STUDIES IN NONLINEAR DIFFERENTIAL EQUATIONS

THESIS SUBMITTED FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN THE FACULTY OF NATURAL SCIENCES

GOA UNIVERSITY

Recommended for the award of
Ph. D. degree.

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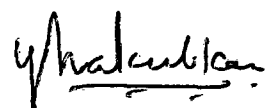
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September 2000

Dedicated to
all my Mathematics TEACHERS,
who inspired me to take

Mathematics as a profession

DECLARATION

I declare that this thesis has been composed by me and has not not been submitted to any other University or Institution for the award of any Degree, Diploma, or any other similar title.
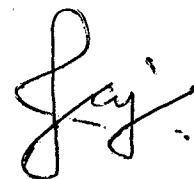
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CERTIFICATE

Certified that the thesis entitled, " **Studies in Nonlinear differential equations** " by Yeshwant S. Valaulikar has been carried out by the candidate under self guidance and above declaration of the candidate is true to the best of my knowledge.

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ACKNOWLEDGEMENT

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Chapter 1

INTRODUCTION

The study of differential Equations is a rich area of mathematical research. It has played a major role in the development of mathematics and its applications for nearly five centuries, especially for the advancement of Physics and Engineering. From the 19th century onwards differential equations have played a significant role in Biology, Ecology and Economics. The real world problems are analysed mathematically with the help of models which are often differential equations. According to M.W.Hirsch and S.Smale, " The importance of ordinary differential equations vis-a-vis other areas of sciences lies in its power to motivate, unify, and give force to these areas."

Linear Differential Equations have been studied in great detail. The existence and uniqueness of solution and properties like boundedness, periodicity, oscillations, non-oscillations, stability, etc. have been already studied in respect of linear differential equations [13, 20]. However, very little is known about the non-linear world of differential equations. The world of non- linear differential equations is very wide and needs to be explored in great detail. One of the reason why the need is felt is that these equations represents several complex physical phenomena. The growth of science in the new millennium could well depend upon the success in resolving and developing the methods to study qualitative properties of non-linear problems. The industrial recognition for nonlinear mathematical models, chosen to solve problems of technology, is growing. It is clear that the attempts to understand the

nonlinear world will dominate a large parts of mathematical research in the years to come.

In many application, it is assumed that the future state of the system under consideration is independent of the past state and determined solely on the present. But now it is known that this assumption leads to first approximation of the true situations. For a better approximation one has to consider the past history of the system. This has given rise to what are called delay differential equations or the differential equations with deviating arguments. In general, these are known as functional differential equations. The general theory and basic results for functional differential equations have by now been thoroughly explored and are available in the form books [9, 25, 26, 28, 33]. Nevertheless, there is a still a need for investigation of special delay equations.

Problem under study

Recently, there has been interest in the study of linear differential equations involving piecewise constant delays. Such equations were first introduced by A.D.Myshkis [56]. The study of differential equations with piecewise constant delays was initiated by the work of Busenburg, Cooke and Wiener. It is seen that equations with piecewise constant delays are closely related to impulse and loaded equations and especially to difference equations with discrete arguments. It has been used to study controllability of discrete systems. Some basic tools such as variation of parameters formula, Gronwall type inequalities, needed in the study of qualitative properties of the solutions of equations involving piecewise constant delays have been already studied.

The aim of this work is to study the nonlinear differential equation with piecewise constant deviating argument,

$$x'(t) = f(t, x(t), x([t])), \quad x(t_0) = x_0,$$

with reference to the following problems.

(A) To develop some basic tools required for further study.

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(B) To use method of quasilinearisation to obtain the solution.

(C) To study the controllability.

(D) To study Boundary value problems.

(E) To study the second order equation

$$x''(t) = f(t, x(t), x([t])), x(t_0) = c_0; x'(t_0) = d_0.$$

The method of quasilinearisation was first propounded by Bellman and Kalaba [8, 10]. They showed that the method is an effective tool in the study of non-linear ordinary differential equations. Subsequently, there were several research papers published in engineering sciences involving the use of method of quasilinearisation to several situations. During the last ten years, this method was further extended to cover several nonlinear situations. The main work was initiated by Lakshmikantham and his colleagues [47 - 54]. The method is used for initial as well as boundary value problems.

The problem of nonlinear control has been studied by using the fixed point method [68]. This powerful tool in modern analysis helps us to construct a suitable control function in the case of nonlinear control problems. Using the results of linear system, control function and controllability matrix for nonlinear system are constructed. Under suitable hypothesis a function space is constructed and a suitable operator is defined on it. Schauder's fixed point theorem is then applied to get the desired result.

Layout of the thesis

In this thesis, an attempt is made to develop the theory of nonlinear differential equations with PCDA. The entire thesis is divided into five chapters. The first chapter gives an introduction of the topic, and the reasons for taking up the study. It also includes the outline of the problems dealt with in the thesis. The chapter ends with a plan of the thesis.

Chapter 2 deals with the survey of the available literature on the equations with PCDA. The survey is divided into three parts. The first part briefly outlines the results on the

linear differential equations with PCDA. This includes results on the existence, uniqueness, oscillations and periodicity of the solution of different types of the equations with PCDA. The second section deals with nonlinear differential equations with PCDA. Finally, the chapter ends with brief description of the techniques in nonlinear analysis, which are used for the work of this study.

Chapter 3 is devoted to the study of first order nonlinear differential equations with PCDA. Here we start with the basic results such as mean value property and inequalities. The existence of unique solution is obtained by using the method of quasilinearisation and its improvement. In the first case, we assume the convexity condition and in the second case under relaxed conditions, a simple procedure to construct the solution sequences is given. Inequalities estimating solutions of two differential equations with PCDA are discussed. Further, solution of the periodic boundary value problem is obtained by using the method of quasilinearisation. The chapter ends with discussion on oscillatory behaviour of the solution of a particular nonlinear equation with PCDA.

In chapter 4, we have taken up the controllability problem for the nonlinear system involving PCDA. The sufficient conditions for complete controllability of the linear as well as nonlinear systems are obtained. An operator on a function space is constructed and Schauder's fixed point theorem is used. Some comparison theorems are discussed.

In the last chapter of the thesis, we introduce second order nonlinear differential equation with PCDA. As prerequisites for the main result, solutions of homogeneous and nonhomogeneous linear equations with PCDA are obtained. The main result of this chapter is the existence of maximal and minimal solutions by using the monotone iterative technique. Oscillatory properties of a particular second order differential equation with PCDA are discussed.

Finally, we present a brief summary of the results obtained in this thesis and give some of the unresolved problems. The thesis ends with a complete bibliography.

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Chapter 2

SURVEY OF EXISTING LITERATURE

2.1 Introduction

This chapter deals with the survey of the literature on differential equations with piecewise constant deviating argument (PCDA). The attention to these types of equations was drawn by Myskhis [56], and since then different types of equations with PCDA and their generalisations have been studied by Busenberg, Cooke, Wiener and others. These equations have been a topic of interest for last few decades. A survey article by Cooke and Wiener[19] gives some of the results on these equations. We plan to give survey of linear equations in section 2 and that of nonlinear equations in section 3. Section 4 gives an account of methods from literature on ordinary differential equations, used in our study.

Equations with PCDA are similar in structures to those found in certain " sequentially continuous " models of disease dynamics. Usually diseases are propagated by two main methods of transmission, namely *horizontal transmission* and *vertical transmission*. When an individual picks up the disease through some form of direct or indirect contact with infected individuals, it is called horizontal form of transmission. The vertical transmission is one in which the disease is passed on to a proportion of the offsprings of the infected parents. Various types of models of vertically transmitted diseases are overviewed in [15,

29]. The analysis of certain models of these diseases which are propagated by invertebrated vectors with generators are discussed in [11]. This article discusses a particular disease known as the Rocky mountain fever. The organism that causes this disease is *Rickettsia ricketsi* which is transmitted to human or other large mammals via contact with infected ones. The mathematical models obtained here is a special case of the general form,

$$x'(t) = F(t, x_t), \quad [t] < t \leq [t] + 1, \quad x_{[t]} = \phi_{[t]},$$

$$\phi_{[t]} = G([t], x_{[t]}), \quad [t] \geq 2 \ ; \ \phi_1 = H.$$

where $x : [0, \infty) \longrightarrow IR^n$, and x_t is the past history function defined by ,

$$x_t(s) = \begin{cases} x(t+s), & s \in [-t, 0] \\ 0, & s < -t. \end{cases}$$

Here $x_t \in PC[0, \infty)$, F, G are functions from $[0, \infty) \times PC[0, \infty) \longrightarrow IR^n$, $H \in PC(-\infty, 0]$, where PC denotes piecewise continuous.

Equations of this type have continuous dynamics for intervals of the form $([t], [t] + 1)$. At integer points these equations have a combination of discrete and continuous dynamics. Such equations also arise in number of models of epidemics.

2.2 Linear differential equations with piecewise constant deviating argument

In this section, we present a brief account of the work done on linear differential equation with piecewise constant deviating argument (PCDA). We begin with one of the simplest scalar initial value problem (IVP).

Consider the IVP,

$$x'(t) = ax(t) + a_0 x([t]), \quad x(0) = c_0. \tag{2.1}$$

where a, a_0, c_0, are constants. $a_0 \neq 0$. $[\cdot]$ denotes greatest integer function. $t \in I = [0, \infty)$. The solution of (2.1) is defined as follows:

Definition 2.2.1 : *A solution of (2.1) on I, is a function x(t) that satisfies the conditions:*

(i) x(t) is continuous on I.

(ii) The derivative x'(t) exists at each point $t \in I$, with the possible exception of the points $[t] \in I$, where one sided derivatives exist.

(iii) Equation (2.1) is satisfied on each interval $[n, n+1) \subset I$ with integral end points.

The following result due to Cooke and Wiener gives the method finding the solution of the equation(2.1).

Theorem 2.2.1 *The IVP (2.1) has on I, a unique solution*

$$x(t) = m(t - [t])m^{[t]}(1)c_0 \tag{2.2}$$

where $m(t) = 1 + a^{-1}(e^{at} - 1)(a + a_0)$.

The following theorem established by Cooke and Wiener in [16] generalises the above result.

Theorem 2.2.2 *The scalar IVP*

$$\begin{aligned} x'(t) &= ax(t) + a_0 x([t]) + a_1 x([t-1]) \\ x(-1) &= c_{-1}; \quad x(0) = c_o \end{aligned} \tag{2.3}$$

has on I, a unique solution,

$$x(t) = c_{[t]}e^{a(t-[t])} + a^{-1}\{a_0 c_{[t]} + a_1 c_{[t]-1}\}(e^{a(t-[t])} - 1)$$

where

$$c_{[t]} = \{\lambda_1^{[t]+1}(c_0 - \lambda_2 c_{-1}) + (\lambda_1 c_{-1} - c_0)\lambda_2^{[t]+1}\}/(\lambda_1 - \lambda_2)$$

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and λ_1, λ_2 are roots of the equation $\lambda^2 - b_0\lambda - b_1 = 0$ with $b_0 = e^a + a^{-1}a_0(e^a - 1)$; and $b_1 = a^{-1}a_1(e^a - 1)$.

The solution is obtained by employing the method of steps, by considering the equation (2.3) on the unit interval $[n, n + 1)$.

In ODE with a continuous vector field the solution exists to the right and the left of the initial t-value. In general, this is not the case for the retarded functional DE.[33] But the solution of the equation (2.1) as well as of the equation (2.3) can be extended backwards on $(-\infty, 0]$. This is achieved by considering the respective equations on the interval $[-n, -n + 1)$. We state the result concerning equation (2.1) as given in [16].

Theorem 2.2.3 *If $m(1) \neq 0$, then the solution of (2.1) has a unique backward continuation on $(-\infty, 0]$ given by the formula (2.2).*

Since the method of steps involves the unit interval with integral end points, one may consider any initial point t_0 as integral point, and pose the IVP. The IVP (2.1) is posed at initial point 0. But it is not necessary that any initial point t_0 be an integer. We can as well posed the problem at non-integral point t_0. This fact is established by the following result.

Theorem 2.2.4 *If $m(1) \neq 0$ and $m(t - [t]) \neq 0$, then the equation (2.1) with initial condition $x(t_0) = x_0$ has on $(-\infty, \infty)$ a unique solution given by,*

$$x(t) = m(t - [t])m^{[t]-[t_0]}(1)m^{-1}(t_0 - [t_0])x_0 \qquad (2.4)$$

where $m(t)$ is as defined in (2.2).

Remark 2.2.1 *(i) If $a = 0$ in equation (2.1), then the solution (2.2) becomes*

$$x(t) = (1 + a_0(t - [t]))(1 + a_0)^{[t]}c_0, \quad t \in I.$$

(ii) If $a_0 = 0$ in equation (2.1), then the solution (2.2) becomes $x(t) = e^{at}c_0$, as expected.

(iii) If $t = 0$, then $x_0 = c_0$ and solution (2.4) reduces to solution (2.2).

The IVP (2.1) is generalised in many directions. One such generalisation is obtained by increasing the number of delay terms, namely;

$$x'(t) = ax(t) + \sum_{i=0}^{N} a_i x([t-i]), \quad a_N \neq 0$$

$$x(i) = c_i, \quad i = 0, 1, 2, ... N.$$

The unique solution of this IVP is obtained by Cooke and Wiener in [16]. Here, the authors also discuss extension to the problem (2.3), namely,

$$X'(t) = AX(t) + A_0 X([t]) + A_1 X([t-1]),$$

$$X(-1) = C_1; \quad X(0) = C_0,$$

where A, A_0, A_1 are $r \times r$ matrices and X is r-vector. This IVP has a unique solution provided the matrices A, $e^A - I$, and A_1 are non singular.

In more general, the equation (2.1) can be studied on a Banach space. Here, there is a need to modify the Definition 2.2.1 of the solution of equation (2.1) as per the one given by Krein in [38]. Cooke and Wiener in [16] have established the existence and uniqueness of solution as well as the exponential growth and backward continuation of the solution. The same paper also discusses the scalar IVP,

$$x'(t) = a(t)x(t) + a_0(t)x([t]) + a_1(t)x([t-1]),$$

$$x(0) = c_0; \quad x(-1) = c_{-1}$$

with continuous coefficients on I. A simple algorithm to compute the solution is given.

Shah and Wiener have studied the advanced differential equation with PCDA. All the retarded equations seen above with deviating arguments $[t-1], ..., [t-N]$ being replaced by the advanced arguments $[t+1], ..., [t+N]$ respectively are considered by them in [61]. Here,

they deal with existence and uniqueness of solution of the IVP, its backward continuation, growth and stability.

It is interesting to investigate the oscillatory behavior of the solution of the equation (2.1) which is caused by the deviating argument and which is not seen in case of ordinary differential equations[43]. It is well known that a solution is said to be oscillatory if it has arbitrarily large zeros. The following result is due to Aftabizadeh and Wiener in [1].

Theorem 2.2.5 *Consider the delay differential inequality*

$$x'(t) + a(t)x(t) + p(t)x([t]) \leq 0. \tag{2.5}$$

where $a(t)$ and $p(t)$ are continuous on I. Assume that

$$\lim_{n \to \infty} Sup \int_n^{n+1} p(t) exp(\int_n^t a(s)ds)dt > 1. \tag{2.6}$$

Then the equation (2.5) has no eventually positive solution.

Under the same condition (2.6), Aftabizadeh and Wiener have established that the delay differential inequality

$$x'(t) + a(t)x(t) + p(t)x([t]) \geq 0, \quad t \in I$$

has no eventually negative solution. Hence, we get the following result proved in the same article.

Corollary 2.2.1 *Subject to condition (2.6), the delay differential equation*

$$x'(t) + a(t)x(t) + p(t)x([t]) = 0.$$

has oscillatory solutions only.

When $a(t) = a$ $p(t) = p$ are constants, the condition (2.6) reduces to $p > \frac{a}{e^a - 1}$, which is a sharp condition. Thus we have the following result from [1].

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Theorem 2.2.6 *If* $p < \frac{a}{e^a - 1}$, *then the delay differential equation*

$$x'(t) + ax(t) + px([t]) = 0 \qquad (2.7)$$

has no oscillatory solutions.

Remark 2.2.2 *When* $p = \frac{a}{e^a - 1}$, *the only solution of the equation (2.7) is* $x(t) = 0$. *Hence, we can conclude that* $p > \frac{a}{e^a - 1}$, *is a necessary and sufficient condition for the equation (2.7) to have oscillatory solutions only.*

The following result on number of zeros is found in [1].

Theorem 2.2.7 *If* $p \geq \frac{a}{e^a - 1}$, *then any solution of the equation (2.7) has one and only one zero in each unit interval* $(n,\ n+1)$.

Further in [1], Aftabizadeh and Wiener have discussed the oscillatory properties of the linear advanced differential equation with deviating arguments and of differential equations with several deviating arguments. In [4], Aftabizadeh, Wiener and Xu have studied the oscillatory and periodic solutions of delay differential equations with PCDA. Here, the equation under consideration is,

$$x'(t) = a(t)x(t) + b(t)x([t-1]) = 0, \qquad (2.8)$$

where $a(t)$, $b(t)$ are continuous functions on I. This paper deals with sufficient conditions under which (2.8) has oscillatory solution. The authors claim that this condition is the 'best possible' in the sense that when a and b are constants the condition reduces to a necessary and sufficient condition. The article also deals with the condition under which the oscillatory solutions of equation (2.8) with $a(t) = a$, $b(t) = b$ are periodic.

In [17] Cooke and Wiener have discussed an equation which is alternately of retarded and advanced type, namely, the equation

$$x'(t) = ax(t) + a_0 x(2[\frac{t+1}{2}]) . \qquad (2.9)$$

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Here the argument deviation $\tau(t) = t - 2(\frac{t+1}{2})$ is negative for $2n - 1 \le t \le 2n$ and positive for $2n < t < 2n + 1$ (n is an integer). Equation (2.9) is of advanced type on $[2n, 2n + 1)$ and of retarded type on $(2n, 2n + 1)$. The method of steps is employed to obtain unique solution of the equation (2.9) on I as well as its unique backward continuation on $(-\infty, 0]$. Furthermore the equation (2.9) with variable coefficients $a(t)$, $a_0(t)$ is examined, and the condition for existence of a unique solution on I is determined and conditions under which all solution are oscillatory are obtained. Oscillatory and periodic properties for generalisations of (2.9) are discussed by Aftabizadeh and Wiener in [2].

In [34], Jayasree and Deo have developed some basic tools needed for the study of qualitative properties of solutions of equations involving PCDA. Let $\mathcal{C}(I)$ denote the space of continuous functions mapping $I = [0, \infty)$ into IR^n . The norm of a $n \times n$ matrix $M = (M_{ij})$ is defined by $|M| = max_j \sum_i |m_{ij}|$. Let E denotes the $n \times n$ identity matrix. Consider the systems

$$X'(t) = A(t)X(t),\tag{2.10}$$

$$Y'(t) = A(t)Y(t) + B(t)Y([t]),\tag{2.11}$$

$$Z'(t) = A(t)Z(t) + B(t)Z([t]) + C(t),\tag{2.12}$$

for $t \ge 0$, with initial conditions,

$$X(0) = Y(0) = Z(0) = C_0,\tag{2.13}$$

and the assumption:

(**H**)   *A, B are $n \times n$ matrices with entries real valued continuous functions of $t \in I$*

C is a n column vector with entries real valued continuous functions for $t \in I$,

x, y, z are n vectors and C_0 is a real constant n column vector.

Let Φ be the fundamental matrix (FM) of (2.10), such that $\Phi(0) = E$, the identity

matrix. Then using the method of iteration, the solution of (2.11) and (2.13) is obtained in [36] as follows.

Theorem 2.2.8 *Let the assumption (H) hold. Then there exists a unique solution to the IVP (2.11) and (2.13) for $t \in I$ and it is given by*

$$
\begin{aligned}
Y(t) \ = \ & \lim_{k}\{\Phi(t,0) + \int_0^t \Phi(t,t_1)B(t_1)\Phi([t_1],0)dt_1 \\
& + \int_0^t \int_0^{[t_1]} \Phi(t,t_1)B(t_1)\Phi([t_1],t_2)B(t_2)\Phi([t_2],0)dt_2dt_1 \\
& + ... + \int_0^t \int_0^{[t_1]} ... \int_0^{[t_{k-1}]} \Phi(t,t_1)B(t_1)\Phi([t_1],t_2)B(t_2)... \\
& \times B(t_k)\Phi([t_k],0)dt_k...dt_2dt_1\}c_0.
\end{aligned}
$$

This result is established by using Banach fixed point theorem. A closed form solution of (2.11) and (2.13) is obtained in [16] by Cooke and Wiener. The following definition is of importance to study the perturbation effects on (2.11).

Definition 2.2.2 *The function*

$$
\begin{aligned}
\Psi(t) \ = \ & \{\Phi(t,[t]) + \int_{[t]}^t \Phi(t,s)B(s)ds\} \\
& \times \prod_{k=[t]}^{1} \{\Phi(k,k-1) + \int_{k-1}^k \Phi(k,s)B(s)ds\}, t \in I,
\end{aligned}
$$

satisfying the matrix IVP, $Y'(t) = A(t)Y(t) + B(t)Y([t])$, $Y(0) = E$ is called the FM solution of the equation (2.11).

The method of variation of parameters (VP) is one of the important techniques in the study of the qualitative properties of the solution. In particular, perturbation theory depends on this method. The VP formula for the equation (2.12) obtained in [36] is given below.

Theorem 2.2.9 *Let $Y(t)$ be the solution of (2.11), (2.13). Let Φ and Ψ be the FM 's of the equations (2.10) and (2.11) respectively. Then the unique solution of (2.12) and (2.13)*

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for $t \in I$, is given by

$$Z(t) = Y(t) + \sum_{k=1}^{[t]} \int_{k-1}^{k} \Psi(t,k)\Phi(k,s)C(s)ds$$

$$+ \int_{[t]}^{t} \Phi(t,s)C(s)ds$$

where

$$\Psi(t,k) = \Psi(t) \cdot \Psi^{-1}(k), \quad k = 0,1,2,...[t], t \in I,$$

and

$$Y(t) = \Psi(t)C_0, \quad \Phi(t,s) = \Phi(t) \cdot \Phi^{-1}(s).$$

The above theorem is obtained by considering the equation (2.11) as the basic equation. One can take equation (2.10) as the basic equation and VP formula can be derived. This is achieved in the next theorem [36].

Theorem 2.2.10 *Let Φ and Ψ be the FM's of (2.10) and (2.11) respectively. Then*

$$Z(n) = Y(n) + \sum_{k=1}^{n} \int_{k-1}^{k} \Psi(n,k)\Phi(k,s)C(s)ds$$

where $n \geq 1$ is an integer, and

$$\Psi(n,k) = \Phi(n,k) + \sum_{r=k+1}^{n} \int_{r-1}^{r} \Psi(r-1,k)\Phi(n,s)B(s)ds, \quad for \ n \geq k$$

$$\Psi(n,k) = E; \quad n = k, \quad n = 1,2,...[t].$$

In [60], Rong and Jialin has obtained the solution of the equation (2.12) using the theory of difference equation, and has compared the behavior of the solution of equation (2.12) to that of the corresponding difference equation.

Integral inequalities play a useful role in the study of the qualitative behavior of the solutions of differential equations. Jayasree and Deo have established the Gronwall type integral inequality in the following theorem in [36].

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Theorem 2.2.11 *Let c_0 be a constant and x, a, $b \in [I, \mathbb{R}^+]$.*

If the inequality

$$x(t) \leq c_0 + \int_0^t a(s)x(s) + b(s)x([s])ds, \ t \in I$$

holds, then for $t \in I$

$$x(t) \ \leq \ c_0 \cdot \prod_{k=1}^{[t]} \{ \ exp(\int_{k-1}^k a(r)dr) + \int_{k-1}^k exp(\int_s^k a(r)dr)b(s)ds \}$$

$$\times \{ exp(\int_{[t]}^t a(r)dr) + \int_{[t]}^t exp(\int_0^t a(r)dr)b(s)ds \} \tag{2.14}$$

Remark 2.2.3 *The right hand side of the inequality (2.14) is in fact a solution of the related IVP, $x'(t) = a(t)x(t) + b(t)x([t])$, $x(0) = c_0$.*

We need the following definitions given in [36].

Definition 2.2.3 *For $n \times n$ matrix $B = (b_{ij})$, define the matrix measure μ of B by $\mu(B) = max_j(b_{ij} + \sum_{i=1}^n |b_{ij}|)$.*

Definition 2.2.4 *A solution $Y(t) = (y_1(t), ..., y_n(t))$ of the system (2.11) existing for $t \in I$ is said to be oscillatory if atleast one of its components has arbitrarily large zeros for $t \geq T$, $0 \leq T < \infty$.*

The following result on oscillatory property is taken from [36].

Theorem 2.2.12 *Let $\mu(\cdot)$ denote the matrix measure. Assume that the matrices $A(t)$ and $B(t)$ in (2.11) are such that*

$$\lim_{m \to \infty} \ sup \ \int_m^{m+1} -\mu(B(s)) \ exp(\int_m^s -\mu(A(r))dr \ ) \ ds \ > \ 1.$$

then every solution of (2.11) is oscillatory.

Aftabizadeh and Wiener [5] have discussed the oscillatory and periodic solutions for a system of two equations with PCDA.

Further in [36], Jayasree and Deo have explored scalar retarded equations with two types of delays namely $(t - \tau)$ and $[t]$. The equations involved are

$$x'(t) = ax(t) + bx(t - \tau)$$

$$y'(t) = ay(t) + by(t - \tau) + cy([t]) \qquad (2.15)$$

$$z'(t) = az(t) + bz(t - \tau) + cz([t]) + f(t). \qquad (2.16)$$

where $\tau > 0$, $t \geq 0$ with initial functions,

$$x(t) = y(t) = z(t) = \phi(t), \quad -\tau \leq t \leq 0.$$

The existence and uniqueness of the solutions of the equations (2.15) and (2.16) have been proved. In [36], Jayasree has obtained similar results for the equation

$$x'(t) = ax(t) + bL(x(t + \theta))$$

$$y'(t) = ay(t) + bL(y(t + \theta)) + cy([t])$$

$$z'(t) = az(t) + bL(z(t + \theta)) + cz([t]) + f(t).$$

with initial conditions $x(t) = y(t) = z(t) = \phi(t)$, $-\tau \leq t \leq 0$,

where L is a linear operator mapping $\mathcal{C}([-\tau, 0], I\!R] \to I\!R$ for which $t \geq 0$, a, b, c, are real constants, ϕ is a continuous real valued function defined on $[-\tau, 0]$, τ being a constant and f is continuous function on I.

