omega)\Theta_\omega = \Theta_\omega,$$

for all $t \in \mathbb{R}$. Here $U_t(\omega)^{-1}$ denotes the inverse of $U_t(\omega)$.

Proposition 4.2.3.1 *Let $U_t(\omega)$ be the strongly continuous, one-parameter group of unitary operators implementing the evolution group $\tau_t(\omega)$ in the representation π_ω on $\mathcal{H}_\omega$. Then, we have*

$$U_t(\omega) = D_{-a}^{-1}(U_t(T_{-a}\omega))D_{-a}.$$

Proof Since

$$\pi_\omega(\tau_t(\omega)(A)) = D_{-a}^{-1}(\pi_{T_{-a}\omega}(\tau_t(T_{-a}\omega)(\alpha_a(A))))D_{-a},$$

we have

$$
\begin{aligned}
& \langle U_t(\omega)(\pi_\omega(A)\Theta_\omega), \pi_\omega(B)\Theta_\omega\rangle_\omega \\
= \;& \langle \pi_\omega(\tau_t(\omega)(A))\Theta_\omega, \pi_\omega(B)\Theta_\omega\rangle_\omega \\
= \;& \langle D_{-a}^{-1}(\pi_{T_{-a}\omega}(\tau_t(T_{-a}\omega)(\alpha_a(A))))D_{-a}\Theta_\omega, D_{-a}^{-1}(\pi_{T_{-a}\omega}(\alpha_a(B)))D_{-a}\Theta_\omega\rangle_\omega \\
= \;& \langle D_{-a}^{-1}(\pi_{T_{-a}\omega}(\tau_t(T_{-a}\omega)(\alpha_a(A)))\Theta_{T_{-a}\omega}), D_{-a}^{-1}(\pi_{T_{-a}\omega}(\alpha_a(B))\Theta_{T_{-a}\omega})\rangle_\omega \\
= \;& \langle D_{-a}^{-1}(U_t(T_{-a}\omega))\pi_{T_{-a}\omega}(\alpha_a(A))\Theta_{T_{-a}\omega}, D_{-a}^{-1}(\pi_{T_{-a}\omega}(\alpha_a(B))\Theta_{T_{-a}\omega})\rangle_\omega \\
= \;& \langle (D_{-a}^{-1}(U_t(T_{-a}\omega))D_{-a})\pi_\omega(A)(D_{-a}^{-1}\Theta_{T_{-a}\omega}), \pi_\omega(B)(D_{-a}^{-1}\Theta_{T_{-a}\omega})\rangle_\omega \\
= \;& \langle (D_{-a}^{-1}(U_t(T_{-a}\omega))D_{-a})\pi_\omega(A)\Theta_\omega, \pi_\omega(B)\Theta_\omega\rangle_\omega .
\end{aligned}
$$

Therefore,

$$\langle U_t(\omega)(\pi_\omega(A)\Theta_\omega), \pi_\omega(B)\Theta_\omega\rangle_\omega = \langle (D_{-a}^{-1}(U_t(T_{-a}\omega))D_{-a})\pi_\omega(A)\Theta_\omega, \pi_\omega(B)\Theta_\omega\rangle_\omega .$$

Since Θ_ω is a cyclic vector for $\pi_\omega(\mathcal{A})$, the above equality implies that

$$U_t(\omega) = D_{-a}^{-1}(U_t(T_{-a}\omega))D_{-a}.$$

△

By virtue of the above proposition we have the following corollary.

Corollary 4.2.3.2 *Let $H(\omega)$ be the generator of the strongly continuous, one–parameter group of unitaries $U_t(\omega)$, which implement the evolution group $\tau_t(\omega)$. If $E_\lambda(\omega)$ are the spectral projections associated with $H(\omega)$, then the spectral projections $E_\lambda(T_{-a}\omega)$ associated with the generator $H(T_{-a}\omega)$ of the unitary group $U_t(T_{-a}\omega)$ can be expressed as $E_\lambda(T_{-a}\omega) = D_{-a}(E_\lambda(\omega))D_{-a}^{-1}$.*

Proof We know from Stone's theorem ([Sim 80](Theorem VIII.8)) that, the spectral family $E_\lambda(T_{-a}\omega)$ associated with the unitary group $U_t(T_{-a}\omega)$ is unique. Hence the proof follows from proposition 4.2.3.1. $\triangle$

Next, we shall show that the spectrum of the generator of the unitary group $U_t(\omega)$, is almost surely independent of ω. To this end, we have the following proposition.

Proposition 4.2.3.3 *Let $H(\omega)$ be the generator of the strongly continuous, one-parameter group of unitaries $U_t(\omega)$. Then the spectrum $\sigma(H(\omega))$ of the generator $H(\omega)$ is almost surely independent of ω.*

Proof Let π_ω denote the representation associated with the $(\tau(\omega), \beta)$–KMS state $\rho(\omega)$, with cyclic vector Θ_ω. The unitary group $U_t(\omega)$ with generator $H(\omega)$ implements $\tau_t(\omega)$ in this representation π_ω. Now, for $f \in L^1(I\!R)$, we have

$$\begin{aligned}
\Psi_\omega(f)\phi &= \int_{-\infty}^{\infty} f(t)U_t(\omega)\phi dt = 0, \quad \forall \phi \in \mathcal{H}_\omega \\
&\Leftrightarrow \int_{-\infty}^{\infty} f(t)U_t(\omega)(\pi_\omega(A)\Theta_\omega)dt = 0, \quad \forall A \in \mathcal{A} \\
&\Leftrightarrow \int_{-\infty}^{\infty} f(t)\pi_\omega(\tau_t(\omega)(A))\Theta_\omega dt = 0, \quad \forall A \in \mathcal{A} \\
&\Leftrightarrow \left(\int_{-\infty}^{\infty} f(t)\pi_\omega(\tau_t(\omega)(A))dt\right)\Theta_\omega = 0, \quad \forall A \in \mathcal{A} \\
&\Leftrightarrow \pi_\omega\left(\int_{-\infty}^{\infty} f(t)\tau_t(\omega)(A)dt\right)\Theta_\omega = 0, \quad \forall A \in \mathcal{A} \\
&\Leftrightarrow \pi_\omega\left(\int_{-\infty}^{\infty} f(t)\tau_t(\omega)(A)dt\right) = 0, \quad \forall A \in \mathcal{A} \\
&\Leftrightarrow \int_{-\infty}^{\infty} f(t)\tau_t(\omega)(A)dt = 0, \quad \forall A \in \mathcal{A}.
\end{aligned}$$

The first step follows from the fact that Θ_ω is a cyclic vector for $\pi_\omega(\mathcal{A})$. The second follows from the definition of $U_t(\omega)$. Since $\rho(\omega)$ is a KMS state, the separating character of the cyclic vector Θ_ω for $\pi_\omega(\mathcal{A})''$, accounts for the penultimate step. We arrive at the final step by virtue of the fact that the representation π_ω is faithful. Now, $\sigma(H(\omega)) = -\{s \in I\!R| \hat{f}(s) = 0, \forall f \in \ker \Psi(\omega)\}$, vide [Bri 77] (Chapter 1, Definition 1.4). Therefore, from the above derivation we have $\sigma(H(\omega)) = -\{s \in I\!R| \hat{f}(s) = 0, \forall f \in \ker \Gamma(\omega)\}$, where $\Gamma(\omega)$ is as defined in theorem 4.1.0.35. Hence, the proof follows from theorem (4.1.0.35). $\triangle$

4.3 Direct Integral von Neumann Algebra.

In the previous section, we saw how each of the $\rho(\omega)$'s gave rise to a representation $\pi_\omega(\mathcal{A})$ of the quasi–local algebra $\mathcal{A}$, on a separable Hilbert space $\mathcal{H}_\omega$. It is seen that, these states are $(\tau(\omega), \beta)$–KMS states with respect to the evolution groups $\tau_t(\omega)$. Now, the representations π_ω associated with these states in turn give rise to an ensemble of von Neumann algebras $\{\pi_\omega(\mathcal{A})\}''_{\omega \in \Omega}$. As these von Neumann algebras correspond to distinct realizations of the quasi–local algebra $\mathcal{A}$, one has to treat them as distinct objects. Therefore, one is obliged to invoke the theory of measurable field of von Neumann algebras. The assumption that the action of the measure preserving group of automorphisms is ergodic, allows us to derive some interesting results concerning the spectra of the generators of the unitaries $U_t(\omega)$, which implement the evolution groups $\tau_t(\omega)$ in the representation π_ω. Moreover, the evolution group

$\tau_t(\omega)$ can be extended to a σ-weakly continuous group of automorphisms $\tilde{\tau}_t(\omega)$ of the von Neumann algebra $\pi_\omega(\mathcal{A})''$ such that,

$$\tilde{\tau}_t(\omega)(S_\omega) = U_t(\omega) S_\omega U_t(\omega)^{-1}, \quad \forall S_\omega \in \pi_\omega(\mathcal{A})''.$$

Since

$$\begin{aligned} \tilde{\tau}_t(\pi_\omega(A)) &= U_t(\omega)\pi_\omega(A)U_t(\omega)^{-1} \\ &= \pi_\omega(\tau_t(\omega)(A)), \end{aligned}$$

the restriction of $\tilde{\tau}_t$ to $\pi_\omega(\mathcal{A})$ is $\tau_t(\omega)$. In the sequel, we impose a measurable structure on the field of Hilbert spaces $\omega \mapsto \mathcal{H}_\omega$, and construct a direct integral Hilbert space $H = \int_\Omega^\oplus \mathcal{H}_\omega dP(\omega)$. We also demonstrate that $\omega \mapsto \pi_\omega(\mathcal{A})''$, is a measurable field of von Neumann algebras and establish the existence of the associated direct integral von Neumann algebra $\int_\Omega^\oplus \pi_\omega(\mathcal{A})'' dP(\omega)$. Further, we establish the existence of a strongly continuous, one-parameter group of unitaries U_t acting on the direct integral Hilbert space $\int_\Omega^\oplus \mathcal{H}_\omega dP(\omega)$ such that, $U_t S U_t^{-1} \in \int_\Omega^\oplus \pi_\omega(\mathcal{A})'' dP(\omega)$. for all $S \in \int_\Omega^\oplus \pi_\omega(\mathcal{A})'' dP(\omega)$. Finally, we construct a faithful, normal state ρ of the direct integral von Neumann algebra $\int_\Omega^\oplus \pi_\omega(\mathcal{A})'' dP(\omega)$, which satisfies the Kubo–Martin–Schwinger condition with respect to the σ-weakly continuous group of automorphisms $\tilde{\tau}_t(S) = U_t S U_t^{-1}$, for every decomposable operator $S \in \int_\Omega^\oplus \pi_\omega(\mathcal{A})'' dP(\omega)$.

