MULTIPLE CONNECTED SUMS OF TORUS LINKS

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DECLARATION

I do hereby declare that this thesis entitled "MULTIPLE CONNECTED SUMS OF TORUS LINKS" submitted to Goa University for the award of the degree of Doctor of Philosophy in Mathematics is a record of original and independent work done by me under the supervision and guidance of Jayanthan, A.J., Reader Department of Mathematics, Goa University, and it has not previously formed the basis for the award of any Degree, Diploma, Associateship, Fellowship or other similar title to any candidate of any University.

LUCAS MIRANDA

CERTIFICATE

This is to certify that this thesis entitled "MULTIPLE CONNECTED SUMS OF TORUS LINKS" submitted to Goa University by Shri Lucas Miranda is a bonafied record of original and independent research work done by the candidate under my guidance. I further certify that this work has not formed the basis for the award of any Degree, Diploma, Associateship, Fellowship or other similar title to any candidate of any other University.

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Reader, Departemnt of Mathematics,
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To my parents

Mr. Jose Cirilo Miranda

and

Mrs. Salvita Britto e Miranda.

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Lucas Miranda, Dept. of Mathematics, Goa University.

Introduction

Knots and braids were used long before the practice of Mathematization began to influence thoughts and actions of Mankind. In fact the use of knots and braids predates those of fire and the wheel by countless aeons. Knots and braids are amongst the oldest artefacts. They have been used all along human civilizations in various activities. These activities range from building houses, bridges and boats to weaving and cloth production, from construction of fishing knots and nets to making a apparel to decorative braiding of bags, belts and wall hangings. Yet, in spite of their long time usage, even to this day there are many (mathematical) aspects of their function such as the genera of a given knot and the possibility of isotoping one knot into another are not well understood.

Knots and braids are Geometrical objects and are rightfully placed in the domain of Topology. But the use of various techniques of other branches of Mathematics such as Combinatorics and Algebra to gain an insight into the subject of knots and braids becomes inevitable due to the intricacies involved. Knot theory has now become a subject in its own right and has grown by leaps and bounds along a multidisciplinary front. It involves a wide diversity of ideas, methods and applications. Since its inception as a proper Mathematical discipline in the second half of the 19^{th} century, knot theory has made important contributions by way of applications in fields as diverse as Quantum Physics, Atomic Modelling and Molecular Biology. Knot theory is Mathematically abstruse and hence its modelling demands amazingly complex Mathematical machinery.

Following is a brief chronological account of the development of Topological Knot Theory [17]. Peter Tait (1831-1901) and his collaborators tabulated and classified knots with crossing number up to ten. A mammoth undertaking which took them six long years for the classification of knots with crossing number ten alone. Finally, they were able to resolve a large number of the alternating 11-crossing knots. The work of Tait and his collaborators involving enumeration of knots was rather empiric. They were unable to develop the subject rigorously for want of a knot invariant. The main problem they confronted was that of isotopy equivalence. The problem of isotopy is established as the central problem in knot theory and it is known as the knot problem. This problem could not be dealt with satisfactorily until the advent of Algebraic Topology.

In the year 1908, Tietze made the crucial conjecture that the Topological structure of the knot complement in $\mathbb{S}^3$ carries all the information about the knot. This conjecture was established only recently in the year 1988 [3].

Henri Poincare (1854-1912) developed the mathematical machinery that enabled the use of Algebraic techniques to distinguish between different n-dimensional complexes. Poincare's techniques were useful in studying knots and 3-Manifolds apart from fuelling research in higher dimensional Topology. The first successful Algebraic Topological invariant attached to a link was the Fundamental Group of the link complement. The Fundamental Group expresses the Topology of the link complement in algebraic language that makes it possible to compare different links by comparing the respective Fundamental Groups. Later a general method of writing down a presentation of the Knot Group using a knot projection was introduced by Wirtinger [13]. Max Dehn in 1960 also published methods for presenting knot groups. Max Dhen also showed by performing Dehn surgery that neither of the oriented Trefoils is isotopic to its mirror image. Applications of the Fundamental Group proved the existence of non-trivial knots and also helped in the verification of knot tables. However, James Waddel Alexander (1888-1971) showed that the Knot Groups are not complete invariants of knots. That is, a knot contains more information than the Knot Group can reveal. The Knot Group determines the knot's complement merely up to homotopy type.

Alexander polynomial was one of the first powerful combinatorial invariants invented in knot theory. Alexander also showed that every link is equivalent to a closed braid. Markov introduced two types of braid moves and showed that every equivalence class of braids determined by the moves resulted in the same link [8].

Ralph Hartzler Fox (1913-1973) developed the so called "Fox Calculus" [2] and provided an alternate meaning to the Alexander polynomial. This resulted in another way of calculating the Alexander polynomial.

John Conway found a polynomial of knots which was actually a disguise of the Alexander polynomial [7]. It was a polynomial that could be calculated directly from a diagram by means of a recursive method using certain Skein relations. These Skein relations obviated the use of computers and helped expand the existing knot tables.

Vaughan Jones constructed a link invariant that came to be known as the Jones polynomial [8]. His work linked knots to Statistical Mechanics and sparked an interaction between knot theory and braid theory in the light of Alexander's and Markov's theorems. Jone polynomial was able to detect and differentiate *more* knots and links as compared to the Alexander/Conway polynomial. It also led to an outburst of discoveries of knot polynomials with more than one variable. Two of such major generalizations were the Homfly polynomial and the Kauffman polynomial. Representation theory of braid groups helped generate old and new invariants using the supported Markov traces.

Vassiliev, using combinatorics, produced a numerical invariant of knots that associates rational number to them [12].

Mathematicians have come a long way in understanding links. However, the baffling problem of finding a complete link invariant (if one exists) still remains. Links and Knots play central role in applied sciences such as genetics, molecular chemistry and statistical mechanics. By themselves they are fascinating geometrical objects and remain far from being fully understood on account of not being easily accessible to existing mathematical machinery. There are different intrinsic and extrinsic characteristics of Links and Knots that one tries to understand using different techniques. To some extent, many of these objects have been distinguished by using different combinatorial, algebraic and geometrical techniques. But till date we do not know of any technique that completely classifies Links and Knots.

Following is a brief layout of the work done in this thesis.

The motivation for the work came from the following facts. Every link can be embedded on an orientable surface. The minimum of the genera of all surfaces on which a given link can be embedded is known as the *genus* of the link. Torus links or genus one links are well understood, where as the higher genus links are not. Our study concerns links generated by multiple connected sums of torus links. A multiple connected sum is a generalization of the concept of connected sum. Connected sums generate infinitely many links and so do multiple connected sums. Since multiple connected sums of torus links are accessible by Combinatorial techniques and torus links are well understood, it is advantageous to study links from this perspective.

A multiple connected sum of g torus links is a link that can be embedded in a surface of genus less than or equal to g. The investigation into multiple connected sums of two torus links throws light on the class of double torus links generated by performing multiple connected sums. Likewise it is possible to study larger multiple connected sums involving more than two torus links and hopefully extend our understanding to all Links and Knots.

To perform a multiple connected sum of two torus links, we must perform a regular cut on each of the two torus links by cutting along a simple arc across the longitudinal strands. These arcs must cut the longitudinal strands at equal number of points on both the torus links. Then the open ends of the strands cut on the two torus links are spliced together in such a way that the arcs along which the cuts were made are identified homeomorphically. This can be done in exactly two distinct ways for a fixed m-cut on each of the two torus links. A multiple connected sum of two torus links L_1 and L_2 is denoted by $L_1 \sharp L_2$. The two ways of splicing the open end points of the m-cuts may results in different links.

There is a naturally associated permutation $\sigma(p, d_i)$ with every oriented torus link $L(p, q)$ having a fixed ordered labelling of it's longitudinal strands, given by the action $\sigma(p, d_i) : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ defined by $\sigma(p, d_i)(x) = (x + d_i) \bmod p$, where $d_i = ((-1)^i q)$ mod $p)$ for $i \in \{1, 2\}$. The value of i depends on the order of labelling of the p longitudinal strands and orientation of the link along which the link is traversed. The cycles of this permutation represent the components of the torus link and induces a permutation associated with a regular cut on the oriented torus link. The permutations associated with the two regular cuts along which a multiple connected sum is formed, induce a permutation associated with the multiple connected sum and is referred to as the *resultant permutation.* Each cycle of the resultant permutation represents a component of the associated multiple connected sum and viceversa. Every double torus link that is a multiple connected sum of two tori admits an unambiguous parametric representation. A method using division algorithm to generate the permutation associated with an oriented torus link is described.

It is shown that the number of components of a multiple connected sum and the number of components of it's *elementary extension* differ by exactly one. The resultant permutation associated with a multiple connected sum formed connecting along "large" regular cuts become cumbersome to compute. To economize on computational time involved in computing the resultant permutation associated with a multiple connected sum formed, a new permutation called the *reduced permutation* is associated with them. These reduced permutations associated with a multiple connected sum formed by connecting along large regular cuts are smaller in size as compared to the corresponding associated resultant permutation. However, the reduced permutation preserves the information regarding the number of components of the corresponding associated multiple connected sum. Also, the reduced permutation can be computed directly from the parameters of the two torus links and the size of the regular cuts without invoking the corresponding resultant permutations.

Theorem Every closed connected orientable 3-Manifold is a quotient space of two handle bodies of equal genera g for some $g \in \mathbb{N}$. □

This theorem is an immediate consequence of the following theorem due to Moise [15].

Theorem Every closed connected orientable 3-Manifold is triangulable. □

A decomposition of a closed connected orientable 3-Manifold into two handle bodies of equal genera g, whenever possible, is called a Heegaard splitting of genus g. Any homeomorphism between the boundaries of two genus g handle bodies generates a closed connected orientable 3-Manifold. The Fundamental Group of a 3-Manifold formed as a quotient space of two handle bodies of equal genera say g has g generators and g relations [11]. The g generators represent the g non-trivial canonical curves of the genus g handle body (the domain of the quotient map) and the g relations are obtained from the images of the g generators under the quotient map. Every such homeomorphism forming a quotient space of two genus g handle bodies, maps the g generators on the boundary of the domain handle body onto g non-separating non-parallel simple closed curves on the boundary of the codomain handle body i.e. the image set of the g generators on the boundary of the domain handle body under the quotient map is a link with g non-separating non-parallel components embedded in the boundary of the codomain handle body. Such a quotient map is characterized by the image set of the g generators upto isotopy. To compute the g relations corresponding to the g components of the image link of the g generators, one needs to know the number of times each component winds around the g generators and their order of occurance. In practice this is a very tedious task as one has to depend heavily on a neat picture of the link. But in the case of Multiple Connected Sums of two torus links having two non-separating non-parallel components embedded in a double torus, there exists a simple algorithm to compute effortlessly a presentation of the Fundamental

Group of the associated 3-Manifold without invoking a picture of the link. Links and closed connected 3-Manifolds are closely related. This fact was brought to light by the Lickorish theorem [12] stated below.

Theorem Let M be a closed connected orientable 3-Manifold. There exist finite sets of disjoint solid tori $T_1', T_2', ..., T_n'$ in M and $T_1, T_2, ..., T_n$ in $\mathbb{S}^3$ such that $M - \bigcup_{i=1}^{n} Int(T_i')$ and $\mathbb{S}^3 - \bigcup_{i=1}^{n} Int(T_i)$ are homeomorphic. □

Double Torus D is the boundary of a handle body of genus two. Mapping Class Group $M(D)$ of a double torus D is the group of isotopy classes of orientation preserving homeomorphisms of the double torus to itself [12]. The longitudes, the meridians and the simple closed curves around the waist handle of a double torus are called the canonical curves of the double torus. A double torus has six canonical curves up to isotopy. *Twists* about the two longitudes, two meridians and any one of the *canonical* curves around the waist handle of the double torus D forms a complete set of generators of the Mapping Class Group $M(D)$ [6]. These generators of $M(D)$ are known as the Lickorish generators. Following is a crucial theorem by Lickorish regarding homeomorphisms between two closed connected orientable surfaces of equal genera [12].

Theorem Let $p_1, p_2, ..., p_n$ be disjoint simple closed curves on a closed connected orientable surface F the union of which does not separate F. Let $q_1, q_2, ..., q_n$ be another set of curves with the same properties. Then there exists a homeomorphism h of F to itself that is in the group generated by twists so that $h(p_i) = q_i$ for each $i = 1, 2, ..., n$.
□

Hence, in principle, for any double torus link L having k components, there exists an element of $M(D)$ that maps L onto a set of k canonical components of D. In particular every orientation preserving homeomorphism between any two genus-two surfaces can be generated up to isotopy using the Lickorish generators in $M(D)$. In other words, a mapping class element that sends parallel (non-parallel respectively) components of the link to parallel (non-parallel respectively) canonical curves of the double torus D can be generated using the Lickorish generators of $M(D)$. Every mapping class element preserves the number of distinct isotopy classes. These facts are true for any closed connected orientable surface. However, there is no known algorithm to arrive at such a mapping class element. In the case of double torus links formed by a multiple connected sum, we provide an algorithm to produce such a mapping class element. This also establishes the fact that the maximum and minimum number of distinct classes of canonical curves that the set of double torus links could be mapped to by a mapping class element are 3 and 1 respectively. Any Multiple Connected Sum that is mapped to two or three distinct classes of canonical curves by a mapping class element must necessarily be a genus two link.

Finally, a General Multiple Connected Sum $L_1\sharp_{m_1}L_2\sharp_{m_2}L_3\sharp_{m_3}\cdots\sharp_{m_{n-1}}L_n$ of n torus links is considered. A General Multiple Connected Sum $L_1\sharp_{m_1}L_2\sharp_{m_2}L_3\sharp_{m_3}\cdots\sharp_{m_{n-1}}L_n$ can be arranged as a chain of $(n-2)$ simple reverse multiple connections of $(n-1)$ submultiple connected sums $L_i''\sharp_{m_i}L_{i+1}'$, $i = 1, 2, ..., n-1$ of two torus links, where $L_1'' = L_1$ and $L_n' = L_n$ and is written as $(L_1\sharp_{m_1}L_2')\bigoplus(L_2''\sharp_{m_2}L_3')\bigoplus\cdots\bigoplus(L_{n-1}''\sharp_{m_{n-1}}L_n)$. The unspliced meridional strands of the $(n-1)$ submultiple connected sums $L_i''\sharp_{m_i}L_{i+1}'$, $i = 1, 2, ..., n-1$ are arranged alternately as shown in figure 3.10. This way of arranging a General Multiple Connected Sum enables us to represent it in an unambiguous parametric form. A scheme to label the longitudinal strands of the general multiple connected sum is established. Once the longitudinal strands of the general multiple connected sum are labelled according to the scheme and a compatible orientation is assigned to it, it is possible to derive a permutation $p(m)$ associated with the general multiple connected sum. This permutation $p(m)$ preserves the information regarding the number of components in the general multiple connected sum. Examples of multiple connected sums of three torus links are considered and their associated permutations are computed from which the number of components of the respective links are obtained by simply counting the number of cycles.

Computing invariants of multiple connected sums has been called off for the moment due to time limitations. However, we would like to undertake the task in our future pursuits and hope to arrive at fruitful results. We wonder whether there exists general concept of connected sum that generates all links and is combinatorially accessible.

Chapter 1

Multiple Connected Sums of two Torus Links

In this chapter, we define "multiple connected sum" of two torus links. Multiple connected sums of two torus links either generate double torus links or torus links. In §1.2 we obtain the permutation naturally associated with a given torus link for a fixed orientation of the link and in §1.3 we obtain the permutation associated with a regular n-cut of an oriented torus link. Then, in §1.4 we use this permutation to deduce the permutation associated with the "n-cut" of the torus link. The permutation associated with a multiple connected sum is given by the composition of the two permutations of the "n-cut" torus links used to form the sum. Also, we present some combinatorial results pertaining to the permutations "respected" by torus links, that throw light on the phenomena of multiple connected sums. Finally, in §1.5 we derive the permutations respected by torus links using division algorithm.

1.1 Preliminaries

In this section, we establish some basic concepts required for the rest of this thesis.

Definition 1.1.1. Let $C_r = \{(x, y, 0) : x^2 + (y - 4r)^2 = 4\}$ be the circle of radius 2 with center $(0, 4r)$ in the $x\ y-plane$, where $r \in \mathbb{Z}$. Set $A_n = \bigcup_{r=0}^{n-1} C_r$. Define $H_n = \{(x, y, z) \in \mathbb{R}^3 : d((x, y, z), A_n) \leq 1\}$. Any subspace of $\mathbb{R}^3$ isotopic to the set H_n is called a *handle body of genus n*.

Definition 1.1.2. A *solid torus* is a handle body of genus 1 and is isotopic to the set $\{(x, y, z) \in \mathbb{R}^3 : ((x^2 + y^2)^{1/2} - 2)^2 + z^2 \leq 1\}$.

This set is obtained by rotating the disc $D_1 = \{(x, y, z) \in \mathbb{R}^3 : (x - 2)^2 + z^2 \leq 1, y = 0\}$ about the circle $S_2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 4, z = 0\}$.

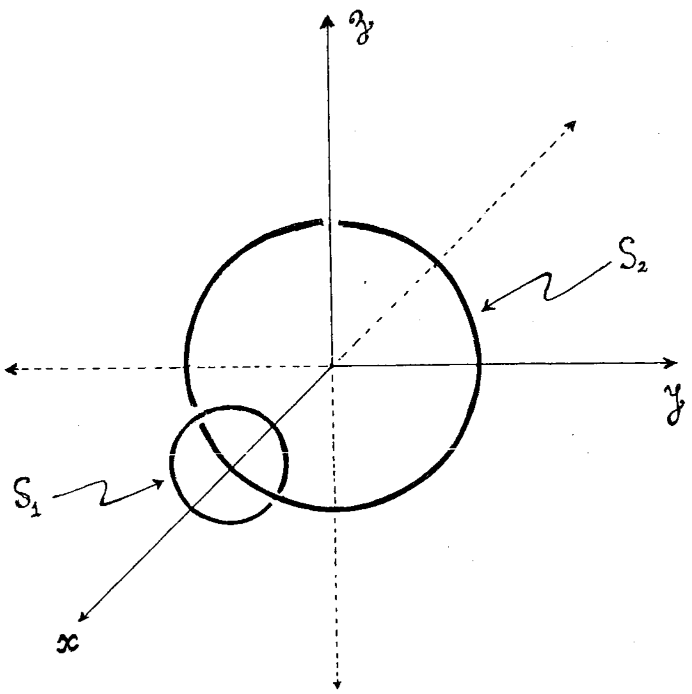


Figure 1.1 Rotation of S_1 about the z-axis along S_2.

Definition 1.1.3. A *Torus* is any topological subspace of $\mathbb{R}^3$ isotopic to the set $\{(x, y, z) \in \mathbb{R}^3 : ((x^2 + y^2)^{1/2} - 2)^2 + z^2 = 1\}$.

This set is obtained by rotating the circle $S_1 : \{(x, y, z) \in \mathbb{R}^3 : (x-2)^2 + z^2 = 1\,,\, y = 0\}$ about the z-axis along the circle $S_2 : \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 4\,,\, z = 0\}$ (figure 1.1). Note that we do not distinguish between any two tori and for all references hereafter, we concentrate our attention on the torus in the definition 1.1.2.

Definition 1.1.4. Boundary of a handle body of genus n is a *surface of genus n.* A torus can also be defined as the boundary of a solid torus.

Definition 1.1.5. A *longitude* of a torus is any simple closed curve embedded on the torus and is isotopic on the torus to the curve $x^2 + y^2 = 4$, $z = 1$ and a *meridian* of a torus is any simple closed curve embedded on the torus and is isotopic on the torus to the curve $(x-2)^2 + z^2 = 1$, $y = 0$.

Remark 1.1.1. The meridian on the boundary torus is null homotopic in the solid torus where as the longitude on the boundary torus is not null homotopic in the solid torus. This is so because the point at infinity is fixed 'outside' the solid torus. This is the the distinction between the longitudinal and meridional curves of a torus.

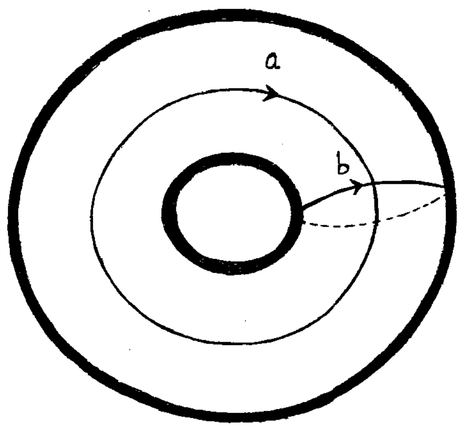


Figure 1.2 Torus with positively oriented longitude a and meridian b.