The differential equations in Banach space with PCDA are considered by Wiener in [64]. The existence and uniqueness of solution of IVP posed at $t = 0$ is established for equations with bounded as well as unbounded operators. The properties of solutions of equations with bounded operators are similar to those of solutions of systems of ordinary differential equations which can be viewed as equations in a finite dimensional space. This article also discusses some results on the asymptotic behavior of the solutions and equations with unbounded delay in case of several argument deviations.

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A linear system of differential equations with PCDA $[t + \frac{1}{2}]$ is studied for oscillations

by Wiener and Cooke in [66]. The existence of a solution is obtained for the equations,

$$x'(t) = Ax(t) + Bx([t + \frac{1}{2}]), \quad x(0) = c_0 ,$$

and

$$x'(t) = Ax(t) + Bx([t + \frac{1}{2}]) + f(t), \quad x(0) = c_0 ,$$

where A , B are constant matrices of appropriate order and f is a locally integrable vector

function. This work also contains the following result.

Theorem 2.2.13 *The problem* $\quad x'(t) = Ax(t) + Bx([t]) + f(t), \quad x(0) = c_0$

has on I *a unique solution*

$$
\begin{aligned}
x(t) &= M(t - [t]) \, M_1^{[t]} \{ c_0 + \sum_{j=1}^{[t]} M_1^{-j} \int_{j-1}^{j} e^{A(j-s)} f(s) \} \\
&+ \int_{[t]}^{t} e^{a(t-s)} f(s) ds ,
\end{aligned}
$$

if the matrices A *and* M *are non singular and* $f(t)$ *is locally integrable. This solution has*

a unique backward continuation on $(-\infty, 0]$ *given by*

$$
\begin{aligned}
x(t) &= M(t - [t]) \, M_1^{[t]} \{ c_0 + \sum_{j=1}^{-[t]} M_1^{j-1} \int_{-j+1}^{-j} e^{A(-j+1-s)} f(s) \} \\
&+ \int_{[t]}^{t} e^{a(t-s)} f(s) ds.
\end{aligned}
$$

where $M(t) = e^{At} + (e^{At} - I)A^{-1}B, \quad$ *and* $\quad M_1 = M(1)$

Further in [66] , the oscillatory and periodic properties of the solutions are discussed in terms of

the eigen values of a certain matrix associated with the system.

Delay differential equations are related to some discrete differential equations arising in

Numerical analysis. Gyori [32] has established some approximating results for the solution

of delay differential equations via differential equations with PCDA.

Theorem 2.3.2 *Assume that* $x'(t) = f(x, \mu)$, *where* $f \in C[\mathbb{R}^2]$, *satisfies the existence and uniqueness conditions in* $\mathbb{R}^2$ *and it's solution can be extended over the interval* I. *Then on* I , *there exists a unique solution of (2.18).*

Further Aftabizadeh and Wiener have proved the following result.

Theorem 2.3.3 *If* $f(x, \mu)$. *is continuous in* $\mathbb{R}^2$, *and the solutions of the equation* $x'(t) = f(x, \mu)$, *can be extended over* I, *then the problem (2.18) has a solution on* I.

In the process of investigating (2.17), Aftabizadeh and Wiener have proved the existence of maximal and minimal solutions by using the well known Monotone iterative method. In order to establish this, the upper and lower solutions of (2.17) are defined and the fundamental result concerning the upper and lower solutions is proved.

Theorem 2.3.4 *Consider the differential inequalities*

$$u'(t) \leq f(t, u(t), u([t]))$$
$$v'(t) \geq f(t, v(t), v([t])) , \quad t > 0,$$

where, $f \in C[\ I \times \mathbb{R} \times \mathbb{R} \ , \ \mathbb{R} \ ]$. *Suppose* $f(t, x, y)$ *is nondecreasing in* y *for each* $(t, x) \in I \times \mathbb{R}$ *and.*

$$f(t, x, y) - f(t, z, y) \leq L(x - z) , \quad whenever \ x \geq z .$$

Then $u(0) \leq v(0) \Rightarrow u(t) \leq v(t)$, *for all* $t \geq 0$.

Further in [3], the authors have prove the existence of the solution of the equation (2.18) on the closed set $\Omega = \{(x, y) : u(t) \leq x \ , \ y \leq v(t) \ , \ t \geq 0 \}$.

Theorem 2.3.5 *Let* $u(t)$ *and* $v(t)$ *be the lower and the upper solution of (2.18) such that* $u(t) \leq v(t)$ *on* I *and* $f \in C(\Omega)$. *Assume that* $x'(t) = f(x, \mu)$, *satisfies the existence conditions on* Ω . *Then there exists a solution* $x(t)$ *of (2.18) such that* $u(t) \leq x(t) \leq v(t)$ *on* I .

The following Lemma is of vital importance for monotone iterative method and has been proved in [3].

Lemma 2.3.1 *Suppose that* $x \in C[\ I, \mathbb{R}]$, *and the derivative* $x'(t)$ *exists at each point* $t \in I$, *with the possible exception of the points* $[t] \in I$, *where one sided derivatives exist. Assume that.*

$$x'(t) \leq Mx(t) \ + \ Nx([t]) \,, \ x(0) \leq 0 \,,$$

where M *and* N *are constants such that*

$$N \geq \frac{-M}{e^{aM} - 1} \, e^{aM} \,, \ 0 < a < 1.$$

Then $x(t) \leq 0$ *on* I .

The well known monotone iterative method proves the existence of minimal and maximal solutions of the equation (2.17) through the construction of monotone sequences of solutions of the corresponding linear delay differential equation. We state the result proved in [3].

Theorem 2.3.6 *Let* $u(t)$ *and* $v(t)$ *be lower and upper solutions of the equation (2.17) such that* $u(t) \leq v(t)$ *on* I . *Suppose that*

$$f(t, x_1, y_1) - f(t, x_2, y_2) \ \geq \ M(x_1 - x_2) \ + \ N(x_2 - y_2), \ t \geq 0,$$

$$for \ \ u(t) \leq x_2(t) \leq x_1(t) \leq v(t) \ \ , \ \ u(t) \leq y_2(t) \leq y_1(t) \leq v(t)$$

$$and \ \ N \geq \frac{-M}{e^{aM} - 1} \, e^{aM} \ \ , \ \ for \ each \ 0 < a < 1.$$

Then there exists monotone sequences $\{u_m(t)\}$ *and* $\{v_m(t)\}$ *with* $u_0(t) = u(t)$, $v_0(t) = v(t)$ *such that* $u_m(t) \to \alpha(t)$, $v_m(t) \to \beta(t)$ *as* $m \to \infty$ *monotonically on* I, *and* $\alpha(t)$, $\beta(t)$ *are minimal and maximal solutions of the equation (2.17) respectively.*

In the proof of the above theorem, the linear delay differential equation constructed is

$$x'(t) = f(t, \eta(t), \eta([t])) \ + \ M\{x(t) - \eta(t)\} \ + \ N\{x([t]) - \eta([t])\}$$

where $\eta(t) \in C[I, I\!R]$ is such that $u(t) \leq \eta(t) \leq v(t)$. The solution of this equation can be obtained by using scalar form of Theorem 2.2.13, which has been proved in [3].

In [34], Jayasree has defined the maximal and the minimal solution of (2.17) and has proved their existence under the hypothesis of Theorem 2.3.1. The following comparison theorem has been established.

Theorem 2.3.7 *Let* $r(t)$ *be the maximal solution of (2.17) on the interval* $[0, a)$, $a > 0$. *Let* $m \in C[I, I\!R]$, $m(0) \leq r(0)$, *and if* $m'(t) \leq f(t, m(t), m([t]))$, $t \in I$, *then* $m(t) \leq r(t)$, $t \in I$.

A useful result to study stability and boundedness property is established in [34].

Theorem 2.3.8 *Let* $\Phi(t)$ *be the fundamental solution of* $x'(t) = a(t)x(t)$ *satisfying* $\Phi(0) = 1$ *and* $\Psi(t)$ *be the fundamental solution of* $y'(t) = a(t)y(t) + b(t)y([t])$ *satisfying* $\Psi(0) = 1$. *Let* $|\Psi(t)| \leq \gamma(t)$, *where* $\gamma(t)$ *is a positive real valued function defined on* I

and let $\gamma(0) = \gamma_0$. *Also suppose that the function* $f(t, z(t)) : I \times I\!R \to I\!R$ *satisfies the inequalities.*

$$|\Psi^{-1}(k)\Phi(k,s)f(s,z(s))| \leq W(s, \frac{|z(s)|}{\gamma(s)}, \frac{|z(k-1)|}{\gamma(k-1)}), \quad k-1 \leq s \leq k, \quad k=1,2,\ldots$$

and

$$|\Phi(t,s)f(s,z(s))| \leq \gamma(t)W(s, \frac{|z(s)|}{\gamma(s)}, \frac{|z([t])|}{\gamma([t])}), \quad k=[t] \leq s \leq t.$$

where $W(t, r(t), r([t]))$ *is monotone increasing function in second and third variable. Let* $r(t, 0, r_0)$ *be a solution of* $r'(t) = W(t, r(t), r([t]))$, $r(0) = r_0$.

Then the solution $z(t, 0, x_0)$ *of the equation*

$$z'(t) = a(t)z(t) + b(t)z([t]) + f(t, z(t))$$

satisfies $|z(t, 0, x_0)| \leq \gamma(t)r(t)$, $t \in I$, *if* $z(t, 0, x_0)$ *is such that* $|x_0| \leq \gamma_0 r_0$.

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In [37] , Jayasree and Deo have discussed the nonlinear variation of parameters formula to obtain the solution of the equation

$$z'(t) = f(t, z(t)) + g(z([t])) + c(t) \qquad (2.19)$$

in terms of the solution of the equations

$$x'(t) = f(t, x(t))$$

$$\text{and} \quad y'(t) = f(t, y(t)) + g(y([t])) \qquad (2.20)$$

with initial conditions

$$x(t_0) = y(t_0) = z(t_0) = x_0, \quad x_0 \in I\!R$$

$$\text{and} \quad x([t_0]) = y([t_0]) = z([t_0]), t_0 \geq 0. \qquad (2.21)$$

Here $f : [t_0, \infty) \times I\!R \to I\!R$, $c : [t_0, \infty) \to I\!R$, are continuous functions and g is a piecewise continuous function on $[t_0, \infty)$. The following two lemmas are of vital importance.

Lemma 2.3.2 *Let* $g(y([t]))$ *be a piecewise continuous function defined on* $I\!R$ *and let* $\frac{\partial g(y([t]))}{\partial y([t])}$ *exists and be piecewise continuous on* $I\!R$. *Then*

$$g(y_2([t])) - g(y_1([t])) = (\int_0^1 \frac{\partial g(sy_2([t]) + (1-s)y_1([t]))}{\partial y([t])} ds)(y_2([t]) - y_1([t])) , \ t \in I .$$

Lemma 2.3.3 *Assume that* $f \in C[[t_0, \infty) \times I\!R, I\!R]$, g *be a piece wise continuous function on* $I\!R$ *and* f, g *possess partial derivatives* $\frac{\partial f(t, y(t))}{\partial y}$ *and* $\frac{\partial g(y([t]))}{\partial y([t])}$.

Denote $H_1(t, t_0, x_0) = \frac{\partial f(t, y(t))}{\partial y}$ *and* $H_2([t], t_0, x_0) = \frac{\partial g(y([t]))}{\partial y([t])}$.

Then $\Psi(t, t_0, x_0) = \frac{\partial y(t, t_0, x_0)}{\partial x_0}$ *exists and is the solution of the variational equation,*

$$z'(t) = H_1(t, t_0, x_0)z(t) + H_2([t], t_0, x_0)z([t]) \qquad (2.22)$$

such that $\Psi(t_0, t_0, x_0) = 1.$

The variational equation (2.22) stated above is linear, and is used to obtain the nonlinear variation of parameters formula following Alekseev's method [55].

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Theorem 2.3.9 *Let $y(t, t_0, x_0)$ be the unique solution of (2.20),(2.21); $\Phi(t, t_0, x_0)$ be the solution of the variational equation (2.22) with $H_2 \equiv 0$ and $\Psi(t, t_0, x_0)$ be the solution of the equation (2.22) existing for $t \geq 0$. Then there exists a unique solution $z(t)$ for (2.19), (2.21) given by*

$$
z(t, t_0, x_0) = \begin{cases}
x_0, & t \in [0, t_0). \\[2mm]
y(t, t_0, x_0) + \int_{t_0}^{t} \Phi(t, s, z(s)) c(s) ds, & t \in [t_0, 1), \\[2mm]
y(t, t_0, x_0) + \int_{t_0}^{1} \Psi(t, 1, y(1)) \Phi(1, s, z(s)) c(s) ds \\[2mm]
\quad + \sum_{k=2}^{[t]} \int_{k-1}^{k} \Psi(t, k, y(k)) \Phi(k, s, z(s)) c(s) ds \\[2mm]
\quad + \int_{[t]}^{t} \Phi(t, s, z(s)) c(s) ds, & t \geq 1.
\end{cases}
\tag{2.23}
$$

The equation (2.23) is obtained on each unit interval, by employing Alekseev's formula for the corresponding equation.

Remark 2.3.1 *(i) If $t_0 = 0$, then (2.23) takes the form,*

$$
z(t, 0, x_0) = y(t, 0, x_0) + \sum_{k=1}^{[t]} \int_{k-1}^{k} \Psi(t, k, y(k)) \Phi(k, s, z(s)) c(s) ds
$$
$$
+ \int_{[t]}^{t} \Phi(t, s, z(s)) c(s) ds.
$$

(ii) If $f(t, z(t)) = a(t)z(t)$ and $g(z([t])) = b(t)z([t])$; $a(t)$, $b(t)$, are continuous functions then (2.23) takes the form ,

$$
z(t, 0, x_0) = y(t, 0, x_0) + \sum_{k=1}^{[t]} \int_{k-1}^{k} \Psi(t, k) \Phi(k, s) c(s) ds
$$
$$
+ \int_{[t]}^{t} \Phi(t, s) c(s) ds,
$$

which is given by Theorem 2.2.9 .

(iii) If $g = 0$, then (2.23) gives Alekseev's formula.

A further generalisation of (2.23) is also established in [34] which is extension of Ladde's result for ordinary differential equation [40].

In [64], Wiener has discussed nonlinear differential equation with PCDA in Banach spaces. Here the equation,

$$x'(t) = A(t)x(t) + f(t, x(t - [t]), x(t - 2[t]), ...x(t - N[t]))$$

has been investigated. As a special case following result is obtained, which is an extension of the results in Theorems 2.3.1, 2.3.2, and 2.3.3.

Theorem 2.3.10 *If $f(x, \lambda) \in C(\mathbb{R}^2)$ is different from zero everywhere and the solution of the equation $x'(t) = f(x, \lambda)$ can be extended over I, then the problem*

$$x'(t) = f(x(t), x([t])), \quad x(0) = c_0, \quad 0 \leq t \leq \infty. \tag{2.24}$$

has a unique solution. If $f(c_0, c_0) = 0$ and $\int f^{-1}(x, c_0)dx$ diverges as $x \to \infty$, then $x = c_0$ is the unique solution of the equation (2.24). If this integral converges, then equation (2.24) has more than one solution.

In [16], Cooke and Wiener have discussed the scalar equation,

$$x'(t) = f(x(t), x([t]), x([t - 1])), \quad x(0) = c_0, \quad x(1) = c_1, \quad t \in I.$$

The existence of unique solution is obtained by assuming the existence of ·\·; solution of the corresponding differential equation with parameters.

Gyori and Ladas [31] have studied a nonlinear equation,

$$x'(t) + \sum_{i=1}^{m} p_i(t)f_i(x([t - k_i])) = 0, \quad t \geq 0,$$

for oscillations of the solutions. The necessary and sufficient conditions are obtained in terms of the solutions of the associated linear equation with PCDA. In [60] Rong and Jialin have studied the periodic solutions of the equation

$$x'(t) = A(t)x(t) + B(t)x([t]) + g(t, \Phi(t), \Phi([t]))$$

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in a Banach space of almost periodic functions.

2.4   Results from ordinary differential equations

In this section we present the summary of the existing results on the first order nonlinear ordinary differential equation, and relevant to the study undertaken. First we shall discuss the methods used to establish the existence of the solution.

Consider the IVP

$$x'(t) = f(t, x(t)),\ x(0) = x_0 \quad t \in J : 0 \le t \le T,\ \ T > 0, \tag{2.25}$$

where $f \in C^1[J \times IR, IR]$. It is well known that, if we assume the Lipschitz condition, then there is a unique solution for the IVP. The theory has been developed further by dropping Lipschitz condition at the cost of uniqueness property. This leads to the concepts of maximal solution, minimal solution, upper solution and lower solution. These solutions play important role in the development of the theory. The method of upper and lower solution yields existence of solutions in a closed set and give rise to the famous Comparison principle. We shall give two important methods of establishing the existence of the solution of the IVP (2.25).

(I) MONOTONE ITERATIVE TECHNIQUE.

Monotone iterative technique (MIT) is a constructive method for establishing the existence of extremal solutions. This method yields monotone sequences converging to solution of (2.25). These sequences are such that each member of these sequences is a solution of a certain linear differential equation. Since these solutions can be computed , the method provides numerical procedure for the computation of solutions. This fact makes MIT advantageous and important. Furthermore MIT can be used to obtain two sided pointwise bounds on the solutions. These bounds are useful in studying qualitative and quantitative behavior

of the solutions. We state the theorem on MIT and proof can be seen in [42].

Theorem 2.4.1 *Let* $f \in C[J \times \mathbb{R}, \mathbb{R}]$, u_0, v_0 *be lower and upper solutions of (2.25) such that* $u_0 \leq v_0$ *on* J. *Further suppose that*

(A): $\quad f(t,x) - f(t,y) \geq -M(x-y)$ *for* $u_0 \leq y \leq x \leq v_0$ *and* $M \geq 0$.

Then there exils monotone sequences $\{u_n\}$, $\{v_n\}$ *such that*

$u_n \rightarrow u$ *and* $v_n \rightarrow v$ *as* $n \rightarrow \infty$ *uniformly and monotonically on* J. u *and* v *are minimal and maximal solution of (2.25), respectively.*

Remark 2.4.1 *If we* $M = 0$ *in condition* **(A)**, *then it is clear that* f *is monotone nondecreasing. We can prove the result similarly by assuming* f *to be monotone nonincreasing instead of condition* **(A)**.

MIT has been applied to functional differential equations [41] as well as periodic BVP [46].

(II) METHOD OF QUASILINEARISATION

The method of quasilinearisation is a well established technique used to obtain approximate solutions to nonlinear differential equations. The method was developed by Bellman and Kalaba.[8, 10]. If we assume that:

(A1): $\quad f(l, x(l))$ *is uniformly convex in* x *for* $0 \leq l \leq T$.

Then the method of quasilinearisation gives a monotonic increasing sequence of approximate solutions converging uniformly to the solution of (2.25). The sequence provides good lower bounds for the solution. It is to be noted that this convergence is quadratic in the following sense.

Definition 2.4.1 *For* $x \in C[J]$, *let* $||x|| = Sup\{x(l) : l \in J\}$, *and suppose that* w_n *is an approximate solution of (2.25), and* x *is a solution of (2.25). Then the sequence* $\{w_n\}$

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converges to x quadratically if there exists $\lambda > 0$ such that

$$||x - w_n|| \leq \lambda \cdot ||x - w_{n-1}||^2.$$

One can prove the dual result giving upper bounds under the assumption :

(**A2**) : $f(t, x(t))$ is uniformly concave in x for $0 \leq t \leq T$.

In the last decade, the method of quasilinearisation has attracted much attention. It has been generalised and extended using the less restrictive assumptions on the function f so that the method can be applied to solve a larger class of problems. In what follows throughout the section, let $\Omega = \{(t, x) : u(t) \leq x(t) \leq v(t), \ t \in J\}$. where $u(t)$, $v(t)$, $x(t)$ are lower solution, upper solution and solution of (2.25) respectively. We state the result from due to Lakshmikantham and Malek [47].

Theorem 2.4.2 : *Assume that $u, v \in C^1[J, \mathbb{R}]$ are lower and upper solutions of (2.25) such that $u(t) \leq v(t)$ on J; and*

(**A3**) : f_x, f_{xx} *exist, and are continuous and satisfy*

$$f_{xx}(t, x) + 2M \geq 0, \ M \geq 0, \quad for \ (t, x) \in J \times \mathbb{R}.$$

Then there exists a monotone sequence $\{w_n(t)\}$ which converges uniformly to the solution $x(t)$ of the equation (2.25) and the convergence is quadratic.

Remark 2.4.2 *(i) In (**A3**) :, the requirement is that, the function $f(t, x) + Mx^2$ should be convex for some $M > 0$.*

(ii)When $M = 0$, $v(t) = x(t)$, which is assumed to exists on J and

$u(t) = w_0$, any constant that satisfies $f(t, w_0) \geq 0$, the above result reduces to the method of quasilinearisation .

In [48] , the method has been extended to show that the monotone sequences can be constructed to obtain lower and upper bounds simultaneously as well as the quadratic convergence by decomposing the function f into a difference of two convex or concave functions.

However the drawback of this extension is that the elements of the sequences are not the solutions of some linear problems. The proof requires an extra condition and is not decisive.

The paper [51] by Lakshmikantham and Koksal has discussed the problem of obtaining a lower approximations which converges quadratically to the unique solution of (2.25) by decomposing f into a sum of convex and Lipschitzian functions. This is established in the following theorem.

Theorem 2.4.3 :*Assume that $u, v \in C^1[J, \mathbb{R}]$ are lower and upper solutions of (2.25) such that $u(t) \leq v(t)$ on J; and*

(A4) : $f \in C[\Omega, \mathbb{R}]$ *admits a decomposition* $f = f_1 + f_2$, f_1 *is uniformly convex in* x *for* $t \in J$ *and* f_2 *is Lipschitzian in* x .

Then there exists a monotone sequence $\{w_n(t)\}$ *which converges uniformly to the solution* $x(t)$ *of the equation (2.25) and the convergence is quadratic.*

Remark 2.4.3 :

(i) If $f_2 = 0$ then , the above theorem reduces to the method of quasilinearisation.

(ii) If f_1 is not Lipschitzian, then we can still prove the convergence of the sequence $\{w_n(t)\}$ to the minimal solution. However, the convergence in this case is not quadratic, but weakly quadratic in the following sense.

Definition 2.4.2 *We say that the sequence $\{w_n\}$ converges to w weakly quadratically, if there exist positive constants λ_1, λ_2, and λ_3, such that*

$$\max_{t \in J} |w(t) - w_n(t)| \leq \lambda_1 \{\lambda_2 + \lambda_3 \cdot \max_{t \in J} |x(t) - w_{n-1}|^2\}.$$

Remark 2.4.4 *If the assumption (A4) in the above theorem is replaced by,*

(A5) : $f \in C[\Omega, \mathbb{R}]$ *admits a decomposition* $f = f_1 + f_2$, f_1 *is uniformly convex in* x *for* $t \in J$ *and* f_2 *is continuous in* x *on* Ω,

then there exists a monotone sequence $\{w_n(t)\}$ which converges to the minimal solution α of the equation (2.25) and the convergence is weakly quadratic.

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In [52], Lakshmikantham et al, has revisited the problem discussed in [48]. It is shown that when the function f is decomposed into the difference of two concave or convex functions, different results can be obtain with same conclusions. They have used a special approach to obtain the elements of the monotone sequences as the solutions of some linear differential equations. Coupled lower and upper solutions are used for this purpose. This method has been improved in [53]. The new method involves first obtaining both lower and upper bounds for the solution in terms of monotone iterates which are the solutions of simpler nonlinear equations. Then the properties of the auxiliary functions in the nonlinear equations are used to adopt an improved procedure which will lead to solutions of some linear differential equations.

In [54], Lakshmikantham has succeeded in avoiding the multistage process obtained in [53] and develop a simple algorithm that provides directly the monotone sequences that are the solutions of linear differential equations. We state this result in the following theorem.

Theorem 2.4.4 *Assume that:*

(H1) $u_0, v_0 \in \mathcal{C}^1[J, \mathbb{R}]$, *are such that* $u_0' \leq f(t, u_0), v_0' \geq f(t, v_0)$

and $u_0(t) \leq v_0(t)$, $t \in J$.

(H2) $f, \phi \in C^{0,2}[\Omega_1, \mathbb{R}]$, $f_{xx}(t,x) + \phi_{xx}(t,x) \geq 0$ *on* Ω_1 *and*

$\phi_{xx}(t,x) > 0$ *on* Ω_1, *where* $\Omega_1 = \{(t,x) : u_0(t) \leq x(t) \leq v_0(t), \ t \in J\}$.

Then there exist monotone sequences $\{u_n(t)\}$, $\{v_n(t)\}$ *which converge uniformly to the unique solution* $x(t)$ *of the equation (2.25) and the convergence is quadratic.*

Remark 2.4.5 *The simple approach obtained in the above result can also be applied in all the previous situations which are also possible to deduce as special cases of this result.*

Lakshmikantham and Shahzad [49] have also extended the generalised quasilinearisation method decomposing the function f into two functions F and G such that $F + \psi$ is concave

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and $G + \phi$ is convex for some concave function ψ and convex function ϕ. The condition **(H2)** in the above theorem is to be replaced by the following, the remaining part of the statement being unchanged.

(H3) $f \in C[\Omega_1, I\!R]$, f admits a decomposition $f = F + G$, where F_x, G_x, F_{xx}, G_{xx} exit and are continuous satisfying $F_{xx}(t,x) + \psi_{xx}(t,x) \leq 0$ and

$G_{xx}(t,x) + \phi_{xx}(t,x) \geq 0$ on Ω_1, where ψ, $\phi \in C[\Omega_1, I\!R]$,

$\phi_x(t,x), \psi_x(t,x), \phi_{xx}(t,x), \psi_{xx}(t,x) \quad \phi_{xx}(t,x) < 0, \quad \phi_{xx}(t,x) > 0$ on Ω_1.