4.3.1 Measurable Field of Hilbert spaces

We begin with the following proposition.

Proposition 4.3.1.1 *The field of separable Hilbert spaces $\omega \mapsto \mathcal{H}_\omega$ is a measurable field of Hilbert spaces.*

Proof To this end, recall that the family of states $\{\rho(\omega)\}$ on $\mathcal{A}$, is such that $\rho(\omega)$ is a $(\tau(\omega), \beta)$–KMS state with respect to the evolution group $\tau_t(\omega)$ and $\omega \mapsto \rho(\omega)(A)$ is a measurable function of ω for all $A \in \mathcal{A}$. Since $(\mathcal{H}_\omega, \pi_\omega, \Theta_\omega)$ is a cyclic representation of $\mathcal{A}$ induced by $\rho(\omega)$, we have $\rho(\omega)(A) = \langle \pi_\omega(A)\Theta_\omega, \Theta_\omega \rangle_\omega$. $\mathcal{A}$, being an uniformly matricial $C^\star$–algebra, there exists a sequence of elements $\{A_n\}$ in $\mathcal{A}$ such that, $\mathcal{A}_0 = \{A_n | \, n \in \mathbb{Z}^+\}$ is dense in $\mathcal{A}$ and hence, for each $\omega \in \Omega$, $\pi_\omega(\mathcal{A}_o)$ is operator–norm dense in $\pi_\omega(\mathcal{A})$. Since Θ_ω is a cyclic vector associated with the representation π_ω on $\mathcal{H}_\omega$, $\{\pi_\omega(A)\Theta_\omega | A \in \mathcal{A}\}$ is dense in $\mathcal{H}_\omega$ for $\omega \in \Omega$. As $\pi_\omega(\mathcal{A}_o)$ is operator–norm dense in $\pi_\omega(\mathcal{A})$, it is easily seen that the sequence of vector fields $\omega \mapsto \pi_\omega(A_i)\Theta_\omega$; $i = 1, 2, \ldots$, is a total sequence in $\mathcal{H}_\omega$ for all $\omega \in \Omega$. Moreover, in view of the assumption that the map $\omega \mapsto \rho(\omega)(A)$ is measurable for all $A \in \mathcal{A}$, it is readily seen that the function $\omega \mapsto \langle \pi_\omega(A_i)\Theta_\omega, \pi_\omega(A_j)\Theta_\omega \rangle_\omega$ is measurable for $i, j = 1, 2, \ldots$. Therefore, it follows from [Dix 81] (Part II, Chapter 1, Prop 4) that, there exists exactly one measurable vector field structure on the $\mathcal{H}_\omega$'s given by a collection of vector fields $\mathcal{F}$ such that, the vector fields $\omega \mapsto \pi_\omega(A_i)\Theta_\omega$, are measurable with respect to this collection $\mathcal{F}$. Therefore the field of Hilbert spaces $\omega \mapsto \mathcal{H}_\omega$ is a measurable field of Hilbert spaces. △

The above fact allows us to define the direct integral Hilbert space $\mathcal{H}$, of all square integrable vector fields in $\mathcal{F}$ over Ω, from the measurable field of

Hilbert spaces $\omega \mapsto \mathcal{H}_\omega$. Here the inner product $\langle .,. \rangle$ on $\mathcal{H}$, is given by

$$\langle \xi, \eta \rangle = \int_\Omega \langle \xi(\omega), \eta(\omega) \rangle dP(\omega),$$

for all square integrable vector fields ξ, $\eta \in \mathcal{F}$. We denote the same by $\int_\Omega^\oplus \mathcal{H}_\omega dP(\omega)$. It is worth mentioning that the Hilbert space $\mathcal{H}$ is a separable Hilbert space. This follows from the fact that P is the completion of a probability measure defined on the Borel sigma algebra generated by the topology of a complete separable metric space Ω.

Next, we aim to show the following.

Proposition 4.3.1.2 *If $U_t(\omega)$ is the strongly continuous, one-parameter group of unitaries implementing the evolution group $\tau_t(\omega)$ in the representation π_ω then, for each $t \in \mathbb{R}$, $\omega \mapsto U_t(\omega)$ is a measurable field of unitary operators. In fact $(t, \omega) \mapsto \langle U_t(\omega)(\pi_\omega(A_i)\Theta_\omega), \pi_\omega(A_j)\Theta_\omega \rangle_\omega$ is jointly measurable in t and ω for $i, j = 1, 2, \ldots$.*

Proof It is clear from the proof of proposition 4.3.1.1 and definition 1 in [Dix 81] (Part II, Chapter 1) that, $\{x_i\}$, where $x_i(\omega) = \pi_\omega(A_i)\Theta_\omega$, is a fundamental sequence of measurable vector fields with values in $\mathcal{H}_\omega$. Therefore, it is easily seen from proposition 1 (Chapter 2, Part (II)) in [Dix 81] that, the above proposition will follow if one can show that $\omega \mapsto \langle U_t(\omega)(\pi_\omega(A_i)\Theta_\omega), \pi_\omega(A_j)\Theta_\omega \rangle_\omega$, is a measurable scalar valued function for $i, j = 1, 2, \ldots$. Since $U_t(\omega)$ implements $\tau_t(\omega)$, we have $U_t(\omega)(\pi_\omega(A_i))\Theta_\omega = \pi_\omega(\tau_t(\omega)(A_i))\Theta_\omega$. Now $(\pi_\omega, \mathcal{H}_\omega, \Theta_\omega)$ being a cyclic representation of $\mathcal{A}$ associated with the state $\rho(\omega)$, we have $\rho(\omega)(A) = \langle \pi_\omega(A)\Theta_\omega, \Theta_\omega \rangle_\omega$ for all

$A \in \mathcal{A}$. Therefore, $\langle U_t(\omega)(\pi_\omega(A_i)\Theta_\omega), \pi_\omega(A_j)\Theta_\omega\rangle_\omega = \rho(\omega)(A_j^*\tau_t(\omega)(A_i))$, for $i, j = 1, 2 \ldots$. Hence, the proof of the proposition follows from (1) in theorem 4.2.2.3. In fact, as a consequence of (1) in theorem 4.2.2.3, we have actually shown that the map $(t, \omega) \mapsto \langle U_t(\omega)(\pi_\omega(A_i)\Theta_\omega), \pi_\omega(A_j)\Theta_\omega\rangle_\omega$ is jointly measurable in t and ω for $i, j = 1, 2, \ldots$. $\triangle$

The following proposition is a consequence of the preceding proposition.

Proposition 4.3.1.3 *Let $\{\xi(\omega)\}$ and $\{\eta(\omega)\}$ be two measurable vector fields in $\mathcal{F}$. Then the map $(t, \omega) \mapsto \langle U_t(\omega)\xi(\omega), \eta(\omega)\rangle_\omega$, is a jointly measurable, scalar valued function of t and ω, for all measurable vector fields ξ and η in $\mathcal{F}$.*

Proof It is seen that $\{x_i\}$, where $x_i(\omega) = \pi_\omega(A_i)\Theta_\omega$, is a fundamental sequence of measurable vector fields. Therefore, it follows from [Dix 81] (Problem 3, Chapter 1, Part II) that, for any measurable vector field ξ in $\mathcal{F}$, there exists a sequence of vector fields ξ_n, of the form $\xi_n(\omega) = \sum_{i=1}^{n} f_i(\omega)\pi_\omega(A_i)\Theta_\omega$, converging to ξ almost everywhere, where the f_i's are complex valued measurable functions on Ω. Clearly, these vector fields are measurable with respect to $\mathcal{F}$. It is readily seen from proposition 4.3.1.2 that, for any two complex valued measurable functions f and g on Ω, the scalar valued function $(t, \omega) \mapsto \langle U_t(\omega)(f(\omega)\pi_\omega(A_i)\Theta_\omega), g(\omega)\pi_\omega(A_j)\Theta_\omega\rangle_\omega$, is jointly measurable in t and ω, for all $i, j = 1, 2, \ldots$. Hence, for any two vector fields ξ and η of the form $\xi(\omega) = \sum_{l=1}^{n} f_l(\omega)\pi_\omega(A_i)\Theta_\omega$ and $\eta(\omega) = \sum_{k=1}^{m} g_k(\omega)\pi_\omega(A_k)\Theta_\omega$ respectively, the scalar valued function $(t, \omega) \mapsto \langle U_t(\omega)\xi(\omega), \eta(\omega)\rangle_\omega$, is jointly measurable, where the f_l's and g_k's are complex valued measurable functions on Ω. Next, we shall show that for all measurable vector fields ξ, η in $\mathcal{F}$, the function $(t,\omega) \mapsto \langle U_t(\omega)\xi(\omega), \eta(\omega)\rangle_\omega$ is jointly measurable. To this end, let $\{\xi_n\}$ and $\{\eta_n\}$ be sequences of vector fields of the form $\xi_n(\omega) = \sum_{i=1}^n f_i(\omega)\pi_\omega(A_i)\Theta_\omega$ and $\eta_n(\omega) = \sum_{j=1}^n g_j(\omega)\pi_\omega(A_j)\Theta_\omega$ respectively, converging to ξ and η almost everywhere, where the f_i's and g_j's are complex valued measurable functions on Ω.