For convenience of argument, we need to orient the objects in our studies. For this purpose, we first fix certain conventions of orientation for the longitude and the meridian of a torus.

Definition 1.1.6. An oriented longitude of a torus is said to be *positively* (*negatively*) *oriented* if it has clockwise (anticlockwise) orientation (figure 1.2) when viewed from the positive *z-axis.* An oriented meridian of a torus is said to be *positively (negatively) oriented* if it is isotopic to the meridian having anticlockwise (clockwise respectively) orientation when viewed from the positive y-axis (figure 1.2).

Definition 1.1.7. Let a be a positively oriented longitude and b be a positively oriented meridian of a torus and p and q be any two relatively prime integers. A *torus knot* $K(p,q)$ is a simple closed curve embedded in the torus and that belongs to the isotopy class $\{a^p = b^q\}$ of the Fundamental Group of the torus. A *torus link* $L(p,q)$ is a collection of d pairwise disjoint torus knots $K(r,s)$ embedded in a torus with $p = dr$ and $q = ds$ where d is the g.c.d. of p and q.

Remark 1.1.2. A torus knot K(p,q) can be obtained as the image of the line segment joining the points $O \equiv (0,0)$ and $P \equiv (p,q)$, under the quotient map from $\mathbb{R}^2$ to the quotient space $\mathbb{R}^2/\mathbb{Z}^2$ that is a torus.

1.2 Permutations associated with torus links

In this section, we show that there exists a natural way of associating a permutation in S_p (S_q, respectively) with a given torus link $L(p,q)$ for a fixed orientation of the link and a fixed order of labelling of the longitudinal (meridional, respectively) strands. In chapter 2, we will see that these permutations play an important role in deriving the permutations associated with the multiple connected sums of torus links that in turn are used to compute the number of components of the multiple connected sums. A torus Link $L(p, q)$ can also be obtained by forming the quotient space of a rectangle with $p+q$ non-intersecting line segments in the rectangle (as shown in figure 1.3) under the quotient map described below.

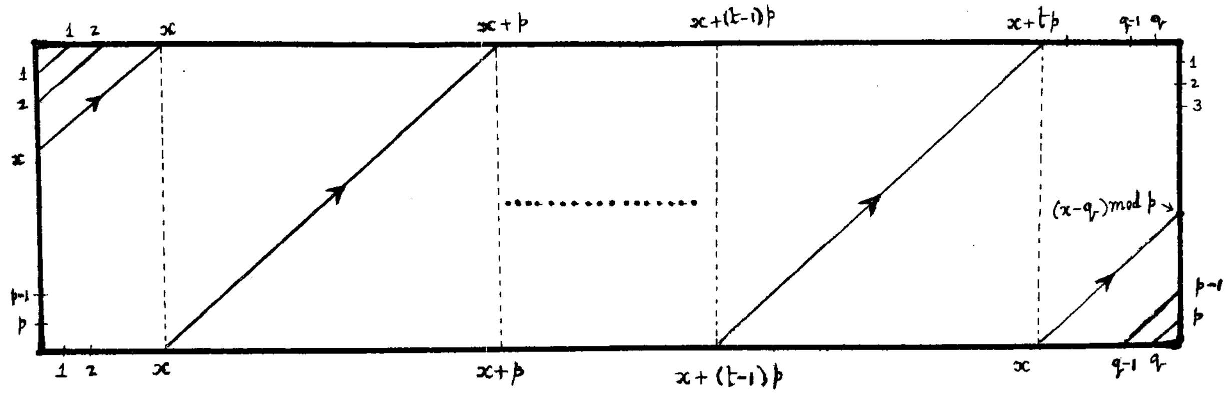


Figure 1.3 Rectangle with $p+q$ non-intersecting line segments.

There are p points marked on each of the two vertical sides and are labelled sequentially by the number 1 to p from top to bottom and q points marked on each of the two horizontal sides and are labelled sequentially by the numbers 1 to q from left to right of the rectangle. These labels on the rectangle are joined in pairs by $p+q$ non-intersecting line segments in the rectangle as shown in the figure 1.3. The quotient map identifies the opposite sides of the rectangle such that the points with identical labels on opposite sides are identified. Under this quotient map, the rectangle becomes a torus and the $p+q$ non-intersecting line segments in the rectangle form either the torus link $L(p, q)$ or $L(q, p)$ depending on the order of identification of the side of the rectangle. In other words, if the identification map is such that the horizontal sides of the rectangle are identified prior to identifying the vertical sides, then the torus link $L(p, q)$ is formed. On the other hand, if the identification map is such that the vertical sides of the rectangle are identified prior to identifying the horizontal sides, then the torus link $L(q, p)$ is formed. These two torus links belong to the same isotopy class namely $\{a^p = b^q\}$ of the Fundamental Groups of the respective tori where a and b are the generators of the Fundamental groups of the two tori. Note that the horizontal sides and vertical sides of the rectangle become the longitude and the meridian respectively of the torus under the former identification, while under the latter identification, they become the meridian and the longitude respectively. The two identifications discussed above produce torus links $L(p, q)$ and $L(q, p)$ in $\mathbb{R}^3$ when $p \neq q$, that are in general not isotopic. Further, any one of these identifications considered above can be performed in yet another two distinct ways depending on the 'location' of the point at infinity. These two distinct ways of identifying will result in two torus links $L_i(p, q)$, $i = 1, 2$ each being a reflection of the other about the xy-plane. To avoid any confusion, we presume the identification map is such that the oriented vertical and horizontal sides of the rectangle in figure (1.3) will be the positively oriented longitudinal and positively oriented meridional curves a and b respectively as seen in figure (1.2). This identification map converts the rectangle into a torus and the $p + q$ line segments lying in it into a torus link $L(p, q)$ with the vertical sides of the rectangle with the labels $\{1, 2, ..., p\}$ forming a meridian m and the horizontal sides with the labels $\{1, 2, ..., q\}$ forming a longitude l (figure 1.4).

Remark 1.2.1. A Heegard genus 1 decomposition of $\mathbb{S}^3$ is a splitting of $\mathbb{S}^3$ into two handle bodies of genus one having a torus as a common boundary surface. Hence, a torus link $L(p, q)$ can be simultaneously embedded on the common boundary of two different solid tori of a Heegaard genus one decomposition of $\mathbb{S}^3$ such that on one of the solid tori the p strands are longitudinal and the q strands are meridional, while on the other solid torus the p strands are meridional and the q strands are longitudinal. Hence, the concepts of "longitude" and "meridian" are notional. However, we fix these notions by fixing the point at infinity as an 'exterior point'.

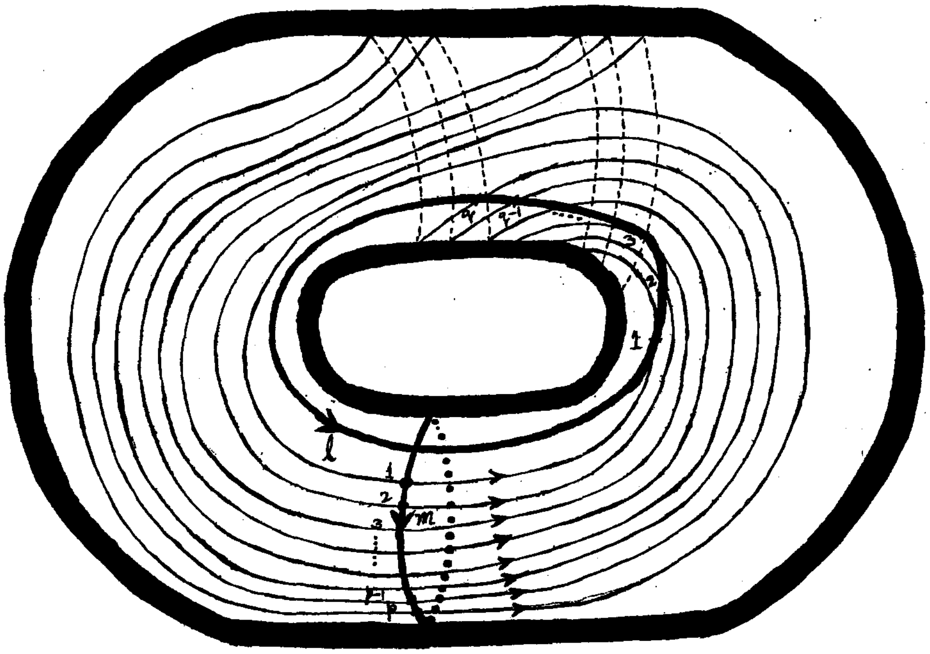


Figure 1.4 The link $L(p, q)$ embedded in a torus.

Now, we develop the naturally associated permutation in S_p with a torus link $L(p, q)$ for a fixed orientation of the link and a fixed sequential order of labelling of the longitudinal strands. By a fixed orientation of the link, we mean that each component of the link must be assigned a compatible orientation (figure 1.4). We travel along each of the components of the link in the direction of orientation and record the labels traversed in the order of their arrival. Each time we travel along a component and record the labels traversed in the order of their arrival, we will have generated the cycle corresponding to that component. This we do once with each of the components of the link and then take the product of these cycles to arrive at a permutation in S_p.

The rectangle with $p+q$ line segments (figure 1.3) is more suitable to generate the naturally associated permutation with the torus link $L(p,q)$ than the link itself. Hence, we deal with the rectangle with $p+q$ line segments to generate the naturally associated permutation with the torus link $L(p,q)$. Denote the set of labels on the vertical sides of the rectangle in figure 1.3, that are the same as the labels on the meridian m of the torus in figure 1.4 by $\mathbb{Z}_p = \{1, 2, ..., p\}$. Let d denote the greatest common divisor of p and q. Now, we start at any label say $x_1 \in \mathbb{Z}_p$ and travel on the torus along the strand of the link passing through the label x_1 in the direction of orientation assigned to the link. After exactly one longitudinal revolution on the torus, we will arrive at the label $(x_1 + d_i) \bmod p \in \mathbb{Z}_p$ where $d_i = ((-1)^i q) \bmod p)$ for $i \in \{1, 2\}$. The value of i depends on the orientation of the link and the order of labelling of the longitudinal strands. This fact can be easily contemplated from the figure 1.3. After making exactly (p/d) such longitudinal revolutions (that will also effect exactly (q/d) meridional revolutions) we will arrive back at the initial label $x_1 \in \mathbb{Z}_p$. In the process, one would have travelled via each of the labels of the cycle representing the component of the link containing the label x_1 in it. Likewise, we travel along all the remaining $(d-1)$ components to generate the corresponding cycles. The d cycles are independent of the choice of the starting point in $\mathbb{Z}_p$. The labels $\{1, 2, ..., d\} \subset \mathbb{Z}_p$ lie on the d distinct components of the link. For the sake of convenience, we begin each of the d cycles with labels from $\{1, 2, ..., d\} \subset \mathbb{Z}_p$. Therefore, for all $j = 1, 2, ..., d$

these d cycles will be as follows.

$(\sigma_{ij}) = (j, (j + d_i)\text{mod}\, p, ..., (j + ((p/d) - 1)d_i)\text{mod}\, p)$.

The permutation $\sigma(p, d_i)$ associated with the torus link $L(p, q)$ for a fixed orientation and a fixed ordered labelling of the longitudinal strands is a product of the disjoint d cycles (σ_{ij}). i.e., $\sigma(p, d_i) = \sigma_{i1} \circ \sigma_{i2} \circ ... \circ \sigma_{id}$.

The two different orientations of a torus link $L(p, q)$ for a fixed labelling give rise to two different permutations that are inverses of each other. Each of these permutations is a function $\sigma(p, d_i) : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ defined by $\sigma(p, d_i)(x) = (x + d_i) \,\text{mod}\, p$.

Remark 1.2.2. (1) Each cycle gives the orbit (trajectory) marked by labels in $\mathbb{Z}_p$ written in the order of their arrival as we travel along the corresponding component of the link for a given orientation.

(2) Two torus links $L_i(p_i, q_i)$, $i = 1, 2$ having the same parameters i.e. $p_1 = p_2$ and $q_1 = q_2$ need not be equivalent in the sense of isotopy in the torus or in $\mathbb{S}^3$. For example the left and right trefoil knots are not isotopic in $\mathbb{S}^3$ and hence in the torus as well, even though they have the same parameters $p_1 = p_2 = 2$ and $q_1 = q_2 = 3$. This distinction occurs because the two trefoil knots are embedded differently in $\mathbb{S}^3$ and in the torus. Hence, it is evident that the notation $L(p, q)$ for a torus link with p longitudinal strands and q meridional strands is inadequate. That is, it fails to describe a torus link completely and unambiguously. To capture this distinction, we must also encode the orientation of the link in its description. With a view to distinguish the two possible ways of embedding a torus link $L(p, q)$ in a torus, we make the following definitions.

To encode the orientation of the torus link $L(p, q)$, we must allow the parameters p and q to take values in the set of integers. First consider a torus knot $K(p, q)$ with a fixed orientation and parameters $p, q > 0$. Represent $K(p, q)$ by a new notation $K(\bar{p}, \bar{q})$ called the *signed notation.* Here $\bar{p} = p$ ($-p$, respectively) if the p longitudinal strands of the torus knot $K(p, q)$ have orientation compatible with (opposite to, respectively) the orientation of the canonical curve a of the torus in figure 1.2 and are said to be *positively*(*negatively*, respectively) oriented longitudinal strands. Further, $\bar{q} = q$ ($-q$, respectively) if the q meridional strands of the torus knot $K(p, q)$ have orientation compatible with (opposite to, respectively) the orientation of the canonical curve b of the torus in figure 1.2 and are said to be *positively* (*negatively*, respectively) oriented meridional strands. Note that the signed notation for a torus knot (that we will soon extend to torus links) takes into account the orientation of the knot of the torus by allowing signed parameters.

A torus knot $K(p, q)$ is said to be *positive* if for a fixed orientation both the longitudinal as well as the meridional strands are oriented either positively or negatively. A torus knot $K(p, q)$ is said to be *negative* if it is not *positive.* Equivalently, a torus knot written in the signed notation $K(\bar{p}, \bar{q})$ with respect to a fixed orientation is *positive* if $\bar{p}\bar{q} > 0$ and is *negative* if $\bar{p}\bar{q} < 0$. Note that this parity of a torus knot is independent of the orientation assigned to the knot. A torus link $L(p, q)$ is said to be *positive* (*negative*, respectively) if for a fixed orientation any one and hence every component is a positive (negative, respectively) torus knot.

Two or more longitudes (meridians) of an oriented torus link are said to have *compatible* orientation if they are either all positively orientated or all negatively oriented. Two components of an oriented torus link are said to have compatible orientation if their respective longitudinal and meridional strands have compatible orientations.

Note that as a convention, we always assign compatible orientation to all the components of a torus link $L(p,q)$. A torus link $L(p,q)$ with the signed notation $L(\bar{p},\bar{q})$ with respect to some fixed compatible orientation of the link is *positive* (*negative*, respectively) if $\bar{p}\bar{q} > 0$ ($\bar{p}\bar{q} < 0$, respectively). Though the signed notation adopted for a torus link completely describes any torus link, it has the following lacuna. It cannot accommodate simultaneously the occurrence of positively and negatively oriented longitudinal or meridional strands. Though such a situation never occurs in torus links, it does occur in double torus links. To overcome this difficulty we consider a more general notation to represent a torus link.

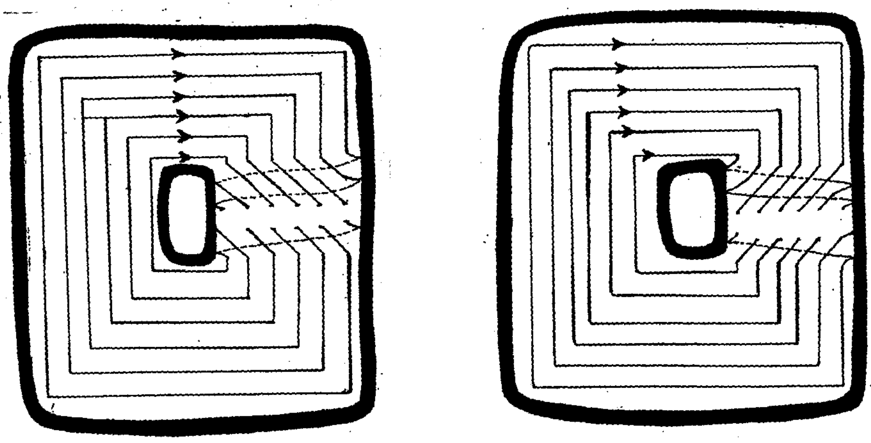
We represent an oriented torus link $L(p,q)$ by four coordinates or a quadruple of non-negative integers written as $((p_1,p_2),(q_1,q_2))$ where the first pair (p_1,p_2) represents the longitudinal strands of the link and the next pair (q_1,q_2) represents the meridional strands of the link. Here p_1 stands for the number of positively oriented longitudinal strands of the link while p_2 stands for the number of negatively oriented longitudinal strands of the link. And q_1 stands for the number of positively oriented meridional strands while q_2 stands for the number of negatively oriented meridional strands. We refer to this notation of an oriented torus link represented by a quadruple of non-negative integers as the *parametric representation* of the oriented torus link. From the parametric notation, we can retrieve our old signed notation by simply writing $L(p_1-p_2,q_1-q_2)$. Once the compatible orientation is assigned to the longitudinal strands of a torus link, the orientations of all the meridians are automatically fixed. Since, by convention, we assign all the components of a torus link compatible orientation, we will have exactly two non-zero parameters in the parametric representation of a torus link. To determine the parametric representation of a general oriented double torus link one would require to depend heavily on a neat diagram of the link. However, in the case of oriented double torus links generated by multiple connected sums, the parametric representation is more convenient (see §3.1).

The following are equivalent definitions of positive and negative torus knots.

Remark 1.2.3. (1) An oriented torus knot $K(p,q)$ is *positive* (*negative*, respectively) if and only if it belongs to the isotopy class $a^{\alpha}b^{\beta}$ in the fundamental group of the torus in which the knot is embedded such that $\alpha\beta > 0$, ($\alpha\beta < 0$, respectively), where a and b are the oriented canonical curves of the torus.

(2) The parametric representation of a positive torus knot $K(p,q)$ will be $((p,0),(q,0))$ or $((0,p),(0,q))$ depending on the orientation assigned to the knot. The parametric representation of a negative torus knot $k(p,q)$ will be $((p,0),(0,q))$ or $((0,p),(q,0))$ depending on the orientation assigned to the knot.

Since more information pertaining to the links can be encoded in the parametric representation in comparison to other notations, we use the parametric representation for torus links henceforth.



(a) (b)

Figure 1.5 (a) positive and (b) negative torus links.

Remark 1.2.4. (1) The torus knots $K_1((1,0),(1,0))$ and $K_2((1,0),(0,1))$ are isotopic in $\mathbb{S}^3$, but are not isotopic on the torus.

(2) If two torus links are isotopic on the torus, then they are isotopic in $\mathbb{S}^3$.

(3) Two torus links are isotopic on the torus if and only if it is possible to orient them so that they posses the same parametric representation.

(4) Let the p longitudinal strands of an oriented torus knot $K(p,q)$ be labelled by the elements of $\mathbb{Z}_p$. Then the number of longitudinal revolution required to be made along the knot in the direction of orientation starting from the label $x \in \mathbb{Z}_p$ to arrive at the label $(x+1) \bmod p \in \mathbb{Z}_p$ is equal to $d_i^{-1} \pmod{p}$. Here $i \in \{1,2\}$ depends on both the orientation of the knot and the direction of labelling of the p longitudinal strands. Here $d_i^{-1} (\bmod\, p)$ is the multiplicative $(\bmod\, p)$ inverse of d_i in $\mathbb{Z}_p$.

(5) Let the p longitudinal strands of an oriented torus knot $K(p,q)$ be labelled by the elements of $\mathbb{Z}_p$. Then the order or place of occurrence of any label $x \in \mathbb{Z}_p$ in the permutation $\sigma(p,d_i)$ associated with the torus knot $K(p,q)$ for some $i \in \{1,2\}$ is given by $(1+(x-1)d_i)(\bmod\, p)$ provided the permutation $\sigma(p,d_i)$ begins with the label $1 \in \mathbb{Z}_p$. Here again i depends on the orientation of the knot and the direction of labelling of the p longitudinal strands.

1.3 Regular n-cuts

Here, we set about building the machinery required for performing a multiple connected sum or an n-connected sum of two torus links $L_i(p_i,q_i)$, $i = 1,2$ denoted by $L_1 \sharp_n L_2$ where n is a non-negative integer. An n-connected sum is basically a generalization of the concept of 'connected sum of knots'.