In the last decade, the method of quasilinearisation and its generalisations are extended to different types of equations. Deo and Sivasundaram [24] has extended the method to functional differential equation, $x'(t) = f(t, x_t)$ where $x_t = x(t + s)$, $-\tau \leq s \leq 0$, $t \in J$. Here the convergence of the monotone sequences is superlinear. Deo and Knoll have extended the method and its generalisations to integro-differential equations [21, 22, 23]. The application of Taylor's theorem to formulate the related linear equations helps in obtaining the convergence of the iterates of the order $k \geq 2$. Lakshmikantham and Neito have applied the method for first order periodic boundary value problem [50]. Stutson and Vatsala [60] have extended the method with f being sum of a nonconvex function, a nonconcave function and a Lipschitz function and have obtain quadratic convergence. A numerical example in support of the result is given.

Finally, we give a brief account of results on control theory. Control theory is relatively a new branch of mathematics developed in th 20th century. It analyses the behaviour of a given system under specified circumstances. The basic results on controllability of a linear system can be found in [6, 7, 14, 57].

Consider the linear system,

$$x'(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0. \tag{2.26}$$

where A is $n \times n$, and B is $n \times m$ continuous matrices.

Definition 2.4.3 *The system (2.26) is said to be completely controllable (c.c.) if for any t_0, any initial state $x(t_0) = x_0$ and any given final state x_f there exists a finite time $t_f > t_0$ and a control $u(t), t_0 \le t \le t_f$, such that $x(t_f) = x_f$.*

We state the necessary and sufficient condition for the c.c. of the system (2.26) as given in [6].

Theorem 2.4.5 *The system (2.26) is c.c. if and only if the $n \times n$ symmetric controllability matrix,*

$$U(t_0, t_f) = \int_{t_0}^{t_f} \Phi(t_0, \tau)B(\tau)B^T(\tau)\Phi^T(t_0, \tau)d\tau.$$

where Φ denotes the fundamental matrix solution of $x'(t) = A(t)x(t)$ and T denotes transpose, is nonsingular. In this case the control function is given by,

$$u(t) = -B^T(t)\Phi^T(t_0, t)U^{-1}(t_0, t_f)\{x_0 - \Phi(t_0, t_f)x_f\},$$

defined on $t_0 \le t \le t_f$, transfers $x(t_0) = x_0$ to $x(t_f) = x_f$.

Yamamoto in [68], has established sufficient condition for the controllability of the nonlinear system. Schauder's fixed point theorem is used to obtain the result. Some comparison theorems are proved which give conditions for the existence of a set satisfying the conditions of the main result.

Chapter 3

FIRST ORDER NONLINEAR DIFFERENTIAL EQUATIONS

3.1   Introduction

In this chapter, we undertake the study of the first order nonlinear differential equations with piecewise constant deviating argument (PCDA). The aim of this study is to develop some basic results useful for further investigation of the equations with PCDA with respect to the properties of its solution. This work continues the one done by Aftabizadeh [1-5], Cooke [15-19], Jayasree [34-37], Wiener[64-67] and others. Sectionwise contents of the chapter are as follows.

Section 2 deals with the preliminaries and notations. The concepts of solution, upper solution, and lower solution are defined. Two lemmas required for the further study are established. These are the simple extensions of the one found in the theory of ordinary differential equations.

In section 3, we establish the existence and uniqueness of the solution of the first order nonlinear differential equation with PCDA. This has been achieved by using the method of quasilinearisation, under two different conditions. In both the cases the convergence of the sequences is quadratic. In the first case, we have used a condition which reduces to convexity condition. In the second case, the method gives an algorithmic approach which

directly yields the construction of monotone sequences.

Section 4 is on inequalities related to equations with PCDA. First we have extended the Gronwall type inequality established by Jayasree and Deo. This is followed by inequalities estimating the solutions of two different equations with PCDA.

Section 5 deals with the investigation of the nonlinear periodic boundary value problem. The basic concepts are defined and the method of quasilinearisation is employed to obtain the existence of a solution. The necessary preliminary results and the solution of the associated linear boundary value problem are obtained.

Finally, in section 6, the oscillatory behaviour of a particular first order nonlinear differential equation with PCDA is discussed. The existence of solution of this equation is also established.

3.2 Notations and Preliminaries

In this section we introduce the notations, concepts and prove some basic results required for the further studies.

Consider the initial value problem (IVP),

$$x'(t) = f(t, x(t), x([t])), \quad x(0) = x_0,$$ (3.1)

where $t \in J = [0, T], T > 0$. $f \in C[J \times IR \times IR, IR]$, $[\cdot]$ is the greatest integer function, and $x_0 \in IR$ is a constant. Equation (3.1) is a piecewise constant delay differential equation because of the presence of the term $x([t])$, [see, 64].

We need to define a solution of the IVP (3.1).

Definition 3.2.1 *A solution of (3.1) on J is a function* $x : J \longrightarrow IR$ *that satisfies the conditions :*

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(i) x(t) is continuous on J,

(ii) the derivative x'(t) exists at each point $t \in J$, with the possible exception of the points $[t] \in J$, where one sided derivatives exist,

(iii) equation (3.1) is satisfied on each interval $J_n = [n, n+1)$ with integral end points.

We define an upper solution and a lower solution for the IVP (3.1).

Definition 3.2.2 *A continuous function $v : J \longrightarrow \mathbb{R}$ is said to be an upper solution of (3.1), if the derivative $v'(t)$ exists at each point $t \in J$, with the possible exception of the points $[t] \in J$, where one sided derivatives exist, and*

$$v'(t) \geq f(t, v(t), v([t])), v(0) \geq x_0.$$

It is said to be a lower solution if the reverse inequalities hold.

The following result is an extension of the lemma in [45] and is required for our further discussion.

Lemma 3.2.1 *Let $f(t, x, y) \in C[J \times \Omega \times \Omega, \mathbb{R}]$, Ω is an open interval in $\mathbb{R}$. Let the partial derivatives f_x, f_y both exist and are continuous for $t \in J$. Then*

$$(i) \; f(t, x_2, y) - f(t, x_1, y) = \int_0^1 [f_x(t, sx_2 + (1-s)x_1, y)](x_2 - x_1)ds.$$

$$(ii) \; f(t, x, y_2) - f(t, x, y_1) = \int_0^1 [f_y(t, x, sy_2 + (1-s)y_1)](y_2 - y_1)ds.$$

Proof :(i) Let $F(s) = f(t, sx_2 + (1-s)x_1, y)$, $0 \leq s \leq 1$.

Then $F(0) = f(t, x_1, y)$; $F(1) = f(t, x_2, y)$, and $\frac{dF}{ds} = f_x(t, sx_2 + (1-s)x_1, y) \cdot (x_2 - x_1)$.

Then on integration from 0 to 1, we get.

$$F(1) - F(0) = \{ \int_0^1 f_x(t, sx_2 + (1-s)x_1, y)ds \}(x_2 - x_1)$$

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This implies

$$f(t, x_2, y) - f(t, x_1, y) = \{\int_0^1 f_x(t, sx_2 + (1-s)x_1, y)ds\}(x_2 - x_1).$$

(ii) Let $G(s) = f(t, x, sy_2 + (1-s)y_1)$, $0 \leq s \leq 1$.

Then $G(0) = f(t, x, y_1)$; $G(1) = f(t, x, y_2)$, and $\frac{dG}{ds} = f_y(t, x, sy_2 + (1-s)y_1) \cdot (y_2 - y_1)$.

Then Therefore on integrating from 0 to 1, we get,

$$G(1) - G(0) = \{\int_0^1 f_y(t, x, sy_2 + (1-s)y_1)ds\}(y_2 - y_1)$$

This implies,

$$f(t, x, y_2) - f(t, x, y_1) = \{\int_0^1 f_y(t, x, sy_2 + (1-s)y_1)ds\}(y_2 - y_1).$$

We now establish an important result concerning upper and lower solutions.

Lemma 3.2.2 *Let u, v be lower and upper solutions of (3.1), respectively, such that*

$$u'(t) \leq f(t, u(t), u([t]))$$

$$v'(t) \geq f(t, v(t), v([t])),$$

$u(0) \leq x_0 \leq v(0)$, where $f \in C[J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$. Suppose that $f(t, x, y)$ is nondecreasing in y for $(t, x) \in J \times \mathbb{R}$ and satisfy the condition

$$f(t, x_1, y_1) - f(t, x_2, y_2) \leq L[(x_1 - x_2) + (y_1 - y_2)], \tag{3.2}$$

whenever $x_1 \geq x_2, y_1 \geq y_2$ and $L > 0$ is a constant. Then $u(t) \leq v(t)$, for all $t \in J$.

Proof : Let $t \in [n.n+1)$, and $u_n(t)$, $v_n(t)$ denote the lower and upper solution respectively on the unit interval $[n, n+1)$.

We show that $u_n(n) \leq v_n(n)$, $n = 0, 1, 2, \ldots \Rightarrow u_n(t) \leq v_n(t)$, for $t \in [n, n+1)$.

Let us first suppose that

$$u_n'(t) \leq f(t, u_n(t), u_n(n))$$

$$v_n'(t) > f(t, v_n(t), v_n(n)), \quad t \in [n, n+1).$$

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and $u_n(n) < v_n(n)$. We have to show that $u_n(t) < v_n(t)$ on $[n, n+1)$. If not , then there exists $t_n \in (n, n+1)$ such that $u_n(t_n) = v_n(t_n)$ and $u_n(t) < v_n(t)$, $\forall\ t \in [n, t_n)$.

For small $h < 0$, $u_n(t_n + h) - u_n(t_n) < v_n(t_n + h) - v_n(t_n)$.

This gives

$$\frac{u_n(t_n + h) - u_n(t_n)}{h} > \frac{v_n(t_n + h) - v_n(t_n)}{h}.$$

and hence , we get $u'_n(t_n) \geq v'_n(t_n)$. From this we conclude that

$$f(t, u_n(t_n), u_n(n)) \geq f(t, v_n(t_n), v_n(n)) .$$

Since $u_n(n) \leq v_n(n)$ and $u_n(t_n) = v_n(t_n)$, the above inequality contradicts the nondecreasing property of f. Hence $u_n(t) < v_n(t)$ on $[n, n+1)$.

Next, define $\rho_n(t) = v_n(t) + \epsilon \cdot e^{2Lt}$, $t \in [n, n+1)$, and $\epsilon > 0$ is sufficiently small. Then $\rho_n(t) > v_n(t)$, $t \in [n, n+1)$. Therefore, using (3.2), we get,

$$f(t, \rho_n(t), \rho_n(n)) - f(t, v_n(t), v_n(n)) \leq L\{(\rho_n(t) - v_n(t)) + (\rho_n(n) - v_n(n))\}$$
$$= L\{\epsilon \cdot e^{2Lt} + \epsilon \cdot e^{2Ln}\}.$$

This yields, $f(t, \rho_n(t), \rho_n(n)) \leq L \cdot \epsilon \cdot \{e^{2Lt} + e^{2Ln}\} + f(t, v_n(t), v_n(n))$.

Hence using definition of $\rho_n(t)$, we get,

$$\begin{aligned}
\rho'_n(t) &= v'_n(t) + 2\epsilon L e^{2Lt} \\
&> f(t, v_n(t), v_n(n)) + 2\epsilon L e^{2Lt}. \\
&> f(t, \rho_n(t), \rho_n(n)) - L\epsilon(e^{2Lt} + e^{2Ln}) + 2\epsilon L e^{2Lt}. \\
&= f(t, \rho_n(t), \rho_n(n)) + L\epsilon(e^{2Lt} - e^{2Ln}) ,\ t \in [n, n+1). \\
&> f(t, \rho_n(t), \rho_n(n)).
\end{aligned}$$

Also , $u'_n(t) \leq f(t, u_n(t), u_n(n))$ and $u_n(n) < \rho_n(n)$, $t \in [n, n+1)$.

Hence it follows that $u_n(t) < \rho_n(t)$, $\forall t \in [n, n+1)$.

Let $\epsilon \to 0$. Then we get, $u_n(t) \leq v_n(t)$, $\forall t \in [n, n+1)$, $n = 0, 1, \ldots$

and this completes the proof.

The next result is an extension of Gronwall's integral inequality for equation with PCDA. Its proof can be found in [34].

Lemma 3.2.3 *Let $c_0 \geq 0$ be a constant, and $u, a, b \in C[J, \mathbb{R}^+]$.*

If

$$u(t) \leq c_0 + \int_0^t [a(s)u(s) + b(s)u([s])] ds, t \in J$$

holds, then for $t \in J$,

$$u(t) \leq c_0 \prod_{k=1}^{[t]} \{ exp(\int_{k-1}^k a(r)dr) + \int_{k-1}^k exp(\int_s^k a(r)dr)b(s)ds \}$$

$$\times \{ exp(\int_{[t]}^t a(r)dr + \int_{[t]}^t exp(\int_{[t]}^t a(r)dr)b(s)ds \}.$$

In particular, when $a(t) = a$, and $b(t) = b$, $t \in J$ are constant functions, then

$$u(t) \leq c_0[(1 + \frac{b}{a})e^a - \frac{b}{a}]^{[t]}[(1 + \frac{b}{a})e^{a(t-[t])} - \frac{b}{a}].$$

3.3 Method of Quasilinearisation

In this section, the well-known method of quasilinearisation is extended to differential equations with PCDA. The technique involves the application of lower and upper solution and differential inequalities. The upper and lower sequences converge to the unique solution quadratically. The quasilinearisation method involves constructions of monotone sequences of approximate solutions to linear differential equations. The concept of lower and upper solutions and related differential inequalities are used in getting the desired results under the relaxed conditions.

In the following theorem, we assume the convexity type condition on f, and obtain the desired result.

Theorem 3.3.1 *Assume that*

(h1) $u_0, v_0 \in C[J, \mathbb{R}]$ are lower and upper solutions of (3.1) respectively such that

$u_0(t) \leq v_0(t)$ *on J;*

(h2) Let $\Omega = \{(t, x, y) : u_0 \leq x \leq v_0,\ u_0 \leq y \leq v_0,\ t \in J\}$ and

$f \in C[\Omega, \mathbb{R}]$ *such that f_x, f_y, exist and are continuous on Ω;*

(h3) $f_x(t, x, y)$ and $f_y(t, x, y)$ are increasing in y for fixed t,x, on Ω

(h4) For each $x \geq y$

$$f(t, x(t), x([t])) \;\geq\; f(t, y(t), y([t])) + f_x(t, y(t), y([t]))(x(t) - y(t))$$

$$+ f_y(t, y(t), y([t]))(x([t]) - y([t])), \tag{3.3}$$

for $(t, x, y) \in \Omega$. Then there exist monotone sequences $\{u_k(t)\}, \{v_k(t)\}$, with $u_k(t) \leq v_k(t)$, which converge uniformly to the unique solution $x(t)$ of (3.1), for $t \in J$.

Proof : Let $u_1(t)$ and $v_1(t)$ be the solutions of the related linear differential equations with PCDA.

$$u_1'(t) \;=\; f(t, u_0(t), u_0([t])) + f_x(t, u_0(t), u_0([t]))(u_1(t) - u_0(t))$$

$$+ f_y(t, u_0(t), u_0([t]))(u_1([t]) - u_0([t])),\ u_1(0) = x_0. \tag{3.4}$$

and

$$v_1'(t) \;=\; f(t, v_0(t), v_0([t]) + f_x(t, v_0(t), v_0([t]))(v_1(t) - v_0(t))$$

$$+ f_y(t, v_0(t), v_0([t]))(v_1([t]) - v_0([t])),\ v_1(0) = x_0, \tag{3.5}$$

such that $u_0(0) \leq x_0 \leq v_0(0)$, where u_0 and v_0 are as defined in (h1).

We first prove that

$$u_0(t) \leq u_1(t) \leq v_1(t) \leq v_0(t),\ for\ all\ t \in J. \tag{3.6}$$

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Let $p(t) = u_0(t) - u_1(t)$. Clearly $p(0) \leq 0$. We have,

$$
\begin{aligned}
p'(t) &= u_0'(t) - u_1'(t) \\
&\leq f(t, u_0(t), u_0([t])) - f(t, u_0(t), u_0([t])) - f_x(t, u_0(t), u_0([t]))(u_1(t) - u_0(t)) \\
&\quad - f_y(t, u_0(t), u_0([t]))(u_1([t]) - u_0([t])) \\
&= f_x(t, u_0(t), u_0([t]))p(t) + f_y(t, u_0(t), u_0([t]))p([t]).
\end{aligned}
$$

Therefore, we can treat $p(t)$ as a lower solution of

$$
z'(t) = f_x(t, u_0(t), u_0([t]))z(t) + f_y(t, u_0(t), u_0([t]))z([t]), \quad z(0) = 0. \tag{3.7}
$$

Further $\tilde{p}(t) \equiv 0$ can be treated as the upper solution of the equation (3.7)
Therefore,

$$
\begin{aligned}
0 &\equiv \tilde{p}'(t) = f_x(t, u_0(t), u_0([t]))\tilde{p}(t) + f_y(t, u_0(t), u_0([t]))\tilde{p}([t]) \\
\tilde{p}(0) &= 0.
\end{aligned}
$$

Hence, applying the special case of Lemma 3.2.2, we get $p(t) \leq 0$, for all $t \geq 0$,
which yields $u_0(t) \leq u_1(t)$ on J.

Similarly, we can show that $v_1(t) \leq v_0(t)$ on J.

Next, let $q(t) = u_1(t) - v_0(t)$. Clearly $q(0) = u_1(0) - v_0(0) = x_0 - v_0 \leq 0$.
and

$$
\begin{aligned}
q'(t) &= u_1'(t) - v_0'(t). \\
&\leq f(t, u_0(t), u_0([t])) + f_x(t, u_0(t), u_0([t]))(u_1(t) - u_0(t)) \\
&\quad + f_y(t, u_0(t), u_0([t]))(u_1([t]) - u_0([t])) - f(t, v_0(t), v_0([t])). \tag{3.8}
\end{aligned}
$$

But since $v_0(t) \geq u_0(t)$, we get using (3.3)

$$
\begin{aligned}
f(t, v_0(t), v_0([t])) &\geq f(t, u_0(t), u_0([t])) + f_x(t, u_0(t), u_0([t]))(v_0(t) - u_0(t)) \\
&\quad + f_y(t, u_0(t), u_0([t]))(v_0([t]) - u_0([t])).
\end{aligned}
$$

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This with(3.8) gives,

$$q'(t) \leq f_x(t, u_0(t), u_0([t]))q(t) + f_y(t, u_0(t), u_0([t]))q([t])$$

since $q(0) \leq 0$, we get $q(t) \leq 0$, for all $t \in J$, and hence we conclude that $u_1(t) \leq v_0(t), t \in J$.

Next to prove that $u_1(t) \leq v_1(t)$, $t \in J$, we note that $u_0(t) \leq u_1(t) \leq v_0(t)$. Hence using (3.3) and (3.4), we get,

$$u_1'(t) \leq f(t, u_1(t), u_1([t]))$$

Similarly, using (3.3) and (3.5), and since $v_0 \geq v_1$, we get,

$$v_1'(t) \geq f(t, v_1(t), v_1([t])).$$

Hence, since $u_1(0) = v_1(0) = x_0$, again by the special case of Lemma 3.2.2, we get $u_1(t) \leq v_1(t)$, $t \in J$. This establishes the inequalities in (3.6).

Now assume that, for $k > 1$,

$$
\begin{aligned}
u_k'(t) &\leq f(t, u_k(t), u_k([t])) \\
v_k'(t) &\geq f(t, v_k(t), v_k([t]))
\end{aligned}
$$

and $u_0(t) \leq u_1(t) \leq ... \leq u_k(t) \leq v_k(t) \leq ... \leq v_1(t) \leq v_0(t)$, $t \in J$. We shall show that

$$u_k(t) \leq u_{k+1}(t) \leq v_{k+1}(t) \leq v_k(t) \ on J. \tag{3.9}$$

where $u_{k+1}(t)$ and $v_{k+1}(t)$ are the solutions of the linear IVP

$$
\begin{aligned}
u_{k+1}'(t) = \ &f(t, u_k(t), u_k([t])) + f_x(t, u_k(t), u_k([t]))(u_{k+1}(t) - u_k(t)) \\
&+ f_y(t, u_k(t), u_k([t]))(u_{k+1}([t]) - u_k([t])), \ u_{k+1}(0) = x_0. \tag{3.10}
\end{aligned}
$$

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and

$$v'_{k+1}(t) = f(t, v_k(t), v_k([t])) + f_x(t, v_k(t), v_k([t]))(v_{k+1}(t) - v_k(t))$$
$$+ f_y(t, v_k(t), v_k([t]))(v_{k+1}([t]) - v_k([t])), \quad v_{k+1}(0) = x_0. \qquad (3.11)$$

Let $p(t) = u_k(t) - u_{k+1}(t)$, then $p(0) = 0$. Therefore, in view of (3.10), we get

$$p'(t) = u'_k(t) - u'_{k+1}(t)$$
$$\leq f_x(t, u_k(t), u_k([t]))p(t) + f_y(t, u_k(t), u_k([t]))p([t]).$$

This fact with $p(0) = 0$ and the application of the special case of Lemma 3.2.2, we get
$p(t) \leq 0$ which means $u_k(t) \leq u_{k+1}(t)$, $t \in J$.

Again, let $q(t) = u_{k+1}(t) - v_k(t)$. Then

$$q'(t) = u'_{k+1}(t) - v'_k(t),$$
$$\leq f(t, u_k(t), u_k([t])) + f_x(t, u_k(t), u_k([t]))(u_{k+1}(t) - u_k(t))$$
$$- f_y(t, u_k(t), u_k([t]))(u_{k+1}([t]) - u_k([t])) - f(t, v_k(t), v_k([t])). \qquad (3.12)$$

Now since $v_k \geq u_k$, using (3.3) and (3.12), we get,

$$q'(t) \leq f_x(t, u_k(t), u_k([t]))q(t) + f_y(t, u_k(t), u_k([t]))q([t]).$$

Since $q(0) = 0$, by the application of special case of Lemma 3.2.2 as before,
we get, $q(t) \leq 0$, for all $t \in J$. Hence $u_{k+1}(t) \leq v_k(t)$, $t \in J$.

Thus we have $u_k(t) \leq u_{k+1}(t) \leq v_k(t)$, $t \in J$. Similarly, we can show that
$u_k(t) \leq v_{k+1}(t) \leq v_k(t)$, $t \in J$.

Next using (3.10) and (3.3), we get,

$$u'_{k+1}(t) \leq f(t, u_{k+1}(t), u_{k+1}([t])).$$

Similarly, we can show that

$$v'_{k+1}(t) \geq f(t, v_{k+1}(t), v_{k+1}([t])).$$

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Hence, since $u_{k+1}(0) \leq x_0 \leq v_{k+1}(0)$, by the special case of Lemma 3.2.2,

we get $u_{k+1}(t) \leq v_{k+1}(t)$, $t \in J$.

Thus we have $u_k(t) \leq u_{k+1}(t) \leq v_{k+1}(t) \leq v_k(t), t \in J$.

Hence, by induction, we get

$$u_0(t) \leq u_1(t) \leq \ldots \leq u_k(t) \leq v_k(t) \leq \ldots \leq v_1(t) \leq v_0(t) \ on \ J.$$

Now assume that $\lim_{k \to \infty} u_k(t) = \alpha(t)$. Then integrating (3.10) on both sides between 0

and t and then taking limits as $k \to \infty$, we get

$$\alpha(t) = x_0 + \int_0^t f(s, \alpha(s), \alpha([s])) ds,$$

which shows that $\alpha(t)$ is a solution of the IVP (3.1). Similarly we can show that $\{v_k(t)\}$ also

converge to a solution $\beta(t)$ of the IVP (3.1). Since solution of (3.1) is unique,

we have, $\alpha(t) = \beta(t) = x(t)$. Thus $\{u_k(t)\}, \{v_k(t)\}$ converge uniformly and monotonically

to the unique solution x(t) of (3.1) on J.

In the next theorem, we show that the convergence of the sequences is quadratic in the

sense of Definition 2.4.1 .

Theorem 3.3.2 *Under the hypothesis of Theorem 3.3.1, the convergence of*

$\{u_k(t)\}, \{v_k(t)\}$ *is quadratic .*

Proof : Define $p_{k+1}(t) = x(t) - u_{k+1}(t) \geq 0$, $k = 0, 1, 2, \ldots$

so that $p_{k+1}(0) = 0$. Therefore,

$$
\begin{aligned}
p'_{k+1}(t) &= x'(t) - u'_{k+1}(t) \\
&= f(t, x(t), x([t])) - \{f(t, u_k(t), u_k([t])) \\
&\quad + f_x(t, u_k(t), u_k([t]))(u_{k+1}(t) - u_k(t)) \\
&\quad + f_y(t, u_k(t), u_k([t]))(u_{k+1}([t]) - u_k([t]))\}
\end{aligned}
$$

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$$= f(t, x(t), x([t])) - f(t, u_k(t), u_k([t]))$$

$$- f_x(t, u_k(t), u_k([t]))(u_{k+1}(t) - u_k(t))$$

$$- f_y(t, u_k(t), u_k([t]))(u_{k+1}([t]) - u_k([t]))$$

$$= f(t, x(t), x([t])) - f(t, u_k(t), x([t]))$$

$$+ f(t, u_k(t), x([t])) - f(t, u_k(t), u_k([t]))$$

$$- f_x(t, u_k(t), u_k([t]))(u_{k+1}(t) - u_k(t))$$

$$- f_y(t, u_k(t), u_k([t]))(u_{k+1}([t]) - u_k([t])) \} .$$

A simple computation using Lemma 3.2.1 and (h3) yields.

$$p'_{k+1}(t) = \int_0^1 [f_x(t, sx(t) + (1-s)u_k(t), x([t])) - f_x(t, u_k(t), u_k([t]))]p_k(t)ds$$

$$+ \int_0^1 [f_y(t, u_k(t), sx([t]) + (1-s)u_k([t])) - f_y(t, u_k(t), u_k([t]))]p_k([t])ds$$

$$+ f_x(t, u_k(t), u_k([t]))(p_{k+1}(t) + f_y(t, u_k(t), u_k([t]))(p_{k+1}([t]))$$

$$\leq \int_0^1 L_1[sp_k(t) + p_k([t])]p_k(t)ds + \int_0^1 L_2 sp_k^2([t])ds$$

$$+ M_1 p_{k+1}(t) + M_2 p_{k+1}([t])$$

where $|f_x(t, x, y)| \leq M_1$, $|f_y(t, x, y)| \leq M_2$ on Ω, and L_1, L_2 are constants as in (3.2). Further simplification yields.

$$p'_{k+1}(t) \leq \frac{L_1}{2}p_k^2(t) + L_1 p_k([t])p_k(t) + \frac{L_2}{2}p_k^2([t])$$

$$+ M_1 p_{k+1}(t) + M_2 p_{k+1}([t]) .$$

Hence,

$$p'_{k+1}(t) \leq L_1 p_k^2(t) + L_3 p_k^2([t]) + M_1 p_{k+1}(t) + M_2 p_{k+1}([t]).$$

$$\leq \max_j L_1 p_k^2(t) + \max_j L_3 p_k^2([t]) + M_1 p_{k+1}(t) + M_2 p_{k+1}([t]).$$

$$\leq L \max_j p_k^2(t), \tag{3.13}$$

where $L, L_i, i = 1, 2, 3$ are suitable constants.