Therefore, for almost every (t,ω) in $\mathbb{R} \times \Omega$,

$$\lim_{n\to\infty} \langle U_t(\omega)\xi_n(\omega), \eta_n(\omega)\rangle_\omega = \langle U_t(\omega)\xi(\omega), \eta(\omega)\rangle_\omega.$$

Since $(t,\omega) \mapsto \langle U_t(\omega)\xi_n(\omega), \eta_n(\omega)\rangle_\omega$ is jointly measurable in t and ω, for all $n \in \mathbb{Z}^+$, the measurability of $(t,\omega) \mapsto \langle U_t(\omega)\xi(\omega), \eta(\omega)\rangle_\omega$ follows easily. $\triangle$

Proposition 4.3.1.2 yields the following corollary.

Corollary 4.3.1.4 *1. For $\lambda \in \mathbb{R}$, if $E_\lambda(\omega)$ are the spectral projections of the generator $H(\omega)$ of $U_t(\omega)$, then $\omega \mapsto E_\lambda(\omega)$ is a measurable field of orthogonal projections.*

2. For each z in $\mathbb{C}$, $\omega \mapsto \langle (R(H(\omega), z))\xi(\omega), \eta(\omega)\rangle_\omega$, is a measurable field of resolvent operators, where $R(H(\omega), z)$ stands for the resolvent $(H(\omega) - zI)^{-1}$.

Proof Let ξ and η be two measurable vector fields in $\mathcal{F}$. Then (2) follows from the fact that, if $\Im z > 0$,

$$\langle (R(H(\omega), z))\xi(\omega), \eta(\omega)\rangle_\omega = i \int_0^\infty e^{izt} \langle e^{-iH(\omega)t}\xi(\omega), \eta(\omega)\rangle_\omega dt$$

and if $\Im z < 0$,

$$\langle (R(H(\omega), z))\xi(\omega), \eta(\omega)\rangle_\omega = -i \int_0^\infty e^{-izt} \langle e^{iH(\omega)t}\xi(\omega), \eta(\omega)\rangle_\omega dt,$$

where $H(\omega)$ is the generator of $U_t(\omega)$ and the integral is a Riemann integral. Now, (1) follows from (2) if we notice that

$$
\begin{aligned}
&\langle E_\lambda(\omega)\xi(\omega), \eta(\omega)\rangle_\omega \\
= &\lim_{\delta\to 0+}\lim_{\epsilon\to 0+}\frac{1}{2\pi i}\int_{-\infty}^{\lambda+\delta}\langle\left((t-i\epsilon-H(\omega))^{-1}-(t+i\epsilon-H(\omega))^{-1}\right)\xi(\omega), \eta(\omega)\rangle_\omega dt.
\end{aligned}
$$

△

It was established in corollary 4.2.3.2 that, if $E_\lambda(\omega)$ are the spectral projections associated with the generator $H(\omega)$ of $U_t(\omega)$, then, for all $a \in \mathbb{Z}^\nu$, we have $D_{-a}(E_\lambda(\omega))D_{-a}^{-1} = E_\lambda(T_{-a}\omega)$. Here $D_{-a} : \mathcal{H}_\omega \to \mathcal{H}_{T_{-a}\omega}$ is the isomorphism constructed in subsection 4.2.3. In view of this fact, we have the following proposition.

Proposition 4.3.1.5 *Let $E_\lambda(\omega)$ be the spectral projection associated with the generator $H(\omega)$ of $U_t(\omega)$. Then, $\dim(E_\lambda(\omega))$ is almost surely constant.*

Proof We know from proposition 1 in [Dix 81] (Chapter 1, Part II) that, there exists a measurable field of orthonormal bases $\{\xi_n\}$ in the collection of measurable vector fields $\mathcal{F}$. Hence,

$$\dim(E_\lambda(\omega)) = \sum_{i=1}^{\infty}\langle E_\lambda(\omega)\xi_i(\omega), \xi_i(\omega)\rangle_\omega.$$

Using the fact that $E_\lambda(T_{-a}\omega) = D_{-a}(E_\lambda(\omega))D_{-a}^{-1}$, it is evident that $\dim(E_\lambda(\omega))$ is an invariant function of ω. Besides, the measurability of the function $\omega \mapsto \dim(E_\lambda(\omega))$ follows from the corollary to proposition 4.3.1.2. Hence, by the ergodicity of the measure preserving group of automorphisms, this invariant measurable function is almost surely constant. △

The fact that the action of the measure preserving group of automorphisms is ergodic, allows us to derive some interesting results pertaining to the spectra of the generators $H(\omega)$ of the unitary groups $U_t(\omega)$. In this connection we have the following proposition.

Proposition 4.3.1.6 *The discrete and essential spectrum of the generator $H(\omega)$ of the unitary group $U_t(\omega)$, are almost surely independent of ω.*

Proof For each $\lambda \in I\!R$ the map $\omega \mapsto E_\lambda(\omega)$ is a measurable field of orthogonal projections associated with the generators $H(\omega)$ of $U_t(\omega)$. Now λ is in the essential spectrum $\sigma_{ess}(H(\omega))$ of $H(\omega)$, if and only if,

$$\dim(E_\nu(\omega) - E_\mu(\omega)) = \infty,$$

for $\mu < \lambda < \nu$. Clearly, the function $\omega \mapsto \dim(E_\nu(\omega) - E_\mu(\omega))$ is an invariant measurable function. So, by ergodicity of the action of the measure preserving group of automorphisms, it is almost surely independent of ω. Hence the essential spectrum $\sigma_{ess}(H(\omega))$ of $H(\omega)$ is almost surely independent of ω. Now, as the discrete spectrum $\sigma_{dis}(H(\omega))$ of $H(\omega)$ is such that $\sigma_{dis}(H(\omega)) = \sigma(H(\omega)) \cap \{\sigma_{ess}(H(\omega))\}^c$, it follows from proposition 4.2.3.3, that the discrete spectrum is also surely independent of ω. $\triangle$

4.3.2 Measurable Field of von Neumann Algebras.

We begin with the definition of a measurable field of von Neumann algebras.

Definition 4.3.2.1 *Let $\omega \mapsto \mathcal{H}'_\omega$ be a measurable field of complex Hilbert spaces over Ω, and for each $\omega \in \Omega$, $\mathcal{A}(\omega)$ be a von Neumann algebra acting* *on $\mathcal{H}'_\omega$. The field of von Neumann algebras, $\omega \mapsto \mathcal{A}(\omega)$ over Ω, is said to be measurable if, there exists a sequence $\omega \mapsto T_1(\omega), \omega \mapsto T_2(\omega), \ldots$ of measurable field of operators such that, $\mathcal{A}(\omega)$ is the von Neumann algebra generated by the $T_i(\omega)$'s almost everywhere.*

On appealing to proposition 4.3.1.1 we demonstrate the following fact.

Proposition 4.3.2.2 *The field of von Neumann algebras $\omega \mapsto \pi_\omega(\mathcal{A})''$ is a measurable field of von Neumann algebras.*