To perform an n-connected sum $L_1 \sharp_n L_2$, we must first cut out an open disc D_i from the torus T_i embedding the link L_i. The open disc D_i cut out from the torus T_i must cut off exactly n simple arcs from the p_i (locally) parallel longitudinal strands of the torus link L_i. Following concepts are required to formulate this idea mathematically.

Definition 1.3.1. A simple arc in the set $L_i \bigcap D_i$ is said to be a *cut out arc.*

Definition 1.3.2. A *boundary component* of $\partial\bar{D}_i$ is the closure of a simple arc in the set $\partial\bar{D}_i \setminus (L_i \bigcap \partial\bar{D}_i)$. The set of all boundary components is denoted by $c(\partial\bar{D}_i)$.

Definition 1.3.3. A point in $L_i \bigcap \partial\bar{D}_i$ is called an *end point* of the cut out arcs.

Definition 1.3.4. Two cut out arcs $A_{i1}, A_{i2} \in L_i \bigcap D_i$ are said to be *adjacent* if there exist two boundary components $B_{i1}, B_{i2} \in c(\partial\bar{D}_i)$ such that $A_{i1} \bigcup A_{i2} \bigcup B_{i1} \bigcup B_{i2}$ bounds a disc in $D_i, i = 1, 2$.

Definition 1.3.5. A cut out arc $A \in L_i \bigcap D_i$ is said to be an extreme arc if it is adjacent to exactly one other cut out arc.

Definition 1.3.6. The cutting out of the open disc D_i from the torus T_i across the p_i longitudinal strands is said to be a *regular meridional n-cut* if

(i) D_i must intersect L_i transversely (not tangentially) at each of the $2n$ end points of the cut out arcs(figure 1.6(a)),

(ii) there must be exactly n cut out arcs, and

(iii) there should not exist a simple arc in $L_i \bigcap (T_i \setminus D_i)$ whose union with a boundary component in $c(\partial\bar{D}_i)$ bounds a disc in $(T_i \setminus D_i)$ (figure 1.6(b) and (c)).

Remark 1.3.1. Note that a *regular longitudinal n-cut* could likewise be performed across the q_i meridional strands of the torus link $L_i, i = 1, 2$. However, they are not combinatorially equivalent. This fact can be verified by simply computing the permutations associated with regular n-cuts for a fixed orientation of the link. The term *n-cut* is used for regular meridional n-cut henceforth as we use only such cuts in multiple connected sums.

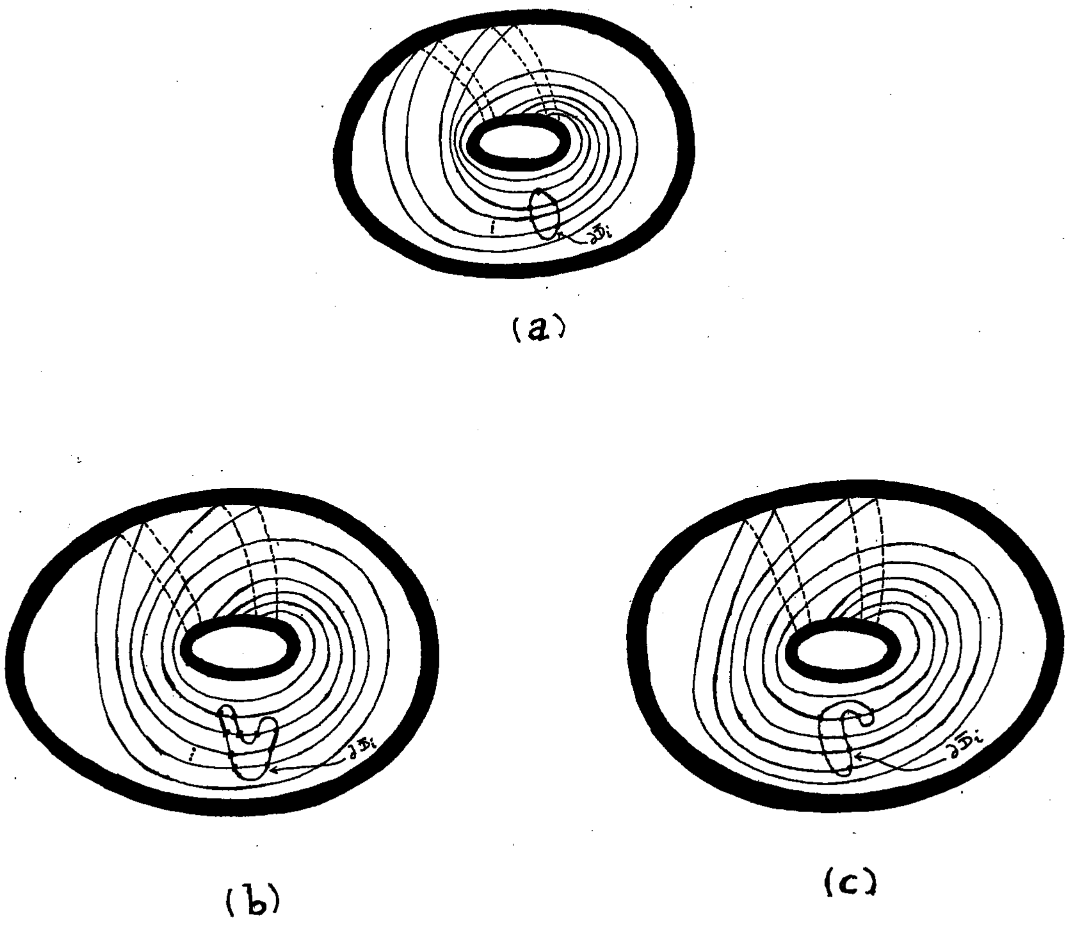
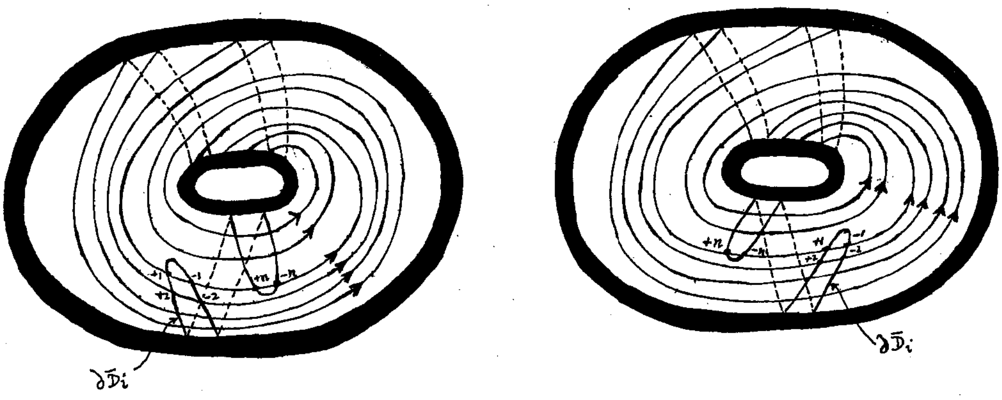


Figure 1.6 Non-regular cuts.

Label the end points of the cut out arcs of an n-cut by the labels from the set $X = \{\pm 1, \pm 2, ..., \pm n\}$ sequentially as follows. The labelling must begin with the labels ± 1 for the end points of one of the extreme arcs of the n-cut and end with the labels $\pm n$ for the end points of the other extreme arc of the n-cut. The *left* end point of each i^{th} arc is to be labelled $+i$ and the *right* end point of the i^{th} arc is to be labelled $-i$ for all i, $1 \leq i \leq n$ (figure 1.7(a) and (b)).



(a) (b)

Figure 1.7 n-cuts.

In the figure 1.7 (a) and (b), the links are oriented in the negative direction of a longitude of the torus. The labelling of the end points is done sequentially in the positive direction of orientation of a meridian of the torus.

An n-cut on an oriented torus link $L(p,q)$ can be performed in two different ways for $n > p$ and for a fixed direction of sequential labelling (along a meridian of the torus) of the end points of the n-cut out arcs. An n-cut of $L(p,q)$ where $n > p$ is said to be a *direct* (regular meridional) n-cut if moving along the strand in the direction of the orientation starting from the end point of the cut out arc labelled $-((l-1)p+r)$, the next end point of a cut out arc encountered will be the one labelled $+(lp+r)$ where $1 \leq r \leq p$, $lp+r \leq n$ and $l \in \mathbb{N}$ (figure 1.7(a)). An n-cut of $L(p,q)$

where $n > p$ is said to be a *reverse* (regular meridional) n-cut for a fixed direction of sequential labelling (along a meridian of the torus) of the end points of the cut out arcs and for a fixed orientation of the link if it is not a direct meridional n-cut. A direct (reverse, respectively) n-cut of an oriented torus link $L(p,q)$ becomes a reverse (direct, respectively) n-cut if the orientation of the link is reversed.

An n-cut on a torus T_i embedding a link $L_i(p_i, q_i)$ in accordance with the conditions (1), (2) and (3) stated above can be equivalently performed by cutting along a simple curve $C_i, i = 1, 2$ homeomorphic to a closed bounded interval across the longitudinal strands p_i. The curve C_i must intersect the torus link L_i at n distinct points. The following concept is required to formulate this idea mathematically.

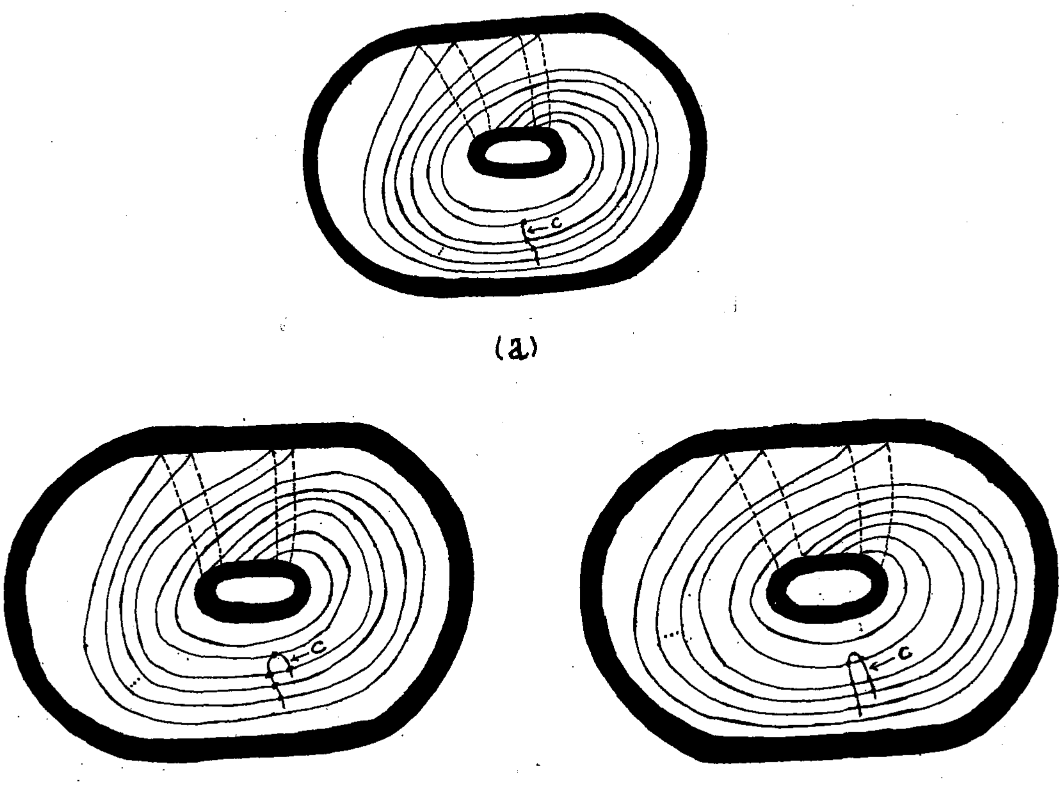


Figure 1.8 Non-regular n-cut along a curve.

Definition 1.3.7. A *component* of C_i is the closure of a simple arc in the set $C_i \setminus (C_i \bigcap L_i)$. The set of components of C_i is denoted by $c(C_i)$.

Definition 1.3.8. The cutting along a simple arc C_i intersecting the torus link L_i at n points is said to be a n-*cut* if

(i) neither of the two end points of C_i should lie on any of the p_i longitudinal strands (figure 1.8(a)),

(ii) C_i must intersect the link transversally (figure 1.8(b)), and

(iii) there should not exist a simple arc in the set $L_i \setminus L_i \bigcap C_i$ and a component of C_i whose union bounds a disc in T_i (figure 1.8(c)).

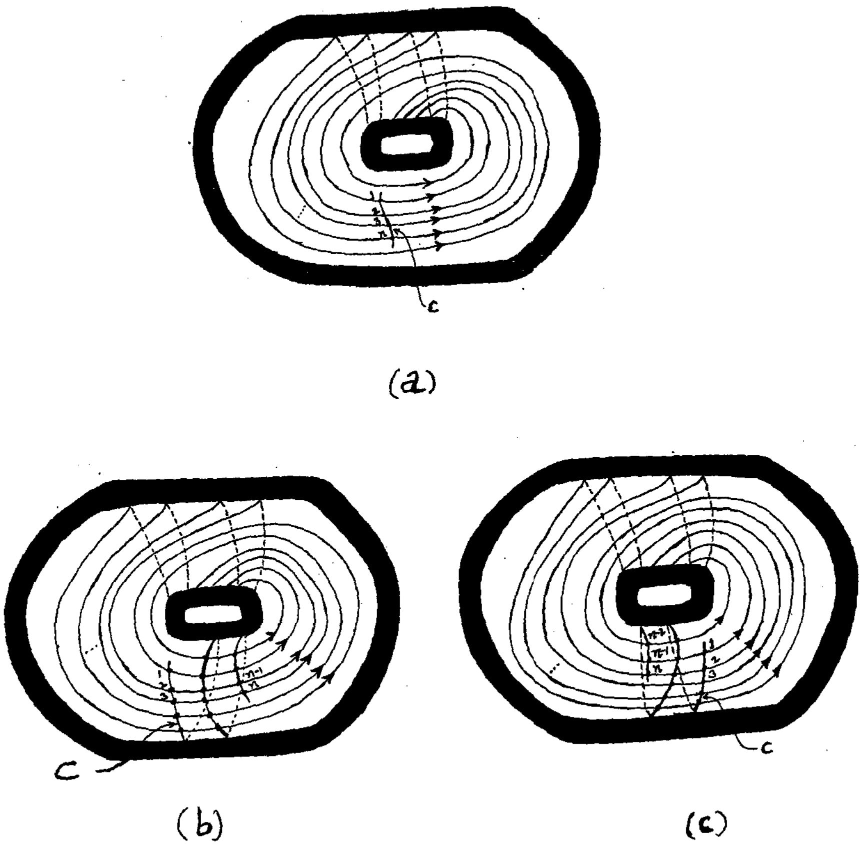


Figure 1.9 n-cuts along a curve.

For $p > 1$ and $1 \leq n \leq p$, an n-cut along a curve on $L(p,q)$ cuts n consecutive strands of the p longitudinal strands at one point each (figure 1.9(a)). For $p > 1$ and $lp < n \leq (l+1)p$ for some $l \in \mathbb{N}$, a n-cut along a curve on $L(p,q)$ cuts the p longitudinal strands at n points. In this case the first $(n - lp)$ strands will be cut at $(l+1)$ points each and the remaining $((l+1)p - n)$ strands will be cut at l points each (figure 1.9(b) and (c)).

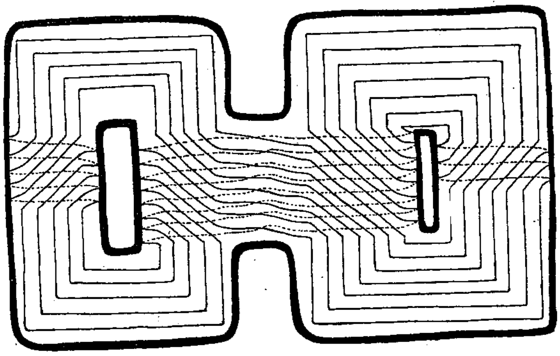
To perform an n-connected sum $L_1 \sharp_n L_2$ also known as a multiple connected sum of two n-cut torus links L_i, $i = 1, 2$, we do the following.

(1) Perform an n-cut on each of the tori T_i containing the link L_i

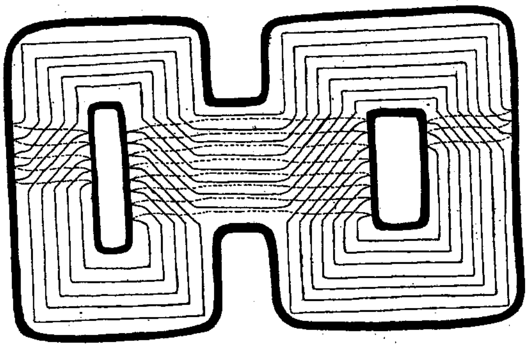
(2) Label the end points of the strands of the link L_i along the n-cut sequentially by the labels $X = \{\pm 1, \pm 2, ..., \pm n\}$ (as explained above) and

(3) Form the quotient space of the two n-cut torus links by identifying the boundaries of $T_i \setminus D_i$, $i = 1, 2$ by either an orientation preserving homeomorphism $h_1 : \partial(T_1 \setminus D_1) \longrightarrow \partial(T_2 \setminus D_2)$ such that $h_1(\pm x) = \mp x$, for each $\pm x \in X$ or an orientation preserving homeomorphism $h_2 : \partial(T_1 \setminus D_1) \longrightarrow \partial(T_2 \setminus D_2)$ such that $h_2(\pm x) = \pm(n - x + 1)$ for each $\pm x \in X$.

These two homeomorphisms ensure that the $2n$ end points of one n-cut torus link are matched with the $2n$ end points of the other n-cut torus link. These two ways of forming the quotient spaces of $T_i \setminus D_i$ by the homeomorphisms h_j, $j = 1, 2$ defined above will in general result in different multiple connected sums for the same pair of n-cut torus links. This fact can be easily realized by computing the number of components of both the resulting multiple connected sums (figure 1.10). In this figure we take two multiple connected sums made of the same torus links $(5, 3)$ and $(7, 5)$ spliced along 8-cuts using the two different homeomorphisms given above. This gives two distinct double torus links.



(a)



(b)

Figure 1.10 Distinct multiple connected sums formed from two 8-cut torus links.

Let an n-cut of an oriented torus link $L(p,q)$, $p < n$ be labelled sequentially in the positive (negative, respectively) direction of a meridian with labels from the set X. We can relabel the same n-cut keeping the orientation fixed by the map $f : X \longrightarrow X$ defined by $f(\pm(n-x+1)) = \mp x$. This relabelling will convert a direct (reverse, respectively) n-cut into a reverse (direct, respectively) n-cut. Also the map f reverses the direction of sequential labelling of the n-cut. This map f from X to itself can be extended to a homeomorphism of the boundary of the n-cut torus (containing X). Then, $h_i \equiv h_j \circ f$ and $(T_1 \setminus D_1) \cup_{h_i} (T_2 \setminus D_2) \equiv (T_1 \setminus D_1) \cup_{f \circ h_j} (T_2 \setminus D_2)$ where $i, j \in \{1, 2\}$ and $i \neq j$. Hence, the relabelling homeomorphism f enables us to construct the two quotient spaces $(T_1 \setminus D_1) \cup_{h_i} (T_2 \setminus D_2)$, $i = 1, 2$ from two n-cut torus links L_1 and L_2 using any one of the homeomorphisms h_j defined above. Therefore, without loss of generality, we will use the homeomorphism h_1 to construct the two quotient spaces together with the relabelling homeomorphism f. Further, we ignore the signs of the labels assigned to the end points of the arcs cut out by the n-cuts to enable us to write the permutation associated with an n-cut of a torus link in S_n. This aspect of an n-cut is discussed in the next section.

1.4 Permutation associated with an n-cut

For an n-cut performed on a torus link $L(p, q)$, the permutation associated with the n-cut is denoted by (i) $\sigma^{(n)}(p, d_i)$ if $n \leq p$ and by (ii) $\sigma_{dir}{}^{(n)}(p, d_i)$ if $n > p$ and the n-cut is direct, and by (iii)$\sigma_{rev}{}^{(n)}(p, d_i)$ if $n > p$ and the n-cut is reverse. The permutation associated with an n-cut of a torus link $L(p, q)$ is derived directly from the permutation $\sigma^{(p)}(p, d_i) = \sigma(p, d_i) \in S_p$ associated with $L(p, q)$ defined above.

Case(1) $n \leq p$.