Let $E = L \max_{t \in J} p_k^2(t)$. Then on integrating (3.13), we get,

$$p_{k+1}(t) \le E.T + \int_0^t [M_1 p_{k+1}(s) + M_2 p_{k+1}([s])]ds, t \in J.$$

By particular case of Lemma 3.2.3, this inequality yields,

$$p_{k+1}(t) \le \frac{E.T}{M_1^2}[(M_1 + M_2)e^{M_1} - M_2]^T[(M_1 + M_2)e^{M_1 T} - M_2].$$

Hence we obtain the estimate,

$$\max_J |x(t) - u_{k+1}(t)| \le C[L \max_J |x(t) - u_k(t)|^2],$$

where $C = \frac{T}{M_1^2}[(M_1 + M_2)e^{M_1} - M_2]^T[(M_1 + M_2)e^{M_1 T} - M_2]$.

Similarly, if we define $q_{k+1}(t) = v_{k+1}(t) - x(t) \ge 0$, so that

$q_{k+1}(0) = 0$, $k = 0, 1, 2, \ldots$ then we can obtain the estimate

$$\max_J |v_{k+1}(t) - x(t)| \le \tilde{C}[\tilde{L} \max_J |v_k(t) - x(t)|^2],$$

where $\tilde{C}, \tilde{L}$ are suitable constants.

This completes the proof.

Remark 3.3.1 *If $f(t, x(t), x([t])) = f(t, x(t))$, then the above results reduce to those in [8, 10].*

We now extend the method of quasilinerisation used in the above theorem to obtain an algorithmic approach which will make possible to construct the monotone sequences. These sequences are of the solutions of the associated linear differential equations with PCDA.

Theorem 3.3.3 *Assume that*

(H1) $u_0, v_0 \in C[J, \mathbb{R}]$ are lower and upper solutions of (3.1) respectively such that

$u_0(t) \le v_0(t)$ on J.

(H2) Let $\Omega = \{(t, x, y) : u_0 \le x \le v_0, u_0 \le y \le v_0, t \in J\}$ and

$f, \phi \in C[\Omega, \mathbb{R}]$ such that f_x, f_y, ϕ_x, ϕ_y, exist and are continuous on Ω;

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(H3) (i) $f_x(t, x, y), f_y(t, x, y)$ *are increasing in y for fixed t, x,*

   (ii) $\phi_x(t, x, y)$ *is increasing in x for fixed t, y,*

   (iii) $\phi_x(t, x, y)$ *is increasing in y for fixed t, x,*

   (iv) $\phi_y(t, x, y)$ *is increasing in y for fixed t, x,*

on Ω.

(H4) Let $F(t, x, y) = f(t, x, y) + \phi(t, x, y)$ *and for each* $x \geq y$

$$
\begin{aligned}
f(t, x(t), x([t])) \geq\ & F(t, y(t), y([t])) + F_x(t, y(t), y([t]))[x(t) - y(t)] \\
& + F_y(t, y(t), y([t]))[x([t]) - y([t])] - \phi(t, x(t), x([t])) \quad (3.14)
\end{aligned}
$$

for $(t, x, y) \in \Omega$.

(H5) $F_y(t, m(t), m([t])) - \phi_y(t, n(t), n([t])) \geq 0$,

   $u_0(t) \leq m(t) \leq n(t) \leq v_0(t), t \in J$.

   Then there exist monotone sequences $\{u_k(t)\}, \{v_k(t)\}$, $t \in J$, *which converge uniformly to the unique solution* $x(t)$ *of (3.1), for* $t \in J$.

Proof : Let $u_1(t)$ and $v_1(t)$ be the solutions of the related linear differential equations with PCDA,

$$
\begin{aligned}
u_1'(t) =\ & f(t, u_0(t), u_0([t])) \\
& + [F_x(t, u_0(t), u_0([t])) - \phi_x(t, v_0(t), v_0([t]))](u_1(t) - u_0(t)) \\
& + [F_y(t, u_0(t), u_0([t])) - \phi_y(t, v_0(t), v_0([t]))](u_1([t]) - u_0([t])) \\
u_1(0) =\ & x_0, \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (3.15)
\end{aligned}
$$

and

$$
\begin{aligned}
v_1'(t) =\ & f(t, v_0(t), v_0([t]) \\
& + [F_x(t, u_0(t), u_0([t])) - \phi_x(t, v_0(t), v_0([t]))](v_1(t) - v_0(t))
\end{aligned}
$$

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$$+[F_y(t, u_0(t), u_0([t])) - \phi_y(t, v_0(t), v_0([t]))](v_1([t]) - v_0([t]))$$

$$v_1(0) = x_0, \tag{3.16}$$

such that $u_0(0) \le x_0 \le v_0(0)$, where u_0 and v_0 are as defined in (H1).

We first prove that

$$u_0(t) \le u_1(t) \le v_1(t) \le v_0(t), \; for \; all \; t \in J. \tag{3.17}$$

Let $p(t) = u_0(t) - u_1(t)$. Clearly $p(0) \le 0$. We have

$$
\begin{aligned}
p'(t) &= u_0'(t) - u_1'(t).\\
&\le f(t, u_0(t), u_0([t])) - f(t, u_0(t), u_0([t]))\\
&\quad -[F_x(t, u_0(t), u_0([t])) - \phi_x(t, v_0(t), v_0([t]))](u_1(t) - u_0(t))\\
&\quad -[F_y(t, u_0(t), u_0([t])) - \phi_y(t, v_0(t), v_0([t]))](u_1([t]) - u_0([t]))\\
&= [F_x(t, u_0(t), u_0([t])) - \phi_x(t, v_0(t), v_0([t]))]p(t)\\
&\quad +[F_y(t, u_0(t), u_0([t])) - \phi_y(t, v_0(t), v_0([t]))]p([t]).
\end{aligned}
$$

Therefore, treating $p(t)$ as a lower solution of

$$
\begin{aligned}
z'(t) &= [F_x(t, u_0(t), u_0([t])) - \phi_x(t, v_0(t), v_0([t]))]z(t)\\
&\quad +[F_y(t, u_0(t), u_0([t])) - \phi_y(t, v_0(t), v_0([t]))]z([t])\\
z(0) &= 0. \tag{3.18}
\end{aligned}
$$

Further $\tilde{p}(t) \equiv 0$ can be treated as the upper solution of the equation (3.18) Therefore,

$$
\begin{aligned}
0 \equiv \tilde{p}'(t) &= [F_x(t, u_0(t), u_0([t])) - \phi_x(t, v_0(t), v_0([t]))]\tilde{p}(t)\\
&\quad +[F_y(t, u_0(t), u_0([t])) - \phi_y(t, v_0(t), v_0([t]))]\tilde{p}([t])\\
\tilde{p}(0) &= 0.
\end{aligned}
$$

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Hence applying the special case of Lemma 3.2.2 and (H5), we get $p(t) \le 0$,

for all $t \ge 0$, which yields $u_0(t) \le u_1(t)$ on J.

Similarly, we can show that $v_1(t) \le v_0(t)$ on J.

Next, let $q(t) = u_1(t) - v_0(t)$. Clearly $q(0) = u_1(0) - v_0(0) = x_0 - v_0(0) \le 0$,

and

$$
\begin{aligned}
q'(t) &= u_1'(t) - v_0'(t). \\
&\le f(t, u_0(t), u_0([t])) \\
&\quad + [F_x(t, u_0(t), u_0([t])) - \phi_x(t, v_0(t), v_0([t]))](u_1(t) - u_0(t)) \\
&\quad + [F_y(t, u_0(t), u_0([t])) - \phi_y(t, v_0(t), v_0([t]))](u_1([t]) - u_0([t])) \\
&\quad - f(t, v_0(t), v_0([t])).
\end{aligned}
\tag{3.19}
$$

But since $v_0(t) \ge u_0(t)$, we get using (3.14)

$$
\begin{aligned}
f(t, v_0(t), v_0([t])) &\ge f(t, u_0(t), u_0([t])) \\
&\quad + [F_x(t, u_0(t), u_0([t]))](v_0(t) - u_0(t)) \\
&\quad + [F_y(t, u_0(t), u_0([t]))](v_0([t]) - u_0([t])) \\
&\quad - [\phi(t, v_0(t), v_0([t])) - \phi(t, u_0(t), u_0([t]))].
\end{aligned}
\tag{3.20}
$$

By the Mean value theorem,

$$
\begin{aligned}
&\phi(t, v_0(t), v_0([t])) - \phi(t, u_0(t), u_0([t])) \\
&= \phi(t, v_0(t), v_0([t])) - \phi(t, u_0(t), v_0([t])) + \phi(t, u_0(t), v_0([t])) - \phi(t, u_0(t), u_0([t])) \\
&= \phi_x(t, \xi, v_0([t]))(v_0(t) - u_0(t)) + \phi_y(t, u_0(t), \eta)(v_0([t]) - u_0([t])).
\end{aligned}
\tag{3.21}
$$

where ξ and η are such that $u_0(t) \le \xi \le v_0(t)$ and $u_0([t]) \le \eta \le v_0([t])$.

Now using (II3)(ii), (II3)(iv), (3.21) yields,

$$
\phi(t, v_0(t), v_0([t])) - \phi(t, u_0(t), u_0([t]))
$$

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$$\leq \quad \phi_x(t, v_0(t), v_0([t]))(v_0(t) - u_0(t)) + \phi_y(t, u_0(t), v_0([t]))(v_0([t]) - u_0([t])). \quad (3.22)$$

Hence in view of (3.20), we obtain

$$
\begin{aligned}
f(t, v_0(t), v_0([t])) \quad \geq \quad & f(t, u_0(t), u_0([t])) \\
& + F_x(t, u_0(t), u_0([t]))(v_0(t) - u_0(t)) + F_y(t, u_0(t), u_0([t]))(v_0([t]) - u_0([t])) \\
& - \phi_x(t, v_0(t), v_0([t]))(v_0(t) - u_0(t)) - \phi_y(t, u_0(t), v_0([t]))(v_0([t]) - u_0([t])).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f(t, v_0(t), v_0([t])) \quad \geq \quad & f(t, u_0(t), u_0([t])) \\
& + [F_x(t, u_0(t), u_0([t])) - \phi_x(t, v_0(t), v_0([t]))](v_0(t) - u_0(t)) \\
& + [F_y(t, u_0(t), u_0([t])) - \phi_y(t, v_0(t), v_0([t]))](v_0([t]) - u_0([t]))
\end{aligned}
$$

$$(3.23)$$

This with (3.19) gives,

$$
\begin{aligned}
q'(t) \quad \leq \quad & [F_x(t, u_0(t), u_0([t])) - \phi_x(t, v_0(t), v_0([t]))]q(t) \\
& + [F_y(t, u_0(t), u_0([t])) - \phi_y(t, v_0(t), v_0([t]))]q([t]).
\end{aligned}
$$

Since $q(0) \leq 0$, we get $q(t) \leq 0$, for all $t \in J$, and hence we conclude that

$$u_1(t) \leq v_0(t), t \in J.$$

Next to prove that $u_1(t) \leq v_1(t)$, $t \in J$, we note that $u_0(t) \leq u_1(t) \leq v_0(t)$. Hence using (3.14) and (3.15), we get,

$$
\begin{aligned}
u_1'(t) \quad \leq \quad & f(t, u_1(t), u_1([t])) + \phi(t, u_1(t), u_1([t])) - \phi(t, u_0(t), u_0([t])) \\
& - \phi_x(t, v_0(t), v_0([t]))](u_1(t) - u_0(t)) - \phi_y(t, v_0(t), v_0([t]))](u_1([t]) - u_0([t])).
\end{aligned}
$$

Employing the mean value theorem and (H3)(ii), (H3)(iv), we get,

$$u_1'(t) \quad \leq \quad f(t, u_1(t), u_1([t]))$$

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$$+\phi_x(t,v_0(t),v_0([t]))](u_1(t)-u_0(t))+\phi_y(t,v_0(t),v_0([t]))](u_1([t])-u_0([t]))$$

$$-\phi_x(t,v_0(t),v_0([t]))](u_1(t)-u_0(t))-\phi_y(t,v_0(t),v_0([t]))](u_1([t])-u_0([t]))$$

$$= f(t,u_1(t),u_1([t])).$$

Similarly, we can show that $v_1'(t) \geq f(t,v_1(t),v_1([t]))$.

Hence, since $u_1(0)=v_1(0)=x_0$, again by the special case of Lemma 3.2.2,

and (H5), we get $u_1(t) \leq v_1(t)$, $t \in J$. This establishes the inequalities in (3.17).

Now assume that, for $k > 1$,

$$u_k'(t) \leq f(t,u_k(t),u_k([t])),$$
$$v_k'(t) \geq f(t,v_k(t),v_k([t])),$$

and $u_0(t) \leq u_1(t) \leq ... \leq u_k(t) \leq v_k(t) \leq ... \leq v_1(t) \leq v_0(t)$, on J.
We shall show that

$$u_k(t) \leq u_{k+1}(t) \leq v_{k+1}(t) \leq v_k(t), \quad on\ J,$$

where $u_{k+1}(t)$ and $v_{k+1}(t)$ are the solutions of the linear IVP,

$$u_{k+1}'(t) = f(t,u_k(t),u_k([t]))$$
$$+[F_x(t,u_k(t),u_k([t]))-\phi_x(t,v_k(t),v_k([t]))](u_{k+1}(t)-u_k(t))$$
$$+[F_y(t,u_k(t),u_k([t]))-\phi_y(t,v_k(t),v_k([t]))](u_{k+1}([t])-u_k([t]))$$
$$u_{k+1}(0) = x_0. \tag{3.24}$$

and

$$v_{k+1}'(t) = f(t,v_k(t),v_k([t]))$$
$$+[F_x(t,u_k(t),u_k([t]))-\phi_x(t,v_k(t),v_k([t]))](v_{k+1}(t)-v_k(t))$$
$$+[F_y(t,u_k(t),u_k([t]))-\phi_y(t,v_k(t),v_k([t]))](v_{k+1}([t])-v_k([t]))$$
$$v_{k+1}(0) = x_0. \tag{3.25}$$

Let $p(t) = u_k(t) - u_{k+1}(t)$, then $p(0) = 0$. Therefore, in view of (3.24), we get

$$
\begin{aligned}
p'(t) &= u'_k(t) - u'_{k+1}(t) \\[4pt]
&\leq [F_x(t, u_k(t), u_k([t])) - \phi_x(t, v_k(t), v_k([t]))]p(t) \\[4pt]
&\quad + [F_y(t, u_k(t), u_k([t])) - \phi_y(t, v_k(t), v_k([t]))]p([t])
\end{aligned}
$$

This fact with $p(0) = 0$ and the application of the special case of Lemma 3.2.2 and (H5), we get, $p(t) \leq 0$ which means $u_k(t) \leq u_{k+1}(t)$, $t \in J$.

Again, let $q(t) = u_{k+1}(t) - v_k(t)$. Then

$$
\begin{aligned}
q'(t) &= u'_{k+1}(t) - v'_k(t). \\[4pt]
&\leq f(t, u_k(t), u_k([t])) \\[4pt]
&\quad + [F_x(t, u_k(t), u_k([t])) - \phi_x(t, v_k(t), v_k([t]))](u_{k+1}(t) - u_k(t)) \\[4pt]
&\quad - [F_y(t, u_k(t), u_k([t])) - \phi_y(t, v_k(t), v_k([t]))](u_{k+1}([t]) - u_k([t])) \\[4pt]
&\quad - f(t, v_k(t), v_k([t])).
\end{aligned}
\tag{3.26}
$$

Employing (3.14), the mean value theorem, and (H3)(ii), (H3)(iv), we arrive at

$$
\begin{aligned}
f(t, v_k(t), v_k([t])) &\geq f(t, u_k(t), u_k([t])) \\[4pt]
&\quad + [F_x(t, u_k(t), u_k([t])) - \phi_x(t, v_k(t), v_k([t]))](v_k(t) - u_k(t)) \\[4pt]
&\quad + [F_y(t, u_k(t), u_k([t])) - \phi_y(t, v_k(t), v_k([t]))](v_k([t]) - u_k([t])).
\end{aligned}
$$

Hence (3.26) gives,

$$
\begin{aligned}
q'(t) &\leq [F_x(t, u_k(t), u_k([t])) - \phi_x(t, v_k(t), v_k([t]))]q(t) \\[4pt]
&\quad + [F_y(t, u_k(t), u_k([t])) - \phi_y(t, v_k(t), v_k([t]))]q([t]).
\end{aligned}
$$

Since $q(0) = 0$, by the application of special case of Lemma 3.2.2 and (H5) as before, we get, $q(t) \leq 0$, for all $t \in J$. Hence $u_{k+1}(t) \leq v_k(t)$, $t \in J$.

Thus we have $u_k(t) \leq u_{k+1}(t) \leq v_k(t)$, $t \in J$.

Similarly, we can show that $u_k(t) \leq v_{k+1}(t) \leq v_k(t)$, $t \in J$.

Next using (3.24) and (3.14), we get,

$$
\begin{aligned}
u'_{k+1}(t) \leq\ & f(t, u_{k+1}(t), u_{k+1}([t])) \\
& + \phi(t, u_{k+1}(t), u_{k+1}([t])) - \phi(t, u_k(t), u_k([t])) \\
& - \phi_x(t, v_k(t), v_k([t]))](u_{k+1}(t) - u_k(t)) \\
& - \phi_y(t, v_k(t), v_k([t]))](u_{k+1}([t]) - u_k([t])).
\end{aligned}
$$

Using the mean value theorem and (H3)(ii), (H3)(iv) we get,

$$
u'_{k+1}(t) \leq f(t, u_{k+1}(t), u_{k+1}([t])).
$$

Similarly, we can show that

$$
v'_{k+1}(t) \geq f(t, v_{k+1}(t), v_{k+1}([t])).
$$

Hence, since $u_{k+1}(0) \leq x_0 \leq v_{k+1}(0)$, by the special case of Lemma 3.2.2 and (H5),

we get, $u_{k+1}(t) \leq v_{k+1}(t)$, $t \in J$. Thus, we have $u_k(t) \leq u_{k+1}(t) \leq v_{k+1}(t) \leq v_k(t)$, $t \in J$.

Hence, by induction, we get,

$$
u_0(t) \leq u_1(t) \leq \dots \leq u_k(t) \leq v_k(t) \leq \dots \leq v_1(t) \leq v_0(t) \ on J.
$$

Now assume that $\lim_{k \to \infty} u_k(t) = \alpha(t)$. Then integrating (3.24) on both sides between 0

and t and then taking limits as $k \to \infty$, we get

$$
\alpha(t) = x_0 + \int_0^t f(s, \alpha(s), \alpha([s])) ds,
$$

which shows that $\alpha(t)$ is a solution of the IVP (3.1). Similarly we can show that

$\{v_k(t)\}$ also converge to a solution $\beta(t)$ of the IVP (3.1). Since solution of (3.1) is unique,

we have

$\alpha(t) = \beta(t) = x(t)$. Thus $\{u_k(t)\}, \{v_k(t)\}$ converge uniformly and monotonically to the unique solution x(t) of (3.1) on J.

Theorem 3.3.4 *Under the hypothesis of Theorem 3.2.3, the convergence of* $\{u_k(t)\}, \{v_k(t)\}$ *is quadratic .*

Proof : Define

$$p_{k+1}(t) = x(t) - u_{k+1}(t) \geq 0$$

$$q_{k+1}(t) = v_{k+1}(t) - x(t) \geq 0, k = 0, 1, 2, \ldots$$

so that $p_{k+1}(0) = 0$ and $q_{k+1}(0) = 0$. Therefore,

$$
\begin{aligned}
p'_{k+1}(t) &= x'(t) - u'_{k+1}(t) \\
&= f(t, x(t), x([t])) - \{f(t, u_k(t), u_k([t])) \\
&\quad + [F_x(t, u_k(t), u_k([t])) - \phi_x(t, v_k(t), v_k([t]))](u_{k+1}(t) - u_k(t)) \\
&\quad + [F_y(t, u_k(t), u_k([t])) - \phi_y(t, v_k(t), v_k([t]))](u_{k+1}([t]) - u_k([t]))\}
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
p'_{k+1}(t) &= F(t, x(t), x([t])) - F(t, u_k(t), x([t])) + F(t, u_k(t), x([t])) - F(t, u_k(t), u_k([t])) \\
&\quad + [F_x(t, u_k(t), u_k([t])) - \phi_x(t, v_k(t), v_k([t]))](p_{k+1}(t) - p_k(t)) \\
&\quad + [F_y(t, u_k(t), u_k([t])) - \phi_y(t, v_k(t), v_k([t]))](p_{k+1}([t]) - p_k([t])) \\
&\quad - [\phi(t, x(t), x([t])) - \phi(t, u_k(t), x([t]))] \\
&\quad - [\phi(t, u_k(t), x([t])) - \phi(t, u_k(t), u_k([t]))].
\end{aligned}
$$

By Lemma 3.2.1 and (II3) yields,

$$
\begin{aligned}
p'_{k+1}(t) &= \int_0^1 [F_x(t, sx(t) + (1-s)u_k(t), x([t])) - F_x(t, u_k(t), u_k([t]))]p_k(t)ds \\
&\quad + \int_0^1 [F_y(t, u_k(t), sx([t]) + (1-s)u_k([t])) - F_y(t, u_k(t), u_k([t]))]p_k([t])ds
\end{aligned}
$$

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$$+[F_x(t, u_k(t), u_k([t])) - \phi_x(t, v_k(t), v_k([t]))](p_{k+1}(t)$$

$$+[F_y(t, u_k(t), u_k([t])) - \phi_y(t, v_k(t), v_k([t]))](p_{k+1}([t])$$

$$+\int_0^1 [\phi_x(t, v_k(t), v_k([t])) - \phi_x(t, sx(t) + (1 - s)u_k(t), x([t]))]p_k(t)ds$$

$$+\int_0^1 [\phi_y(t, v_k(t), v_k([t])) - \phi_y(t, u_k(t), sx([t]) + (1 - s)u_k([t]))]p_k([t])ds$$

$$\leq \int_0^1 L_1[sp_k(t) + p_k([t])]p_k(t)ds + \int_0^1 L_2sp_k^2([t])ds$$

$$+(M_1 + N_1)p_{k+1}(t) + (M_2 + N_2)p_{k+1}([t])$$

$$+\int_0^1 L_3[(v_k(t) - x(t))$$

$$+ (1 - s)(x(t) - u_k(t)) + (v_k([t]) - x([t]))]p_k(t)ds$$

$$+\int_0^1 L_4[v_k(t) - x(t)) + (x(t) - u_k(t))$$

$$+ (v_k([t]) - x([t])) + (1 - s)(x([t]) - u_k([t]))]p_k([t])ds$$

where $|F_x(t, x, y)| \leq M_1$, $|F_y(t, x, y)| \leq M_2$, $|\phi_x(t, x, y)| \leq N_1$, $|\phi_y(t, x, y)| \leq N_2$ on Ω, and L_1, L_2, L_3, L_4 are constants as in (3.2). Further simplification yields,

$$p_{k+1}'(t) \leq \frac{L_1}{2}p_k^2(t) + L_1p_k([t])p_k(t) + \frac{L_2}{2}p_k^2([t])$$

$$+Mp_{k+1}(t) + Np_{k+1}([t])$$

$$+L_3q_k(t)p_k(t) + \frac{L_3}{2}p_k^2(t) + L_3q_k([t])p_k(t) + L_4q_k(t)p_k([t])$$

$$+L_4p_k(t)p_k([t] + L_4q_k([t])p_k([t]) + \frac{L_4}{2}p^2k([t]).$$

where $M = M_1 + N_1, N = M_2 + N_2$.

Hence,

$$p_{k+1}'(t) \leq Q_1p_k^2(t) + Q_2q_k^2(t) + Q_3p_k^2([t])$$

$$+Q_4q_k^2([t]) + Mp_{k+1}(t) + Np_{k+1}([t]).$$

$$\leq \max_j Q_1p_k^2(t) + \max_j Q_2q_k^2(t) + \max_j Q_3p_k^2([t])$$

$$+ \max_j Q_4q_k^2([t])] + Mp_{k+1}(t) + Np_{k+1}([t]).$$

$$\leq Q\max_j p_k^2(t) + \tilde{Q}\max_j q_k^2(t)$$

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$$+ M p_{k+1}(t) + N p_{k+1}([t]). \tag{3.27}$$

where $Q, \tilde{Q}, Q_i, i = 1, 2, 3, 4$ are suitable constants.

Let $E = Q \max_J p_k^2(t) + \tilde{Q} \max_J q_k^2(t)$. Then on integrating (3.27) we get

$$p_{k+1}(t) \leq E.T + \int_0^t [M p_{k+1}(s) + N p_{k+1}([s])] ds, t \in J$$

By particular case of Lemma 3.2.3, this inequality yields,

$$p_{k+1}(t) \leq \frac{E.T}{M^2} [(M + N)e^M - N]^T [(M + N)e^{MT} - N]$$

Hence we obtain the estimate,

$$max_J |x(t) - u_{k+1}(t)| \leq C[Q \max_j |x(t) - u_k(t)|^2 + \tilde{Q} \max_j |v_k(t) - x(t)|^2.$$

where $C = \frac{T}{M^2} [(M + N)e^M - N]^T [(M + N)e^{MT} - N]$.

Similarly, we can obtain

$$max_J |v_{k+1}(t) - x(t)| \leq \tilde{C}[R \max_j |v_k(t) - x(t)|^2 + \tilde{R} \max_j |x(t) - u_k(t)|^2,$$

where $\tilde{C}, R, \tilde{R}$, are suitable constants.

This completes the proof.