Proof Let $\mathcal{A}_0$ be defined as in proposition 4.3.1.1. It has been shown that $\omega \mapsto \mathcal{H}_\omega$ is a measurable field of Hilbert spaces on which the $\pi_\omega(\mathcal{A})''$'s act. Consider the sequence $\omega \mapsto \pi_\omega(A_1), \omega \mapsto \pi_\omega(A_2), \ldots$, of fields of operators on $\mathcal{H}_\omega$, where $A_i \in \mathcal{A}_0$. As observed earlier in proposition 4.3.1.1, the vector fields $\{x_i\}$, where $x_i(\omega) = \pi_\omega(A_i)\Theta_\omega$, form a fundamental sequence of measurable vector fields. Therefore, it follows from the measurability of $\omega \mapsto \langle \pi_\omega(A)\Theta_\omega, \Theta_\omega\rangle_\omega$ for all $A \in \mathcal{A}$, and proposition 1 (Part II, Chapter 2) in [Dix 81] that, $\omega \mapsto \pi_\omega(A_i)$ are measurable fields of operators with respect to $\mathcal{F}$. We have to show that for almost every $\omega \in \Omega$, the von Neumann algebra $\pi_\omega(\mathcal{A})''$ is generated by the $\pi_\omega(A_i)$'s. By definition, this amounts to showing that for almost every $\omega \in \Omega$, $\pi_\omega(\mathcal{A})''$ is the smallest von Neumann algebra containing $\pi_\omega(\mathcal{A}_o)$. i.e., showing that for almost every $\omega \in \Omega$, $\pi_\omega(\mathcal{A})''$ is the smallest von Neumann algebra containing $\pi_\omega(\mathcal{A}_0) \cup \pi_\omega(\mathcal{A}_0)^*$, since the von Neumann algebras containing $\pi_\omega(\mathcal{A}_0)$ are just those containing $\pi_\omega(\mathcal{A}_0) \cup \pi_\omega(\mathcal{A}_0)^*$. This is tantamount to showing that for almost every $\omega \in \Omega$, $\pi_\omega(\mathcal{A})''$ is the smallest von Neumann algebra containing the *–algebra $\mathcal{G}(\pi_\omega(\mathcal{A}_o))$ generated by $\pi_\omega(\mathcal{A}_0)$ (smallest *–algebra containing $\pi_\omega(\mathcal{A}_0)$). This follows from the fact that the smallest von Neumann algebra containing $\pi_\omega(\mathcal{A}_0) \cup \pi_\omega(\mathcal{A}_0)^*$ is precisely the smallest von Neumann algebra containing $\mathcal{G}(\pi_\omega(\mathcal{A}_0))$. Since $\mathcal{G}(\pi_\omega(\mathcal{A}_0))''$ is the smallest von Neumann algebra containing $\mathcal{G}(\pi_\omega(\mathcal{A}_0))$, it amounts to showing that,

$$\pi_\omega(\mathcal{A})'' = \mathcal{G}(\pi_\omega(\mathcal{A}_0))''.$$

Since $\{\pi_\omega(A)\Theta_\omega | A \in \mathcal{A}_0\}$ is dense in $\mathcal{H}_\omega$, for all $\omega \in \Omega$, it follows from theorem (10) in [Em 72] (Page 116) that,

$$\mathcal{G}(\pi_\omega(\mathcal{A}_0))'' = \overline{\mathcal{G}(\pi_\omega(\mathcal{A}_0))}^w,$$

where for any subset $\mathcal{N} \subseteq \mathcal{H}$, $\overline{\mathcal{N}}$ denotes the closure with respect to the operator norm topology and $\overline{\mathcal{N}}^w$ with respect to the weak operator topology. Next, $\pi_\omega(\mathcal{A}_0)$ is operator–norm dense in $\pi_\omega(\mathcal{A})$. Therefore,

$$\overline{\mathcal{G}(\pi_\omega(\mathcal{A}_0))} = \pi_\omega(\mathcal{A}).$$

Further,

$$\pi_\omega(\mathcal{A}) = \overline{\mathcal{G}(\pi_\omega(\mathcal{A}_0))} \subseteq \overline{\mathcal{G}(\pi_\omega(\mathcal{A}_0))}^w.$$

This implies that,

$$\overline{\pi_\omega(\mathcal{A})}^w \subseteq \overline{\mathcal{G}(\pi_\omega(\mathcal{A}_0))}^w.$$

Hence,

$$\overline{\pi_\omega(\mathcal{A})}^w = \overline{\mathcal{G}(\pi_\omega(\mathcal{A}_0))}^w = \mathcal{G}(\pi_\omega(\mathcal{A}_0))''.$$

Now, the von Neumann algebra $\pi_\omega(\mathcal{A})''$ is the weak closure of $\pi_\omega(\mathcal{A})$. Hence, we conclude that $\pi_\omega(\mathcal{A})''$ is the smallest von Neumann algebra containing $\mathcal{G}(\pi_\omega(\mathcal{A}_0))$, for almost every $\omega \in \Omega$. Thus, for almost every $\omega \in \Omega$, $\pi_\omega(\mathcal{A})''$ is generated by the measurable field of operators $\omega \mapsto \pi_\omega(A_i)$. This proves the proposition conclusively. $\triangle$

In proposition 4.3.2.2, we demonstrated that $\omega \mapsto \pi_\omega(\mathcal{A})''$ is a measurable field of von Neumann algebras. Since the quasi-local algebra $\mathcal{A}$ is simple, the cylic representation π_ω is a faithful representation of $\mathcal{A}$. Therefore, the measurable fields of operators $\omega \mapsto \pi_\omega(A_i)$ which generate $\pi_\omega(\mathcal{A})''$, are essentially bounded. Thus, they define a sequence of decomposable operators $\int_\Omega^\oplus \pi_\omega(A_i)dP(\omega)$, on the direct integral Hilbert space $\mathcal{H} = \int_\Omega^\oplus \mathcal{H}_\omega dP(\omega)$. Therefore, it follows from [Dix 81] ((i) in Prop 1, Chapter 3, Part II) that, the set $\mathcal{M}$ of all decomposable operators

$$T = \int_\Omega^\oplus T(\omega)dP(\omega),$$

where $T(\omega) \in \pi_\omega(\mathcal{A})''$ almost everywhere, is a von Neumann algebra, which by definition is a decomposable von Neumann algebra [Dix 81] (See Part II,Chapter 3, Definition 2), and denoted as

$$\mathcal{M} = \int_\Omega^\oplus \pi_\omega(\mathcal{A})''dP(\omega).$$

Thus, we have succeeded in constructing a direct integral von Neumann algebra $\mathcal{M}$, from the representations π_ω of the quasi-local algebra $\mathcal{A}$. This was achieved by putting a measurable structure on the Hilbert fields $\omega \mapsto \mathcal{H}_\omega$, using the cyclicity of the representations π_ω. This at the same time allowed the field of von Neumann algebras $\omega \mapsto \pi_\omega(\mathcal{A})''$ to acquire a measurable structure. The direct integral von Neumann algebra $\mathcal{M}$ constructed by us is generated by the set of all diagonalisable operators $\mathcal{N}$ and the countable family $\{\int_\Omega^\oplus \pi_\omega(A_i)dP(\omega)\}$ of decomposable operators.

4.3.3 Automorphism Group of the Direct Integral von Neumann Algebra

Next, we shall construct a σ–weakly continuous, one–parameter group of automorphisms $\tilde{\tau}_t$, of the direct integral von Neumann algebra $\mathcal{M}$. We first construct a strongly continuous, one–parameter group of unitaries U_t, on the direct integral Hilbert space $\mathcal{H}$. We know that there exists a strongly continuous, one–parameter group of unitaries $U_t(\omega)$ on the Hilbert space $\mathcal{H}_\omega$, which implements the evolution group $\tau_t(\omega)$. It has already been established in proposition 4.3.1.2 that, for each fixed $t \in \mathbb{R}$, $\omega \mapsto U_t(\omega)$ is a measurable field of unitary operators on $\mathcal{H}_\omega$. Clearly, the measurable field of unitary operators $\omega \mapsto U_t(\omega)$ is essentially bounded for each $t \in \mathbb{R}$. In view of this, the measurable field of unitary operators $\omega \mapsto U_t(\omega)$ defines a one–parameter family of decomposable operators $U_t = \int_\Omega^\oplus U_t(\omega)dP(\omega)$ on $\mathcal{H}$. Moreover, it has been demonstrated in proposition 4.3.1.3 that, for any two measurable vector fields in $\mathcal{F}$, the scalar valued function $(t,\omega) \mapsto \langle U_t(\omega)\xi(\omega), \eta(\omega)\rangle_\omega$ is jointly measurable in t and ω. In the discussion that follows, we demonstrate that for each $t \in \mathbb{R}$, the decomposable operator U_t is an unitary operator on the direct integral Hilbert space $\mathcal{H}$ and that, the one–parameter family of decomposable operators $\{U_t\}$, is indeed a strongly continuous one–parameter group of unitary operators on the direct integral Hilbert space $\mathcal{H} = \int_{\Omega}^{\oplus} \mathcal{H}_\omega$. To this end, we have the following proposition.

Proposition 4.3.3.1 *U_t is an unitary operator on the direct integral Hilbert space $\mathcal{H}$, of square integrable vector fields, for each $t \in \mathbb{R}$.*

Proof For square integrable vector fields ξ, $\eta \in \mathcal{F}$, we have

$$\langle U_t\xi, U_t\eta\rangle = \int_{\Omega} \langle U_t(\omega)\xi(\omega), U_t(\omega)\eta(\omega)\rangle_\omega dP(\omega),$$

where, $\langle .,. \rangle$, denotes the inner product on the direct integral Hilbert space $\mathcal{H}$, of all square integrable vector fields. Since $\omega \mapsto U_t(\omega)$ is a measurable field of unitary operators, it follows from proposition 1 (Chapter 2, Part II) in [Dix 81] that, $\omega \mapsto U_t(\omega)^*$ is also a measurable field of unitary operators. $U_t(\omega)$ being an unitary operator, we have

$$\langle U_t(\omega)U_t(\omega)^*\xi(\omega), \eta(\omega)\rangle_\omega = \langle \xi(\omega), \eta(\omega)\rangle_\omega,$$

and

$$\langle U_t(\omega)^*U_t(\omega)\xi(\omega), \eta(\omega)\rangle_\omega = \langle \xi(\omega), \eta(\omega)\rangle_\omega.$$

Now it follows from the properties of decomposable operators [Dix 81] (Proposition 3, Chapter 2, Part II) that,

$$U_t^* = \int_{\omega}^{\oplus} U_t(\omega)^* dP(\omega) \quad and \quad U_tU_t^* = \int_{\omega}^{\oplus} U_t(\omega)U_t(\omega)^* dP(\omega).$$

Therefore, for ξ, $\eta \in \mathcal{H}$, we have

$$\begin{aligned} \langle U_tU_t^*\xi, \eta\rangle &= \int_{\Omega} \langle U_t(\omega)U_t(\omega)^*\xi(\omega), \eta(\omega)\rangle_\omega dP(\omega) \\ &= \int_{\Omega} \langle \xi(\omega), \eta(\omega)\rangle_\omega dP(\omega) \\ &= \langle \xi, \eta\rangle. \end{aligned}$$

Similarly, since

$$U_t^* U_t = \int_\omega^\oplus U_t(\omega)^* U_t(\omega) dP(\omega),$$

we have

$$\begin{aligned}
\langle U_t^* U_t \xi, \eta \rangle &= \int_\Omega \langle U_t(\omega)^* U_t(\omega)\xi(\omega), \eta(\omega)\rangle_\omega dP(\omega) \\
&= \int_\Omega \langle \xi(\omega), \eta(\omega)\rangle_\omega dP(\omega) \\
&= \langle \xi, \eta \rangle.
\end{aligned}$$

The proposition now follows readily from the above equalities. △

We are now in a position to demonstrate the following.