In this case, the permutation $\sigma^{(n)}(p, d_i)$ associated with the n-cut is derived directly from $\sigma(p, d_i)$ by deleting all the terms greater than n and preserving the order of the terms left behind.

Case(2) $n > p$.

In this case, the following two subcases arise:

Subcase(a) The meridional n-cut is direct.

Let $\sigma^{(p)}(p, d_i) = \sigma(p, d_i)$. Then $\sigma_{dir}{}^{(n)}(p, d_i)$ is defined by induction on n as

$$\sigma_{dir}^{(kp+r)}(p, d_i)(x) = \begin{cases} kp + r & \text{if } x = (k-1)p + r \\ (r + d_i) mod\, p & \text{if } x = kp + r \\ \sigma_{dir}{}^{(kp+r-1)}(p, d_i)(x) & \text{otherwise} \end{cases}$$

Subcase(b) The meridional n-cut is reverse.

Let $\sigma^{(p)}(p, d_i) = \sigma(p, d_i)$. Then $\sigma_{dir}{}^{(n)}(p, d_i)$ is defined by induction on n as

$$\sigma_{rev}{}^{(kp+r)}(p, d_i)(x) = \begin{cases} kp + r & \text{if } x = (r - d_i) \bmod p \\ (k-1)p + r & \text{if } x = kp + r \\ \sigma_{rev}{}^{(kp+r-1)}(p, d_i)(x) & \text{otherwise} \end{cases}$$

Note that $\sigma_{dir}{}^{(n)}(p, d_i) = \sigma_{rev}{}^{(n)}(p, d_j)^{-1}$ where $d_x = ((-1)^x q) \bmod p$ and $i \neq j$.

Definition 1.4.1. A permutation $\sigma \in S_n$, $n \in \mathbb{N}$ is said to be *respected* by a torus link $L(p, q)$ if there exists an n-cut of the torus link such that the permutation associated with it is σ.

Definition 1.4.2. An $(m + 1)$-cut on a torus link $L(p, q)$ is called an *elementary extension* of an m-cut on the same torus link $L(p, q)$ if either (a) $m \leq p$ or (b) $m > p$ and the m-cut and $(m+1)$-cut are either both direct (or both reverse) cuts for a fixed orientation of the link.

Note that for $m \neq p$ the elementary extension of an m-cut is unique and for $m = p$ there are exactly two distinct elementary extensions for the m-cut.

Definition 1.4.3. An n-cut on $L(p, q)$ is said to be an *extension* of an m-cut on $L(p, q)$ for $n > m$ if there exist $(m+1)$-cut, $(m+2)$-cut,...,$(n-1)$-cut on $L(p, q)$ such that each $(i+1)$-cut is an elementary extension of the i-cut for $m \leq i < n$.

Remark 1.4.1. (1) Given a permutation $\sigma \in S_n$ for $n > 3$, there may or may not exist any torus link that respects it. For example, there does not exist a 7-cut on any torus link $L(p, q)$ that respects the cyclic permutation $\rho = (1, 5, 2, 6, 3, 4, 7) \in S_7$ for following reasons. Consider the two cases (a) If $p \geq 7$, then the difference between the first and second terms of ρ implies that $d_i = 4$. Now we must have the third term of ρ equal to $9(mod p) = 2$ implying thereby that $p = 7$. Hence the term following the term 3 in ρ should have been 7, but that is not the case. (b) If $p < 7$, then it can be verified that ρ is neither a direct nor a reverse extension of the permutation $\sigma(p, d_i)$ for all the possible values of p.
(2) If a permutation $\sigma \in S_n$ is respected by a torus link, then it is respected by infinitely many distinct torus links.

We state below some elementary combinatorial results pertaining to permutation that could be associated with n-cuts of torus links and do come handy later.

Lemma 1.4.1. *Let $r_1, r_2, r_3 \in \mathbb{N}$ be such that $1 \leq r_1 < r_2 < r_3$ with r_2 relatively prime to r_3. Then $-r_3(\operatorname{mod} r_2) = r_1$ if and only if $\sigma(r_2, r_1) = \sigma^{(r_2)}(r_3, r_2)$ and ${\sigma_{dir}}^{(r_3)}(r_2, r_1) = \sigma(r_3, r_2)$.* □

Lemma 1.4.2. *Let $r, r_1, r_2 \in \mathbb{N}$ such that $r < r_1 < r_2$ and r is relatively prime to r_i for $i = 1, 2$. Then $r_1(\operatorname{mod} r) = r_2(\operatorname{mod} r)$ if and only if $\sigma(r_1, r) = \sigma^{(r_1)}(r_2, r)$ and ${\sigma_{dir}}^{(r_2)}(r_1, r) = \sigma(r_2, r)$.* □

Corollary 1.4.3. *Let r and s be relatively prime positive integers such that $1 \leq r < s$, $t(n) = r + (sn)$ and $u(n) = 2r + (s(2n-1))$. Then $\sigma^{(t(n))}(u(n), t(n)) = \sigma^{(t(n))}(u(n+k), t(n+k))$ for all $n, k \in \mathbb{N}$.* □

Lemma 1.4.4. *Let $p > q$ be relatively prime numbers. Then $\sigma^{(q)}(p, -q(\bmod p)) = \sigma(q, p(\bmod q))$ and $\sigma^{(q)}(p, q(\bmod p)) = \sigma(q, -p(\bmod q))$. Further, $\sigma(p, -q(\bmod p)) = {\sigma_{rev}}^{(p)}(q, p(\bmod q))$ and $\sigma(p, q(\bmod p)) = {\sigma_{dir}}^{(p)}(q, -p(\bmod q))$.* □

Corollary 1.4.5. *Let p, q be relatively prime positive integers. Then $\sigma(p, q(\bmod p)) = \sigma^{(p)}(p+kq, q(\bmod (p+kq)))$ and $\sigma(p, -q(\bmod p)) = \sigma^{(p)}(p+kq, -q(\bmod (p+kq)))$. Also, ${\sigma_{rev}}^{(p+kq)}(p, q(\bmod p)) = \sigma(p+kq, q(\bmod (p+kq)))$ and ${\sigma_{dir}}^{(p+kq)}(p, -q(\bmod p)) = \sigma(p+kq, -q(\bmod (p+kq)))$ for any $k \in \mathbb{N}$.* □

1.4.1 Associated Permutation using Division Algorithm

Given any two positive integers r_k and r_{k-1} such that $r_k \geq r_{k-1}$, by division algorithm of integers, we can find a unique sequence of integers $r_0 < r_1 < r_2 <, ..., < r_k$ such that $-r_{i+1}(\bmod r_i) = r_{i-1}$ for all $i = 1, 2, ..., k-1$ where r_0 is the greatest common divisor of r_k and r_{k-1}. Given an oriented torus link $L(p, q)$ with $p > q$, we get a similar unique sequence of positive integers terminating at the greatest common divisor of p and q say r_0. To arrive at the sequence, take p as the first term of the sequence, d_j as the second term of the sequence and then using the recurrence relation stated above derive the unique sequence. The term $d_j = (-1)^j q \bmod p$, where $j \in \{1, 2\}$ depends on the orientation of $L(p, q)$ as well as the order of labelling of the longitudinal strands. From this sequence we can extract the permutation $\sigma(p, d_j)$ associated with $L(p, q)$. Note that for any three consecutive terms r_{i-1}, r_i and r_{i+1} of the sequence with $2 \leq i \leq k-1$, we can extend the permutation $\sigma^{(r_i)}(r_i, r_{i-1})$ respected by $L(p, q)$ to the permutation $\sigma^{(r_{i+1})}(r_{i+1}, r_i)$ respected by $L(p, q)$. This can be achieved using the formula $\sigma^{(r_{i+1})}(r_{i+1}, r_i) = \sigma_{dir}^{(r_{i+1})}(r_i, r_{i-1})$ (see Lemma 1.4.1.). Hence the permutation $\sigma^{(r_1)}(r_1, r_0)$ can be extended to $\sigma(p, d_j)$ by induction.

Chapter 2

Permutation and the Fundamental Group of a Manifold associated with a multiple connected sum

In §2.1, we observe that the number of components in a multiple connected sum and the number of components in its elementary extension differ by one. We derive a permutations in $S_{p_1+p_2}$ (in $S_{max\{p_1,p_2\}}$, respectively) from the resultant permutation associated with direct (reverse, respectively) multiple connected sum $L_1\sharp_m L_2$ of the torus links $L_i(p_i, q_i)$, $i = 1, 2$ for any $m \in \mathbb{N}$. We call it the reduced permutation and denote it by $p(m)$. From the reduced permutation we can compute the number of components in the double torus link formed by multiple connected sum. Finally, in §2.2, we present a scheme to derive a presentation of the fundamental group of any genus two 3-Manifolds associated with a double torus link having two non-separating components and which is generated by a multiple connected sum of two torus links [4].

2.1 Number of components in a multiple connected sum

The m-connected sum or a multiple connected sum $L_1\sharp_m L_2$ of two torus links $L_i = L_i(p_i, q_i)$, $i = 1, 2$ is generally a double torus link. We say a torus link $L(p, q)$ is m-cut meridionally if the m-cut is made across the p longitudinal strands and is labelled either along the positive or negative direction of the meridian. From now on, whenever we deal with a multiple connected sum it will mean that both the torus links involved in it are m-cut meridionally for some $m \in \mathbb{N}$ unless stated otherwise and will be simply refered to as m-cuts. The permutation associated with an m-cut oriented torus link L_i will be denoted by ${\sigma_i}^{(m)}$ (instead of ${\sigma_i}^m(p_i, d_{ij})$ where $d_{ij} = (-1)^j q_i(\mod p_i)$ and $j \in \{1, 2\}$) irrespective of the fact that it could be a direct or a reverse m-cut for $m > p_i$. The permutation associated with $L_1\sharp_m L_2$ is called the resultant permutation and is denoted by $\sigma(L_1\sharp_m L_2)$ and is given by the composition ${\sigma_2}^{(m)} \circ {\sigma_1}^{(m)}$, where $\sigma_i^{(m)}$ is the permutation associated with the m-cut on L_i in $L_1\sharp_m L_2$ for $i = 1, 2$ with respect to the induced orientation.. The number of components in $L_1\sharp_m L_2$ is denoted by $n(L_1\sharp_m L_2)$ and is equal to the number of pairwise disjoint cycles in $\sigma(L_1\sharp_m L_2)$. This fact is obvious because each cycle in each ${\sigma_i}^{(m)}$ represents an orbit (component) of the torus link $L_i(p_i, q_i)$, $i = 1, 2$ and hence each cycle in $\sigma(L_1\sharp_m L_2) = {\sigma_2}^{(m)} \circ {\sigma_1}^{(m)}$ represents a component of $L_1\sharp_m L_2$ and vice versa.

Remark 2.1.1. $L_1\sharp_m L_2$ will be either a genus one or genus zero link if $m = p_1 = p_2$ where $L_i(p_i, q_i), i = 1, 2$.

Definition 2.1.1. $L_1\sharp_{m+1} L_2$ is called an *elementary extension* of $L_1\sharp_m L_2$ if the $(m+1)$-cuts on the torus links L_i in $L_1\sharp_{m+1} L_2$ are elementary extensions of the m-cuts on the torus links L_i in $L_1\sharp_m L_2$ for $i = 1, 2$ respectively.

If $p_1 \neq m \neq p_2$, then the elementary extension of $L_1\sharp_m L_2$ is unique. If $p_1 = m \neq p_2$, or $p_1 \neq m = p_2$, then there are exactly two elementary extensions of $L_1\sharp_m L_2$. Finally, if $m = p_1 = p_2$, then there are exactly four elementary extensions of $L_1\sharp_m L_2$.

Definition 2.1.2. $L_1\sharp_n L_2$ is said to be an *extension* of $L_1\sharp_m L_2$ for $n > m$, if there exists a sequence $L_1\sharp_{m+1} L_2$, $L_1\sharp_{m+2} L_2$,...,$L_1\sharp_{n-1} L_2$ such that $L_1\sharp_{i+1} L_2$ is an elementary extension of $L_1\sharp_i L_2$ for each $m \leq i < n$.

Lemma 2.1.1. *Let L_1 and L_2 be any two torus links and $L_1\sharp_{(m+1)} L_2$ be an elementary extension of $L_1\sharp_m L_2$ for some $m \in \mathbb{N}$. Then $n(L_1\sharp_{(m+1)} L_2) = n(L_1\sharp_m L_2) \pm 1$.*

Proof Let $n(L_1\sharp_m L_2) = k$. To extend $L_1\sharp_m L_2$ to $L_1\sharp_{(m+1)} L_2$ as an elementary extension, we need to cut the two torus links L_1 and L_2 at $(m+1)^{th}$ points say A_1 and A_2 respectively in $L_1\sharp_m L_2$. Then the two open ends each at A_1 and A_2 on either torus links are joined across the waist handle to arrive at $L_1\sharp_{(m+1)} L_2$.
Case(1) A_1 and A_2 lie on the same component of $L_1\sharp_m L_2$.

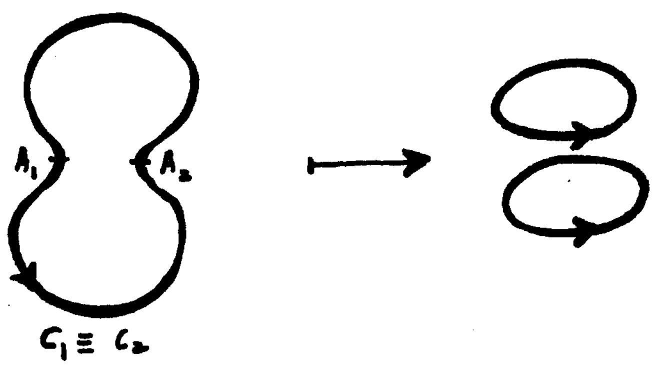


Figure 2.1 $n(L_1\sharp_{(m+1)} L_2) = n(L_1\sharp_m L_2) + 1$ for $C_1 = C_2$.

Here, the process involved in arriving at the elementary extension $L_1\sharp_{(m+1)}L_2$ of $L_1\sharp_m L_2$ is similar to that of cutting an oriented knot (as other components of the link are not playing any role) at two different places and then joining the open end points of one cut to the open end points of the other. This process splits the component into two components (figure 2.1). Therefore, $n(L_1\sharp_{(m+1)}L_2) = n(L_1\sharp_m L_2) + 1$.

Case(2) A_1 and A_2 lie on two different components of $L_1\sharp_m L_2$.

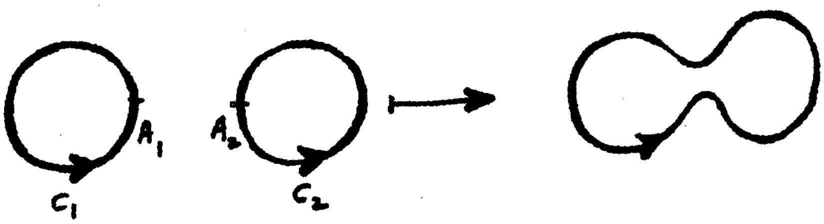


Figure 2.2 $n(L_1\sharp_{(m+1)}L_2) = n(L_1\sharp_m L_2) - 1$ for $C_1 \neq C_2$.

In this case, the process involved in arriving at the elementary extension $L_1\sharp_{(m+1)}L_2$ of $L_1\sharp_m L_2$ is similar to that of a connected sum of two oriented knots. This process fuses the two components into one (figure 2.2). Therefore, $n(L_1\sharp_{(m+1)}L_2) = n(L_1\sharp_m L_2) - 1$. □

2.1.1 Reduced Permutations

In this section, we describe an algorithm to associate a permutation $p(m) \in S_p$ with $L_1\sharp_m L_2$ where $L_i = L_i(p_i, q_i)$ and $p \leq p_1 + p_2$. This permutation $p(m)$ carries the information of $n(L_1\sharp_m L_2)$. In fact, the number of pairwise disjoint cycles in $\sigma(L_1\sharp_m L_2)$ is the same as the number of pairwise disjoint cycles in $p(m)$. The permutation $p(m)$

is called the *reduced permutation* associated with $L_1\sharp_m L_2$. The algorithm describes how to obtain the associated reduced permutation $p(m)$ without invoking $\sigma(L_1\sharp_m L_2)$ for "large" m. This reduces the time involved in computing $n(L_1\sharp_m L_2)$. As we have already observed earlier, there are two distinct ways of forming an m-connected sum of two torus links L_i, $i = 1, 2$. A multiple connected sum $L_1\sharp_m L_2$ of two torus links L_1 and L_2 is said to be *reverse* if for a fixed orientation of $L_1\sharp_m L_2$, the induced orientations of the links L_1 and L_2 are either both positive or both negative in the longitudinal direction. $L_1\sharp_m L_2$ is called *direct* if it is not reverse. Note that in an oriented direct (reverse, respectively) multiple connected sum $L_1\sharp_m L_2$ with $m \geq p_1 + p_2$ ($m > \max\{p_1, p_2\}$, respectively) the m-cuts on torus links will be both direct or both reverse (neither both direct nor both reverse, respectively) with respect to the orientations induced from $L_1\sharp_m L_2$. These two ways of forming $L_1\sharp_m L_2$ of torus links L_1 and L_2 for a fixed m generally results in different links. This fact can be verified by computing their respective number of components.

We are interested in computing $p(m)$ for the two types of multiple connected sums mentioned above. These permutations will be easier to compute being smaller in size in comparison with the respective resultant permutations for large values of m. The number of components of the multiple connected sum can also be computed from the reduced permutations because they have the same number of cycles as the resultant permutations. We deal with the different types of connected sums for obtaining the reduced permutation below.

Type(1) Let $L_1\sharp_m L_2$ be an oriented direct multiple connected sum of two torus links $L_i(p_i, q_i)$. Without loss of generality, let the orientation of $L_1\sharp_m L_2$ be such that the induced orientation of the link L_1 be positive in the longitudinal direction.

Case(a) $m \leq p_1 + p_2$.

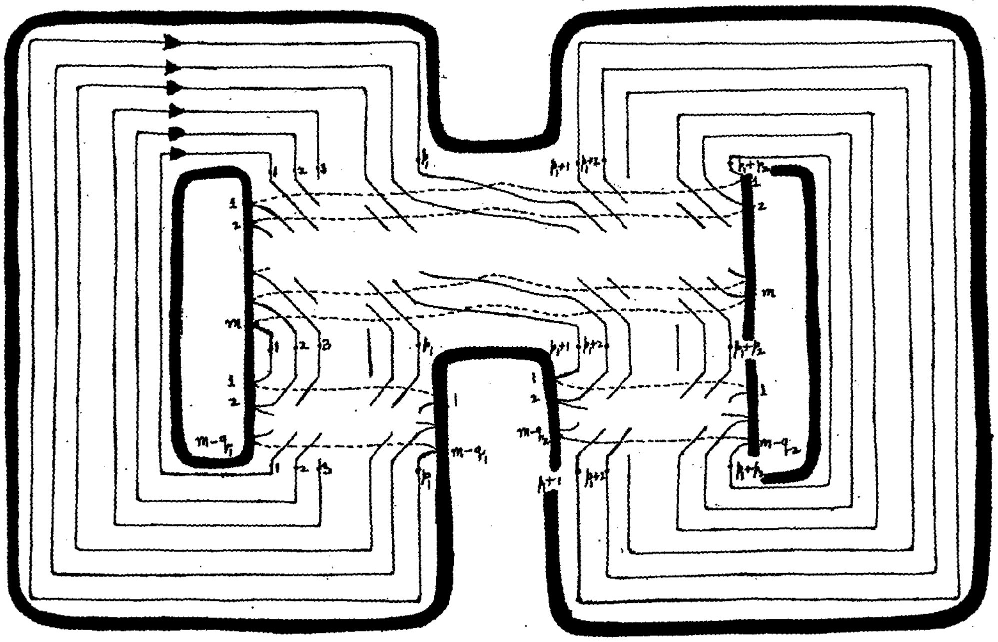


Figure 2.3 Direct Multiple Connected Sum $L_1 \sharp_m L_2$.

Here the reduced permutation $p(m)$ associated with $L_1 \sharp_m L_2$ is $\sigma(L_1 \sharp_m L_2) = \sigma_2^{(m)} \circ \sigma_1^{(m)}$ where $\sigma_i^{(m)} = \sigma_i^{(m)}(p_i, d_{ij})$ and $d_{ij} = (-1)^j q_i \bmod p_i$ for $i = 1, 2$ and $j = 1$. Further, $\sigma_i^{(m)}$ is the permutation associated with the m-cut on L_i in $L_1 \sharp_m L_2$ for $i = 1, 2$ with respect to the induced orientation. The number of components $n(L_1 \sharp_m L_2)$ is given by the number of cycles present in $\sigma(L_1 \sharp_m L_2)$.