Remark 3.3.2 *If $f(t, x(t), x([t])) = f(t, x(t))$, then the above results reduce to those in [54]. Further in addition, if $\phi \equiv 0$, then condition (3.14) is equivalent to*

$$f(t, x) \geq f(t, y) + f_x(t, x)(x - y), for \ x \geq y,$$

which infact can be obtained from $f_{xx}(t, x) \geq 0$, that is f is a convex function as given in [10].

3.4 Inequalities:

In this section we establish some inequalities, which play a vital role in study of differential equations with deviating arguments. These inequalities are simple extensions to the exsisting ones in ordinary differential equations as well as functional differential equations [44]. The section starts with simple extensions to generalisation of Gronwall Bellman integral inequality to differential equations with PCDA obtained by Jayasree and Deo [36]. The two sided estimates related to solutions of two systems is obtained which will be useful in studying certain stability properties of the systems.

We first recall the inequality established by Jayasree and Deo[36].

Lemma 3.4.1 *Let c_0 be a constant and x, a, $b \in C[I, \mathbb{R}^+]$. If the inequality*

$$x(t) \leq c_0 + \int_0^t a(s)x(s) + b(s)x([s])ds, \ t \in I$$

holds, then for $t \in I = [0, \infty)$

$$x(t) \leq c_0 \cdot \prod_{k=1}^{[t]} \{ exp(\int_{k-1}^k a(r)dr) + \int_{k-1}^k exp(\int_s^k a(r)dr)b(s)ds \}$$

$$\times \{ exp(\int_{[t]}^t a(r)dr) + \int_{[t]}^t exp(\int_0^t a(r)dr)b(s)ds \} . \quad (3.28)$$

The following theorem is an extension to the above lemma .

Theorem 3.4.1 *Let x, a, $b \in C[I, \mathbb{R}^+]$. Let $f(t)$ be positive, continuous, and monotonic nondecreasing on I. If the inequality,*

$$x(t) \leq f(t) + \int_0^t a(s)x(s) + b(s)x([s])ds, \ t \in I$$

holds, then for $t \in I$,

$$x(t) \leq f(t) \cdot \prod_{k=1}^{[t]} \{ exp(\int_{k-1}^k a(r)dr) + \int_{k-1}^k exp(\int_s^k a(r)dr)b(s)ds \}$$

$$\times \{ exp(\int_{[t]}^t a(r)dr) + \int_{[t]}^t exp(\int_0^t a(r)dr)b(s)ds \} .$$

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Proof : Consider the inequality,

$$x(t) \leq f(t) + \int_0^t a(s)x(s) + b(s)x([s])ds, \ t \in I.$$

Since $f(t)$ is positive , dividing by $f(t)$ throughout, we get

$$\frac{x(t)}{f(t)} \leq 1 + \int_0^t a(s)\frac{x(s)}{f(t)} + b(s)\frac{x([s])}{f(t)}ds.$$

Now , $[s] \leq s$, f is nondecreasing $\Rightarrow f([s]) \leq f(s)$ and hence $\frac{1}{f(s)} \leq \frac{1}{f([s])}$. Also $\frac{1}{f(t)} \leq \frac{1}{f(s)}$. Therefore letting , $r(t) = \frac{x(t)}{f(t)}$ in the above inequality, we get

$$r(t) \leq 1 + \int_0^t a(s)r(s) + b(s)r([s])ds.$$

By using above Lemma 3.4.1, we get

$$r(t) \leq \prod_{k=1}^{[t]} \{ exp(\int_{k-1}^k a(r)dr) + \int_{k-1}^k exp(\int_s^k a(r)dr)b(s)ds\}$$
$$\times \{exp(\int_{[t]}^t a(r)dr) + \int_{[t]}^t exp(\int_0^t a(r)dr)b(s)ds, \}$$

which gives the desired result.

We shall now establish some inequalities which are helpful in comparative study of two different equations.

Consider the equations with PCDA.

$$x'(t) = f(t, x(t), x([t])), \quad x(0) = x_0 \tag{3.29}$$

and

$$y'(t) = g(t, y(t), y([t])), \quad y(0) = y_0 \tag{3.30}$$

where $t \in I = [0, \infty)$, f and g are defined and continuous real valued functions on $I \times IR \times IR$, $x_0, y_0 \in IR$, $[t]$ denotes the integer valued function.

We now establish the result of fundamental importance for subsequent discussions in this section. To do this we employ the well known method given in [45] and one used by Aftabizadeh and Wiener in [3].

Theorem 3.4.2 *Let $F(t, u, v) : I \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, non-decreasing in v for each fixed (t, u). Assume that*

$$x'(t) \leq F(t, x(t) + w(t), x([t]) + w([t])) \tag{3.31}$$

and

$$y'(t) > F(t, y(t) + w(t), y([t]) + w([t])) \tag{3.32}$$

for $t \in I$ and $w(t) : I \longrightarrow \mathbb{R}$ is a continuous function.

Then $\qquad x(0) < y(0) \quad$ *implies* $\quad x(t) < y(t) \quad$ *for* $t \in I$.

Proof We shall prove the assertion on the unit interval $[n, n+1)$, $n = 0, 1, 2, 3 \ldots$

Let $x_n(t), y_n(t)$ and $w_n(t)$ be satisfying (3.31) and (3.32) on $[n, n+1)$, namely;

$$x'_n(t) \leq F(t, x_n(t) + w_n(t), x_n([t]) + w_n([t])), \tag{3.33}$$

and

$$y'_n(t) > F(t, y_n(t) + w_n(t), y_n([t]) + w_n([t])). \tag{3.34}$$

We prove that

$$x_n(n) \leq y_n(n) \Rightarrow x_n(t) \leq y_n(t), \quad t \in [n, n+1). \tag{3.35}$$

If the assertion in (3.35) is false, then the set $S = \{t \in [n, n+1)/ y_n(t) \leq x_n(t)\}$ is not empty. Let $t_n = \inf S$, then $t_n \in S$ and by (3.33) $t_n > n$. Hence, we have

$$x_n(t_n) = y_n(t_n) \quad \text{at} \quad t = t_n \quad \text{and} \quad x_n(t) < y_n(t), \quad \text{for} \quad t \in [n, t_n).$$

For sufficiently small $h < 0$, we have,

$$\frac{x_n(t_n + h) - x_n(t_n)}{h} > \frac{y_n(t_n + h) - y_n(t_n)}{h}$$

which implies that $x'(t_n) > y'_n(t_n)$. Therefore using (3.33) and (3.34), we get,

$$F(t_n, x_n(t_n) + w_n(t_n), x_n(n) + w_n(n)) > F(t_n, y_n(t_n) + w_n(t_n), y_n(n) + w_n(n)).$$

This contradicts the non decreasing character of F. Hence the set S is empty and we have $x_n(t) < y_n(t)$ on $[n, n+1)$, $n = 0, 1, 2, \ldots$. Therefore $x(t) < y(t)$ for $t \in I$.

Remark 3.4.1 *The conclusion of the above theorem holds even when (3.31) and (3.32) are replaced by* $x'(t) < F(t, x(t) + w(t), x([t]) + w([t]))$,

and $y'(t) \geq F(t, y(t) + w(t), y([t]) + w([t]))$ *respectively, other assumptions remaining the same .*

We make use of Theorem 3.4.2 to prove the following inequality.

Theorem 3.4.3 *: Let* $G(t, u, v)$, $H(t, u, v) : I \times \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *be continuous and nondecreasing in* v *for each fixed* (t, u) *. Assume that* G *and* H *satisfy the inequality,*

$$G(t, m(t) + w(t), m([t]) + w([t])) \leq m'(t) \leq H(t, m(t) + w(t), m([t]) + w([t])) \quad (3.36)$$

where $t \in I$, *and* $m(t) : I \longrightarrow \mathbb{R}^+$ *is a continuous function with* $m(0) = m_0$ *. Also for* $t \geq 0$, *assume that* $\beta(t)$ *and* $\alpha(t)$ *are the maximal and the minimal solutions of the equations,*

$$\beta'(t) = H(t, \beta(t) + w(t), \beta([t]) + w([t])), \quad \beta(0) = \beta_0 \quad (3.37)$$

and

$$\alpha'(t) = G(t, \alpha(t) + w(t), \alpha([t]) + w([t])), \quad \alpha(0) = \alpha_0 , \quad (3.38)$$

respectively. Then $\alpha_0 \leq m_0 \leq \beta_0$ *implies*

$$\alpha(t) \leq m(t) \leq \beta(t), \quad for \quad t \in I. \quad (3.39)$$

Proof : Let $\alpha(t, \epsilon)$ be a solution of the equation

$$\alpha' = G(t, \alpha(t) + w(t), \alpha([t]) + w([t])) - \epsilon$$

$$\alpha(0, \epsilon) = \alpha_0 - \epsilon , \quad \epsilon > 0.$$

Then $\alpha'(t, \epsilon) < G(t, \alpha(t, \epsilon) + w(t), \alpha([t], \epsilon) + w([t]))$.

Using (3.36), we get, $m'(t) \geq G(t, m(t) + w(t), m([t]) + w([t]))$.

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Observe that $\alpha_0 - \epsilon \le \alpha_0 \le m_0$. Since G satisfies all the conditions of Theorem 3.4.2, using the Remark 3.4.1, we have, $\alpha(t, \epsilon) \le m(t)$.

Let $\epsilon \longrightarrow 0$, we get $\alpha(t) \le m(t)$, which is the left side of (3.39). The right side of (3.39) can be proved similarly, and this completes the proof.

Next we prove a result which can be used for comparative study of the properties of solutions of (3.29) and (3.30).

Theorem 3.4.4 *Let G, H be as in Theorem 3.4.3. Suppose that the functions f, g in (3.29) and (3.30) respectively satisfy the inequality,*

$$|y(t) - x(t)| + hG\left(t, |y(t) - x(t)| + |x(t)|, |y([t]) - x([t])| + |x([t])|\right)$$

$$\le |y(t) - x(t) + h\{g(t, y(t), y([t])) - f(t, x(t), x([t]))\}|$$

$$\le |y(t) - x(t)| + hH\left(t, |y(t) - x(t)| + |x(t)|, |y([t]) - x([t])| + |x([t])|\right). \tag{3.40}$$

For small $h > 0$ and $t \in I$, let $\beta(t)$ and $\alpha(t)$ be the maximal and minimal solutions of

$$\beta'(t) = H\left(t, \beta(t) + x(t), \beta([t]) + x([t])\right), \quad \beta(0) = \beta_0 \tag{3.41}$$

and

$$\alpha'(t) = G\left(t, \alpha(t) + x(t), \alpha([t]) + x([t])\right), \quad \alpha(0) = \alpha_0, \tag{3.42}$$

respectively. If $x(t)$, $y(t)$ are solutions of (3.29) and (3.30) respectively, then

$$\alpha_0 \le |y_0 - x_0| \le \beta_0 \quad \Rightarrow \quad \alpha(t) \le |y(t) - x(t)| \le \beta(t) \quad, for \ t \in I.$$

Proof : We shall prove the result on the unit interval $[n, n+1)$, $n = 0, 1, 2, \ldots$.

Let $x_n(t)$, $y_n(t)$, $\beta_n(t)$, $\alpha_n(t)$ be satisfying (3.29), (3.30), (3.41), (3.42) respectively on $[n, n+1)$, $n = 0, 1, 2, \ldots$. We shall show that

$$\alpha_n(n) \le |y_n(n) - x_n(n)| \le \beta_n(n) \quad \Rightarrow \quad \alpha_n(t) \le |y_n(t) - x_n(t)| \le \beta_n(t)$$

Define $p_n(t) = |y_n(t) - x_n(t)|$, then $p_n(n) = |y_n(n) - x_n(n)|$. Using (3.40), we get

$$
\begin{aligned}
p_n(t+h) &= |y_n(t+h) - x_n(t+h)| \\
&= |y_n + h\, g(t, y_n(t), y_n(n)) + \epsilon_1 h - x_n(t) - hf(t, x_n(t), x_n(n)) - \epsilon_2 h| \\
&\leq |y_n(t) - x_n(t) + h\{g(t, y_n(t), y_n(n)) - f(t, x_n(t), x_n(n))\}| + |\epsilon_1 h| + |\epsilon_2 h| \\
&\leq |y_n(t) - x_n(t)| + h.H\left(t, |y_n(t) - x_n(t)| + |x_n(t)|, |y_n(n) - x_n(n)| + |x_n(n)|\right) \\
&\quad + |\epsilon_1 h| + |\epsilon_2 h| \\
&= p_n(t) + h.H\left(t, p_n(t) + |x_n(t)|, p_n(n) + |x_n(n)|\right) + |\epsilon_1 h| + |\epsilon_2 h|
\end{aligned}
$$

where ϵ_1, $\epsilon_2 \longrightarrow 0$ as $h \longrightarrow 0$. Therefore we get

$$
\begin{aligned}
p_n'(t) &= \lim_{h \to 0} \frac{p_n(t+h) - p_n(t)}{h} \\
&\leq H\left(t, p_n(t) + |x_n(t)|, p_n(n) + |x_n(n)|\right), \quad t \in [n, n+1).
\end{aligned}
\tag{3.43}
$$

Using Theorem 3.4.3 and (3.43), we get $p_n(t) \leq \beta_n(t)$. Similaraly we can obtain $\alpha_n(t) \leq p_n(t)$. This gives us,

$$
\alpha_n(t) \leq |y_n(t) - x_n(t)| \leq \beta_n(t) \text{ for } t \in [n, n+1), \quad n = 0, 1, 2, \dots .
$$

Therefore, $\alpha(t) \leq |y(t) - x(t)| \leq \beta(t)$ for $t \in I$.

3.5 Nonlinear periodic boundary value problem

This section deals with the nonlinear periodic boundary value problem (PBVP). We have employed the method of quasilinearisation to prove the existence of solution. The associated linear PBVP is also discussed.

Consider the nonlinear PBVP,

$$
x'(t) = f(t, x(t), x([t])),
\tag{3.44}
$$

$$
x(0) = x(2\pi)
\tag{3.45}
$$

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where $t \in \mathcal{J} = [0, 2\pi]$. $f \in C[\mathcal{J} \times I\!R \times I\!R, I\!R]$, $[\cdot]$ denotes the greatest integer function. Equation (3.44) is a differential equation with piecewise constant deviating argument and the existence of solution for the initial value problem has been established by Aftabizadeh and Wiener in [3].

We introduce the following definition of a solution of the PBVP (3.44) and (3.45).

Definition 3.5.1 *A solution of (3.44) and (3.45) on $\mathcal{J}$ is a function $x : \mathcal{J} \to I\!R$ that satisfies the following conditions :*

(i) $x(t)$ is continuous on $\mathcal{J}$.

(ii) The derivative $x'(t)$ exists at each point $t \in \mathcal{J}$, with the possible exception of the points $[t] \in \mathcal{J}$, where one sided derivatives exist.

(iii) Equation (3.44) is satisfied on each interval $\mathcal{J}_n = [n, n+1)$ with integral end points.

(iv) $x(t)$ satisfies the condition (3.45).

We now define the classical upper and lower solutions for the PBVP (3.44) and (3.45).

Definition 3.5.2 *A continuous function $u : \mathcal{J} \longrightarrow I\!R$ is said to be a lower solution of the PBVP (3.44) and (3.45), if the derivative $u'(t)$ exists at each point $t \in \mathcal{J}$, with the possible exception of the points $[t] \in \mathcal{J}$, where one sided derivatives exist, and*

$$u'(t) \leq f(t, u(t), u([t])), \quad u(0) \leq u(2\pi) . \tag{3.46}$$

It is said to be an upper solution, if the reversed inequalities in (3.46) hold.

Usually the classical lower and upper solutions are ordered. If u, v are lower and upper solutions of the PBVP (3.44) and (3.45) respectively, then either $u \leq v$ or $v \leq u$ on $\mathcal{J}$.

In the following discussion of this section, we assume that

(H1) u, $v \in C[\mathcal{J}, I\!R]$ are lower and upper solutions of (3.44) and (3.45) respectively such that $u(t) \leq v(t)$ on $\mathcal{J}$.

(H2) For $f \in C[\mathcal{J} \times I\!R \times I\!R, I\!R]$ and ($H1$), there exits atleast one solution $x(t)$ of the PBVP (3.44) and (3.45) such that $u(t) \leq x(t) \leq v(t)$.

(H3) Whenever ($H2$) holds, let

$$S = \{(t, x, y) \in \mathcal{J} \times I\!R \times I\!R : u(t) \leq x(t) \leq v(t) \ , u([t]) \leq y(t) \leq v[t] \ \}.$$

(H4) $f(t, x, y) \in C[S, I\!R]$ such that the partial derivatives $f_x(t, x, y), f_y(t, x, y)$ exist, are continuous on S and $|f_x(t, x, y)| \leq m_1;$ $|f_y(t, x, y)| \leq m_2$ on S, for some positive constants m_1, m_2

(H5) For (t, x_1, y_1), $(t, x_2, y_2) \in S$ such that $x_1 \geq x_2$, $y_1 \geq y_2$,

$$
\begin{aligned}
f(t, x_1, y_1) - f(t, x_2, y_2) \geq \ & f_x(t, x_2, y_2)(x_1 - x_2) - M_1(x_1 - x_2)^2 \\
& + f_y(t, x_2, y_2)(y_1 - y_2) - M_2(y_1 - y_2)^2,
\end{aligned}
$$

where M_1, $M_2 \geq 0$ are constants.

For $u(t) \leq x_2 \leq x_1 \leq v(t);$ $u([t]) \leq y_2 \leq y_1 \leq v([t])$, define the function $g : S \times I\!R \times I\!R \times I\!R \times I\!R \to I\!R$ by

$$
\begin{aligned}
g(t, x_1, x_2, y_1, y_2) \ = \ & f(t, x_2, y_2) \\
& + \ \{f_x(t, x_2, y_2) + 2M_1 x_2\}(x_1 - x_2) - M_1(x_1^2 - x_2^2) \\
& + \ \{f_y(t, x_2, y_2) + 2M_2 y_2\}(y_1 - y_2) - M_2(y_1^2 - y_2^2). \quad (3.47)
\end{aligned}
$$

From (3.47), we get .

$$g(t, x_1, x_1, y_1, y_1) = f(t, x_1, y_1). \qquad (3.48)$$

Next define the function $F : S \times I\!R \times I\!R \to I\!R$ by

$$F(t, x, y) = f(t, x, y) + M_1 x^2 + M_2 y^2, \qquad (3.49)$$

where M_1, M_2 are as in $(H5)$.

We need to prove the following result.

Theorem 3.5.1 *Suppose that the hypotheses $(H1)$ to $(H5)$ are satisfied. Then for any*

x_1, x_2, y_1, y_2, w *such that* $u(t) \leq w(t) \leq v(t)$, $t \in \mathcal{J}$, $u(t) \leq x_2 \leq x_1 \leq v(t)$;

and $u([t]) \leq y_2 \leq y_1 \leq v([t])$ *we have,*

$$(i)\ f(t, x_1, y_1) - f(t, x_2, y_2) \geq \{f_x(t, x_2, y_2) + 2M_1 x_2\}(x_1 - x_2) - M_1(x_1^2 - x_2^2)$$

$$+\{f_y(t, x_2, y_2) + 2M_2 y_2\}(y_1 - y_2) - M_2(y_1^2 - y_2^2)\ (3.50)$$

and

$$(ii)\ g(t, x(t), w(t), x([t]), w([t])) - g(t, y(t), w(t), y([t]), w([t]))$$

$$\leq N_1(x(t) - y(t)) + N_2(x([t]) - y([t]))\quad (3.51)$$

where $N_1, N_2 \geq 0$ *and* $u(t) \leq y(t) \leq x(t) \leq v(t)$ *and* $u(t) \leq w(t) \leq v(t)$; $t \in S$

Proof :We have from $(H5)$,

$$f(t, x_1, y_1) - f(t, x_2, y_2)$$

$$\geq f_x(t, x_2, y_2)(x_1 - x_2) - M_1(x_1 - x_2)^2$$

$$+ f_y(t, x_2, y_2)(y_1 - y_2) - M_2(y_1 - y_2)^2$$

$$= \{f_x(t, x_2, y_2) + 2M_1 x_2\}(x_1 - x_2) - 2M_1 x_2(x_1 - x_2) - M_1(x_1^2 - 2x_1 x_2 + x_2^2)$$

$$+ \{f_y(t, x_2, y_2) + 2M_2 y_2\}(y_1 - y_2) - 2M_2 y_2(y_1 - y_2) - M_2(y_1^2 - 2y_1 y_2 + y_2^2).$$

$$= \{f_x(t, x_2, y_2) + 2M_1 x_2\}(x_1 - x_2) - M_1(x_1^2 - x_2^2)$$

$$+\{f_y(t, x_2, y_2) + 2M_2 y_2\}(y_1 - y_2) - M_2(y_1^2 - y_2^2).$$

This proves (i).

Next to prove (ii), using (3.47), we get,

$$g(t, x(t), w(t), x([t]), w([t])) - g(t, y(t), w(t), y([t]), w([t]))$$

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$$= f(t, w(t), w([t]))$$

$$+ \{f_x(t, w(t), w([t])) + 2M_1 w(t)\}(x(t) - w(t)) - M_1(x^2(t) - w^2(t))$$

$$+ \{f_y(t, w(t), w([t])) + 2M_2 w([t])\}(x([t]) - w([t])) - M_2(x^2([t]) - w^2([t]))$$

$$- f(t, w[t], w([t]))$$

$$- \{f_x(t, w(t), w([t])) + 2M_1 w(t)\}(y(t) - w(t)) + M_1(y^2(t) - w^2(t))$$

$$- \{f_y(t, w(t), w([t])) + 2M_2 w([t])\}(y([t]) - w([t])) + M_2(y^2([t]) - w^2([t]))$$

$$= f_x(t, w(t), w([t]))(x(t) - y(t)) + 2M_1 w(t)(x(t) - y(t)) - M_1(x^2(t) - y^2(t))$$

$$+ f_y(t, w(t), w([t]))(x([t]) - y([t])) + 2M_2 w([t])(x([t]) - y([t])) - M_2(x^2([t]) - w^2([t]))$$

$$\leq \{m_1 + 2M_1(w(t) - \eta(t))\}(x(t) - y(t)) + \{m_2 + 2M_2(w([t]) - \eta([t]))\}(x([t]) - y([t]))$$

$$\leq N_1(x(t) - y(t)) + N_2(x([t]) - y([t])),$$

where $\eta(t)$ is such that $u(t) \leq \eta(t) \leq v(t)$, $t \in \mathcal{J}$; and

$N_1 = m_1 + 2M_1 \ Sup\{|w(t) - z(t)|\};$ $N_2 = m_2 + 2M_2 \ Sup\{|w([t]) - z([t])|\},$

$u(t) \leq w(t) \leq v(t);$ $u(t) \leq z(t) \leq v(t)$, $t \in \mathcal{J}$. This completes the proof.

Remark 3.5.1 *The assertion (i) in Theorem 3.5.1 implies that*

$$f(t, x(t), x([t])) \geq g(t, x(t), y(t), x([t]), y([t])) \tag{3.52}$$

for $y(t) \leq x(t)$.

Theorem 3.5.2 *The nonlinear PBVP*

$$x'(t) = g(t, x(t), u(t), x([t]), u([t])); \quad x(0) = x(2\pi); \ t \in \mathcal{J} \tag{3.53}$$

has atleast one solution $x(t)$ such that $u(t) \leq x(t) \leq v(t)$ where $u(t)$, $v(t)$ are lower and upper solutions of (3.44), respectively.

Proof : For $t \in \mathcal{J}$, we have by using (3.48),

$$u'(t) \leq f(t, u(t), u([t])) = g(t, u(t), u(t), u([t]), u([t]))$$

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and, by using (3.52).

$$v'(t) \geq f(t, v(t), v([t])) \geq g(t, v(t), v(t), v([t]), v([t]))$$

Hence, u, v are lower and upper solutions of (3.53) respectively.

Therefore, by analogous of $(H2)$, there exists a solution $x(t)$ of (3.53)

such that $u(t) \leq x(t) \leq v(t)$.

We require the following lemma concerning solution of the linear PBVP.

Lemma 3.5.1 *The linear PBVP*

$$x'(t) + ax(t) + bx([t]) = h(t); \quad x(0) = x(2\pi); \quad t \in \mathcal{J} \tag{3.54}$$

a, b, *are constants,* $a \neq 0$, *has a unique solution*

$$x(t) = \{c_0 \, \lambda^{[t]}(1) + \sum_{i=1}^{[t]} \lambda^{[t]-i}(1) \, \gamma(i-1, i)\} \, \lambda(t - [t]) + \gamma([t], t) \tag{3.55}$$

where $x(0) = c_0$, *provided*

$$c_0 = \frac{1}{1 - \lambda^6(1)\lambda(2\pi - 6)} \{\lambda(2\pi - 6) \sum_{i=1}^{6} \lambda^{6-i}(1) \, \gamma(i-1, i) + \gamma(6, t)\} \tag{3.56}$$

where

$$\lambda(t) = e^{-at} + (e^{-at} - 1)ba^{-1}; \quad \lambda(1) = e^{-a} + (e^{-a} - 1)ba^{-1}; \tag{3.57}$$

$$\gamma(i-1, i) = \int_{i-1}^{i} e^{-a(t-s)} h(s)ds; \quad i = 1, 2, ...[t]. \tag{3.58}$$

Proof : Let $x_n(t)$ be solution of the equation (3.54) on $[n, n+1)$, satisfying

the condition $x(n) = c_n$, $n = 0, 1, ...$ Then, we have

$$x_n(t) = c_n \lambda(t - n) + \gamma(n, t),$$

where λ, and γ are as defined in (3.57) and (3.58), respectively.

Let $t \to n + 1$, then we have the recurrence relation,

$$c_{n+1} = c_n \lambda(1) + \gamma(n, n+1),$$

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which yields.

$$c_n = c_0 \lambda^n(1) + \sum_{i=1}^{n} \lambda^{n-i}(1)\gamma(i-1,i).$$

Hence the solution of the equation (3.54) is given by ,

$$x(t) = \{c_0\ \lambda^{[t]}(1) + \sum_{i=1}^{[t]} \lambda^{[t]-i}(1)\ \gamma(i-1,i)\}\ \lambda(t-[t]) + \gamma([t],t).$$

The boundary conditions $x(0) = x(2\pi) = c_0$ yields,

$$c_0 = \frac{1}{1 - \lambda^6(1)\lambda(2\pi - 6)}\{\lambda(2\pi - 6)\sum_{i=1}^{6} \lambda^{6-i}(1)\ \gamma(i-1,i) + \gamma(6,t)\}$$

Hence the proof.