Theorem 4.3.3.2 *The one–parameter family of unitary operators $\{U_t\}$ on the direct integral Hilbert space $\mathcal{H}$, is in fact a strongly continuous, one–parameter group of unitary operators on $\mathcal{H}$.*

Proof For $t_1, t_2 \in \mathbb{R}$, and $\xi, \eta \in \mathcal{H}$, we have

$$\langle U_{t_1+t_2}\xi, \eta \rangle = \int_\Omega \langle U_{t_1+t_2}(\omega)\xi(\omega), \eta(\omega)\rangle_\omega dP(\omega).$$

It follows from the properties of decomposable operators [Dix 81] (Proposition 3, Chapter 2, Part II) that,

$$U_{t_1} U_{t_2} = \int_\Omega^\oplus U_{t_1}(\omega) U_{t_2}(\omega) dP(\omega).$$

Therefore, since $U_t(\omega)$ is an unitary group, we have

$$\begin{aligned}
\langle U_{t_1+t_2}\xi, \eta \rangle &= \int_\Omega \langle U_{t_1}(\omega) U_{t_2}(\omega)\xi(\omega), \eta(\omega)\rangle_\omega dP(\omega) \\
&= \langle U_{t_1} U_{t_2} \xi, \eta \rangle.
\end{aligned}$$

This shows that the family of unitary operators $\{U_t\}$ on $\mathcal{H}$, is a one-parameter group of unitaries with identity $\int_\Omega^\oplus U_o(\omega)dP(\omega)$.

In order to establish that the one-parameter group of unitaries U_t on $\mathcal{H}$, is strongly continuous, it is enough to show that the function $t \mapsto \langle U_t\xi, \eta\rangle$, is continuous in t, for every square integrable vector field ξ, η in $\mathcal{F}$. Recall that for each $t \in I\!R$, $\omega \mapsto \langle U_t(\omega)\xi(\omega), \eta(\omega)\rangle_\omega$ has been shown to be a measurable field of unitary operators in proposition 4.3.1.2. We also have

$$|\langle U_t(\omega)\xi(\omega), \eta(\omega)\rangle_\omega| \leq \|\xi(\omega)\|\|\eta(\omega)\|,$$

and

$$\int_\Omega \|\xi(\omega)\|\|\eta(\omega)\|dP(\omega) \leq \left(\int_\Omega \|\xi(\omega)\|^2 dP(\omega)\right)^{\frac{1}{2}} \left(\int_\Omega \|\eta(\omega)\|^2 dP(\omega)\right)^{\frac{1}{2}} < \infty.$$

Hence, it follows from the dominated convergence theorem that, for all square integrable vector fields ξ, η in $\mathcal{F}$,

$$\begin{aligned}\lim_{t\to 0}\langle U_t\xi, \eta\rangle &= \lim_{t\to 0}\int_\Omega \langle U_t(\omega)\xi(\omega), \eta(\omega)\rangle_\omega dP(\omega)\\ &= \int_\Omega \lim_{t\to 0}\langle U_t(\omega)\xi(\omega), \eta(\omega)\rangle_\omega dP(\omega).\end{aligned}$$

Moreover, $U_t(\omega)$ is a strongly continuous, one-parameter group of unitary operators. Therefore,

$$\begin{aligned}\lim_{t\to 0}\langle U_t\xi, \eta\rangle &= \int_\Omega \langle \xi(\omega), \eta(\omega)\rangle_\omega dP(\omega)\\ &= \langle \xi, \eta\rangle.\end{aligned}$$

Since U_t has been endowed with a group structure, this proves the strong continuity of the one-parameter group of unitaries U_t, on the direct integral Hilbert space $\mathcal{H}$, conclusively. $\triangle$

Next, for

$$S = \int_{\Omega}^{\oplus} S(\omega) dP(\omega) \in \mathcal{M},$$

define

$$\tilde{\tau}_t(S) = U_t S U_t^{-1}.$$

Now, $\omega \mapsto U_t(\omega)S(\omega)U_t(\omega)^{-1}$ is an essentially bounded measurable field of operators for each $t \in I\!R$. It follows from the properties of decomposable operators that,

$$U_t S U_t^{-1} = \int_{\Omega}^{\oplus} U_t(\omega)S(\omega)U_t(\omega)^{-1} dP(\omega).$$

Since $U_t(\omega)S(\omega)U_t(\omega)^{-1} \in \pi_\omega(\mathcal{A})''$, we have $U_t S U_t^{-1} \in \mathcal{M}$. Thus, it follows from the strong continuity of U_t that $\tilde{\tau}_t$ is a σ–weakly continuous, one–parameter group of automorphisms of the decomposable von Neumann algebra $\mathcal{M}$.

4.3.4 Construction of a KMS State of the Direct Integral von Neumann Algebra

Finally, we establish the existence of a faithful, normal $(\tilde{\tau}, \beta)$–KMS state of $\mathcal{M}$. Now, the state $\rho(\omega)$ which can be written as a vector state $\rho(\omega)(A) = \langle \pi_\omega(A)\Theta_\omega, \Theta_\omega \rangle_\omega$ in the representation π_ω, on a separable Hilbert space $\mathcal{H}$, is a $(\tau(\omega), \beta)$–KMS state. Therefore, it follows from corollary 5.3.4, in [Rob 81] and theorem 4.12 in [Hug 72] that, $\rho(\omega)$ can be easily extended to a faithful, normal $(\tilde{\tau}(\omega), \beta)$–KMS state $\tilde{\rho}(\omega)$, of the von Neumann algebra $\pi_\omega(\mathcal{A})''$, where $\tilde{\rho}(\omega)(S) = \langle S\Theta_\omega, \Theta_\omega \rangle_\omega$, for $S \in \pi_\omega(\mathcal{A})''$ and $\tilde{\tau}_t(\omega)$ is the σ–weakly continuous group of automorphisms of $\pi_\omega(\mathcal{A})''$. Clearly, the restriction of $\tilde{\rho}(\omega)$

to $\pi_\omega(\mathcal{A})$ gives the state $\rho(\omega)$ on $\mathcal{A}$.

Let us now construct a state $\tilde{\rho}$ of the von Neumann algebra $\mathcal{M}$ from the field of states $\omega \mapsto \tilde{\rho}(\omega)$, on $\pi_\omega(\mathcal{A})''$. Such a field of states on the von Neumann algebras $\pi_\omega(\mathcal{A})''$ is said to be a measurable field if, $\omega \mapsto \tilde{\rho}(\omega)(T(\omega))$ is a measurable function of ω for every measurable field of operators $\omega \mapsto T(\omega)$. Since, $\omega \mapsto \Theta_\omega$ is a measurable vector field with respect to $\mathcal{F}$, it is clear from the definition of $\tilde{\rho}(\omega)$ that, $\omega \mapsto \tilde{\rho}(\omega)$ is a measurable field of states on $\pi_\omega(\mathcal{A})''$. Define

$$\tilde{\rho}\left(\int_\Omega^\oplus S(\omega)dP(\omega)\right) = \int_\Omega \tilde{\rho}(\omega)(S(\omega))dP(\omega),$$

for all decomposable operators $\omega \mapsto S(\omega)$ in $\mathcal{M}$. Let $\alpha \in \mathbb{C}$, and $\omega \mapsto S(\omega)$, $\omega \mapsto S_1(\omega)$, $\omega \mapsto S_2(\omega)$ define elements in $\mathcal{M}$. It follows from the properties of decomposable operators (Proposition 3, Page 182) in [Dix 81]) that,

$$\begin{aligned}
&\tilde{\rho}\left(\int_\Omega^\oplus S_1(\omega)dP(\omega) + \int_\Omega^\oplus S_2(\omega)dP(\omega)\right) \\
&= \tilde{\rho}\left(\int_\Omega^\oplus (S_1(\omega) + S_2(\omega))dP(\omega)\right) \\
&= \int_\Omega \tilde{\rho}(\omega)\left(S_1(\omega) + S_2(\omega)\right)dP(\omega) \\
&= \int_\Omega \tilde{\rho}(\omega)\left(S_1(\omega)\right)dP(\omega) + \int_\Omega \tilde{\rho}(\omega)\left(S_2(\omega)\right)dP(\omega) \\
&= \tilde{\rho}\left(\int_\Omega^\oplus S_1(\omega)dP(\omega)\right) + \tilde{\rho}\left(\int_\Omega^\oplus S_2(\omega)dP(\omega)\right)
\end{aligned}$$

Also,

$$\begin{aligned}
\tilde{\rho}\left(\alpha\int_\Omega^\oplus S(\omega)dP(\omega)\right) &= \int_\Omega \tilde{\rho}(\omega)\left(\alpha S(\omega)\right)dP(\omega) \\
&= \alpha\int_\Omega \tilde{\rho}(\omega)S(\omega)dP(\omega) \\
&= \alpha\tilde{\rho}\left(\int_\Omega^\oplus S(\omega)dP(\omega)\right)
\end{aligned}$$

Hence, $\tilde{\rho}$ is a linear functional on $\mathcal{M}$. Since,

$$\tilde{\rho}\left(\int_\Omega^\oplus I_\omega dP(\omega)\right) = \int_\Omega \tilde{\rho}(\omega)\,(I_\omega)\,dP(\omega)$$
$$= 1,$$

where $I(\omega)$ is the identity operator on $\mathcal{H}_\omega$, $\tilde{\rho}$ is a state.