Case(b) $m \geq p_1 + p_2$.

Here $p(m)$ has "length" equal to $p_1 + p_2$ and is derived in the following way without using the labels of the m-cuts in $L_1 \sharp_m L_2$.

Let $A = \{1, 2, ..., p_1 + p_2\}$, $B = \{1, 2, ..., p_1\}$ and $C = \{1, 2, ..., p_2\}$. We label the

$p_1 + p_2$ longitudinal strands of L_1 and L_2 by the elements in A as shown in figure 2.3. The elements of the set B are used for labelling the p_1 longitudinal strands of L_1 and the elements of the set $A \setminus B$ are used for labelling the p_2 longitudinal strands of L_2. Our aim here is to derive $p(m)$ associated with $L_1 \sharp_m L_2$ in terms of the labels of the set A. To derive $p(m)$, we must traverse through the $p_1 + p_2$ longitudinal strands of L_1 and L_2 in $L_1 \sharp_m L_2$ and record the labels on the longitudinal strands in the order of their arrival without repeating any label. Note that every component of the direct multiple connected sum must wind around at least one of the longitudes of the double torus in which it is embedded. This is so because in a direct $L_1 \sharp_m L_2$ with $m > p_1 + p_2$ the m-cuts are either both direct or both reverse with respect to orientations induced from $L_1 \sharp_m L_2$. Hence it follows (figure 2.4) that the number of cycles in $\sigma(L_1 \sharp_m L_2)$ that is equal to the number of cycles in $p(m)$ cannot exceed $p_1 + p_2$, and every component of $L_1 \sharp_m L_2$ must contain at least one strand labelled by the set A. The reduced permutation is a bijection $p(m)$: $A \longrightarrow A$ and is derived as follows.

In the figure 2.3, the rectangle R induces the function $\Phi : A \longrightarrow A$ defined by $\Phi(i) = (i + t(m)) \mod (p_1 + p_2)$ where $t(m) = -m \mod (p_1 + p_2)$. The rectangle R_1 induces the function $\Psi_1 : B \longrightarrow B$ defined by $\Psi_1(i) = (i + r(m)) \mod p_1$ where $r(m) = (m - q_1) \mod p_1$. And the rectangle R_2 induces the function $\Psi_2 : C \longrightarrow C$ defined by $\Psi_2(i) = (i + s(m)) \mod p_2$ where $s(m) = (m - q_2) \mod p_2$. The reduced permutation $p(m)$ is given by

$$p(m)(i) = \begin{cases} (\Psi_1 \circ \Phi)(i) & \text{if } 1 \leq \Phi(i) \leq p_1 \\ (\Psi_2(\Phi(i) - p_1)) + p_1 & \text{if } p_1 + 1 \leq \Phi(i) \leq p_1 + p_2 \end{cases}$$

$$= \begin{cases} ((i+t(m)) \mod (p_1+p_2)+r(m)) \mod p_1, \\ \qquad if\ 1 \leq (i+t(m)) \mod (p_1+p_2) \leq p_1 \\ (([i+t(m)] \mod (p_1+p_2)-p_1+s(m)) \mod p_2+p_1, \\ \qquad if\ p_1+1 \leq (i+t(m)) \mod (p_1+p_2) \leq (p_1+p_2) \end{cases} \quad ...(2.1).$$

For an oriented direct multiple connected sums $L_1\sharp_m L_2$ of two torus links L_i, $i = 1, 2$ where $m \in Y = \{l, l+1, ...\}$ and $l = p_1+p_2$ consider the function $F : Y \longrightarrow \mathbb{N}$ defined by $F(m) = n(L_1\sharp_m L_2)$. It is clear from the above reduced permutation formula (2.1) that the function F is periodic and has a period equal to a divisor of $l.c.m.(p_1, p_2, p_1 + p_2)$. Hence the number of components of the direct multiple connected sum $L_1\sharp_m L_2$ is periodic with period equal to a divisor of $\mathrm{lcm}(p_1, p_2, p_1+p_2)$.

The following lemma shows that there exists a recurrence relation relating consecutive reduced permutations $p(l+r)$ and $p(l+r-1)$ associated with oriented direct multiple connected sums $L_1\sharp_{(l+r)}L_2$ and $L_1\sharp_{(l+r-1)}L_2$ respectively for $l = p_1+p_2$ and $r \in N$. Here the direct multiple connected sums $L_1\sharp_{(l+r)}L_2$ and $L_1\sharp_{(l+r-1)}L_2$ are compatibly oriented and the former multiple connected sum is an elementary extension of the latter.

Lemma 2.1.2. *Let $L_1\sharp_{(l+r)}L_2$ be an elementary extension of a direct multiple connected sum $L_1\sharp_{(l+r-1)}L_2$ where $l = p_1+p_2$ and $r \in N$ having the associated reduced permutations $p(l+r)$ and $p(l+r-1)$ respectively with respect to some fixed compatible orientations of the multiple connected sums. Then $p(l+r) = p(l+r-1) \circ (s, t)$ where $s, t \in \{1, 2, ..., p_1+p_2\}$ such that $(s-r+1) \mod l = 1$ and $(t-r+1) \mod l = p_1+1$. Further, $p(l+r) = p(l)\prod_{i=1}^{r}(s_i, t_i)$ where $L_1\sharp_{(l+r)}L_2$ is an extension of $L_1\sharp_{(l)}L_2$ and $s_i, t_i \in \{1, 2, ..., p_1+p_2\}$ are such that $(s_i-(i-1)) \mod l = 1$ and $(t_i-(i-1)) \mod l = p_1+1$.*

Proof It follows from the equation (2.1) of reduced permutation that $p(l+r)(i) = p(l+r-1)(i)$ for all i such that $2 \leq (i-r+1) \mod l \leq p_1$ and for all $p_1+2 \leq (i-r+1) \mod l \leq l$. Hence, it suffices to show that (1) $p(l+r)(s) = p(l+r-1)(t)$ and (2) $p(l+r)(t) = p(l+r-1)(s)$.

But, $p(l+r)(s) = p(l+r)(s) = (p_1+r-q_2)(\mod p_2)+p_1 = p(l+r-1)(t) = p(l+r-1)(t)$ and $p(l+r)(t) = p(l+r)(t) = (p_2+r-q_1) \mod p_1 = p(l+r-1)(s) = p(l+r-1)(s)$. Hence,the recurrence relation $p(l+r) = p(l+r-1) \circ (s,t)$ holds. Therefore, $p(l+r) = p(l)\prod_{i=1}^{r}(s_i,t_i)$ follows by repeatedly applying the above recurrence relation for each elementary extension from $p(l)$ to $p(l+r)$. □

The next lemma together with the Lemma 2.1.2 gives a special case of Lemma 2.1.1.

Lemma 2.1.3. *If $\sigma \in S_n$ has r number of cycles for some $r \in \mathbb{N}$, then $\sigma \circ (i,j)$ will have $r \pm 1$ cycles for $i,j \in \{1,2,...,n\}$ and $i \neq j$.*

Proof: Suppose $\sigma = \prod_{i=1}^{r}\sigma_i$ be the cyclic decomposition of σ as a product (composition) of disjoint cycles.

Case (1) Both i and j appear in the same cycle $\sigma_s = (n_1,n_2,...,n_{m_s})$, $1 \leq s \leq r$.

Suppose $i = n_u$ and $j = n_v$ where $1 \leq u < v \leq m_s$. Then,

$\sigma_s \circ (i,j) = (n_1,...,n_{u-1},i,n_{u+1},...,n_{v-1},j,n_{v+1},...,n_{m_s}) \circ (i,j)$

$= (n_1,...,n_{u-1},j,n_{v+1},...,n_{m_s}) \circ (n_{u+1},...,n_{v-1},i)$.

Since the other cycles in the decomposition of σ are unaffected, the number of the cycles in $\sigma \circ (i,j)$ is equal to $r+1$.

Case (2) i and j appear in two different cycles of σ

Let $\sigma_s = (n_1,n_2,...,n_{m_s})$ and $\sigma_t = (p_1,p_2,...,p_{m_t})$ where $1 \leq s < t \leq r$ and, $i = n_u$ and $j = p_v$ for some $1 \leq u \leq m_s$ and $1 \leq v \leq m_t$. Then, $\sigma_s \circ \sigma_t \circ (i,j)$

$= (n_1, ..., n_{u-1}, i, n_{u+1}, ..., n_{m_s}) \circ (p_1, ..., p_{v-1}, j, p_{v+1}, ..., p_{m_t}) \circ (i, j)$

$= (n_1, ..., n_{u-1}, j, p_{v+1}, ..., p_{m_t}, p_1, ..., p_{v-1}, i, n_{u+1}, ..., n_{m_s})$. Hence, the number of the cycles in $\sigma \circ (i, j)$ is reduced by one and is equal to $(r - 1)$. □

Type (2): Let $L_1 \sharp_m L_2$ be an oriented reverse multiple connected sum of $L_1 = L_1(p_1, q_1)$ and $L_2 = L_2(p_2, q_2)$. Without loss of generality, let the orientation of $L_1 \sharp_m L_2$ be such that the induced orientation of both the links L_i, $i = 1, 2$ be positive in the longitudinal direction.

Case (a) $m \leq p_2$ and $p_1 < p_2$.

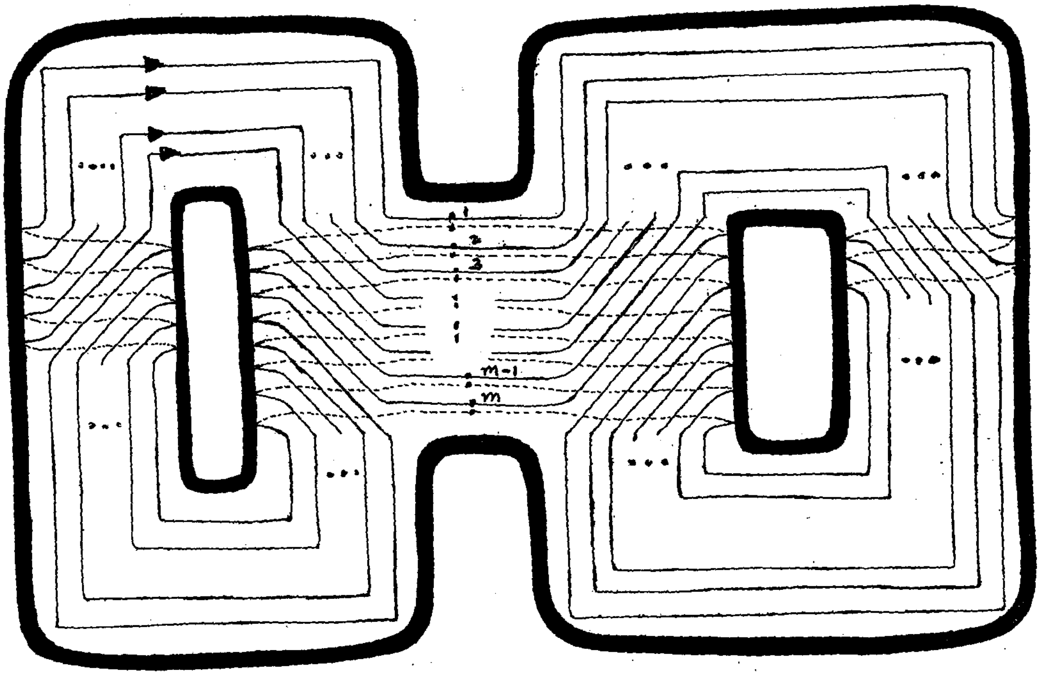


Figure 2.4 Reverse Multiple Connected Sum $L_1 \sharp_m L_2$.

Here the reduced permutation $p(m)$ associated with $L_1 \sharp_m L_2$ is $\sigma(L_1 \sharp_m L_2) = \sigma_2^{(m)} \circ \sigma_1^{(m)}$, where $\sigma_1^{(m)} = \sigma_1^{(m)}(p_1, d_{1j})$, $\sigma_2^{(m)} = \sigma_2^{(m)}(p_2, d_{2k})$ and $d_{il} = (-1)^l q_i \bmod p_i$ for

$l \in \{1,2\}$ and $j \neq k$. Further, $\sigma_i^{(m)}$ is the permutation associated with the m-cut on L_i in $L_1 \sharp_m L_2$ for $i = 1,2$ with respect to the induced orientation. The number of components $n(L_1 \sharp_m L_2)$ is given by the number of cycles present in $\sigma(L_1 \sharp_m L_2)$.

Case (b) $m > p_2 > p_1$.

Here $p(m)$ has 'length' p_2 and is derived in the following way using the labels of the m-cuts in $L_1 \sharp_m L_2$.

Let $A = \{1,2,...,m\}$, $A_0 = \{1,2,...,p_2 - p_1, m - p_1 + 1,...,m\}$, $A_1 = \{m - p_2 + 1,...,m\}$, $A_2 = \{m - p_2 + 1,...,m - p_1\}$, $A_3 = \{1,2,...,p_2 - p_1\}$ and $A_4 = \{1,2,...,p_2\}$. Denote $\sigma(L_1 \sharp_m L_2)$ by σ^m for the sake of brevity. The action of σ^m on A is the bijection $\sigma^{(m)} : A \longrightarrow A$ given by $\sigma^{(m)}(x)$

$$= \begin{cases} m - \{m - (x + p_1 + q_2) \text{mod } p_2\}\overline{\text{mod } p} & \text{if } 1 \leq x \leq p_2 - p_1 \\ x - (p_2 - p_1) & \text{if } p_2 - p_1 + 1 \leq x \leq m - p_1 \\ m - \{m - ((x - q_1)(\text{mod } p_1) + q_2) \text{mod } p_2\}\overline{\text{mod } p_2} & \text{if } m - p_1 + 1 \leq x \leq m \end{cases}$$

where the range of $\text{mod}\, p_2 = \{1,2,...,p_2\}$ and of $\overline{\text{mod } p_2} = \{0,1,2,...,p_2 - 1\}$.

Let $\sigma_0 = \sigma^{(m)}|_{A_0}$ and $\sigma_1 = \sigma^{(m)}|_{A_2}$ be the restrictions. The map $\sigma_2 : A_1 \longrightarrow A_1$ is a bijection deduced from σ_0 and σ_1 and is given by

$$\sigma_2(x) = \begin{cases} m - \{m - [x \text{ mod } (p_2 - p_1) + p_1 + q_2] \text{ mod } p_2\} \overline{\text{mod } p_2} \\ \qquad if \;\; m - p_2 + 1 \leq x \leq m - p_1 \\ m - \{m - [(x - q_1) \text{ mod } p_1 + q_2] \text{ mod } p_2\} \overline{\text{mod } p_2} \\ \qquad if \;\; m - p_1 + 1 \leq x \leq m \end{cases}$$

Finally, $p(m) : A_4 \longrightarrow A_4$ is the bijection defined by

$$p(m)(x) = \begin{cases} p_2 - \{m - [(x + m - p_2) \text{ mod } (p_2 - p_1) + p_1 + q_2] \text{ mod } p_2\} \overline{\text{mod } p_2} \\ \qquad if \;\; 1 \leq x \leq p_2 - p_1, \\ p_2 - \{m - [(x + m - p_2 - q_1)(\text{mod } p_1) + q_2] \text{ mod } p_2\} \overline{\text{mod } p_2} \\ \qquad if \;\; p_2 - p_1 + 1 \leq x \leq p_2 \end{cases} \tag{2.2}$$

For a reverse multiple connected sums $L_1\sharp_m L_2$ of two torus links L_i with $p_2 \geq p_1$ and $m \in Y_0 = \{p_2, p_2+1, ...\}$, consider the function $F : Y_0 \longrightarrow \mathbb{N}$ defined by $F(m) = n(L_1\sharp_m L_2)$. It is clear from the reduced permutation formula (2.2) that the function F is periodic and has a period equal to a divisor of l.c.m.$(p_1, p_2, p_2 - p_1)$. Hence the number of components of the reverse multiple connected sum $L_1\sharp_m L_2$ is periodic with period equal to a divisor of $\mathrm{lcm}(p_1, p_2, p_2 - p_1)$.

Case (c): $m > p_1 = p_2$.

In this case we do not derive any reduced permutation $p(m)$ associated with $L_1\sharp_m L_2$, instead we show that the knowledge of $\sigma^{(p_1)} = \sigma_2^{(p_1)} \circ \sigma_1^{(p_1)}$ suffices to compute $n(L_1\sharp_m L_2)$. Denote $\sigma(L_1\sharp_m L_2) = \sigma_2^{(m)} \circ \sigma_1^{(m)}$ by $\sigma^{(m)}$ for the sake of brevity, where $\sigma_i^{(m)}$ denotes the permutation associated with $L_i(p_i, q_i), i = 1, 2$ with respect to the induced orientation from $L_1\sharp_m L_2$. This $\sigma^{(m)}$ is used in Lemma 2.1.4, to prove that the knowledge of $\sigma^{(p_1)}|_{A_4}$ suffices for the purpose of finding $n(L_1\sharp_m L_2)$ for any $m > p_1 = p_2$ where $A_4 = \{1, 2, ..., p_1\}$.

Let $A = \{1, 2, ..., m\}$, $A_1 = \{1, 2, ..., m - p_1\}$ and $A_2 = \{m - p_1 + 1, ..., m\}$. The action of $\sigma^{(m)}$ on A is the bijection $\sigma^{(m)} : A \longrightarrow A$ given by

$$\sigma^{(m)}(x) = \begin{cases} x & \text{if } 1 \leq x \leq m - p_1 \\ m - \{m - (x - q_1 + q_2) \bmod p_1\} \overline{\bmod\ p_1} & \text{if } m - p_1 + 1 \leq x \leq m \end{cases} \qquad ...2.3$$

Here the range of $\bmod p_1 = \{1, 2, ..., p_1\}$ and the range of $\overline{\bmod\ p_1} = \{0, 1, 2, ..., p_1 - 1\}$.

Lemma 2.1.4. *Let $L_1\sharp_m L_2$ be a reverse multiple connected sum with $m > p_1 = p_2$ then $n(L_1\sharp_m L_2) = (m - p_1) + n(L_1\sharp_{p_1} L_2)$.*

Proof: To prove this result it suffices to prove (1) $\sigma^{(m)}|_{A_1}(x) = x$ for $x \in A_1$ and (2) $\sigma^{(m)}|_{A_2}(m - p_1 + x) = (m - p_1) + \sigma^{(p_1)}|_{A_1}(x)$ for $x \in A_1$. The first equation is obvious. We now prove the second equation. We have $\sigma^{(p_1)}(x) = (x + q_2 - q_1)$

mod $p_1 = y$ (say). Here $\sigma^{(p_1)}$ denotes the composition $\sigma_2 \circ \sigma_1$ of the permutations $\sigma_1(p_1, d_{1i})$ and $\sigma_2(p_2, d_{2j})$ for $i, j \in \{1,2\}$ and $i \neq j$ associated with the two p_1-cuts on the torus links $L_1(p_1, q_1)$ and $L_2(p_2, q_2)$ in the reverse multiple connected sum $L_1 \sharp_{p_1} L_2$. Then,

$$\begin{aligned}
\sigma^{(m)}(m - p_1 + x) &= (\sigma_2^{(m)} \circ \sigma_1^{(m)})(m - p_1 + x) \\
&= \sigma_2^{(m)}(\{(m - p_1 + x)(\text{mod } p_1) + (p_1 - q_1)\}\text{mod } p_1) \\
&= \sigma_2^{(m)}((m - q_1 + x)\text{mod } p_1) \\
&= \{(m - q_1 + x)(\text{mod } p_1) - p_2 + q_2\}(\text{mod } p_2) + m_0 p_1 \\
&= (m + q_2 - q_1 + x)(\text{mod } p_1) + m_0 p_1 \\
&= (m + y)(\text{mod } p_1) + m_0 p_1 \\
&= (m - p_1) + y
\end{aligned}$$

Here m_0 is a non-negative integer chosen so that $\sigma^{(m)}(m - p_1 + x) \in \{m - p_1 + 1, ..., m\}$. The last equality $(m + y)(\text{mod } p_1) + m_0 p_1 = (m - p_1) + y$ holds, since the quantities on either side of the equation belong to the set $\{m - p_1 + 1, ..., m\}$ and $((m + y)(\text{mod } p_1) + m_0 p_1)\text{mod } p_1 = ((m - p_1) + y)\text{mod } p_1$. □

2.2 Fundamental Group of Genus two 3-Manifolds

Here, we provide simple schemes to obtain a presentation of the Fundamental Group of genus two 3-Manifolds [4] associated with links generated by multiple connected sums of two torus links. The scheme for direct multiple connected sums of two torus links $L_i(p_i, q_i)$, $i = 1,2$ uses $p(m)$ if $m \geq p_1 + p_2$ or the permutation $\sigma_i^{(m_i)}$ where $m_i = \max\{m, p_i\}$, $i = 1,2$ if $m < p_1 + p_2$. On the other hand, the scheme for reverse multiple connected sums uses $p(m)$ if $m \geq \max\{p_1, p_2\}$ or the permutation $\sigma_i^{(m_i)}$ where $m_i = \max\{p_i, m\}$, $i = 1,2$ if $m < \max\{p_1, p_2\}$.