We are now in position to prove the main result of this section. The existence of solution of the PBVP (3.44) and (3.45) is established. For this purpose the method of quasilinearisation is employed to obtain a sequence of approximate solutions converging to the required solution.

Theorem 3.5.3 *For the nonlinear PBVP (3.44) and (3.45) satisfying the hypotheses* $(H1) - (H5)$, *there exists* $w_j \in C^1[\mathcal{J}, \mathbb{R}]$, $j = 0,1,2,...$ *such that the sequence* $\{w_j\}$ *is monotone and converges uniformly to a solution of the PBVP (3.44) and (3.45).*

Proof : First we set $w_0(t) = u(t)$, the lower solution of the PBVP (3.44) and (3.45). Then by Theorem 3.5.2 , there exits a solution w_1 of (3.53) such that $w_0(t) \le w_1(t) \le v(t)$, where $v(t)$ is the upper solution of the PBVP (3.44) and (3.45). Thus $w_1(t)$ satisfies the equation,

$$w'(t) = g(t, w(t), w_0(t), w([t]), w_0([t])); \quad w(0) = w(2\pi); \quad t \in \mathcal{J}$$

Suppose that we have constructed $w_j(t)$, $j \ge 1$ such that

$$u(t) = w_0(t) \le w_1(t) \le w_2(t) \le ... \le w_j(t) \le v(t)$$

on $\mathcal{J}$, and w_j 's are the solutions of the equations

$$w'(t) = g(t, w(t), w_{j-1}(t), w([t]), w_{j-1}([t])); \quad w(0) = w(2\pi); \quad j \ge 1. \qquad (3.59)$$

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Hence we get $w_j'(t) = g(t, w_j(t), w_{j-1}(t), w_j([t]), w_{j-1}([t]))$

Also, $v'(t) \geq f(t, v(t), v([t])) \geq g(t, v(t), w_{j-1}(t), v([t]), w_{j-1}([t]))$ $j \geq 1$;

which means that $w_j(t)$ is a lower solution and $v(t)$ is a upper solution of the equation (3.59), respectively. Again by employing Theorem 3.5.2, we get a solution $w_{j+1}(t)$ of (3.59) such that $w_j(t) \leq w_{j+1}(t) \leq v(t)$.   Hence, the sequence $\{w_j(t)\}$ is increasing and it has a pointwise limit, say $w(t)$.

Next let $w_{j+1}(t)$ be the solution of the linear equation

$$w_{j+1}'(t) + a w_{j+1}(t) + b w_{j+1}([t]) = h_j(t), \tag{3.60}$$

where a, b are constants, $a \neq 0$, $t \in \mathcal{J}$; $w_{j+1}(0) = w_{j+1}(2\pi)$ and

$$
\begin{aligned}
h_j(t) &= a\, w_{j+1}(t) + b w_{j+1}([t]) + f(t, w_j(t), w_j([t])) \\
&\quad + \{f_x(t, w_j(t), w_j([t])) + 2M_1 w_j(t)\}\{w_{j+1}(t) - w_j(t)\} - M_1\{w_{j+1}^2(t) - w_j^2(t)\} \\
&\quad + \{f_y(t, w_j(t), w_j([t])) + 2M_2 w_j([t])\}\{w_{j+1}([t]) - w_j([t])\} - M_2\{w_{j+1}^2([t]) - w_j^2([t])\}.
\end{aligned}
$$

The solution of the equation (3.60) is given by (3.55), and this shows that $\{w_j\}$ is bounded in $\mathcal{C}^1(\mathcal{J})$. Hence, the sequence $\{w_j\}$ converges uniformly to w.

As $j \to \infty$, $h_j(t) \to h(t) = aw(t) + bw([t]) + f(t, w(t), w([t]))$.

Hence taking limit, as $j \to \infty$,   equation (3.60) gives

$$w'(t) = f(t, w(t), w([t])), \quad w(0) = w(2\pi),$$

which shows that $w(t)$ is a solution of the PBVP (3.44) and (3.45).

3.6   Oscillatory behaviour.

In this section, we shall establish the result concerning the oscillatory property of the solution of a nonlinear first order differential equation with PCDA. It is known that a solution is said to be oscillatory if it has arbitrary large zeros. We first establish the existence of the solution.

Consider the IVP.

$$x'(t) + a(t)x(t) = f(t, x([t])), \quad x(0) = c_0, \quad t \in I = [0, \infty), \tag{3.61}$$

where $a \in C[I, I\!R]$, $f \in C[I \times I\!R, I\!R]$, $c_0 \in I\!R$. Equation (3.61) is a nonlinear differential equation with PCDA. The solution of (3.61) can be defined as in Definition 3.2.1 . We prove the following result.

Theorem 3.6.1 *The equation (3.61) has a solution on* I .

Proof :Let $t \in [n, n+1)$ and $x_n(t)$ be the solution of the equation (3.61) on the unit interval $[n, n+1)$ with $x_n(n) = x(n) = c_n$.

Then $x_n'(t) + a(t)x_n(t) = f(t, c_n)$. Its solution is given by.

$$x_n(t) = c_n \; exp\Big(-\int_n^t a(s)ds\Big) + exp\Big(-\int_n^t a(s)ds\Big) \cdot \int_n^t f(s, c_n) \; exp\Big(\int_n^s a(r)dr\Big)ds.$$

Let $E(n, t) = exp\big(-\int_n^t a(r)dr\big)$ and $F(n, t) = \int_n^t f(s, c_n) \; exp\big(\int_n^s a(r)dr\big)ds$.

Then, we get. $x_n(t) = c_n E(n, t) + E(n, t)F(n, t)$.

Let $t \to n+1$, and since $x_n(n+1) = c_{n+1}$, we get the recurrence relation.

$$c_{n+1} = c_n E(n, n+1) + E(n, n+1)F(n, n+1),$$

$$\text{OR} \quad c_n = c_{n-1}E(n-1, n) + E(n-1, n)F(n-1, n), \quad n = 1, 2, \dots \tag{3.62}$$

Repeated use of the recurrence relation (3.62) yields.

$$
\begin{aligned}
c_n &= c_0 \cdot \prod_{j=0}^{n-1} E(j, j+1) \;+\; \prod_{j=0}^{n-1} E(j, j+1)F(0, 1) \\
&+ \prod_{j=1}^{n-1} E(j, j+1)F(1, 2) \;+ \dots + \prod_{j=n-1}^{n-1} E(j, j+1)F(n-1, n).
\end{aligned}
$$

Now, $\displaystyle\prod_{j=k}^{n-1} E(j, j+1) = exp\Big(-\int_k^n a(s)ds\Big)$, for $k = 1, 2, \dots, n-1$,

and hence we get,

$$c_n = c_0 exp\left(-\int_0^n a(s)ds\right) + \sum_{k=0}^{n-1} exp\left(-\int_k^n a(s)ds\right)F(k,k+1).$$

Thus, the solution $x(t)$ of the equation (3.61) is given by

$$
\begin{aligned}
x(t) = \quad & c_0 exp\left(-\int_0^t a(s)ds\right) \\
& + \sum_{k=0}^{[t]-1} exp\left(-\int_k^t a(s)ds\right) \cdot \left\{\int_k^{k+1} f(s,c_k)exp\left(\int_k^s a(r)dr\right)ds\right\} \\
& + exp\left(-\int_{[t]}^t a(s)ds\right) \cdot \int_{[t]}^t f(s,c_{[t]})exp\left(\int_{[t]}^s a(r)dr\right)ds.
\end{aligned}
\tag{3.63}
$$

We shall consider a particular case of the equation (3.61).

Corollary 3.6.1 *The solution of the IVP,*

$$x'(t) + ax(t) = g(x([t])), \quad x(0) = c_0, \quad t \in I = [0,\infty), a \neq 0, \tag{3.64}$$

where a is a constant, $g \in C[\mathbb{R}, \mathbb{R} \setminus \{0\}]$, $c_0 \in \mathbb{R}$. is given by

$$x(t) = c_0 e^{-at} + \sum_{k=0}^{[t]-1} e^{-a(t-k)} g(c_k)\frac{e^a - 1}{a} + \frac{1 - e^{-a(t-[t])}}{a}, \tag{3.65}$$

where $c_k = x(k)$, $k = 0, 1, ..., n$.

Proof : Take $a(t) = a$, and $f(t, x([t])) = g(x([t]))$ in Theorem 3.6.1, then (3.63) gives (3.65). Hence the proof.

Remark 3.6.1 *When $a = 0$, (3.64) reduces to $x'(t) = g(x([t])$ and its solution on $[n, n+1)$ is given by, $x_n(t) = c_n + g(c_n)(t - n)$.*

Now we shall study the oscillatory behaviour of the solution of equation (3.64). The following result gives a necessary condition under which the solution (3.65) has a zero in each unit interval.

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Theorem 3.6.2 *If the solution $x(t)$ of the equation (3.64) has a zero in the unit interval $(n, n+1)$, then,*

$$\frac{1-e^a}{a} < \frac{c_n}{g(c_n)} < 0, \tag{3.66}$$

where $x_n(n) = c_n$.

If (3.66) is not satisfied then (3.64) has no zero on $(n, n+1)$.

Proof : Let $x_n(t)$ be the solution of the equation (3.64) on the unit interval $(n, n+1)$. Then from (3.65),

$$x_n(t) = c_n \, e^{-a(t-n)} + \frac{g(c_n)}{a}(1 - e^{-a(t-n)}), \quad a \neq 0.$$

Suppose that $x_n(t)$ has a zero at $t_n \in (n, n+1)$, then

$$c_n e^{-a(t_n-n)} + \frac{g(c_n)}{a}\{1 - e^{-a(t_n-n)}\} = 0. \tag{3.67}$$

This on simplification yields,

$$e^{a(t_n-n)} = 1 - \frac{ac_n}{g(c_n)}.$$

Case(i): Suppose $a > 0$. Then $1 < 1 - \frac{ac_n}{g(c_n)} < e^a$.

This implies $\frac{c_n}{g(c_n)} < 0$ and $\frac{1-e^a}{a} < \frac{c_n}{g(c_n)}$.

Case(ii): Suppose $a < 0$. Then $e^a < 1 - \frac{ac_n}{g(c_n)} < 1$.

This again leads to $\frac{c_n}{g(c_n)} < 0$ and $\frac{1-e^a}{a} < \frac{c_n}{g(c_n)}$.

Hence, we get

$$\frac{1-e^a}{a} < \frac{c_n}{g(c_n)} < 0.$$

Remark 3.6.2 *(i) Observe that from (3.67), we get $t_n = n + \frac{1}{a}log\{1 - \frac{ac_n}{g(c_n)}\}$.*

(ii) When $a = 0$, using Remark 3.6.1, we get $t_n = n - \frac{c_n}{g(c_n)}$ and the condition (3.66) reduces to $-1 < \frac{c_n}{g(c_n)} < 0$.

(iii) If $g(x([t])) = -p \cdot x([t])$, p is a constant, then (3.66) reduces to $p > \frac{a}{e^a-1}$, which is a necessary and sufficient condition obtained in [1].

Chapter 4

CONTROLLABILITY

4.1 Introduction

This chapter is concerned with the controllability of a nonlinear system involving piecewise constant deviating argument(PCDA). The control theory is a discipline of increasing applications. It is the area of applications dealing with basic principles underlying the analysis and design of control systems. Controllability theory attempts to define and isolate the theoretical limits to which a system can be controlled. The important problem here is that to compel or control the system to behave in some desired fashion. In elementary differential equation, the nonhomogeneous term (or the perturbed term) is a fixed specified function of independent variable. If this term is made to vary arbitrarily, then the system behaviour will changed. This change in behaviour is studied under the controllability problem.

The Controllability of nonlinear systems is a problem of wide interest. There are different approaches to study this problem [58]. Most of them are established techniques of the nonlinear analysis. Among these, the fixed point method is widely used. Yamamoto has obtained the results for ordinary differential equation by using Schauder's fixed point in [68].

In this chapter, we apply the fixed point method to study the controllability of a nonlinear system with PCDA. The controllability problem is transformed to a fixed point problem of a nonlinear operator in some function space. The Schauder's fixed point theorem is used to get the desired result. As a preliminary requirement, result for the linear case is proved.

Section 2 contains notations and preliminaries required for the further development.

In section 3, we discuss the nonlinear system with PCDA. First we obtain a sufficient condition for the controllability by using the approach of Yamamoto[68]. An operator is constructed on a Banach space of vector valued continuous function, and controllability problem is transformed into an existence of a fixed point. Finally, in section 4, we establish conditions for existence of a set over which fixed point will exist. These results are called comparison theorems.

4.2 Notations and Preliminaries

In this section, we consider the system,

$$z'(t) = A(t)z(t) + B(t)z([t]) + C(t)u(t), z(0) = z_0 \tag{4.1}$$

$t \in \overline{J} = [0, t_f], \quad z, z_0 \in I\!\!R^n, u \in I\!\!R^m,$

where $A(t), B(t)$, are $n \times n$ continuous matrices and $C(t)$ is a $n \times m$ continuous matrix on $\overline{J}$. $[\cdot]$ is the greatest integer function. Equation (4.1) is a differential equation with PCDA because of the presence of the term $z([t])$. The solution of (4.1) can be defined in a similar way as we have done in Chapter 2 and 3.

Let $\Phi(t)$ be the Fundamental matrix (FM) of the system

$x'(t) = A(t)x(t), \ x(0) = z_0, \quad x \in I\!\!R^n$ satisfying $\Phi(0) = E$, an identity matrix of

order $n \times n$, and $\Phi(t,s) = \Phi(t)\Phi^{-1}(s)$. Let $\Psi(t)$ be the FM of the system

$y'(t) = A(t)y(t) + B(t)y([t]), y(0) = z_0, \quad y \in I\!\!R^n$ satisfying $\Psi(0) = E$ and

$\Psi(t,k) = \Psi(t)\Psi^{-1}(k), \quad k = 0, 1, \cdots [t], t \in \overline{J}.$ The solution of this system is

given by $y(t) = \Psi(t,0)z_0.$

We need the following Lemma which is deduced from variation of parameters formula proved in [36].

Lemma 4.2.1 *The unique solution of (4.1) is given by*

$$z(t) = \Psi(t,0)z_0$$

$$+ \Psi(t,0) \sum_{k=1}^{[t]} \int_{k-1}^{k} \Psi(0,k)\Phi(k,s)C(s)u(s)ds$$

$$+ \Phi(t,0) \int_{[t]}^{t} \Phi(0,s)C(s)u(s)ds. \tag{4.2}$$

We have the following definition of controllability.

Definition 4.2.1 *The system (4.1) is said to be controllable from $(0, z_0)$ to (t_f, z_f), if for some control function $u(t)$, $t \in \overline{J}$, the solution $z(t)$ of (4.1) satisfying $z(0) = z_0$ also satisfies $z(t_f) = z_f \in \overline{J}$, where t_f and z_f are preassigned terminal time and state, respectively.*

If the system (4.1) is controllable for all z_0 at $t = 0$ and for all z_f at $t = t_f$, then it is said to be completely controllable (c.c.) on $\overline{J}$.

We now establish the sufficient condition for the controllability of the linear system (4.1).

Theorem 4.2.1 *Consider the control problem (4.1) whose unique solution $z(t)$ is given by (4.2). If the matrix $\mathbf{U}(0, t_f)$ defined by*

$$\mathbf{U}(0, t_f) = \begin{cases} \mathbf{U}(0, [t_f]) = \sum_{k=1}^{[t_f]} \int_{k-1}^{k} \Psi(0,k)\Phi(k,s)C(s) \\ \qquad \times [C^T(s)\Phi^T(k,s)\Psi^T(0,k)]ds, \quad on\ [0, [t_f]] \\ \mathbf{U}([t_f], t_f) = \int_{[t_f]}^{t_f} \Phi(0,s)C(s)C^T(s)\Phi^T(0,s)ds, \quad on\ [[t_f], t_f] \end{cases} \tag{4.3}$$

is nonsingular, where T denotes the transpose, then the system (4.1) is c.c..

In this case one of the control functions which transfers the system from $(0, z_0)$ to (t_f, z_f) is given by

$$u(t) = \begin{cases} -C^T(t)\Phi^T(k,t)\Psi^T(0,k)\mathbf{U}^{-1}(0, [t_f])[\frac{z_0}{2} - \Psi(0, t_f)\frac{z_f}{2}], \quad on\ [k-1, k] \\ -C^T(t)\Phi^T(0,t)\mathbf{U}^{-1}([t_f], t_f) \\ \qquad \times [\Phi^{-1}(t_f, 0)\Psi(t_f, 0)\frac{z_0}{2} - \Phi(0, t_f)\frac{z_f}{2}] \quad on\ [[t_f], t_f]. \end{cases} \tag{4.4}$$

where $k = 1, 2, \cdots, [t_f]$.

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Proof : Since $\mathbf{U}(0, t_f)$ is non-singular, the control function u(t) given by (4.4) is well defined. Using (4.2), we get

$$
\begin{aligned}
z(t_f) &= \Psi(t_f, 0)z_0 + \Psi(t_f, 0)\sum_{k=1}^{[t_f]}\int_{k-1}^{k}\Psi(0, k)\Phi(k, s)C(s) \\
&\quad [-C^T(s)\Phi^T(k, s)\Psi^T(0, k)\mathbf{U}^{-1}(0, [t_f])](\frac{z_0}{2} - \Psi(0, t_f)\frac{z_f}{2})ds, \\
&\quad + \Phi(t_f, 0)\int_{[t_f]}^{t_f}\Phi(0, s)C(s)[-C^T(s)\Phi^T(0, s)\mathbf{U}^{-1}([t_f], t_f)] \\
&\quad (\Phi^{-1}(t_f, 0)\Psi(t_f, 0)\frac{z_0}{2} - \Phi(0, t_f)\frac{z_f}{2})ds \\
&= \Psi(t_f, 0)z_0 - \Psi(t_f, 0)\mathbf{U}(0, [t_f])\mathbf{U}^{-1}(0, [t_f])(\frac{z_0}{2} - \Psi(0, t_f)\frac{z_f}{2}) \\
&\quad - \Phi(t_f, 0)\mathbf{U}([t_f], t_f)\mathbf{U}^{-1}([t_f], t_f)(\Phi^{-1}(t_f, 0)\Psi(t_f, 0)\frac{z_0}{2} - \Phi(0, t_f)\frac{z_f}{2}). \\
&= \Psi(t_f, 0)z_0 - \Psi(t_f, 0)\frac{z_0}{2} + \frac{z_f}{2} - \Psi(t_f, 0)\frac{z_0}{2} + \frac{z_f}{2} \\
&= z_f,
\end{aligned}
$$

as required. Hence, the system (4.1) is c.c.

Remark 4.2.1 *The control function* u(t) *defined by (4.4) is a piecewise continuous function. However, if we assume that,*

$(\mathcal{H}1)$ $\Psi(0, k) = \Psi(0, k+1)\Phi(k+1, k),$ $k = 1, 2, ..., [t_f] - 1.$

$(\mathcal{H}2)$ $\Psi^T(0, [t_f])\mathbf{U}^{-1}(0, [t_f])\Psi(0, t_f) = \Phi^T(0, [t_f])\mathbf{U}^{-1}([t_f], t_f)\Phi(0, t_f)$

then, these conditions ensure that the left hand side and right hand side limits match at each of the integer points, making u(t) *continuous.*

4.3 The Nonlinear system

In this section, we obtain a sufficient conditions for the controllability of the nonlinear system.

Consider the control process described by the nonlinear equation,

$$
\begin{aligned}
z'(t) &= A(t, z(t), u(t))z(t) + B(t, z(t), u(t))z([t]) + C(t, z(t), u(t))u(t) + g(t, z(t), u(t)) \\
z(0) &= z_0, \tag{4.5}
\end{aligned}
$$

$t \in \overline{J} = [0, t_f]$, $z, z_0, g \in I\!\!R^n, u \in I\!\!R^m$. $A(t, z, u), B(t, z, u)$ are $n \times n$ matrices

and $C(t, z, u)$ is a $n \times m$ matrix. The matrix functions $A(t, z, u), B(t, z, u), C(t, z, u)$ are

all continuous with respect to their arguments.

The related linear control system is given by,

$$z'(t) = A(t, w(t), v(t))z(t) + B(t, w(t), v(t))z([t]) + C(t, w(t), v(t))u(t) + g(t, w(t), v(t)),$$

$$z(0) = z_0, \tag{4.6}$$

where $w = w(t), v = v(t)$ are continuous functions of appropriate dimensions as z and u

respectively. Observe that $A(t, w, v), B(t, w, v), C(t, w, v)$ and $g(t, w, v)$ are functions

of time t. Hence, by Lemma 4.2.1, the solution of the system (4.6) is given by

$$z(t) = \Psi(t, 0, w, v)z_0$$

$$+ \Psi(t, 0, w, v) \sum_{k=1}^{[t]} \int_{k-1}^{k} \Psi(0, k, w, v)\Phi(k, s, w, v)[C(s, w, v)u(s) + g(s, w, v)]ds$$

$$+ \Phi(t, 0, w, v) \int_{[t]}^{t} \Phi(0, s, w, v)[C(s, w, v)u(s) + g(s, w, v)]ds \tag{4.7}$$

where $\Phi(t, t, w, v) = E$, $\Phi(t, 0, w, v)\Phi(0, s, w, v) = \Phi(t, s, w, v)$,

$\Psi(t, t, w, v) = E$, $\Psi(t, 0, w, v)\Psi(0, k, w, v) = \Psi(t, k, w, v)$.

Theorem 4.3.1 *Consider the linear control system (4.6) whose solution $z(t)$ is given by*

(4.7). If the matrix,

$\mathbf{U}(0, t_f, w, v)$

$$= \begin{cases} \mathbf{U}(0, [t_f], w, v) = \sum_{k=1}^{[t_f]} \int_{k-1}^{k} \Psi(0, k, w, v)\Phi(k, s, w, v)C(s, w, v) \\ \qquad \times [C^T(s, w, v)\Phi^T(k, s, w, v)\Psi^T(0, k, w, v)]ds, \quad on \; [0, [t_f]] \\ \\ \mathbf{U}([t_f], t_f, w, v) = \int_{[t_f]}^{t_f} \Phi(0, s, w, v) \\ \qquad \times [C(s, w, v)C^T(s, w, v)\Phi^T(0, s, w, v)]ds, \qquad on \; [[t_f], t_f] \end{cases} \tag{4.8}$$

is nonsingular, then the control process (4.6) is c.c.. In this case one of the control functions

which steers the state (4.7) to a preassigned z_f at time t_f is given by

$$u(t) = u(t, 0, z_0, t_f, z_f, w, v)$$

$$
= \begin{cases}
\begin{aligned}
& -C^T(t,w,v)\Phi^T(k,t,w,v)\Psi^T(0,k,w,v)\mathbf{U}^{-1}(0,[t_f],w,v) \\
& \times[\tfrac{z_0}{2} - \Psi(0,t_f,w,v)\tfrac{z_f}{2} \\
& + \sum_{k=1}^{[t_f]}\int_{k-1}^{k}\Psi(0,k,w,v)\Phi(k,s,w,v)g(s,w,v)ds] \, , \qquad \text{on } [k-1,k], \\[2em]
& -C^T(t,w,v)\Phi^T(0,t,w,v)\mathbf{U}^{-1}([t_f],t_f,w,v) \\
& \times\{\Phi^{-1}(t_f,0,w,v)\Psi(t_f,0,w,v)\tfrac{z_0}{2} - \Phi(0,t_f,w,v)\tfrac{z_f}{2} \\
& + \int_{[t_f]}^{t_f}\Phi(0,s,w,v)g(s,w,v)ds\} \, , \qquad \text{on } [[t_f],t_f] \, ,
\end{aligned}
\end{cases}
\tag{4.9}
$$

where $k = 1, 2, \cdots, [t_f]$. *The control function* $u(t)$ *is continuous if the following conditions are satisfied.*

$(\mathcal{H}3)$ $\Psi(0,k,w,v) = \Psi(0,k+1,w,v)\Phi(k+1,k,w,v), \quad k = 1,2,...,[t_f]-1.$

$(\mathcal{H}4)$ $\Psi^T(0,[t_f],w,v)\mathbf{U}^{-1}(0,[t_f],w,v)\Psi(0,t_f,w,v)$

$\qquad = \Phi^T(0,[t_f],w,v)\mathbf{U}^{-1}([t_f],t_f,w,v)\Psi(0,t_f,w,v)$

Proof : Since $\mathbf{U}(0,t_f,w,v)$ is non-singular, the control function $u(t) = u(t,0,z_0,t_f,z_f,w,v)$ given by (4.9) is well defined. Using (4.7) and (4.9), we get.

$$
\begin{aligned}
z(t_f) = \ & \Psi(t_f,0,w,v)z_0 \\
& + \ \Psi(t_f,0,w,v)\sum_{k=1}^{[t_f]}\int_{k-1}^{k}\Psi(0,k,w,v)\Phi(k,s,w,v)C(s,w,v) \\
& \quad \{-C^T(s,w,v)\Phi^T(k,s,w,v)\Psi^T(0,k,w,v)\mathbf{U}^{-1}(0,[t_f],w,v) \\
& \quad [(\frac{z_0}{2} - \Psi(0,t_f,w,v)\frac{z_f}{2}) \\
& + \ \sum_{k=1}^{[t_f]}\int_{k-1}^{k}\Psi(0,k,w,v)\Phi(k,s,w,v)g(\tau,w,v)d\tau]\}ds \\
& + \ \Psi(t_f,0,w,v)\sum_{k=1}^{[t_f]}\int_{k-1}^{k}\Psi(0,k,w,v)\Phi(k,s,w,v)g(s,w,v)ds \\
& + \ \Phi(t_f,0,w,v)\int_{[t_f]}^{t_f}\Phi(0,s,w,v)C(s,w,v) \\
& \quad \{-C^T(s,w,v)\Phi^T(0,s,w,v)\mathbf{U}^{-1}([t_f],t_f,w,v) \\
& \quad [\Phi^{-1}(t_f,0,w,v)\Psi(t_f,0,w,v)\frac{z_0}{2} - \Phi(0,t_f,w,v)\frac{z_f}{2})
\end{aligned}
$$

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$$- \int_{[t_f]}^{t_f} \Phi(0, \tau, w, v) g(\tau, w, v) d\tau] ds \}$$

$$+ \quad \Phi(t_f, 0, w, v) \int_{[t_f]}^{t_f} \Phi(0, s, w, v) g(s, w, v) ds$$

$$= \quad \Psi(t_f, 0, w, v) z_0 - \Psi(t_f, 0, w, v) \mathbf{U}(0, [t_f], w, v) \mathbf{U}^{-1}(0, [t_f], w, v)$$

$$[\frac{z_0}{2} - \Psi(0, t_f, w, v) \frac{z_f}{2}$$

$$+ \sum_{k=1}^{[t_f]} \int_{k-1}^{k} \Psi(0, k, w, v) \Phi(k, s, w, v) g(s, w, v) ds]$$

$$+ \Psi(t_f, 0, w, v) \sum_{k=1}^{[t_f]} \int_{k-1}^{k} \Psi(0, k, w, v) \Phi(k, s, w, v) g(s, w, v) ds$$

$$- \Phi(t_f, 0, w, v) \mathbf{U}([t_f], t_f, w, v) \mathbf{U}^{-1}([t_f], t_f, w, v)$$

$$[\Phi^{-1}(t_f, 0, w, v) \Psi(t_f, 0, w, v) \frac{z_0}{2} - \Phi(0, t_f, w, v) \frac{z_f}{2})$$

$$+ \quad \int_{[t_f]}^{t_f} \Phi(0, s, w, v) g(s, w, v) ds]$$

$$+ \Phi(t_f, 0, w, v) \int_{[t_f]}^{t_f} \Phi(0, s, w, v) g(s, w, v) ds$$

$$= \quad \Psi(t_f, 0, w, v) z_0 - \Psi(t_f, 0, w, v) \frac{z_0}{2} + \frac{z_f}{2} - \Psi(t_f, 0, w, v) \frac{z_0}{2} + \frac{z_f}{2}$$

$$= \quad z_f,$$

as required. Hence the system (4.5) is c.c.