Theorem 4.3.4.1 *Let $\tilde{\rho}$ be the state constructed above. Then $\tilde{\rho}$ is a faithful, normal state of the decomposable von Neumann algebra $\mathcal{M}$.*

Proof Let $\omega \mapsto S(\omega)$ define a decomposable operator in $\mathcal{M}^+$, where $\mathcal{M}^+$ is the set of all positive elements in $\mathcal{M}$. Put $S = \int_\Omega^\oplus S(\omega)dP(\omega)$. Suppose we have $\tilde{\rho}(S) = 0$, then it follows from the definition of $\tilde{\rho}$ that $\int_\Omega \tilde{\rho}(\omega)\,(S(\omega))\,dP(\omega) = 0$. Since $\omega \mapsto \tilde{\rho}(S(\omega))$ is a non negative measurable function of ω, we have $\tilde{\rho}(\omega)\,(S(\omega)) = 0$, almost everywhere. Therefore, $S(\omega) = 0$ almost everywhere, since the $\tilde{\rho}(\omega)$'s are faithful. Next, we show that the state $\tilde{\rho}$ is a normal state on $\mathcal{M}$. We know that the $\tilde{\rho}(\omega)$'s are normal states. Let $\{S_\lambda\}$ be an increasing net of elements in $\mathcal{M}^+$ with supremum $S \in \mathcal{M}^+$. Let us denote the collection of all diagonalisable operators on $\mathcal{H}$ by $\mathcal{Z}$. Clearly, $\mathcal{Z} \subseteq \mathcal{M} \subseteq \mathcal{Z}'$. Since $\mathcal{Z}'$ is a σ-finite von Neumann algebra [Dix 81] (See Proposition 7, Chapter 2, Part II) of all decomposable operators on $\mathcal{H}$, it follows that $\mathcal{M}$ is also σ-finite. Therefore, it follows from the corollary to proposition 1 in [Dix 81] (Part I, Chapter 3) that, one can extract an increasing sequence $S_n = \int_\Omega^\oplus S_n(\omega)dP(\omega)$ from the net $\{S_\lambda\}$ with supremum S. Now, there exists an increasing sequence of integers $\{n_k\}$ such that, $S_{n_k}(\omega)$ is an increasing sequence converging strongly to $S(\omega)$ almost everywhere. This is a trivial consequence of proposition 4, in [Dix 81] (Chapter 2, Part II). Hence, $\tilde{\rho}(\omega)(S_{n_k}(\omega))$ is an increasing sequence converging to $\tilde{\rho}(\omega)(S(\omega))$ almost everywhere. Therefore, it follows from the definition of $\tilde{\rho}$ and the monotone convergence theorem that, $\tilde{\rho}(S_{n_k})$ is an increasing sequence converging to $\tilde{\rho}(S)$. This proves conclusively that $\tilde{\rho}$ is a normal state on $\mathcal{M}$. $\triangle$

Now all that remains to be shown is that the state $\tilde{\rho}$ is $(\tilde{\tau},\beta)$–KMS state. Before, we establish this fact, let us give an equivalent definition of a KMS state which we shall have the occasion to use.

Definition 4.3.4.2 *Let $\tilde{\rho}$ be a state on a von Neumann algebra $\mathcal{R}$ and $\tilde{\tau}_t$ a σ–weakly continuous, one–parameter group of automorphisms of the von Neumann algebra $\mathcal{R}$. Then, $\tilde{\rho}$ is said to be a $(\tilde{\tau},\beta)$–KMS state if,*

$$\int_{-\infty}^{\infty} f_{-\beta}(t)\tilde{\rho}(A\tilde{\tau}_t B)dt = \int_{-\infty}^{\infty} f_o(t)\tilde{\rho}(\tilde{\tau}_t(B)A)dt,$$

for all $A, B \in \mathcal{R}$ and all $\hat{f}$ infinitely differentiable with compact support. In the above equality, $f_\gamma(t) = \int_{-\infty}^{\infty} \hat{f}(s)e^{(t+i\gamma)}ds$, for $\gamma = 0$ and $-\beta$.

Theorem 4.3.4.3 *The state $\tilde{\rho}$ constructed above is a $(\tilde{\tau},\beta)$–KMS state of the direct integral von Neumann algebra $\mathcal{M}$.*

Proof For $S = \int_\Omega^\oplus S(\omega)dP(\omega)$, $T = \int_\Omega^\oplus T(\omega)dP(\omega) \in \mathcal{M}$,

$$\begin{aligned}
& \int_{-\infty}^{\infty} f_{-\beta}(t)\tilde{\rho}(T\tilde{\tau}_t(S))dt \\
= & \int_{-\infty}^{\infty} f_{-\beta}(t)\tilde{\rho}(T(U_t S U_t^{-1}))dt \\
= & \int_{-\infty}^{\infty} f_{-\beta}(t)\left(\int_\Omega \tilde{\rho}(\omega)(T(\omega)(U_t(\omega)S(\omega)U_t(\omega)^{-1}))dP(\omega)\right)dt
\end{aligned}$$

$$= \int_{-\infty}^{\infty} f_{-\beta}(t) \left(\int_{\Omega} \langle T(\omega)(U_t(\omega)S(\omega)U_t(\omega)^{-1})\Theta_\omega, \Theta_\omega \rangle_\omega dP(\omega) \right) dt.$$

It follows readily from the measurable structure imposed on the vector field of Hilbert spaces that, $\omega \mapsto \Theta_\omega$ is a measurable field of vectors. Moreover, both $\omega \mapsto S(\omega)$ and $\omega \mapsto T(\omega)$ are essentially bounded measurable fields of operators. Therefore, it follows from the remark made at the end of proposition 1 in [Dix 81] (Part II, Chapter 2) and the definition of measurable fields of operators that, both $\omega \mapsto S(\omega)\Theta_\omega$ and $\omega \mapsto T(\omega)^*\Theta_\omega$ are measurable vector fields. Since

$$\langle T(\omega)(U_t(\omega)S(\omega)U_t(\omega)^{-1})\Theta_\omega, \Theta_\omega \rangle_\omega = \langle U_t(\omega)(S(\omega)\Theta_\omega), T(\omega)^*\Theta_\omega \rangle_\omega,$$

it follows from proposition 4.3.1.3 that,

$$(t, \omega) \mapsto \langle T(\omega)(U_t(\omega)S(\omega)U_t(\omega)^{-1})\Theta_\omega, \Theta_\omega \rangle_\omega,$$

is jointly measurable in t and ω. By the Paley–Weiner theorem, $f_{-\beta}(t)$ is an integrable function of t. Moreover, the $S(\omega)$'s, and $T(\omega)$'s are essentially bounded in norm. Therefore, invoking Fubini's theorem for scalar valued functions on $\mathbb{R} \times \Omega$ and using the fact that $\tilde{\rho}(\omega)$ is a $(\tilde{\tau}(\omega), \beta)$–KMS state on $\pi_\omega(\mathcal{A})''$, where $\tilde{\tau}_t(\omega)(A) = U_t(\omega)AU_t(\omega)^{-1}$, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} f_{-\beta}(t)\tilde{\rho}(T\tilde{\tau}_t(S)) \\
&= \int_{\Omega} \left(\int_{-\infty}^{\infty} f_{-\beta}(t) \langle T(\omega)(U_t(\omega)S(\omega)U_t(\omega)^{-1})\Theta_\omega, \Theta_\omega \rangle_\omega dt \right) dP(\omega) \\
&= \int_{\Omega} \left(\int_{-\infty}^{\infty} f_{-\beta}(t)\tilde{\rho}(\omega)(T(\omega)\tilde{\tau}_t(\omega)(S(\omega)))dt \right) dP(\omega) \\
&= \int_{\Omega} \left(\int_{-\infty}^{\infty} f_o(t)\tilde{\rho}(\omega)(\tilde{\tau}_t(\omega)(S(\omega))T(\omega))dt \right) dP(\omega) \\
&= \int_{\Omega} \left(\int_{-\infty}^{\infty} f_o(t) \langle (U_t(\omega)S(\omega)U_t(\omega)^{-1})T(\omega)\Theta_\omega, \Theta_\omega \rangle_\omega dt \right) dP(\omega).
\end{aligned}$$