An orientable 3-Manifold is a quotient space of (and hence can be decomposed into) two handle bodies each of genus g for some non-negative integer g [15]. A decomposition of an orientable 3-Manifold into two handle bodies of genus g, whenever it is possible is called *Heegaard splitting* of genus g. The smallest g such that the manifold admits a Heegaard splitting of genus g is known as the *Heegaard genus* of the 3-Manifold. Note that an orientable 3-Manifold which is decomposable into two genus g handle bodies can obviously be reconstructed by gluing back the two genus g handle bodies along their boundaries. This gluing of the boundaries of the two handle bodies of genus g is done by a homeomorphism of their boundaries. Any orientation preserving homeomorphism between the boundaries of two genus g handle bodies generates an orientable 3-Manifold as a quotient space of the two handle bodies. Isotopic homeomorphisms between the boundaries of two genus g handle bodies generate homeimorphic 3-Manifold. Any such homeomorphism between two compact connected orientable surfaces is completely determined up to isotopy by the way the canonical curves are mapped [12].

The fundamental Group of an orientable 3-Manifold formed as a quotient space of two genus g handle bodies has g *generators* and g *relations* [11]. The g generators represent the g non-trivial canonical curves of any one of the two genus g handle bodies forming the quotient space. The g relations are obtained from the images of these generators under the homeomorphism onto the other handle body. Any such homeomorphism maps the g non-trivial canonical curves on the boundary of one genus g handle body onto the boundary of the other genus g handle body. This image set is a link with g non-separating components embedded in the boundary of the genus g handle body. To compute the g relations corresponding to the g components of the image link, one needs to know the number of times each components winds around the g canonical curves and their order of occurrence. However, in practice, this is a very tedious task because we need to depend heavily on a neat diagram of the link. In particular, this is true of genus two 3-Manifolds. But in the case of multiple connected sums having two components which are non-separating in the double torus, we provide some simple algorithms to compute a presentation of the fundamental group of associated genus two 3-Manifolds.

Suppose $L_1\sharp_m L_2$ is a link with two components that are non-separating on the double torus. By Lickorish theorem, there exists an orientation preserving homeomorphism of the double torus mapping these two components to two canonical curves (m_1 and m_2 figure 3.1, chapter 3) on the double torus that are contractible on the genus two handle body. Any 3-manifold obtained by such a homeomorphism is called a 3-manifold associated with $L_1\sharp_m L_2$.

Let $L_1\sharp_m L_2$ be embedded on the boundary ∂H_2 of a genus two handle body H_2. Each component of $L_1\sharp_m L_2$ is an element of the $\pi_1(\partial H_2)$. The fundamental group $\pi_1(H_2)$ has two generators that are also the generators of the Fundamental Group of any associated genus two 3-Manifold. They are denoted by x and y (figure 2.5). Homotope H_2 into the wedge of two circles. Each component of the image of $L_1\sharp_m L_2$ under this homotopy is an element of $\pi_1(H_2)$. If $L_1\sharp_m L_2$ has exactly two non-separating components, then the image of $L_1\sharp_m L_2$ under the homotopy are precisely the two relations of the Fundamental Group of genus two 3-Manifold associated with $L_1\sharp_m L_2$ as they are mapped to the (contractible) canonical meridional disk on the other handle body by the identification homeomorphism. These relations can be obtained from the two cycles in $\sigma(L_1\sharp_m L_2)$. This can be done by traversing the terms in the two cycles of $\sigma(L_1\sharp_m L_2)$ in the order of their occurrence and thereby building the words representing the corresponding relations of the presentation in the following way.

Taking the image of a point under a cyclic permutation is called a *move* at that point. If we make a move at the point j_t to the point j_{t+1} in a cyclic permutation $(..., j_t, j_{t+1}, ...)$, we say that we have *left* the point j_t or *arrived* at the point j_{t+1}. Each time we make a move we ought to find out whether the corresponding strand has been traversed along x or y and accordingly record the corresponding label along with the direction. We provide below schemes for obtaining a presentation of the Fundamental Group of the genus two 3-Manifold associated with $L_1\sharp_m L_2$ with two components.

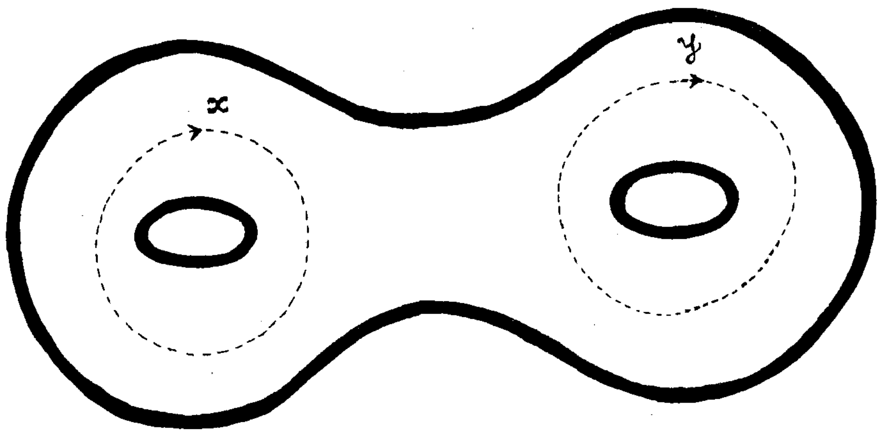


Figure 2.5 The two generators x and y of the Fundamental Group.

Case(1) $L_1\sharp_m L_2$ is an oriented direct multiple connected sum with $n(L_1\sharp_m L_2) = 2$, the two components are non-seperating and $p_1 \leq p_2$.

Without loss of generality we assume that the orientation of $L_1\sharp_m L_2$ is such that the induced orientation of L_1 is positive in the longitudinal direction.

Subcase (a): $p_1 \leq m \leq p_2$.

We traverse the two components of $L_1\sharp_m L_2$ and construct the corresponding relations by recording the generators x or y^{-1} according as the strand of $L_1\sharp_m L_2$ moves along the generator x or y^{-1} respectively. Here, we traverse the components of $L_1\sharp_m L_2$ while making moves along the terms of the permutations ${\sigma_1}^{(m)}$ and ${\sigma_2}^{(p_2)}$. All the possible kinds of moves are illustrated below in three steps.

Without loss of generality, we begin with the first entry in ${\sigma_1}^{(m)}$ and with the empty relation.

Step (1): Suppose we are at the entry j_1 in the cycle ${\sigma_1}^{(m)} = (..., j_1, j_2, ...)$. The next move depends on the the following two possibilities. (i) If $m - p_1 < j_1 \leq m$, then postfix the relation by the generator x while moving to the immediate next entry j_2 in ${\sigma_1}^{(m)}$ and then go to the step (2). (ii) If $1 \leq j_1 \leq m - p_1$, then do not change the relation while moving to the next entry j_2 in ${\sigma_1}^{(m)}$ and then go to step (2).

Explanation: Arriving at a label of the set $\{m - p_1 + 1, ..., m\}$ in the permutation ${\sigma_1}^{(m)}$ indicates that a strand of the link $L_1\sharp_m L_2$ has just been traversed along the generator x in its direction, and leaving a label of the set $\{1, ..., m - p_1\}$ in the cycle ${\sigma_1}^{(m)}$ indicates that a strand of the link $L_1\sharp_m L_2$ is being traversed half way around the waist handle of the double torus. This is the reason for postfixing x to the relation in the former case and leaving the relation unchanged in the latter case.

Step (2): If we are at the entry j_2 in ${\sigma_2}^{(p_2)} = (..., j_2, j_3, ...)$, then postfix the relation by the generator y^{-1} while moving to the next entry j_3 in ${\sigma_2}^{(p_2)}$ and go to step (3).

Explanation: Leaving a label in the permutation ${\sigma_2}^{(p_2)}$ indicates that a strand of the link $L_1\sharp_m L_2$ is being traversed along the generator y in the direction opposite to it. This is the reason for postfixing y^{-1} to the relation.

Note that the permutation ${\sigma_2}^{(m)}$ associated with the m-cut torus link L_2 is extended to the permutation ${\sigma_2}^{(p_2)}$ in case $p_2 > m$ even though the strands labelled by the elements of the set $\{m+1, ..., p_2\}$ are not cut in the connected sum. This is done for keeping track of all the p_2 longitudinal strands of L_2 that contribute to the relations as they wind along the generator y.

Step (3): If we are at the entry j_3, then the move in this step is performed as follows: (i) If $m < j_3 \leq p_2$, then proceed as in step (2). (ii) If $1 \leq j_3 \leq m$, then proceed as in step (1).

We continue this process until we return to the entry of σ_1^m we started with. To obtain the second relation, we must follow the same procedure with the other component.

Subcase (b): $m < min\{p_1, p_2\}$, or $\max\{p_1, p_2\} < m < p_1 + p_2$

In this case, we proceed in a similar way as in subcase (a).

Subcase (c): $m \geq p_1 + p_2$.

In this case also traverse the two components of $L_1\sharp_m L_2$ and construct the corresponding relations by recording the letters x or y^{-1} each time we move along a strand of the link that winds around the generator x in its direction or along y opposite to its direction respectively. We traverse the components of $L_1\sharp_m L_2$ while performing moves along the terms of the associated $p(m)$ given by the formula (2.1) and generate the corresponding relations in the following way.

Let $(j_1, j_2, j_3, ..., j_r)$ be the cycle in $p(m)$ representing one of the components of $L_1\sharp_m L_2$. Then the corresponding relation $z_1.z_2...z_r$ is given by the rule:

$$z_k = \begin{cases} x & \text{if } 1 \leq j_k \leq p_1 \\ y^{-1} & \text{if } p_1 + 1 \leq j_k \leq p_1 + p_2 \end{cases}$$

for all $k = 1, 2, ..., r$.

Explanation: Arriving at a label of the set $\{1, ..., p_1\}$ in $p(m)$ indicates that a strand of the link $L_1\sharp_m L_2$ has just been traversed along the generator x in its direction and leaving a label of the set $\{p_1+1, ..., p_1+p_2\}$ in $p(m)$ indicates that a strand of the link $L_1\sharp_m L_2$ is being traversed along the the generator y in the direction opposite to it. Thus we take x and y^{-1} respectively in the relation at these instances.

We illustrate the procedure by an example. Consider the direct multiple connected sum $L_1\sharp_{32} L_2$ of the two torus links $L_1 = ((5,0),(3,0))$ and $L_2((7,0),(4,0))$. The reduced permutation $p(32)$ associated with $L_1\sharp_{32} L_2$ is $(1,10,6,2,11,7,3,12,8,4)(5,9)$. Hence, a presentation of $\pi_1(M)$ of the genus two 3-Manifold M associated with $L_1\sharp_{32} L_2$ is $\pi_1(M) = \{x, y : xy^{-2}xy^{-2}xy^{-2}x,\ xy^{-1}\} = \mathbb{Z}_2$.

Case(2) $L_1\sharp_m L_2$ is an oriented reverse multiple connected sum with $n(L_1\sharp_m L_2) = 2$, the two components are non-seperating and $p_1 \leq p_2$.

Without loss of generality, let the orientation of the multiple connected sum be such that the induced orientations of L_1 and L_2 will be positive in the longitudinal direction.

Subcase (a) $p_1 \leq m \leq p_2$ and $p_1 \neq p_2$.

We traverse the two components of $L_1\sharp_m L_2$ while making moves along the terms of the permutations ${\sigma_1}^{(m)}$ and ${\sigma_2}^{(p_2)}$ in three steps as illustrated below.

Without loss of generality, we begin at the first entry in σ_1 and with the empty relation.

Step (1): Suppose we are at an entry j_1 in the cycle ${\sigma_1}^{(m)} = (..., j_1, j_2, ...)$. (i) If $m - p_1 < j_1 \leq m$, then we postfix the relation by the generator x. Move to the next entry j_2 in ${\sigma_1}^{(m)}$ and then go to the step (2). (ii) If $1 \leq j_1 \leq m - p_1$, then do not make any change to the relation while moving to the next entry j_2 in ${\sigma_1}^{(m)}$ and then go to the step (2).

Explanation: Arriving at a label of the set $\{m-p_1+1, ..., m\}$ in the permutation ${\sigma_1}^{(m)}$ indicates that a strand of $L_1\sharp_m L_2$ has just been traversed along the generator x in the direction of its orientation. Also leaving a label of the set $\{1, ..., m-p_1\}$ in the permutation ${\sigma_1}^{(m)}$ indicates the fact that a strand of $L_1\sharp_m L_2$ is being traversed half way around the waist handle of the double torus. In the latter case, the strand did not traverse along either of the generators of the fundamental group and therefore no letter has been postfixed to the relation.

Step (2): If we are at the entry j_2 in ${\sigma_2}^{(p_2)} = (..., j_2, j_3, ...)$ postfix the relation by the generator y. Then move to the next entry j_3 in ${\sigma_2}^{(p_2)}$ and then go to the step (3).

Explanation: Leaving a label in the permutation ${\sigma_2}^{(p_2)}$ indicates that a strand of $L_1\sharp_m L_2$ is being traversed around the generator y in the direction of the orientation.

Note that the permutation ${\sigma_2}^{(m)}$ associated with the m-cut torus link L_2 is extended to the permutation ${\sigma_2}^{(p_2)}$ in case $p_2 > m$, even though the strands labelled by the elements of the set $\{m+1, ..., p_2\}$ are not cut and joined in the multiple connected sum. This is done for keeping track with all the p_2 longitudinal strands of the torus link L_2 that contribute to the relations as they wind around the generator y.

Step(3): The third move is performed on the basis of the following two possibilities. (i) If $m < j_3 \leq p_2$, then proceed as in step (2). (ii) If $1 \leq j_3 \leq m$, then proceed as in step (1).

We continue this process until we return to the initial entry we started with. To obtain the second relation, we repeat the same procedure with the second component by performing moves in the entries of ${\sigma_1}^{(m)}$ and ${\sigma_2}^{(p_2)}$.

Subcase (b): $m < p_1 \leq p_2$.

This is similar to the subcase (a) and hence the corresponding scheme to compute the fundamental group of the associated 3-Manifold can be deduced likewise.

Subcase (c) $p_1 < p_2 \leq m$.

We traverse the two components of $L_1 \sharp_m L_2$ while performing moves along the terms of $p(m)$ associated with $L_1 \sharp_m L_2$ given by the formula (2.2) and generate the corresponding relations in the following way. Let $(j_1, j_2, ..., j_r)$ be a cycle in the permutation $p(m)$. Then the corresponding relation $z_1.z_2...z_r$ is defined as

$$z_k = \begin{cases} xy & \text{if } 1 \leq j_k \leq p_1 \\ y & \text{if } p_1 + 1 \leq j_k \leq p_2 \end{cases}$$

for all $k = 1, 2, ..., r$.

Explanation: Arriving at a label of the set $\{1, ..., p_1\}$ in $p(m)$ indicates that a strand of $L_1 \sharp_m L_2$ has just been traversed along the generator x in its direction. Leaving a label of the set $\{1, ..., p_2\}$ in $p(m)$ indicates that a strand of $L_1 \sharp_m L_2$ is being traversed along the generator y in its direction.

Subcase (d) $p_1 = p_2 = m$.

We traverse the two cycles of $\sigma(L_1 \sharp_m L_2)$ and generate the corresponding relations in the following way.

If $(j_1, j_2, ..., j_r)$ is a cycle in the permutation $p(m)$, the the corresponding relation $z_1.z_2...z_r$ is defined as $z_k = xy$ for all $k = 1, 2, ..., r$. Hence the two relations of the fundamental group will be

$$z_1.z_2...z_r = (xy)^r \text{ and } (xy)^{m-r}.$$

Explanation: Arriving at a label of the set $\{1, ..., m\}$ in $p(m)$ indicates that a strand of $L_1 \sharp_m L_2$ has just been traversed along the generator x in its direction, and leaving a label of the set $\{1, ..., m\}$ in $p(m)$ indicates that a strand of L_2 is being traversed along the generator y in the direction of orientation.

Subcase (e) $p_1 = p_2 = m - 1$.

One of the two components of $L_1 \sharp_m L_2$ has the cycle of unit length i. e. (1) (containing only the first label) representing it in $\sigma(L_1 \sharp_m L_2)$ that in turn represents a trivial relation in the fundamental group presentation of the associated 3-Manifold. The other relation is generated from the cycle in $\sigma(L_1 \sharp_m L_2)$ of length p_1 in the following way. Let $(j_1, j_2, ..., j_{p_1})$ be the cycle of length p_1 in the permutation $\sigma(L_1 \sharp_m L_2)$. The corresponding relation is $z_1.z_2...z_{p_1}$ where $z_k = xy$ for all $k = 1, 2, ..., p_1$. Hence, the only relation of the fundamental group is $z_1.z_2...z_{p_1} = (xy)^{p_1}$.

Explanation: There is only one non-trivial relation in this case as the component of $L_1 \sharp_m L_2$ passing through the label 1 winds only around the waist handle. The non-trivial relation is generated using the following fact. Arriving at a label of the set $\{2, ..., m\}$ in $p(m)$ indicates that a strand of $L_1 \sharp_m L_2$ has just been traversed along the generator x in its direction, and leaving a label of the set $\{2, ..., m\}$ in $p(m)$ indicates that a strand of $L_1 \sharp_m L_2$ is being traversed along the generator y in the direction of orientation.

Chapter 3

Mapping Class Elements

3.1 Mapping Class Elements

In §3.1.1. a parametric representation of a shift of an m-cut on a torus link and also an unambiguous parametric representation for a multiple connected sum is provided. In §3.1.2. the effective changes in the parameters governing $L_1 \sharp_m L_2$ when each of the twists $l_1^{\pm 1}$, $l_2^{\pm 1}$, $m_1^{\pm 1}$, $m_2^{\pm 1}$ and $\gamma^{\pm 1}$ is applied to $L_1 \sharp_m L_2$ under specific conditions on the parameters are observed. In §3.1.3. we present the algorithm to generate a mapping class element associated with a multiple connected sum. Finally, in §3.2. an unambiguous parametric representation of a general multiple connected sum is given and permutations associated with multiple connected sums of three torus links are presented.

A *double torus* is the boundary of a genus 2 handle body. *Mapping class group* of a double torus is the group of isotopy classes of homeomorphisms of the double torus to itself [12]. Let D be a double torus and C be a simple closed curve embedded in D. Consider a tubular neighbourhood (annulus) A around the simple closed curve C. Parameterize it by $\mathbb{S}^1 \times [0, 1]$ as $(e^{\theta \imath}, t)$.

Definition 3.1.1. A *twist* τ about C is any homeomorphism isotopic to the homeomorphism $\tau : D \longrightarrow D$ defined as $\tau_{|_{D \setminus A}}$ is identity and $\tau_{|_A}(e^{\theta\imath}, t) = (e^{(\theta+2\pi t)\imath}, t)$.

Definition 3.1.2. Two simple closed curves in D are said to be *twist equivalent* if there exists a mapping class element of D that sends one to the other.