The condition $(\mathcal{H}3)$ ensures the continuity of the control function at the integer endpoint of the each unit interval, except at the point $[t_f]$, where the continuity of $u(t)$ is ensured by the condition $(\mathcal{H}4)$.

We shall now employ the fixed point technique to establish the controllability of the nonlinear system.

Let $C[\overline{J}, \mathbb{R}^{n+m}]$ denote the Banach space of (n+m) dimensional continuous functions on $\overline{J}$. Consider the operator,

$$T : C[\overline{J}, \mathbb{R}^{n+m}] \mapsto C[\overline{J}, \mathbb{R}^{n+m}]$$

defined by $T(w, v) = (z, u)$, where z, u, w, v are as in (4.5) and (4.6).

Theorem 4.3.2 *If there exits a closed bounded convex subset S of $C[\overline{J}, \mathbb{R}^{n+m}]$ such that the*

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operator T is invariant for S, then the system (4.5) satisfying (4.8) is c.c. on $\overline{J}$.

Proof : Choose vectors w, v so as to agree with z, u given by (4.7) and (4.9). Then these vectors are also solutions for the system (4.5). This guarantees the controllability of the system (4.5). Thus the controllability of system (4.5), becomes a problem of existence of a fixed point for (4.7) and (4.9). If there is atleast one set of fixed point for (4.7) and (4.9), then this solution is also a fixed point for (4.9) and (4.10) given by

$$
\begin{aligned}
z(t) \;=\; & \Psi(t, 0, w, v) \\
& [z_0 + \sum_{k=1}^{[t]} \int_{k-1}^{k} \Psi(0, k, w, v)\Phi(k, s, w, v)C(s, w, v)v(s)ds \\
& + \sum_{k=1}^{[t]} \int_{k-1}^{k} \Psi(0, k, w, v)\Phi(k, s, w, v)g(s, w, v)ds] \\
& + \Phi(t, 0, w, v)[\int_{[t]}^{t} \Phi(0, s, w, v)C(s, w, v)v(s)ds \\
& + \int_{[t]}^{t} \Phi(0, s, w, v)g(s, w, v)ds.]
\end{aligned}
\tag{4.10}
$$

The operator T is continuous on $\mathcal{C}[\overline{J}, I\!R^{n+m}]$. Let S be a closed bounded convex subset of $\mathcal{C}[\overline{J}, I\!R^{n+m}]$ and T be invariant for S.

Therefore $T(w, v) = (z, u) \in S$ for any $(w, v) \in S$. Using (4.9) and (4.10) we conclude that $T(S)$ is bounded and equicontinuous. Hence by Schauder's fixed point theorem we conclude that there exists atleast one fixed point of T.

4.4 Comparison Theorems

In this section, we derive results based on comparison principle [13]. Here we try to examine the conditions under which there exits a set which satisfies the conditions of Theorem 4.3.2 Let $\|.\|$ be a norm on some Banach space and $|.|$ be the Euclidean norm. Define the set

$$
S = \{(w, v) \in \mathcal{C}[\overline{J}, I\!R^{n+m}] \stackrel{\ni}{_t} |w(t)| \le \alpha(t), |v(t)| \le \beta(t)\}
$$

We set the following conditions:

$(A1)$ $\|\Phi(t, s, w, v)\| \le M_1 > 0$, for all $t, s \in \overline{J}$.

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(A2) $\|\Psi(t,0,w,v)\| \leq M_2 > 0$, for all $t \in \overline{J}$.

(A3) $\|C(s,w,v)\| \leq M_3 > 0$, for $0 \leq s \leq t, t \in \overline{J}$.

(A4) $|z_0| = M_4 > 0$.

(A5) $\|\Psi(0,k,w,v)\Phi(k,s,w,v)\| \leq m_k$,

   for $k = 1,2,\cdots,[t_f]$, $0 \leq s \leq k \leq t$, $t \in \overline{J}$.

(A6) $Max\{m_k\} = M_5$,   for $k = 1,2,\cdots,[t_f]$.

(A7) $|g(t,z,u)| \leq h(t,|z|,|u|)$, where $h(t,\alpha(t),\beta(t))$ is a continuous function

   of its arguments, and nondecreasing for any $\alpha(t),\beta(t) > 0$.

(A8) $\|C^T(t,w,v)\Phi^T(k,t,w,v)\Psi^T(0,k,w,v)U^{-1}(0,[t_f],w,v)$

   $\times[\frac{z_0}{2} - \Psi(0,t_f,w,v)\frac{z_f}{2}\| \leq n_k$

   for $k = 1,2,\cdots,[t_f]$, $0 \leq s \leq k \leq t$, $t \in \overline{J}$.

(A9) $\|C^T(t,w,v)\Phi^T(k,t,w,v)\Psi^T(0,k,w,v)U^{-1}(0,[t_f],w,v)$

   $\times\Psi(0,k,w,v)\Phi(k,s,w,v)\| \leq p_k$

   for $k = 1,2,\cdots,[t_f]$, $0 \leq s \leq k \leq t$, $t \in \overline{J}$.

(A10) $b_0 = max\{n_k\}$, $b_1 = max\{p_k\}$;

   for $k = 1,2,\cdots,[t_f]$, $0 \leq s \leq k \leq t$, $t \in \overline{J}$.

(A11) $\|C^T(t,w,v)\Phi^T(0,t,w,v)U^{-1}([t_f],t_f,w,v)$

   $\times[\Phi^{-1}(t_f,0,w,v)\Psi(t_f,0,w,v)\frac{z_0}{2} - \Phi(0,t_f,w,v)\frac{z_f}{2}]\| \leq b_0', t \in \overline{J}$.

(A12) $\|C^T(t,w,v)\Phi^T(0,t,w,v)U^{-1}([t_f],t_f,w,v)\Phi(0,s,w,v)\| \leq b_1', t,s \in \overline{J}$.

We have the following result.

Theorem 4.4.1 *If there exits atleast one pair $(\alpha(t),\beta(t))$ such that the inequalities,*

$$\alpha(t) \geq a_0 + a_1 \int_0^{[t]} \beta(s)ds + a_2 \int_0^{[t]} h(s,\alpha(s),\beta(s))ds$$
$$+a_3 \int_{[t]}^t \beta(s)ds + a_4 \int_{[t]}^t h(s,\alpha(s),\beta(s))ds, \quad t \in \overline{J}. \qquad (4.11)$$

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and

$$\beta(t) \geq \begin{cases} b_0 + b_1 \int_0^{[t_f]} h(s, \alpha(s), \beta(s))ds, & on \ [0, [t_f]] \\ b_0' + b_1' \int_{[t_f]}^{t_f} h(s, \alpha(s), \beta(s))ds, & on \ [[t_f], t_f]. \end{cases} \qquad (4.12)$$

are satisfied for any constants $a_0, b_0, b_0' > 0$, *and for some constants* a_1, a_2, a_3, a_4,
$\Lambda_1, b_1' > 0$, *then the system (4.5) satisfying (A1) to (A12) and (4.8) is c.c. on* $\overline{J}$.

Proof : Using (4.10) and conditions (A1) to (A7), we get,

$$|z(t)| \leq \|\Psi(t, 0, w, v)\|$$

$$[\|z_0\| + \sum_{k=1}^{[t]} \int_{k-1}^{k} \|\Psi(0, k, w, v)\Phi(k, s, w, v)\| \|C(s, w, v)\| |v(s)| ds$$

$$+ \sum_{k=1}^{[t]} \int_{k-1}^{k} \|\Psi(0, k, w, v)\Phi(k, s, w, v)\| |g(s, w(s), v(s))| ds]$$

$$+ \|\Phi(t, 0, w, v)\| [\int_{[t]}^{t} \|\Phi(0, s, w, v)\| \|C(s, w, v)\| |v(s)| ds$$

$$+ \int_{[t]}^{t} \|\Phi(0, s, w, v)\| |g(s, w(s), v(s))|] ds.$$

$$\leq M_2 [M_4 + \sum_{k=1}^{[t]} \int_{k-1}^{k} m_k \, M_3 \, |v(s)| ds + \sum_{k=1}^{[t]} \int_{k-1}^{k} m_k \, h(s, |w(s)|, |v(s)|) ds]$$

$$+ \int_{[t]}^{t} M_1 \, M_3 \, |v(s)| ds + \int_{[t]}^{t} M_1 \, h(s, |w(s)|, |v(s)|) ds.$$

$$\leq M_2 \, M_4 + M_2 \, M_3 \, M_5 \int_0^{[t]} \beta(s) ds + M_2 \, M_5 \int_0^{[t]} h(s, \alpha(s), \beta(s)) ds$$

$$+ M_1 \, M_3 \int_{[t]}^{t} \beta(s) ds + M_1 \int_{[t]}^{t} h(s, \alpha(s), \beta(s)) ds.$$

$$= a_0 + a_1 \int_0^{[t]} \beta(s) ds + a_2 \int_0^{[t]} h(s, \alpha(s), \beta(s)) ds$$

$$+ a_3 \int_{[t]}^{t} \beta(s) ds + a_4 \int_{[t]}^{t} h(s, \alpha(s), \beta(s)) ds, \quad t \in \overline{J},$$

$$\leq \alpha(t),$$

where $a_0 = M_2 M_4$, depends on initial value z_0 and a_i' s are suitable constants defined by system parameters M_1, M_2, M_3, M_5.

Next using (4.9) and (A7) to (A10), we get,

$$|u(t)| \leq \|C^T(t, w, v)\Phi^T(k, t, w, v)\Psi^T(0, k, w, v)\mathbf{U}^{-1}(0, [t_f], w, v)$$

$$\times [\frac{z_0}{2} - \Psi(0,t_f,w,v)\frac{z_f}{2}\|$$

$$+ \sum_{k=1}^{[t_f]} \int_{k-1}^{k} \|C^T(t,w,v)\Phi^T(k,t,w,v)\Psi^T(0,k,w,v)$$

$$\times \mathbf{U}^{-1}(0,[t_f],w,v)\Psi(0,k,w,v)\Phi(k,s,w,v)\|\|g(s,w,v)|ds$$

$$\leq b_0 + b_1 \int_0^{[t_f]} h(s,\alpha(s),\beta(s))ds$$

$$\leq \beta(t) \qquad \text{on } [0,[t_f]],$$

and, similarly, using (4.9), (A7), (A11), (A12), we can obtain,

$$|u(t)| \leq b_0' + b_1' \int_{[t_f]}^{t_f} h(s,\alpha(s),\beta(s))ds$$

$$\leq \beta(t) \qquad \text{on } [[t_f],t_f] ,$$

where b_0, b_0' depend on both initial and terminal value z_0, z_f. b_1, b_1' are constants defined by system parameters. This implies that $(z,u) \in S$ and thus existence of S is established. Hence, by Theorem 4.3.2 system (4.5) is c.c. on $\overline{J}$.

We can simplify Theorem 4.3.3 by making the nonlinear function g independent of z. Consider the system,

$$z'(t) = A(t,z(t),u(t))z(t) + B(t,z(t),u(t))z([t]) + C(t,z(t),u(t))u(t) + g(t,u(t))$$

$$z(0) = z_0, \qquad\qquad\qquad\qquad (4.13)$$

$t \in \overline{J} = [0,t_f]$, $z,z_0,g \in I\!\!R^n, u \in I\!\!R^m$. $A(t,z,u), B(t,z,u)$ are $n \times n$ matrices and $C(t,z,u)$ is a $n \times m$ matrix. The matrix functions $A(t,z,u), B(t,z,u), C(t,z,u)$ are all continuous with respect to their arguments. Let us suppose that condition (A7) is replaced by,

$(A7)'$ $|g(t,u)| \leq h(t,|u|)$, where $h(t,\beta(t))$ is a continuous function of its arguments, and nondecreasing for any $\beta(t) > 0$.

We have the following result.

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Theorem 4.4.2 *If there exists atleast one nonnegative solution $\beta(t)$ of the inequality*

$$\beta(t) \geq \begin{cases} b_0 + b_1 \int_0^{[t_f]} h(s, \beta(s))ds, & on \ [0, [t_f]] \\ b_0' + b_1' \int_{[t_f]}^{t_f} h(s, \beta(s))ds, & on \ [[t_f], t_f]. \end{cases} \tag{4.14}$$

for any $b_0, b_0' > 0$ and for some constant b_1, b_1' then the system (4.13) satisfying the conditions Theorem 4.3.3 with (A7) replaced by (A7)', is c.c. on $\overline{J}$.

Proof : Since g is independent of z, we have

$$\alpha(t) \geq a_0 + a_1 \int_0^{[t]} \beta(s)ds + a_2 \int_0^{[t]} h(s, \beta(s))ds$$
$$+ a_3 \int_{[t]}^t \beta(s)ds + a_4 \int_{[t]}^T h(s, \beta(s))ds,$$

and

$$\beta(t) \geq \begin{cases} b_0 + b_1 \int_0^{[t_f]} h(s, \beta(s))ds, & on \ [0, [t_f]] \\ b_0' + b_1' \int_{[t_f]}^{t_f} h(s, \beta(s))ds, & on \ [[t_f], t_f]. \end{cases} \tag{4.15}$$

Now, if the inequality (4.15) has a solution $\beta(t)$, then the first inequality always has a solution sufficiently large. Hence, the result.

Chapter 5

SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

5.1 Introduction

In this chapter, we deal with second order nonlinear differential equations with piecewise constant deviating argument (PCDA). The second order equations with PCDA has been a topic of interest during the last decade, and is not yet explored completely. Some particular equations are discussed by the authors in [39, 67]. It is known that equations with PCDA represent a hybrid of continuous and discrete dynamical systems and combine the properties of both differential and difference equations. These hybrid systems are of interest for those working in control theory and biomedical field. The methods of obtaining the results are similar to those applied for first order equations with PCDA.

Section 2 deals with the existence of solution of the nonlinear equation, under condition that the corresponding first order ordinary differential equation with parameters has a solution.

In section 3, we obtain the solution of a second order linear differential equation with PCDA. We also establish the linear variation of parameters formula. From these we deduce the particular cases required for the monotone iterative technique.

The next section contains the main result of this chapter. Here we establish the existence

of the maximal and minimal solution by using monotone iterative technique. An important lemma dealing with an inequality required for monotone method is proved.

Finally, section 5 is concerned with a particular second order nonlinear differential equation with PCDA. We obtain ~~the~~ existence of solution and state the necessary condition for the solution to have a zero on each unit interval with integral end points. An example is constructed in support of the result.

5.2 Existence of solution

In this section, we prove ~~the~~ existence of solution for the general second order nonlinear differential equation with PCDA. We employ the method used by Aftabizadeh [3] for the first order equation.

Consider the nonlinear equation,

$$x''(t) = f(x(t), x([t])), \quad x(0) = c_0, \quad x'(0) = d_0 \tag{5.1}$$

where $[\cdot]$ denotes the greatest integer function, f is a continuous function on $I\!R \times I\!R$, and $t \in I = [0, \infty)$.

We need the following definition.

Definition 5.2.1 *A solution of the equation(5.1) on I is a function $x(t)$ that satisfies the conditions*

(i) $x(t)$ is continuously differentiable on I.

(ii) $x''(t)$ exists at each point $t \in I$ with the possible exception of points $[t] \in I$ where f has one sided derivatives.

(iii) Equation(5.1) is satisfied on each unit interval $[n, n+1)$ with integral end points.

The equation(5.1) contains an argument that is constant on interval with integral end points. Continuity of the solution at a point joining any two consecutive intervals leads to recurrence relations for solution at such points. Hence, the solution is determined by the finite set of initial data, rather than by an initial function.

Along with equation(5.1), we consider the ordinary differential equation with parameters

$$x'(t) = F(x(t), \lambda, \mu,).$$ (5.2)

If $F(x(t), \lambda, \mu)$ is continuous and different from zero on a set S, then on S there exists a general integral,

$$G(x(t), \lambda, \mu) = t + h(\lambda, \mu)$$ (5.3)

with an arbitrary function $h(\lambda, \mu)$.

We have the following result on existence and uniqueness of solution of (5.1).

Theorem 5.2.1 *Assume that*

(i) $F(x, \lambda, \mu) \in \mathcal{C}(\mathbb{R}^3)$ is different from zero on a set S.

(ii) Equation (5.2) satisfies existence and uniqueness conditions in $\mathbb{R}^3$ and its solution can be extended over I.

(iii) Equation (5.8) has a unique solution with respect to c_{n+1}.

(iv) The system of difference equations (5.9) and (5.10) is uniquely solvable.

Then the IVP (5.1) has a unique solution on I.

Proof Let $x_n(t)$ be a solution of the equation(5.1) on the interval $[n, n+1)$ satisfying the conditions $x(n) = c_n$ and $x'(n) = d_n$. Then we have from (5.1)

$$x_n''(t) = f(x_n(t), c_n).$$

This on integration yields,

$$x'_n(t) = d_n + \int_n^t f(x_n(s), c_n)\, ds. \qquad (5.4)$$

Let

$$F(x_n(t), c_n, d_n) = d_n + \int_n^t f(x_n(s), c_n)\, ds. \qquad (5.5)$$

Using (5.4) and (5.5) we get.

$$x'_n(t) = F(x_n(t), c_n, d_n). \qquad (5.6)$$

The equation (5.6) has solution

$$G(x_n(t), c_n, d_n) = t + h(c_n, d_n).$$

Put $t = n$, to get

$$G(c_n, c_n, d_n) = n + h(c_n, d_n),$$

and therefore we get,

$$G(x_n(t), c_n, d_n) - G(c_n, c_n, d_n) = t - n.$$

This can be written in the form,

$$\int_{c_n}^{x_n(t)} \frac{dx}{F(x, c_n, d_n)} = t - n. \qquad (5.7)$$

At $t = n + 1$, we have

$$\int_{c_n}^{c_{n+1}} \frac{dx}{F(x, c_n, d_n)} = 1. \qquad (5.8)$$

By (iii), (5.8) has a unique solution with respect to c_{n+1} hence we have,

$$c_{n+1} = \phi(c_n, d_n), \quad n = 0, 1, 2, \ldots \qquad (5.9)$$

for some function ϕ. Similarly, from (5.6), we can obtain the relation,

$$d_{n+1} = F(c_{n+1}, c_n, d_n), \quad n = 0, 1, 2, \ldots \qquad (5.10)$$

By (iv), the system of difference equations consisting of (5.9) and (5.10) can be uniquely solved with the initial values c_0, d_o known. Substituting these in (5.7), we can find the solution $x_n(t, c_n, d_n)$ of (5.1) on the interval $[n, n+1)$.

5.3　The Linear Equations

In this section, we establish the results required for the use of monotone iterative technique in the next section, which is the main result of this chapter. We first prove the existence of a solution of the IVP for the second order linear differential equation with PCDA. Then we obtain the variation of parameters formula for the associated non-homogeneous equation with PCDA.

Consider the linear IVP

$$
\begin{aligned}
x''(t) &= a_1 x'(t) + a_2 x(t) + a_3 x([t]), \\
x(0) &= c_0, \quad x'(0) = d_0 ,
\end{aligned}
\tag{5.11}
$$

where a_1, a_2, a_3 are constants $a_2 \neq 0$, $a_3 \neq 0$, $c_0, d_0 \in I\!R$.

We have the following result,

Lemma 5.3.1 *The IVP (5.11) has a unique solution on* I . *The solution on the interval* $[n, n+1)$ *satisfying* $x(n) = c_n$ *and* $x'(n) = d_n$ *is given by the equation(5.15) below.*

Proof : Let $x_n(t)$ be the solution of the IVP (5.11) on the interval $[n, n+1)$. Let $x(i) = c_i$ and $x'(i) = d_i$, for $i = 0, 1, 2, 3, ...$, then we have

$$
\begin{aligned}
x_n''(t) &= a_1 x_n'(t) + a_2 x_n(t) + a_3 c_n , \\
x(n) &= c_n, \quad x'(n) = d_n .
\end{aligned}
\tag{5.12}
$$

If m_1 and m_2 are two distinct roots of the equation $m^2 - a_1 m - a_2 = 0$, then by using variation of parameters formula for second order linear ODE with constant coefficients, the solution of the equation(5.12) is given by

$$
\begin{aligned}
x_n(t) &= \alpha_n e^{m_1 t} + \beta_n e^{m_2 t} \\
&\quad - a_2^{-1} a_3 c_n \left[1 + \frac{m_2}{m_1 - m_2} e^{m_1(t-n)} - \frac{m_1}{m_1 - m_2} e^{m_2(t-n)} \right] ,
\end{aligned}
\tag{5.13}
$$

where α_n and β_n are arbitrary constants. Differentiating, we get

$$x'_n(t) = m_1 \alpha_n e^{m_1 t} + m_2 \beta_n e^{m_2 t}$$
$$-a_2^{-1} a_3 c_n \left[\frac{m_1 m_2}{m_1 - m_2} e^{m_1(t-n)} - \frac{m_1 m_2}{m_1 - m_2} e^{m_2(t-n)} \right]. \qquad (5.14)$$

Letting $t = n$, these equations give.

$$c_n = \alpha_n e^{m_1 n} + \beta_n e^{m_2 n}$$

and

$$d_n = \alpha_n m_1 e^{m_1 n} + \beta_n m_2 e^{m_2 n} .$$

Solving for α_n, β_n and substituting in equations (5.13) and (5.14), on simplification we get

$$x_n(t) = c_n \left\{ \frac{-m_2(1 + a_2^{-1} a_3)}{m_1 - m_2} e^{m_1(t-n)} + \frac{m_1(1 + a_2^{-1} a_3)}{m_1 - m_2} e^{m_2(t-n)} - a_2^{-1} a_3 \right\}$$
$$+ d_n \left\{ \frac{e^{m_1(t-n)}}{m_1 - m_2} - \frac{e^{m_2(t-n)}}{m_1 - m_2} \right\}, \qquad (5.15)$$

and

$$x'_n(t) = c_n \left\{ \frac{-m_2 m_1(1 + a_2^{-1} a_3)}{m_1 - m_2} e^{m_1(t-n)} + \frac{m_1 m_2(1 + a_2^{-1} a_3)}{m_1 - m_2} e^{m_2(t-n)} \right\}$$
$$+ d_n \left\{ \frac{m_1 e^{m_1(t-n)}}{m_1 - m_2} - \frac{m_2 e^{m_2(t-n)}}{m_1 - m_2} \right\}, \qquad (5.16)$$

respectively. Put $t = n + 1$, in equation(5.15), then we have

$$c_{n+1} = c_n \left\{ \frac{-m_2(1 + a_2^{-1} a_3)}{m_1 - m_2} e^{m_1} + \frac{m_1(1 + a_2^{-1} a_3)}{m_1 - m_2} e^{m_2} - a_2^{-1} a_3 \right\}$$
$$+ d_n \left\{ \frac{e^{m_1}}{m_1 - m_2} - \frac{e^{m_2}}{m_1 - m_2} \right\}$$

which can be written in the form

$$c_{n+1} = c_n k_1 + d_n k_2,$$

where k_1, k_2 are suitable constants. Similarly, letting $t = n + 1$ in equation(5.16), we get

$$d_{n+1} = c_n l_1 + d_n l_2$$

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where l_1, l_2 are suitable constants.

Let $v_n = \begin{pmatrix} c_n \\ d_n \end{pmatrix}$, then the above two recurrence relations give,

$$v_{n+1} = \begin{pmatrix} k_1 & k_2 \\ l_1 & l_2 \end{pmatrix} v_n. \tag{5.17}$$

a homogeneous difference equation. We look for a non zero solution of the equation (5.17) in the form $v_n = r\lambda^n$, where r is a constant column vector. This implies that λ satisfies the equation

$$det(A - \lambda I) = 0, \quad \text{where } A = \begin{pmatrix} k_1 & k_2 \\ l_1 & l_2 \end{pmatrix}.$$

This yields, $\lambda^2 - (k_1 + l_2)\lambda + k_1 l_2 - l_1 k_2 = 0$. Assuming that this equation has two roots, λ_1, λ_2 ($\lambda_1 \neq \lambda_2$), we get the general solution of the difference equation (5.17) as $v_n = r_1 \lambda_1^n + r_2 \lambda_2^n$ with r_i are constant column vectors depending upon λ_i, $i = 1, 2$. The r_i's can be found by using initial conditions.

If $\lambda_1 = \lambda_2 = \lambda$ then, $v_n = rn\lambda^n$. The solution $x_n(t)$ is then obtained by substituting the components of v_n in the equation(5.15).

The case $m_1 = m_2$ can be dealt with similarly and we can obtain the solution x_n . This completes the proof.

Remark 5.3.1 *In the following results we shall consider only the case $m_1 > m_2$.*

We need the following simple deduction from Lemma 5.3.1 . We state it without proof.

Lemma 5.3.2 *The solution of*

$$x''(t) = Mx(t) + Nx([t]), \quad x(0) = c_0, \quad x'(0) = d_0,$$

where $M, N \neq 0$, are constants, exists on 1 .