Arguing as above, one can show that both $\omega \mapsto T(\omega)\Theta_\omega$ and $\omega \mapsto S(\omega)^*\Theta_\omega$ are measurable vector fields. Since

$$\langle (U_t(\omega)S(\omega)U_t(\omega)^{-1})T(\omega)\Theta_\omega, \Theta_\omega\rangle_\omega = \langle U_t(\omega)^{-1}(T(\omega)\Theta_\omega), S(\omega)^*\Theta_\omega\rangle_\omega,$$

it follows from proposition 4.3.1.3 that,

$$(t,\omega) \mapsto \langle U_t(\omega)S(\omega)U_t(\omega)^{-1}T(\omega)\Theta_\omega, \Theta_\omega\rangle_\omega,$$

is jointly measurable. Again, by the Paley–Weiner theorem, $f_0(t)$ is an integrable function of t. Hence, on applying Fubini's theorem a second time, we get

$$\begin{aligned}
& \int_{-\infty}^{\infty} f_{-\beta}(t)\tilde{\rho}(T\tilde{\tau}_t(S)) \\
= & \int_{-\infty}^{\infty} f_o(t)\left(\int_\Omega \langle (U_t(\omega)S(\omega)U_t(\omega)^{-1})T(\omega)\Theta_\omega, \Theta_\omega\rangle_\omega dP(\omega)\right) dt \\
= & \int_{-\infty}^{\infty} f_o(t)\left(\int_\Omega \tilde{\rho}(\omega)((U_t(\omega)S(\omega)U_t(\omega)^{-1})T(\omega))dP(\omega)\right) dt \\
= & \int_{-\infty}^{\infty} f_o(t)\tilde{\rho}((U_tSU_t^{-1})T)dt \\
= & \int_{-\infty}^{\infty} f_o(t)\tilde{\rho}(\tilde{\tau}_t(S)T)dt
\end{aligned}$$

This proves conclusively that, $\tilde{\rho}$ is a $(\tilde{\tau},\beta)$–KMS state on the direct integral von Neumann algebra $\mathcal{M}$. △

Since the family of KMS states $\{\rho(\omega)\}$ of $\mathcal{A}$ is not unique, the $(\tilde{\tau},\beta)$–KMS state $\tilde{\rho}$ is by no means unique. However, in view of theorem 4.2.2.2, above the critical temperature T_c, there is an unique family of KMS states $\{\rho(\omega)\}$, which determines the KMS state $\tilde{\rho}$ on $\mathcal{M}$ as constructed above.

Chapter 5

Summary

This final chapter of the thesis is devoted to a discussion on the results obtained in chapters (3) and (4) and their implications. Some of the open problems which remain unresolved are identified. Before proceeding any further it is worth recalling the aims and objectives of the work undertaken here.

The purpose of this investigation was to understand and explain the behaviour of a quantum spin glass through its dynamics. Spin glasses have always been something of a mystery. They are among the least understood systems even in equilibrium statistical mechanics. In particular, their low temperature regime and critical behaviour are extremely complex.

Traditionally, quantum spin glasses have been studied as systems of quantum spins interacting through random interactions. These models are essentially Ising-type models with random coupling. The coupling coefficients are assumed to be independent, identically distributed random variables. Extensive investigations on the existence of the thermodynamic limit have been made by van Hemmen et al [Ent 83, Hem 83]. The almost sure existence of the free energy of an infinite spin system on a lattice with random interactions has been established. This is a generalization of the result of Khanin and Sinai [Sin 79] in the classical case. An alternate model of a quantum spin glass can be based on the realization that, the magnetic ions in a spin glass are randomly distributed at lattice sites. The spins therefore may be considered to be located at the vertices of an infinite connected graph with countably infinite number of vertices. Here, one caricatures a quantum spin glass as a quantum spin system on an infinite connected graph with countably infinite number of vertices. This model may be regarded as a quantum analogue of the systems studied by Preston and others [Pre 74]. But, inspite of the fact that a quantum spin glass admits a natural dynamics, this aspect has not been investigated.

In this thesis, we have attempted the study of the dynamics of a quantum spin glass with the help of both these models, namely, a quantum spin system on an infinite connected graph having countably infinite number of vertices with deterministic interactions of the nearest neighbour type and a quantum spin system on an infinite lattice $\mathbb{Z}^\nu$ with random interactions. The problem to which we have addressed ourselves is that of explaining the behaviour of a quantum spin glass through the dynamics of these spin systems and the associated KMS states.

In the case of the quantum spin system on an infinite graph, the global dynamics has been established. This was achieved by constructing a strongly continuous, one-parameter group of *-automorphisms τ_t of the quasi-local algebra $\mathcal{A}$ associated with the spin system. As expected, the existence of an equilibrium state which is by no means unique, has been established. The equilibrium state ρ was obtained as the thermodynamic limit of the local Gibbs states ρ_Λ. It was also shown that ρ satisfies the Kubo–Martin–Schwinger(KMS) condition with respect to the time evolution group τ_t.

However, all attempts to establish the maximum entropy principle for the infinite spin system were thwarted due to the absence of spatial homogeneity. In fact, one failed to establish the existence of mean entropy and free energy for the infinite system. The problem of establishing the existence of mean entropy and free energy for the infinite system as well as that of establishing the maximum entropy principle remains open.

The other model studied was a quantum spin system on an infinite lattice $\mathbb{Z}^\nu$, with random interactions. Here we have established the existence of a family of strongly continuous, one-parameter groups of *–automorphisms $\{\tau_t(\omega)\}$ of the quasi-local algebra $\mathcal{A}$ associated with the spin system, where ω lives in a probability space $(\Omega, \mathcal{S}, P)$. These automorphism groups $\tau_t(\omega)$ determine the evolution of the infinite spin system. The joint measurability of the map $(t,\omega) \mapsto \tau_t(\omega)(A)$ for all $A \in \mathcal{A}$, has been proved. Some interesting algebraic properties of the generator $\overline{\delta}(\omega)$ of these automorphism groups have been derived. The notion of ergodicity of a measure preserving group of automorphisms of Ω, is used to prove the almost sure independence of the Arveson spectrum $Sp(\tau(\omega))$ of the evolution group $\tau_t(\omega)$. Next, the existence of a family of $(\tau(\omega),\beta)$–KMS states $\{\rho(\omega)\}$ has been established for all $\beta \in \mathbb{R}\backslash\{0\}$. They have been shown to satisfy the condition $\rho(\omega)(A) = \rho(T_{-a}\omega)(\alpha_a(A))$ for all $A \in \mathcal{A}$ and all $a \in \mathbb{Z}^\nu$, where α_a is the action of the lattice $\mathbb{Z}^\nu$ on the quasi–local algebra $\mathcal{A}$. We assume that there exists one such family of $(\tau(\omega), \beta)$–KMS states $\{\rho(\omega)\}$, where $\omega \mapsto \rho(\omega)(A)$ is measurable for all $A \in \mathcal{A}$. It has been shown that the spin system on an infinite lattice with random interactions exhibits a phase structure. In fact, it has been established that there exists an unique KMS state $\rho(\omega)$, above a certain a critical temperature T_c almost surely independent of ω. There is a close connection between the Arveson spectrum of $\tau_t(\omega)$, and the spectrum of the generator of the unitary group $U_t(\omega)$ which implements $\tau_t(\omega)$ in the cyclic representation π_ω induced by the $(\tau(\omega), \beta)$–KMS state $\rho(\omega)$. This fact has been exploited to prove the almost sure independence of the spectrum of the generator of $U_t(\omega)$.

Now, the cyclic representations π_ω induced by the $(\tau(\omega), \beta)$–KMS states $\rho(\omega)$, which satisfy the conditions mentioned above gives rise to an ensemble of von Neumann algebras $\{\pi_\omega(\mathcal{A})''\}$, where each of these von Neumann algebras acts on a separable Hilbert space $\mathcal{H}_\omega$. As these von Neumann algebras correspond to distinct realizations of the quasi–local algebra $\mathcal{A}$, they are treated as distinct objects. This establishes a need to invoke the theory of measurable fields of von Neumann algebras. Using the cyclicity of π_ω, we have constructed a collection of measurable vector fields $\mathcal{F}$, which endows the field of separable Hilbert spaces $\omega \mapsto \mathcal{H}_\omega$ with a measurable structure. Equipped with this structure, we have shown that for each $t \in \mathbb{R}$, $\omega \mapsto U_t(\omega)$

is a measurable field of unitary operators. Further, the joint measurability of $(t,\omega) \mapsto \langle U_t(\omega), \xi(\omega), \eta(\omega)\rangle_\omega$ for all ξ, $\eta \in \mathcal{F}$ is established. We have also derived some interesting ergodic properties of the spectra of generators $H(\omega)$ of the unitary groups $U_t(\omega)$.

In the final part of the thesis we have constructed a direct integral $\mathcal{M}$ from the measurable field of von Neumann algebras $\omega \mapsto \pi_\omega(\mathcal{A})''$. The existence of a strongly continuous, one–parameter group of unitaries U_t on the direct integral Hilbert space $\mathcal{H}$ constructed from the measurable field of Hilbert spaces $\omega \mapsto \mathcal{H}_\omega$, has been established. This group of unitaries in turn gives rise to a σ–weakly continuous group of automorphisms $\tilde{\tau}_t$ of $\mathcal{M}$. From the measurable field of KMS states $\omega \mapsto \tilde{\rho}(\omega)$, which are extensions of the KMS states $\rho(\omega)$ to the von Neumann algebras $\{\pi_\omega(\mathcal{A})''\}$, a faithful normal $(\tilde{\tau}, \beta)$–KMS state $\tilde{\rho}$ of $\mathcal{M}$ has been constructed.