Let $M(D)$ denote the mapping class group of D. A curve in a double torus is called a *canonical curve* if it is isotopic to one of the six curves denoted by l_1, l_2, m_1, m_2, γ and δ (figure 3.1). Note that l_1 and l_2 are the longitudes and m_1 and m_2 are meridians. γ is a simple closed curve around the waist handle of the genus two handle body whose boundary is the double torus D. δ is a simple closed curve around the waist handle of the complement handle body whose boundary is also the double torus D in the genus two Heegaard splitting of $\mathbb{S}^3$. Mapping class group $M(D)$ of the double torus D is generated by the isotopy classes of the five twists known as the *Lickorish generators* [6] about the five canonical curves l_1, l_2, m_1, m_2 and γ. We denote the Lickorich generators about these canonical curves by the same symbols l_1, l_2, m_1, m_2 and γ. Each Lickorish generator is a twist performed about the canonical curve denoted by the same symbol. The set of twists $\{l_1 \,, l_2 \,, m_1 \,, m_2 \,, \delta\}$ is another set of Lickorish generators for $M(D)$. This follows from the fact that γ can be generated from the latter set of generators. In our algorithm, we use the twists $\{l_1^{\pm1} \,, l_2^{\pm1} \,, m_1^{\pm1} \,, m_2^{\pm1} \,, \gamma^{\pm1}\}$ depending on the specifications (mentioned later in this section) of the parameters governing $L_1 \sharp_m L_2$ at that instant and reduce the multiple connected sum to canonical curves in finite number of steps. However, it is possible to use the twists $\{l_1^{\pm1} \,, l_2^{\pm1} \,, m_1^{\pm1} \,, m_2^{\pm1} \,, \delta^{\pm1}\}$ instead and reduce a multiple connected sum to canonical curves in finite number of steps. For any double torus link having k components embedded in the double torus D, there exists a mapping class element of D that maps it into a set of k canonical curves of D [12]. It is known that a double torus link when reduced to canonical curves using a mapping class element will have a maximum of 3 isotopy classes of canonical curves. This also shows that any double torus link with more than three components will have parallel components. Also any double torus link that is mapped to 2 or 3 distinct isotopy classes of canonical curves by a mapping class element must be a genus two link. Self homeomorphisms of a double torus send separating (non-separating, respectively) components of a link embedded in it to separating (non-separating, respectively) components. Hence, the number of non-separating classes of components of a link will be equal to the number of distinct classes of canonical curves to which the link is reduced to by a mapping class element. However, we are not aware of any algorithm that can produce such a mapping class element for any given arbitrary double torus link. In §3.1.3. of this chapter, we provide an algorithm to generate a mapping class element of D that sends a given multiple connected sum $L_1 \sharp_m L_2$ having k components to a set of k canonical curves of D.

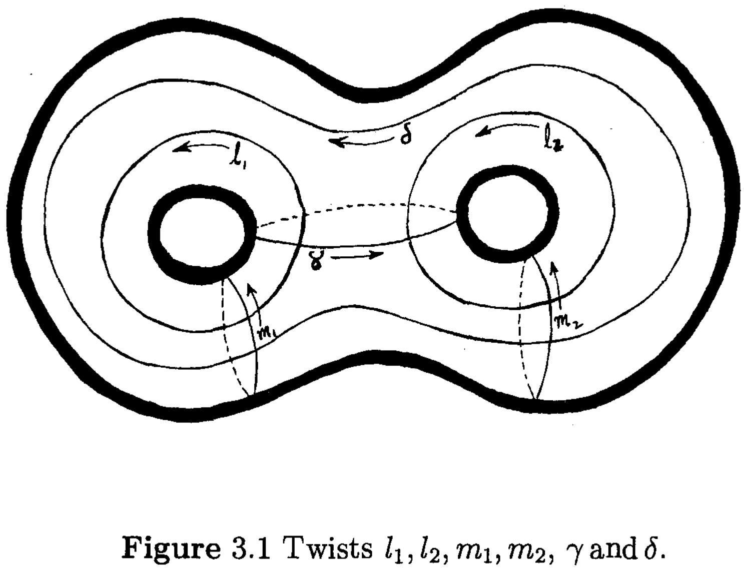


Figure 3.1 Twists l_1, l_2, m_1, m_2, γ and δ.

3.1.1 Parametric Representation of Multiple Connected sums

An m-cut on a torus link $L(p,q)$ can be deformed isotopically from across the p longitudinal strands to the q meridional strands so that the cut becomes isotopic to an arc of the longitude of the torus as shown in figures 3.2, 3.3 and 3.4. Such a deformation of an m-cut is called a *shift*. An m-cut on a torus link $L(p,q)$ is said to be *compatible* if its shift across the q meridional strands cuts some or all of the original q strands as in figure 3.2 and 3.3. An m-cut on $L(p,q)$ is said to be *non-compatible* if it is not compatible as in figure 3.4. Recall that $L(p,q)$ can be represented by four parameters as explained in chapter 1. The 4-tuple parametric representation of a torus link encodes, besides the number of longitudinal and meridional strands, the positivity or negativity of the torus link. This is significant since a positive torus link need not be isotopic in $\mathbb{R}^3$ to a negative torus link even if they possess the same number of longitudinal and meridional strands. An m-cut positive torus link $L(p,q)$ is represented by $[\overline{(p,0)},(q,0);m]$ where the overline over the term $(p,0)$ indicates the cut is performed across the p longitudinal strands, and for $m > p$ this representation fails to distinguish between the two types of cuts namely direct and reverse for a given orientation of the link. However, the shift of these cuts resolves the ambiguity and the m-cut positive torus link is now represented by (1) $[(p,0),\overline{(q,0)};m]$ when $m < q$ and the cut is compatible as in figure 3.2, (2) $[(p,0),\overline{(m,m-q)};m]$ when $m < q$ and the cut is compatible as in figure 3.3, and (3) $[(p,0),\overline{(m+q,m)};m]$ when the cut is non-compatible as in figure 3.4.

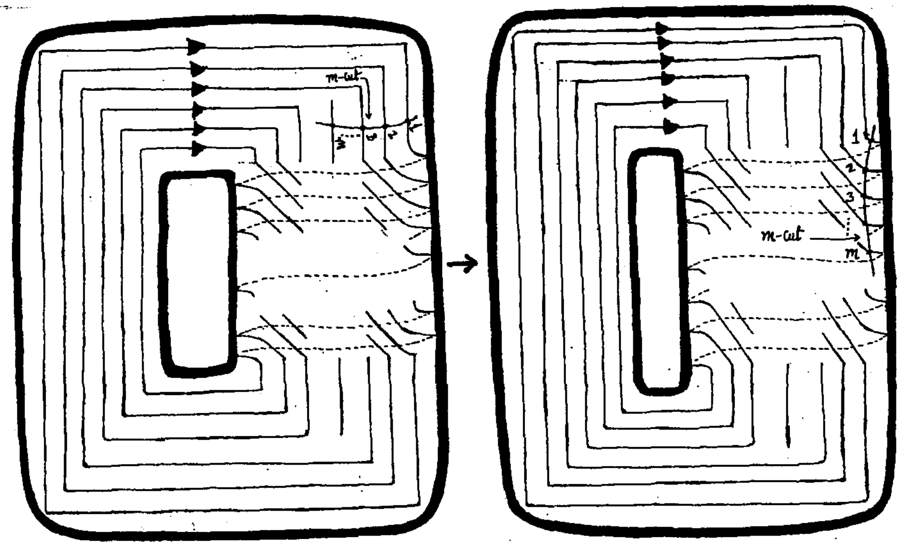


Figure 3.2 $[\overline{(p,0)},(q,0);m] \longrightarrow [(p,0),\overline{(q,0)};m]$ where $m \leq q$

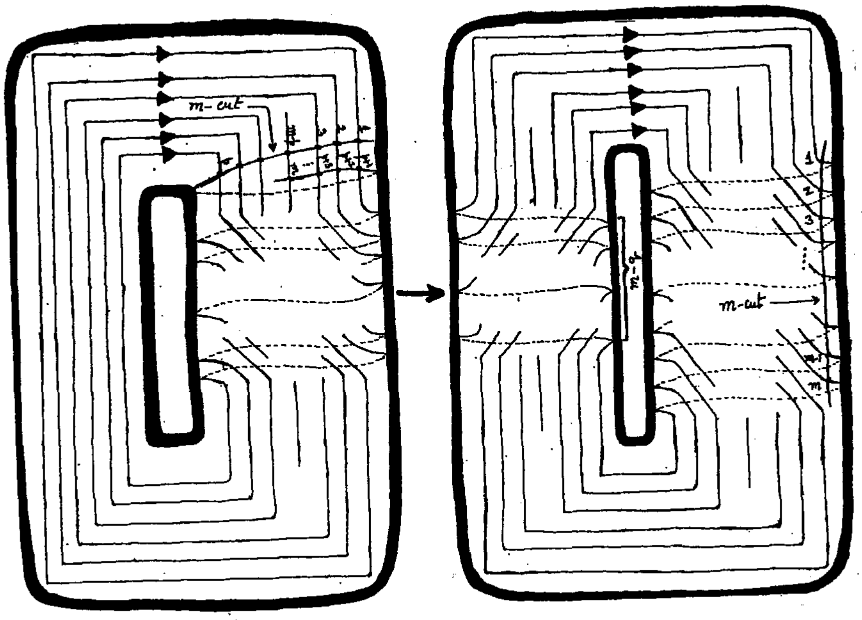


Figure 3.3 $[\overline{(p,0)},(q,0);m] \longrightarrow [(p,0),\overline{(m,m-q)};m]$ where $m > max\{p,q\}$

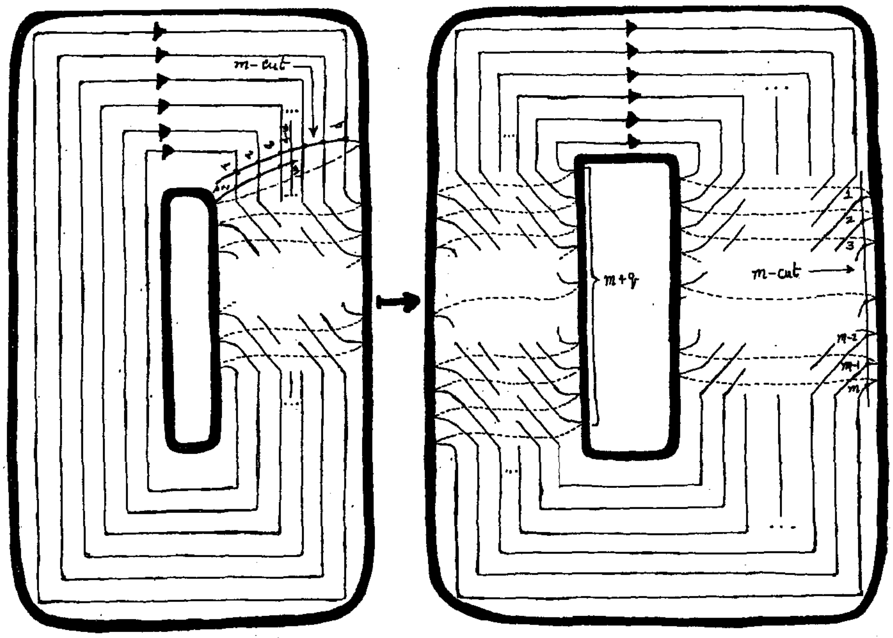


Figure 3.4 $[\overline{(p,0)},(q,0);m] \longrightarrow [(p,0),\overline{(m+q,m)};m]$ where $m > p$

A shift of an m-cut, being a deformation of the cut itself, does not change the cut nor the link it cuts. The two types of m-cuts possible on $L(p,q)$ when $m > p$, namely compatible and non-compatible m-cuts, are basically direct and reverse m-cuts for a fixed orientation of the link and for a labelling in a fixed direction along a meridian. It can be seen that these two types of m-cuts are distinct in general, in the sense that two non-isotopic m-connected sums can be generated from two torus links $L_i, i = 1,2$ using the two types of m-cuts. An illustration of this fact is given soon after establishing the parametric representation of a multiple connected sum below. The parametric representation of the shift of a meridional m-cut of a torus link is unambiguous. The parametric representation of $L_1\sharp_m L_2$ is generated by combining the parametric representations of the shifts of the two torus links $L_i, i = 1, 2$ involved in it together with m the size of the m-cuts. This parametric representation of a multiple connected sum is also unambiguous. Henceforth, in the parametric representations of the multiple connected sums, shifts of meridional m-cuts will be used and we will refrain from inserting the over line over the meridional parameters for the sake of brevity. There should not be any confusions on the part of the reader as we stick to the conventions set above unless otherwise stated. A parametric representation of an oriented $L_1\sharp_m L_2$ involves nine parameters, first four pertaining to the link $L(p_1, q_1)$, next four pertaining to the link $L(p_2, q_2)$ and the last one pertains to the cut. The last parameter is equal to half the number of strands running across the waist handle of the double torus in which $L_1\sharp_m L_2$ is embedded. In the parametric representation for a multiple connected sum, there can be at most seven non-zero parameters because the m-cuts when shifted on both the torus links involved in a multiple connected sum leave the longitudinal strands unaltered. A multiple connected sum having at most five non-zero parameters in the parametric representation is called a *standard* multiple connected sum.

A multiple connected sum $L_1\sharp_m L_2$ of any two m-cut positive torus links $L_1 = L(p_1, q_1)$ and $L_2 = L(p_2, q_2)$ has one of the following type:

1. $[(p_1, 0), (q_1, 0); (p_2, 0), (q_2, 0); m]$ when the m-cuts on $L(p_1, q_1)$ and $L(p_2, q_2)$ are compatible and $m \leq min\{q_1, q_2\}$.
2. $[(p_1, 0), (m, m - q_1); (p_2, 0), (q_2, 0); m]$ when the m-cuts on $L(p_1, q_1)$ and $L(p_2, q_2)$ are compatible and $q_1 < m \leq q_2$.
3. $[(p_1, 0), (q_1, 0); (p_2, 0), (m, m - q_2); m]$ when the m-cuts on $L(p_1, q_1)$ and $L(p_2, q_2)$ are compatible and $q_2 < m \leq q_1$.

4. $[(p_1,0),(m+q_1,m);(p_2,0),(q_2,0);m]$ when the m-cut on $L(p_1,q_1)$ is non-compatible, the m-cut on $L(p_2,q_2)$ is compatible and $m \leq q_2$.

5. $[(p_1,0),(m+q_1,m);(p_2,0),(m,m-q_2);m]$ when the m-cut on $L(p_1,q_1)$ is non-compatible, the m-cut on $L(p_2,q_2)$ is compatible and $m > q_2$.

6. $[(p_1,0),(m,m-q_1);(p_2,0),(m+q_2,m);m]$ when the m-cut on $L(p_1,q_1)$ is compatible, the m-cut on $L(p_2,q_2)$ is non-compatible and $m > q_1$.

7. $[(p_1,0),(q_1,0);(p_2,0),(m+q_2,m);m]$ when the m-cut on $L(p_1,q_1)$ is compatible, the m-cut on $L(p_2,q_2)$ is non-compatible and $m \leq q_1$.

8. $[(p_1,0),(m,m-q_1);(p_2,0),(m,m-q_2);m]$ when the m-cuts on $L(p_1,q_1)$ and $L(p_2,q_2)$ are compatible and $m > max\{q_1,q_2\}$.

9. $[(p_1,0),(m+q_1,m);(p_2,0),(m+q_2,m);m]$ when the m-cuts on $L(p_1,q_1)$ and $L(p_2,q_2)$ are non-compatible.

An m-connected sum $L_1\sharp_m L_2$ may not be isotopic to an m-connected sum $L'_1\sharp_m L_2$, where L_1 and L'_1 are same links but one of them is m-cut compatibly and the other is m-cut non-compatibly and whereas L_2 is m-cut of the same type in both the multiple connected sums. This fact can be realized by computing the number of components in the two resulting multiple connected sums. For example consider $[(5,0),\overline{(7,3)};(4,0),\overline{(7,4)};7]$ and $[(5,0),\overline{(7,3)};(4,0),\overline{(10,7)};7]$. In the former multiple connected sum the torus links $(5,4)$ and $(4,3)$ are both 7-cut compatibly, while in the latter the torus link $(5,4)$ is 7-cut compatibly and the torus link $(4,3)$ is 7-cut non-compatibly. The two resulting multiple connected sums are not isotopic as the number of components in first multiple connected sum is 5 while that of the second is 1. Following diagrams of distinct multiple connected sums are represented by unambiguous parametric representations.

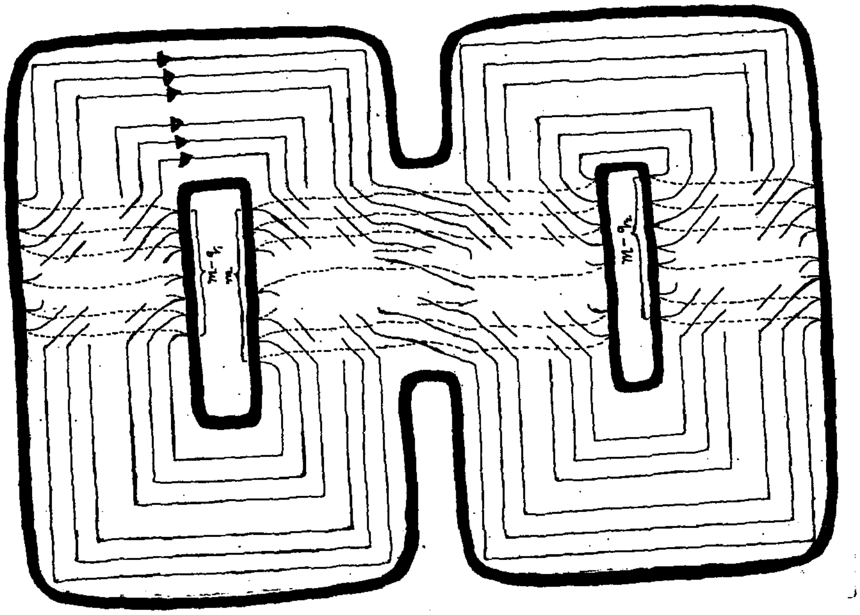


Figure 3.5 (a) $[(p_1, 0), (m, m - q_1); (p_2, 0), (m, m - q_2); m]$.

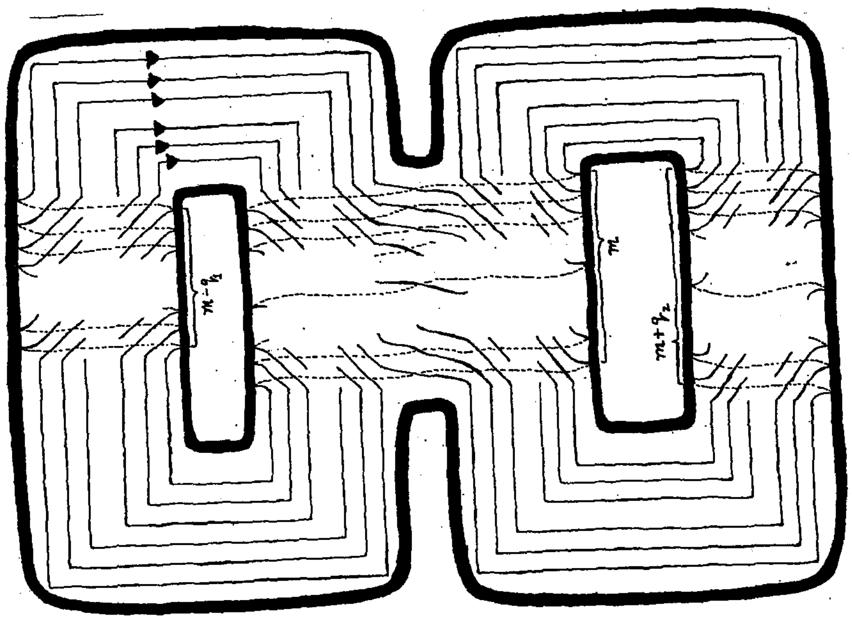


Figure 3.5 (b) $[(p_1, 0), (m, m - q_1); (p_2, 0), (m, m + q_2); m]$.

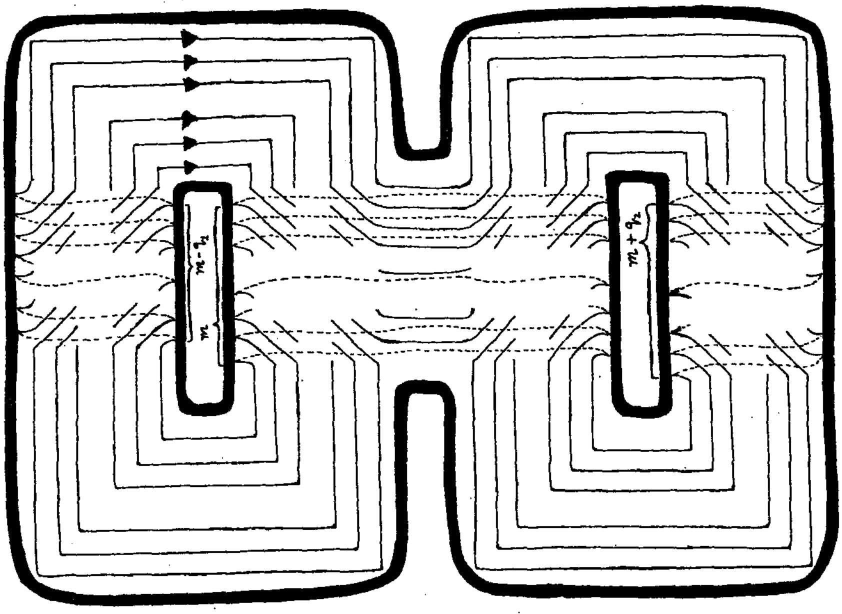


Figure 3.5 (c) $[(p_1, 0), (m, m - q_1); (p_2, 0), (m - q_2, m); m]$.