On the interval $[n, n+1)$, the solution is given by:

$$x_n(t) = c_n \left\{ \frac{(1 + M^{-1}N)}{2} e^{\sqrt{M}(t-n)} + \frac{(1 + M^{-1}N)}{2} e^{-\sqrt{M}(t-n)} - M^{-1}N \right\}$$
$$+ d_n \frac{1}{2\sqrt{M}} \left\{ e^{\sqrt{M}(t-n)} - e^{-\sqrt{M}(t-n)} \right\}. \tag{5.18}$$

We now establish a useful tool in the study of properties of the solutions of differential equations, namely the variation of parameters formula. First we state the result on non-homogeneous difference equation from [27].

Lemma 5.3.3 *The unique solution of the IVP*

$$y(n+1) = A(n)y(n) + g(n), \quad y(n_0) = y_0$$

is given by

$$y(n, n_0, y_0) = \left(\prod_{i=n_0}^{n-1} A(i) \right) y_0 + \sum_{r=n_0}^{n-1} \left(\prod_{i=r+1}^{n-1} A(i) \right) g(r)$$

If A is a constant matrix, and $n_0 = 0$, then we have

$$y(n, 0, y_0) = A^n y_0 + \sum_{r=0}^{n-1} A^{n-r-1} g(r)$$

We now prove the variation of parameters formula.

Lemma 5.3.4 *(Variation of Parameters formula)*

The Linear non-homogeneous equation

$$x''(t) = a_1 x'(t) + a_2 x(t) + a_3 x([t]) + f(t) \tag{5.19}$$

satisfying $x(0) = c_0$, $x'(0) = d_0$, where $f(t)$ is a continuous function on I,

a_2, $a_3 \neq 0$ has on I a unique solution.

The solution $x_n(t)$ on $[n, n+1)$ satisfying $x(n) = c_n$, $x'(n) = d_n$

is given by the equation(5.21) below.

Proof : Let $x_n t$ be the solution of the equation (5.19) on $[n, n+1)$, then we have

$$
\begin{aligned}
x_n''(t) &= a_1 x_n'(t) + a_2 x_n(t) + a_3 c_n + f(t) \\
x(n) &= c_n, \quad x'(n) = d_n
\end{aligned}
\tag{5.20}
$$

If m_1, m_2 $(m_1 \neq m_2)$ are roots of the equation $m^2 - a_1 m - a_2 = 0$, then using variation of parameters formula, we get

$$x_n(t) = c_n \left\{ \frac{-m_2(1 + a_2^{-1} a_3)}{m_1 - m_2} e^{m_1(t-n)} + \frac{m_1(1 + a_2^{-1} a_3)}{m_1 - m_2} e^{m_2(t-n)} - a_2^{-1} a_3 \right\}$$
$$+ d_n \left\{ \frac{e^{m_1(t-n)} - e^{m_2(t-n)}}{m_1 - m_2} \right\} \qquad (5.21)$$
$$+ \frac{1}{m_1 - m_2} \int_n^t \left[e^{m_1(t-s)} - e^{m_2(t-s)} \right] f(s) ds,$$

and

$$x_n'(t) = c_n \left\{ \frac{-m_1 m_2(1 + a_2^{-1} a_3)}{m_1 - m_2} e^{m_1(t-n)} + \frac{m_2 m_1(1 + a_2^{-1} a_3)}{m_1 - m_2} e^{m_2(t-n)} \right\}$$
$$+ d_n \left\{ \frac{m_1 e^{m_1(t-n)} - m_2 e^{m_2(t-n)}}{m_1 - m_2} \right\} \qquad (5.22)$$
$$+ \frac{1}{m_1 - m_2} \int_n^t \left[m_1 e^{m_1(t-s)} - m_2 e^{m_2(t-s)} \right] f(s) ds.$$

Letting $t = n + 1$, we obtain

$$c_{n+1} = c_n k_1 + d_n k_2 + \psi_1(n),$$
$$d_{n+1} = c_n l_1 + d_n l_2 + \psi_2(n), \qquad (5.23)$$

where k_1, k_2, l_1, l_2 are suitable constants as in Lemma 5.3.1 and

$$\psi_1(n) = \frac{1}{m_1 - m_2} \int_n^{n+1} \left[e^{m_1(n+1-s)} - e^{m_2(n+1-s)} \right] f(s) ds,$$

$$\psi_2(n) = \frac{1}{m_1 - m_2} \int_n^{n+1} \left[m_1 e^{m_1(n+1-s)} - m_2 e^{m_2(n+1-s)} \right] f(s) ds.$$

Taking $v_n = \begin{pmatrix} c_n \\ d_n \end{pmatrix}$, we get from equation(5.23), a non-homogeneous difference equation,

$$v_{n+1} = A v_n + \psi(n), \qquad v(0) = v_0,$$

where $A = \begin{pmatrix} k_1 & k_2 \\ l_1 & l_2 \end{pmatrix}$, and $\psi(n) = \begin{pmatrix} \psi_1(n) \\ \psi_2(n) \end{pmatrix}$.

By using lemma 5.3.3, we get

$$v_n = A^n v_0 + \sum_{r=0}^{n-1} A^{n-r-1} \psi(r)$$

The solution $x_n(t)$ is obtained by substituting the component of v_n in equation(5.21). .

5.4 Monotone Iterative Technique

In this section we apply the monotone iterative technique to prove the existence of minimal and maximal solutions for the second order nonlinear differential equation with PCDA. We first define the concepts of upper and lower solution and prove an inequality result required for the monotone method.

Consider the nonlinear equation,

$$x''(t) = f(t, x(t), x([t])), \quad x(0) = c_0, \quad x'(0) = d_0 \tag{5.24}$$

where $f \in C[I \times I\!R \times I\!R, I\!R]$. We have the following definition.

Definition 5.4.1 *A continuous function $u(t)$ on I is said to be a lower solution of (5.24) if x'' exists at each point $t \in I$ with the possible exception of points $[t] \in I$ where one sided derivatives exist, and*

$$u''(t) \le f(t, u(t), u([t])), \quad u(0) \le c_0, \quad u'(0) \le d_0. \tag{5.25}$$

It is said to be an upper solution if the reversed inequalities hold.

We need the following Lemma, which is deduced from Lemma 5.3.2 .

Lemma 5.4.1 *Suppose that $x \in C[I, I\!R]$ and the derivative $x''(t)$ exists at each point $t \in I$ with the possible exception of the points $[t] \in I$ where one-sided derivatives exist. Assume that*

$$x''(t) \le M x(t) + N x([t]), \quad x(0) = c_0 \le 0 \quad x'(0) = d_0 \le 0 \tag{5.26}$$

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where M and N are constants such that

$$N \geq \frac{-M\cosh(\sqrt{M}.\nu)}{[\cosh(\sqrt{M}.\nu) - 1]}; \quad 0 < \nu < 1; \quad M > 0. \tag{5.27}$$

Then $x(t) \leq 0$ on I.

Proof : For $t \in [n, n+1)$, $\quad n = 0, 1, 2...$, consider

$$x_n^{''}(t) \leq Mx_n(t) + Nc_n, \quad x_n(n) = c_n \leq 0 \quad x_n'(n) = d_n \leq 0.$$

Using equation(5.18) and condition(5.27), we get $x_n(t) \leq 0$, $t \in [n, n+1)$. Using continuity of the solution this yields $x(t) \leq 0$ for $t \geq 0$.

We are now in position to prove the main result of this chapter, by using the monotone iterative technique. This method is constructive, and yields monotone sequences converging to solutions of (5.24). These sequences are such that, each of its members is a solution of a linear equation with PCDA. The advantage of the technique is that these solutions can be computed explicitly.

Theorem 5.4.1 *Let u_0 and v_0 be the lower and upper solution of equation(5.24) respectively such that $u_0(t) \leq x(t) \leq v_0(t)$ on I, where $x(t)$ is the solution of (5.24) existing on I. Suppose that*

$$(H1) \quad f(t, x_1, y_1) - f(t, x_2, y_2) \geq M(x_1 - x_2) + N(y_1 - y_2), \quad t \geq 0,$$

for $u_0(t) \leq x_2(t) \leq x_1(t) \leq v_0(t)$, $\quad u_0(t) \leq y_2(t) \leq y_1(t) \leq v_0(t)$ and

$$N \geq \frac{-M\cosh(\sqrt{M}.\nu)}{[\cosh(\sqrt{M}.\nu) - 1]}, \quad 0 < \nu < 1, \quad M > 0$$

Then there exists monotonic sequences $\{u_m(t)\}$ and $\{v_m(t)\}$, with $u_0(t)$ and $v_0(t)$ as lower and upper solutions respectively and such that

$$u_m \longrightarrow u(t), \quad v_m(t) \longrightarrow v(t) \text{ as } m \longrightarrow \infty \text{ monotonically on } I.$$

$u(t)$ and $v(t)$ are minimal and maximal solution of the equation(5.24) respectively.

Proof : For any $w \in [I, \mathbb{R}]$ such that $u_0(t) \le w(t) \le v_o(t)$, consider the linear equation,

$$x''(t) = f(t, w(t), w([t])) + M\{x(t) - w(t)\} + N\{x([t]) - w([t])\} \qquad (5.28)$$

with $x(0) = c_0$, $x'(0) = d_0$.

For every such $w(t)$, there exists a unique solution $x(t)$ of equation(5.28) on I.

Define a map T by $Tw = x$, where x is the unique solution of equation (5.28). This map is used to define the sequences $\{u_m(t)\}$ and $\{v_m(t)\}$. We need to prove the following:

(a) $u_0 \le Tu_0$, $v_0 \ge Tv_0$

(b) T is a monotonic operator on the segment

$$[u_0, v_0] = \{x \in C[I, \mathbb{R}] : u_0(t) \le x(t) \le v_0(t)\}$$

Proof of (a): Let $Tu_0 = u_1$ where u_1 is a unique solution of the equation(5.28) with $w = u_0$, namely;

$$u_1''(t) = f(t, u_0(t), u_0([t])) + M\{u_1(t) - u_0(t)\} + N\{u_1([t]) - u_0([t])\}, \qquad (5.29)$$

$u_1(0) = c_0$, $u_1'(0) = d_0$

Let $p(t) = u_1(t) - u_0(t)$. On each unit interval $[n, n+1)$, $n = 0, 1, 2...$, we have,

$p_n(t) = u_{1,n}(t) - u_{0,n}(t)$, where $u_{1,n}(t)$ satisfies equation(5.29) on $[n, n+1)$,

when $u_0(t) = u_{0,n}(t)$, $u_{1,n}(n) = c_n$, $u_{1,n}'(n) = d_n$ and $u_{0,n}(t)$ satisfying equation(5.25) on $[n, n+1)$. Then we have

$$
\begin{aligned}
p_n''(t) &= u_{1,n}''(t) - u_{0,n}''(t) \\
&\ge u_{1,n}''(t) - f(t, u_{0,n}(t), u_{0,n}(n)) \\
&= M\{u_{1,n}(t) - u_{0,n}(t)\} + N\{u_{1,n}(n) - u_{0,n}(n)\} \\
&= Mp_n(t) + Np_n(n).
\end{aligned}
$$

Note that

$$p_n(n) = u_{1,n}(n) - u_{0,n}(n) = c_n - u_{0,n}(n) \ge 0,$$

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. and

$$p'_n(n) = u'_{1,n}(n) - u'_{0,n}(n) = d_n - u'_{0,n}(n) \geq 0 \ .$$

. By Lemma 5.4.1, we get $p_n(t) \geq 0$ for $t \in [n, n+1)$, $n = 0, 1, 2, \dots$ and hence $P(t) \geq 0$ on I. This shows that $u_1(t) \geq u_0(t)$ or $u_0(t) \leq u_1(t) = Tu_0(t)$, where u_1 satisfy the equation(5.28) with $w = u_0$.

Similarly, by letting $Tv_0 = v_1$, where v_1 satisfying the equation (5.28) with $w = v_0$ and proceeding as above , we can show that $v_0 \geq v_1 = Tv_0$.

<u>Proof of (b)</u>: Let $w_1, w_2 \in C[I, I\!R]$ such that

$$u_0(t) \leq w_1(t) \leq w_2(t) \leq v_0(t).$$

Suppose that $x_1 = Tw_1$ and $x_2 = Tw_2$. Set $q(t) = x_2(t) - x_1(t)$. So that on each unit interval $[n, n+1)$, we have

$$q''_n(t) = x''_{2,n}(t) - x''_{1,n}(t),$$

where notations are as described above . Using equation (5.28), we get

$$
\begin{aligned}
q''_n(t) = \ & f(t, w_{2,n}(t), w_{2,n}(n)) - f(t, w_{1,n}(t), w_{1,n}(n)) \\
& + M\{x_{2,n}(t) - w_{2,n}(t)\} - M\{x_{1,n}(t) - w_{1,n}(t)\} \\
& + N\{x_{2,n}(n) - w_{2,n}(n)\} - N\{x_{1,n}(n) - w_{1,n}(n)\}.
\end{aligned}
$$

Using condition (III), we get.

$$
\begin{aligned}
q''_n(t) \geq \ & M\{w_{2,n}(t) - w_{1,n}(t)\} + N\{w_{2,n}(n) - w_{1,n}(n)\} \\
& + M\{x_{2,n}(t) - w_{2,n}(t)\} + N\{x_{2,n}(n) - w_{2,n}(n)\} \\
& - M\{x_{1,n}(t) - w_{1,n}(t)]\} - N\{x_{1,n}(n) - w_{1,n}(n)\}.
\end{aligned}
$$

On simplification we get

$$q''_n(t)) \geq M\{x_{2,n}(t) - x_{1,n}(t)\} + N\{x_{2,n}(n) - x_{1,n}(n)\}$$

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$$= M q_n(t) + N q_n(n).$$

Also we have $q_n(n) \geq 0$, $\quad q_n'(n) \geq 0$. Therefore, again by making use of lemma 5.4.1, $q_n(t) \geq 0$ for $t \in [n, n+1)$, $\quad n = 0, 1, 2...$, and hence $q(t) \geq 0$ on I. This shows that $x_2(t) \geq x_1(t)$ or $T w_2 \geq T w_1$, for w_1, $w_2 \in [u_0, v_0]$ and $w_1 \leq w_2$. Thus (b) is proved .

Now define the sequences $u_m = T u_{m-1}$ and $v_m = T v_{m-1}$, where u_m and v_m satisfy the equations

$$u_m''(t) = f(t, u_{m-1}(t), u_{m-1}([t])) + M\{u_m(t) - u_{m-1}(t)\} + N\{u_m([t]) - u_{m-1}([t])\}.$$
$$u_m(0) = c_0, \quad u_m'(0) = d_0 \,,$$

and

$$v_m''(t) = f(t, v_{m-1}(t), v_{m-1}([t])) + M\{v_m(t) - v_{m-1}(t)\} + N\{v_m([t]) - v_{m-1}([t])\}$$
$$v_m(0) = c_0, \quad v_m'(0) = d_0 \,,$$

respectively. Proceeding as in the above arguments and using induction, we get

$$u_0(t) \leq u_1(t) \leq ... \leq u_m(t) \leq v_m(t) \leq ... \leq v_1(t) \leq v_0(t), \quad t \geq 0$$

. It then follows that $\lim_{m \to \infty} u_m(t) = u(t)$ and $\lim_{m \to \infty} v_m(t) = v(t)$ uniformly and monotonically, and u and v are solutions of the equation

$$x''(t) = f(t, x(t), x([t])), \quad x(0) = c_0, \quad x'(0) = d_0 \,.$$

. In order to show that $u(t)$ and $v(t)$ are minimal and maximal solutions of the equation(5.24), it is required to show that if $x(t)$ is any solution of the equation(5.24) satisfying $u_0(t) \leq x(t) \leq v_0(t)$ on I then, $u_0(t) \leq u(t) \leq x(t) \leq v(t) \leq v_0(t)$ on I .

Let for some m, $u_m \leq x \leq v_m$ on I . Set $p(t) = x(t) - u_{m+1}(t)$, so that

$$p''(t) = x''(t) - u_{m+1}''(t)$$

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$$= f(t, x(t), x([t])) - f(t, u_m(t), u_m([t]))$$

$$-M\{u_{m+1}(t) - u_m(t)\} - N\{u_{m+1}(t) - u_m([t])\}$$

which, using condition $(H1)$ yields,

$$p''(t) \geq M\{x(t) - u_m(t)\} + N\{x([t]) - u_m([t])\}$$

$$-M\{u_{m+1}(t) - u_m(t)\} - N\{u_{m+1}([t]) - u_m([t])\}$$

$$= M\{x(t) - u_{m+1}(t)\} + N\{x([t]) - u_{m+1}([t])\}$$

$$= Mp(t) + Np([t])$$

Since $p(0) = 0$, by Lemma 5.4.1, $p(t) \geq 0$, which implies that $x(t) \geq u_{m+1}(t)$ on I.

Similarly we can show that $x(t) \leq v_{m+1}(t)$, and hence $u_{m+1}(t) \leq x(t) \leq v_{m+1}(t)$ on I.

This proves, by induction, that $u_m(t) \leq x(t) \leq v_m(t)$ on I for all m.

Taking limit as $m \to \infty$ we conclude that $u(t) \leq x(t) \leq v(t)$ on I.

This completes the proof.

5.5 Oscillatory behaviour

This section deals with the oscillatory behaviour of the solution of a nonlinear second order differential equation with PCDA. The solution is said to be oscillatory if it has arbitrarily large number of zeros. We first establish the existence of the solution of the equation.

Consider the IVP

$$x''(t) + a(t)f(x([t])) = 0, \quad x(0) = c_0 \ ; \ x'(0) = d_0 \tag{5.30}$$

where $f \in C[I\!R, I\!R]$, $a \in C[I, I\!R]$, $I = [0, \infty)$.

We have the following theorem.

Theorem 5.5.1 *Equation (5.30) has a solution on I.*

Proof : We use the method of steps. Let $t \in [n, n+1)$, and $x_n(t)$ be the solution of (5.30) on the interval $[n, n+1)$, . satisfying $x_n(n) = c_n$, and $x_n'(n) = d_n$.

Then, we have

$$x_n''(t) + a(t)f(c_n) = 0,$$

and hence,

$$
\begin{aligned}
x_n'(t) &= d_n - \int_n^t a(s)f(c_n)ds, \\
x_n(t) &= c_n + d_n(t-n) - \int_n^t (t-s)a(s)f(c_n)ds.
\end{aligned}
\tag{5.31}
$$

Letting $t \to n+1$, these yields,

$$
\begin{aligned}
c_{n+1} &= c_n + d_n - \int_n^{n+1} (n+1-s)a(s)f(c_n)ds, \\
d_{n+1} &= d_n - \int_n^{n+1} a(s)f(c_n)ds.
\end{aligned}
$$

Let

$$
\begin{aligned}
\psi_1(n) &= \int_n^{n+1} (n+1-s)a(s)f(c_n)ds, \\
\psi_2(n) &= \int_n^{n+1} a(s)f(c_n)ds.
\end{aligned}
$$

Taking $v_n = \begin{pmatrix} c_n \\ d_n \end{pmatrix}$, we get , a non-homogeneous difference equation,

$$v_{n+1} = Av_n + \psi(n), \qquad v(0) = v_0,$$

where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $\psi(n) = \begin{pmatrix} \psi_1(n) \\ \psi_2(n) \end{pmatrix}$.

By using Lemma 5.3.3, we get

$$v_n = A^n v_0 + \sum_{r=0}^{n-1} A^{n-r-1}\psi(r)$$

The solution $x_n(t)$ is obtained by substituting the component of v_n in equation(5.31).

Remark 5.5.1 *When* $a(t) = a$, *a constant*, *the solution* $x_n(t)$ *of the equation (5.30) is given by*,

$$x_n(t) = c_n + d_n(t-n) - af(c_n)\frac{(t-n)^2}{2},$$

and the two recurrence relations are

$$c_{n+1} = c_n + d_n - \frac{af(c_n)}{2},$$

$$d_{n+1} = d_n - af(c_n).$$

We shall now study the oscillatory behaviour of the solution of the equation (5.30) when $a(t) = a$, a constant.

Consider the IVP,

$$x''(t) + af(x([t])) = 0, \quad x(0) = c_0 \ ; \ x'(0) = d_0 \tag{5.32}$$

where $f \in C[IR, IR \setminus \{0\}]$, $t \in I = [0, \infty)$, $a \neq 0$ is a constant.

We have the following result.

Theorem 5.5.2 *If the solution of equation (5.32) has a zero on unit interval* $(n, n+1)$, *then*

$$2(c_n + d_n) < af(c_n) \quad and \quad d_n^2 + 2ac_nf(c_n) \geq 0. \tag{5.33}$$

where $x_n = c_n$, *and* $x'_n = d_n$.

Proof : Let $x_n(t)$ be the solution of the equation (5.32) on the interval $(n, n+1)$. Then

$$x_n(t) = c_n + d_n(t-n) - af(c_n)\frac{(t-n)^2}{2},$$

Let $t_n \in (n, n+1)$ be a zero of $x_n(t)$. Then

$$c_n + d_n(t_n - n) - af(c_n)\frac{(t_n - n)^2}{2} = 0.$$

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This yields,

$$t_n = n + \frac{d_n \pm \sqrt{d_n^2 + 2ac_nf(c_n)}}{af(c_n)}$$

This implies

$$0 < \frac{d_n \pm \sqrt{d_n^2 + 2ac_nf(c_n)}}{af(c_n)} < 1 \quad \text{and} \quad d_n^2 + 2ac_nf(c_n) \geq 0. \qquad (5.34)$$

<u>Case(i): $af(c_n) > 0$</u> Then the first inequality in (5.34) yields

$$0 < d_n \pm \sqrt{d_n^2 + 2ac_nf(c_n)} < af(c_n)$$

and on simplification, we get,

$$2(c_n + d_n) < af(c_n)$$

<u>Case(ii): $af(c_n) < 0$</u> Then the first inequality in (5.34) yields,

$$0 > d_n \pm \sqrt{d_n^2 + 2ac_nf(c_n)} > af(c_n)$$

and on simplification, we get,

$$2(c_n + d_n) < af(c_n)$$

Thus in both the cases, along with the second inequality in (5.34), we get the same condition. Hence the result.

Example : Consider the IVP

$$x''(t) + x^2([t]) = 0 \ , \ x(0) = 1 \ , \ x'(0) = -\frac{1}{2} \ .$$

Its solution $x_0(t)$ on $(0,1)$ is given by $x_0(t) = 1 - \frac{t}{2} - \frac{t^2}{2}$, and it has no zeros on $(0,1)$. Here, the first inequality in (5.33) does not hold.

SUMMARY

The aim of this thesis is to study nonlinear differential equations with piecewise constant deviating argument (PCDA). The differential equations with PCDA consolidate several properties of the continuous dynamical systems generated by delay differential equations, and of discrete dynamical systems generated by difference equations. An attempt is made to build up the theory of differential equations with PCDA where the argument is the greatest integer function $[t]$. In general, the results of the theory of ordinary differential equations (ODE) are extended in a suitable manner to get corresponding results in delay differential equations. In some respects this requires new ideas and novel approach. Equations with PCDA, being a relatively new topic and of interest for last two decades, there is an opportunity to extend the known results of ODE to equations with PCDA in particular and delay differential equations in general. We briefly summarise below the work done in this thesis and point out some directions for future course of study on differential equations with PCDA.

In the very first chapter, a general introduction of the topic has been given and the problems taken for study are mentioned. The next chapter deals with a brief survey of the present status of the work done on linear as well as nonlinear differential equations with PCDA. It also includes the results from ODE and nonlinear analysis relevant to the work done in the thesis.

In Chapter three, we have discussed the first order differential equations with PCDA. Some simple extensions of results concerning mean value property, and upper and lower solutions are proved. The existence and uniqueness of the solution of

$$x'(t) = f(t, x(t), x([t])), \quad x(0) = x_0, \quad t \in J = [0, T], \ T > 0. \qquad (5.35)$$

is obtained by using the method of quasilinearisation. The condition imposed on the func-

tion f is a convexity type condition [See (H4) of Theorem 3.3.1]. Monotone sequences of approximate solutions are constructed which converge uniformly to the unique solution of (5.35). In the next result, this condition is relaxed, by assuming that $f + \phi$, for some continuous function ϕ, satisfy the convexity type condition [See (H4) of Theorem 3.3.3]. In both the cases, it is shown that the convergence of the monotone sequences is quadratic. Further, some inequalities are proved, which may be useful in the stability theory. We have employed the method of quasilinearisation to prove the existence of solution of the nonlinear periodic boundary value problem

$$x'(t) = f(t, x(t), x([t])), \quad x(0) = x(2\pi), \quad t \in [0, 2\pi]. \tag{5.36}$$

Here a monotone sequence of solutions of some nonlinear equations converges uniformly to a solution of (5.36). The associated linear periodic boundary value problem is also discussed. Finally, we have obtained a necessary condition for a solution of a nonlinear equation with PCDA to have a zero in each unit interval $[n, n+1)$.

Chapter four is devoted to the controllability of a nonlinear system. Sufficient conditions are obtained for both nonlinear as well as the corresponding linear system. The result is established by constructing a nonlinear operator on some function space and then using Schauder's fixed point theorem. Some comparison theorems giving properties of the state as well as control function are obtained. In the last chapter, we introduce a second order nonlinear differential equation with PCDA. The main result of this chapter is the existence of maximal and minimal solutions of the equation

$$x''(t) = f(t, x(t), x([t])), \quad x(0) = c_0, \quad x'(0) = d_0, \quad t \in [0, \infty). \tag{5.37}$$

The monotone iterative technique is used to obtain monotone sequences converging to maximal and minimal solutions. These sequences are constructed by using the solutions of some linear equations. Existence of unique solution of linear as well as nonlinear equations are also established. The chapter ends with a discussion on oscillatory behavior of a second order

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equation. Conditions relating to having zeros of the solution in the unit interval $[n, n+1)$ are obtained.

Problems for further study

Differential equations with PCDA have been found useful in several areas of application. Hence it is required that, they are studied in detail. As mentioned above, there appears to be ample opportunity to study these equations with respect to properties such as asymptotic behavior, periodicity, anti periodicity, stability, etc.. Boundary value problems are not yet discussed fully. Observability of nonlinear system and null controllability can also be studied. Other problems of control theory include stability and optimality. One can also study Integral equations with PCDA and Integro-differential equations with PCDA.

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