The problem that remains to be resolved in this particular model is that of establishing that the transport coefficients of the spin system are almost surely constant. One would expect this to be generally true on physical grounds.

The other problem that remains open is that of establishing a connection between the spectra of the generator of $U_t(\omega)$ and that of the generator of the unitary group U_t on the direct integral Hilbert space $\mathcal{H}$.

Bibliography

[And 75] Edwards E. F. and Anderson P. W. *J. Phys.*, Vol **F5**, Page 965, 1975.

[Bin 86] Binder K. and Young A. P. *Spin glasses:Experimental facts, theoretical concepts and open questions*, Reviews of Modern Physics, Vol 58, Page 801, 1986.

[Bir 87] Birmann M. S. and Solomjak M. Z. *Spectral Theory of Self–Adjoint Operators in Hilbert Space*, D. Reidel Publishing Company, Boston, 1987.

[Bra 76] Bratteli O. *Self–Adjointness of Unbounded Derivations on C^*–Algebras*, Instituto Nazionale di Alta Matematica, Symposia Matematica, Vol 20, 1976.

[Bra 80] Bratteli O. and Kishimoto A. *Generation of Semi–groups and Two Dimensional quantum lattice systems*, Journal of Functional Analysis, Vol 35, Pages 344–368, 1980.

[Bri 77] Brinke G. *On K.M.S Evolution and Liouville Operators in Quantum Statistical Mechanics*, PhD Thesis, University of Gronigen, 1977.

[Bro 77] Brown A. and Pearcy C. *Introduction to Operator Theory–Elements of Functional Analysis*, Springer–Verlag, New York, 1977.

[Car 91] Carlson J. M. , Chayes J. T. ,Chayes L. , Sethna J. P. and Thouless D. J. *J. Stat. Phys.*, Vol 61, Page 1069, 1991.

[Cha 86] Chayes J. T. ,Chayes L. , Sethna J. P. and Thouless D. J. *Comm. Math. Phys.*, Vol 106, Page 41, 1986.

[Cho 66] Choquet G. *Topology*, Translated by Amiel Feinstein, Academic Press, New York, 1966.

[Con 78] Conway J. B. *Functions of One Complex Variable*, Springer International Edition, Narosa Publishing House, New–Delhi, 1978.

[Dix 81] Dixmier J. *Von Neumann Algebras*, North–Holland Publishing Company, Amsterdam, 1981.

[Dun 57] Dunford N. and Schwartz J. T. *Linear Operators*, Part I, *General Theory*, Interscience Publishers, Inc, New York, 1957.

[Dun 63] Dunford N. and Šchwartz J. T. *Linear Operators*, Part II, *Spectral Theory*, Interscience Publishers, A Division of John Wiley and Sons, New York, 1963.

[Em 72] Emch G. G. *Algebraic Methods in Statistical Mechanics and Quantum Field Theory*, Interscience Monographs and Texts in Physics and Astronomy, Edited by Marshak R. E. Wiley Interscience, New York, 1972.

[Emc 72] Emch G. G. *The $C^\star$–Algebraic Approach to Phase Transitions* in *Phase transitions and Critical Phenomena*, Vol 1, Edited by Domb C. and Green M. S. Academic Press, New York, 1972.

[Em 84] Emch G. G. *Mathematical and Conceptual Foundations of 20^{th} Century Physics*, North–Holland Mathematics Studies, Notas de Mathematica (100), New York, 1984.

[Ent 83] van Enter A. C. D. and van Hemmen J. L. , Journal of Statistical Physics, Vol 32, Page 141, 1983.

[Ent 84] van Enter A. C. D. and van Hemmen J. L. , *Phys. Rev.*, Vol **A29**, Page 355, 1984.

[Hal 87] Halmos P. R. *Measure Theory*, Springer International Student Edition, Narosa Publishing House, New–Delhi, 1987.

[Hem 83] van Hemmen J. L. in *Heidelberg Colloquium on Spin Glasses* Edited by van Hemmen J.L. and Morgestern I. Lecture Notes in Physics, Vol 192, Springer–Verlag, 1983.

[Hil 72] Hille E. *Methods in Classical and Functional Analysis*, Addison–Wesley Publishing Company, Reading, Massachusetts, 1972.

[Hil 57] Hille E. and Phillips R. *Functional Analysis and Semi-Groups*, American Mathematical Society Colloquium Publications, Vol 31, Revised Edition, Providence, Rhode–Island, 1957.

[Hug 72] Hugenholtz N. M. *States and Representations in Statistical Mechanics*, in *Mathematics of Contemporary Physics*, Edited by Streater R. F. Academic Press, Inc, New York, 1972.

[Kad 83] Kadison R. V. and Ringrose J. R. *Fundamentals of the Theory of Operator Algebras*, Vol 1, Academic Press, Inc, New York, 1983.

[Kad 86] Kadison R. V. and Ringrose J. R. *Fundamentals of the Theory of Operator Algebras*, Vol 2, Academic Press, Inc, New York, 1986.

[Kat 76] Kato T. *Perturbation Theory for Linear Operators*, Springer–Verlag, New York, 1976.

[Kir 82] Kirsch W. and Martinelli F. *On the Ergodic Properties of the Spectrum of General Random Operators*, J. Reine Angew. Math, Pages 141–156, Vol 334, New York 1982.

[Lan 70] Lanford O. E. *Quantum Spin Systems*, in *Cargèse Lectures in Physics*, Vol 4, Edited by Kastler D. Gordon and Breach, Science Publishers, Inc, New York, 1970.

[Mez 84] Mézard M. , Parisi G. , Sourlas N. , Toulouse G. and Virasoro M. *Phys. Rev. Lett.*, Vol 52, Page 1156, 1984; *J. Phys.* Vol 45, Page 843, 1984.

[Mun 92] Munkres J. R. *Topology–A First Course*, Prentice Hall of India, Private Limited, New–Delhi, 1992.

[Nad 95] Nadkarni M. G. *Ergodic Theory*, Hindustan Book Agency, Delhi, 1995.

[Par 80] Parisi G. *J. Phys.*, Vol **A13**: Page 1101. 1980; Vol **A13**. Page 1887, 1980; *Phys. Rev. Lett.*. Vol 50, Page 1946, 1983.

[Ped 79] Pedersen G. *C^*–Algebras and Their Automorphism Groups*, Academic Press, New York, 1979.

[Pre 74] Preston J. C. *Gibbs Measures on Countable Sets*, Cambridge University Press, Cambridge, 1974.

[Rob 76] Bratteli O. and Robinson D. W. *Unbounded Derivations of C^*-Algebras 2*, Communications in Mathematical Physics, Vol 46, Pages 11–30, 1976.

[Rob 87] Bratteli O. and Robinson D. W. *Operator Algebras and Quantum Statistical Mechanics*, Vol 1, Springer–Verlag, New York, 1987.

[Rob 81] Bratteli O. and Robinson D. W. *Operator Algebras and Quantum Statistical Mechanics*, Vol 2, Springer–Verlag, New York, 1981.

[Roy 88] Royden H. L. *Real Analysis*, Macmillan Publishing Company, New York, 1988.

[Rud 87] Rudin W. *Real and Complex Analysis*, McGraw–Hill Book Company, New York, 1987.

[Rue 69] Ruelle D. *Statistical Mechanics–Rigorous Results*, W. A. Benjamin, Inc, New York, 1969.

[Sak 91] Sakai S. *Operator Algebras in Dynamical Systems– The Theory of Unbounded Derivations in C^*–Algebras*, Cambridge University Press, Cambridge, 1991.

[Sew 86] Sewell G. L. *Quantum Theory of Collective Phenomena* Clarendon Press, Oxford, 1986.

[She 78] Sherrington P. and Kirkpatrick S. *Phys. Rev.*, Vol **B17**, Page 4384, 1978.

[Sim 80] Simon B. and Reed M. *Methods of Modern Mathematical Physics*, I: Functional Analysis, Academic Press, New York, 1980.

[Sin 79] Khanin K. M. and Sinai Ya. G. *J. Stat. Phys.*, Vol 20, Page 573, 1979.

[Sto 74] Stormer E. *Spectra of Ergodic Transformations*, Journal of Functional Analysis, Vol 15, Pages 202–214, 1974.

[Thi 80] Thirring W. *A course in Mathematical Physics–Quantum Mechanics of Large Systems*, Vol 4, Springer–Verlag, New York, 1980.

[Tho 86] Thouless D. J. *Phys. Rev. Lett.* Vol 56, Page 1082, 1986.

[Tit 91] Titchmarsh E. C. *The Theory of Functions*, Oxford University Press, Delhi, 1991.

[Top 71] Topping M. D. *Lectures on Von Neumann Algebras*, Van Nostrand Reinhold Mathematical Studies, vol 36, London, 1971.

[Wei 80] Weidmann J. *Linear Operators in Hilbert Spaces*, Springer–Verlag, New York, 1980.

[Weis 93] Weismann M. B. *What is a Spin glass? A glimpse via mesoscopic noise*, Reviews of Modern Physics, Vol 65. Pages 829–840, 1993.

[Win 70] Winnik M. *Aspects of the Kubo–Martin–Schwinger Condition*, in *Cargèse Lectures in Physics*, Vol 4, Edited by Kastler D. Gordon and Breach, Science Publishers, Inc. New York, 1970.