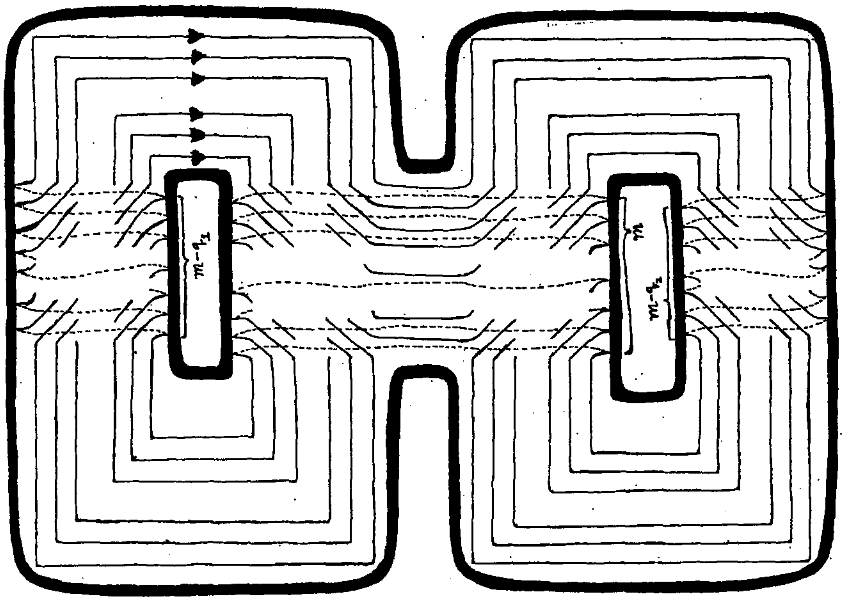


Figure 3.5 (d) $[(p_1, 0), (m, m - q_1); (p_2, 0), (m + q_2, m); m]$.

3.1.2 Twist Transformations

We state below the *transformations* of parametric representations of multiple connected sums when the twists $l_1^{\pm 1}, l_2^{\pm 1}, m_1^{\pm 1}, m_2^{\pm 1}$ and $\gamma^{\pm 1}$ are applied to them under specific conditions on the parameters. The conditions stated in the transformations suffice for our purpose of developing an algorithm to generate the mapping class element associated with a multiple connected sum. We prove some of these twist transformations below and the rest can be proved in an analogous way.

1. $m_1([(p_1, 0), (q_1, 0); (p_2, 0), (q_2, 0); m])$
   $= [(p_1, 0), (q_1 - p_1, 0); (p_2, 0), (q_2, 0); m]$ when $q_1 - m \geq p_1$.

Proof:

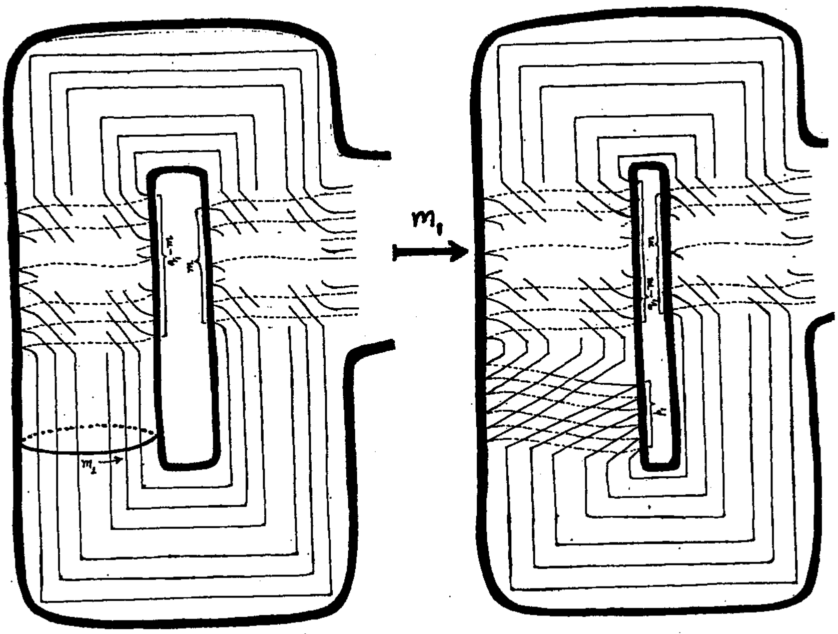


Figure 3.6 m_1 twist on $[(p_1, 0), (q_1, 0); (p_2, 0), (q_2, 0); m]$, $q_1 - m \geq p_1$.

(See figure 3.6) If $q_1 - m \geq p_1$, then apply the m_1 twist to the double torus embedding $[(p_1, 0), (q_1, 0); (p_2, 0), (q_2, 0); m]$ to obtain $[(p_1, 0), (q_1, p_1); (p_2, 0), (q_2, 0); m]$. Then, we get rid of $2p_1$ meridional strands by isotoping them in the direction of orientation of the canonical curve m_1 to a part of the canonical curve l_1 . The remaining $q_1 - p_1$ meridional strands are left unaltered. This results in $[(p_1, 0), (q_1 - p_1, 0); (p_2, 0), (q_2, 0); m]$. □

2. $m_1^{-1}([(p_1, 0), (0, q_1); (p_2, 0), (0, q_2); m])$
   $= [(p_1, 0), (0, q_1 - p_1); (p_2, 0), (0, q_2); m]$ when $q_1 - m \geq p_1$.

3. $m_2([(p_1, 0), (q_1, 0); (p_2, 0), (q_2, 0); m])$
   $= [(p_1, 0), (q_1, 0); (p_2, 0), (q_2 - p_2, 0); m]$ when $q_2 - m \geq p_2$.

4. $m_2^{-1}([(p_1, 0), (0, q_1); (p_2, 0), (0, q_2); m])$
   $= [(p_1, 0), (0, q_1); (p_2, 0), (0, q_2 - p_2); m]$ when $q_2 - m \geq p_2$.

5. $m_1^{-1}([(p_1, 0), (m, m - q_1); (p_2, 0), (q_2, 0); m])$
   $$= \begin{cases} [(p_1, 0), (m, m - q_1 - p_1); (p_2, 0), (q_2, 0); m] & \text{if } m - q_1 \geq p_1 \\ [(p_1, 0), (p_1 + q_1, 0); (p_2, 0), (q_2, 0); m] & \text{if } m - q_1 \leq p_1 \end{cases}$$

6. $m_1([(p_1, 0), (m - q_1, m); (p_2, 0), (q_2, 0); m])$
   $$= \begin{cases} [(p_1, 0), (m - q_1 - p_1, m); (p_2, 0), (q_2, 0); m] & \text{if } m - q_1 \geq p_1 \\ [(p_1, 0), (0, p_1 + q_1); (p_2, 0), (q_2, 0); m] & \text{if } m - q_1 \leq p_1 \end{cases}$$

7. $m_1^{-1}([(p_1, 0), (m, m + q_1); (p_2, 0), (q_2, 0); m])$
   $$= \begin{cases} [(p_1, 0), (m, m + q_1 - p_1); (p_2, 0), (q_2, 0); m] & \text{if } m + q_1 \geq p_1 \\ [(p_1, 0), (p_1 - q_1, 0); (p_2, 0), (q_2, 0); m] & \text{if } m + q_1 \leq p_1 \end{cases}$$

8. $m_1([(p_1, 0), (m + q_1, m); (p_2, 0), (q_2, 0); m])$
   $$= \begin{cases} [(p_1, 0), (m + q_1 - p_1, m); (p_2, 0), (q_2, 0); m] & \text{if } m + q_1 \geq p_1 \\ [(p_1, 0), (0, p_1 - q_1); (p_2, 0), (q_2, 0); m] & \text{if } m + q_1 \leq p_1 \end{cases}$$

9. $m_2^{-1}([(p_1,0),(q_1,0);(p_2,0),(m,m-q_2);m])$
$$= \begin{cases} [(p_1,0),(q_1,0);(p_2,0),(m,m-q_2-p_2);m] & \text{if } m-q_2 \geq p_2 \\ [(p_1,0),(q_1,0);(p_2,0),(p_2+q_2,0);m] & \text{if } m-q_2 \leq p_2 \end{cases}$$

10. $m_2([(p_1,0),(q_1,0);(p_2,0),(m-q_2,m);m])$
$$= \begin{cases} [(p_1,0),(q_1,0);(p_2,0),(m-q_2-p_2,m);m] & \text{if } m-q_2 \geq p_2 \\ [(p_1,0),(q_1,0);(p_2,0),(0,p_2+q_2);m] & \text{if } m-q_2 \leq p_2 \end{cases}$$

11. $m_2^{-1}([(p_1,0),(q_1,0);(p_2,0),(m,m+q_2);m])$
$$= \begin{cases} [(p_1,0),(q_1,0);(p_2,0),(m,m+q_2-p_2);m] & \text{if } m+q_2 \geq p_2 \\ [(p_1,0),(q_1,0);(p_2,0),(p_2-q_2,0);m] & \text{if } m+q_2 \leq p_2 \end{cases}$$

12. $m_2([(p_1,0),(q_1,0);(p_2,0),(m+q_2,m);m])$
$$= \begin{cases} [(p_1,0),(q_1,0);(p_2,0),(m+q_2-p_2,m);m] & \text{if } m+q_2 \geq p_2 \\ [(p_1,0),(q_1,0);(p_2,0),(0,p_2-q_2);m] & \text{if } m+q_2 \leq p_2 \end{cases}$$

13. $l_1([(p_1,0),(q_1,0);(p_2,0),(q_2,0);m])$
$$= \begin{cases} [(p_1-q_1,0),(q_1,0);(p_2,0),(q_2,0);m] & \text{if } p_1 \geq q_1 \\ [(0,q_1-p_1),(q_1,0);(p_2,0),(q_2,0);m] & \text{if } p_1 \leq q_1 \end{cases}$$

Proof: Case (1) $p_1 \geq q_1$

(See figure 3.7 (a)) If $p_1 \geq q_1$, apply the l_1 twist to $[(p_1,0),(q_1,0);(p_2,0),(q_2,0);m]$ to obtain $[(p_1,q_1),(q_1,0);(p_2,0),(q_2,0);m]$. Then, we get rid of $2q_1$ longitudinal strands by isotoping them in the direction of orientation of the canonical curve l_1 to a part of the meridional curve m_1. The remaining p_1-q_1 longitudinal strands are left unaltered. This results in $[(p_1-q_1,0),(q_1,0);(p_2,0),(q_2,0);m]$.

Case(2) $p_1 \leq q_1$

(See figure 3.7(b)) If $p_1 \leq q_1$, apply the l_1 twist to the double torus embedding $[(p_1,0),(q_1,0);(p_2,0),(q_2,0);m]$ to obtain $[(p_1,q_1),(q_1,0);(p_2,0),(q_2,0);m]$. Then, we get rid of $2p_1$ longitudinal strands by isotoping them in the direction of orientation of the canonical curve l_1 to a part of the meridional curve m_1. The remaining q_1-p_1 longitudinal strands are left unaltered. This results in $[(0,p_1-q_1),(q_1,0);(p_2,0),(q_2,0);m]$.

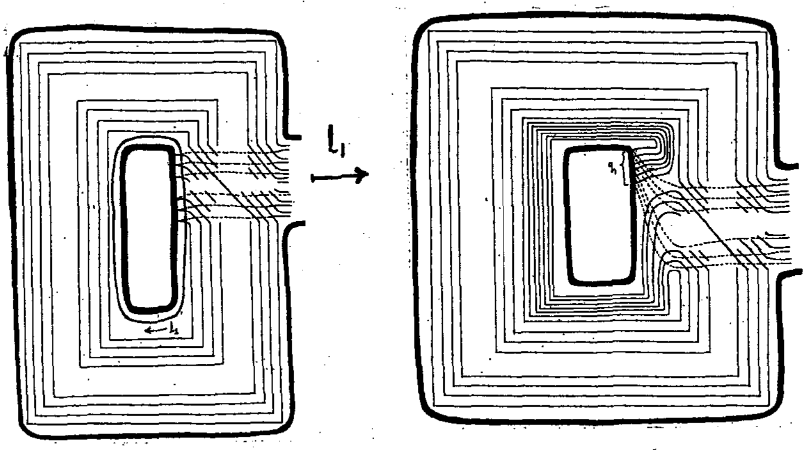


Figure 3.7(a) l_1 twist on $[(p_1,0),(q_1,0);(p_2,0),(q_2,0);m]$, $p_1 \geq q_1$.

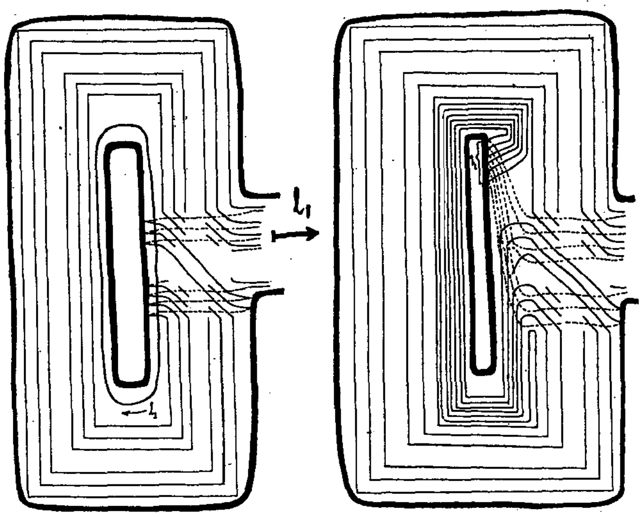


Figure 3.7(b) l_1 twist on $[(p_1,0),(q_1,0);(p_2,0),(q_2,0);m]$, $p_1 \leq q_1$.

□

14. $l_1^{-1}([(p_1,0),(0,q_1);(p_2,0),(0,q_2);m])$
$$= \begin{cases} [(p_1-q_1,0),(0,q_1);(p_2,0),(0,q_2);m] & \text{if } p_1 \geq q_1 \\ [(0,q_1-p_1),(0,q_1);(p_2,0),(0,q_2);m] & \text{if } p_1 \leq q_1 \end{cases}$$

15. $l_2([(p_1,0),(q_1,0);(p_2,0),(q_2,0);m])$
$$= \begin{cases} [(p_1,0),(q_1,0);(p_2-q_2,0),(q_2,0);m] & \text{if } p_2 \geq q_2 \\ [(p_1,0),(q_1,0);(0,q_2-p_2),(q_2,0);m] & \text{if } p_2 \leq q_2 \end{cases}$$

16. $l_2^{-1}([(p_1,0),(0,q_1);(p_2,0),(0,q_2);m])$
$$= \begin{cases} [(p_1,0),(0,q_1);(p_2-q_2,0),(0,q_2);m] & \text{if } p_2 \geq q_2 \\ [(p_1,0),(0,q_1);(0,q_2-p_2),(0,q_2);m] & \text{if } p_2 \leq q_2 \end{cases}$$

17. $\gamma([(p_1,0),(q_1,0);(p_2,0),(q_2,0);m])$
$$= [(p_1,0),(q_1-m,p_1+p_2-m);(p_2,0),(q_2-m,p_1+p_2-m);p_1+p_2-m]$$
$$\text{if } m \leq min\{p_1+p_2,q_1,q_2\}.$$

Proof Case (1) $m \geq p_1$.

(See figure 3.8(a)) If $m \leq min\{p_1+p_2,q_1,q_2\}$ and $m \geq p_1$, then apply the γ twist to the double torus embedding $[(p_1,0),(q_1,0);(p_2,0),(q_2,0);m]$ to obtain $[(p_1,0),(q_1,p_1+p_2);(p_2,0),(q_2,p_1+p_2);m+p_1+p_2]$. Next, we isotope p_1+p_2 strands over the waist handle in the direction of orientation of the canonical curve γ of the double torus towards the inner side of the right hand torus to arrive at $[(p_1,0),(q_1,p_1+p_2);(p_2,0),(q_2,p_1+p_2);m+p_2]$. Further, m strands of these p_1+p_2 strands are isotoped from below the waist handle of the torus again in the direction of orientation of the canonical curve γ of the double torus towards the inner side of the left hand torus to arrive at $[(p_1,0),(q_1-m,p_1+p_2-m);(p_2,0),(q_2-m,p_1+p_2-m);p_2]$. Finally, $m-p_1$ strands of these m strands are isotoped towards the right hand torus from over the waist handle in the direction of the orientation of the canonical curve γ of the double torus to arrive at $[(p_1,0),(q_1-m,p_1+p_2-m);(p_2,0),(q_2-m,p_1+p_2-m);p_1+p_2-m]$.

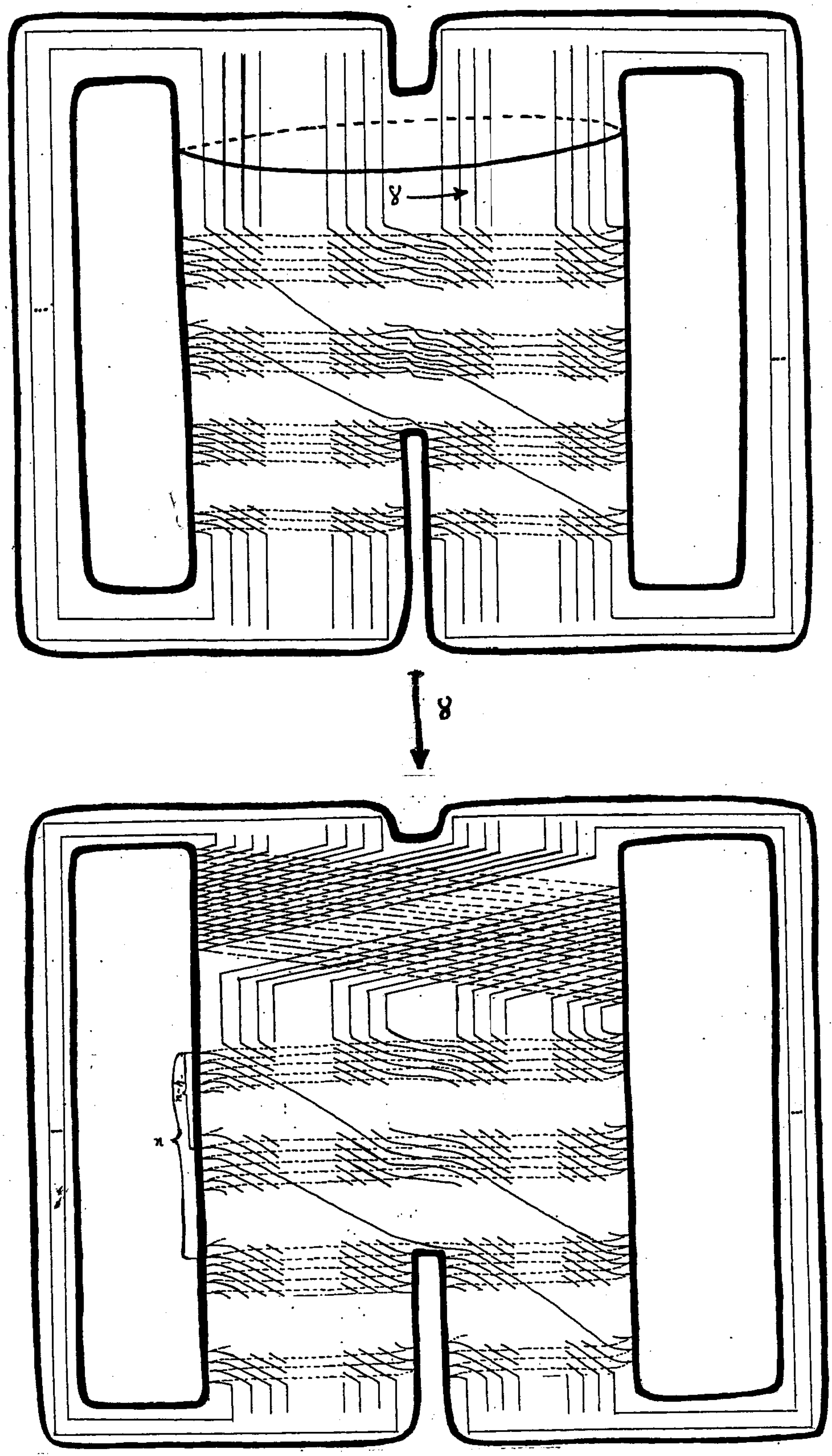


Figure 3.8(a) γ twist on $[(p_1, 0), (q_1, 0); (p_2, 0), (q_2, 0); m]$, $m \leq min\{p_1 + p_2, q_1, q_2\}$.

Case (2) $m \leq p_1$.

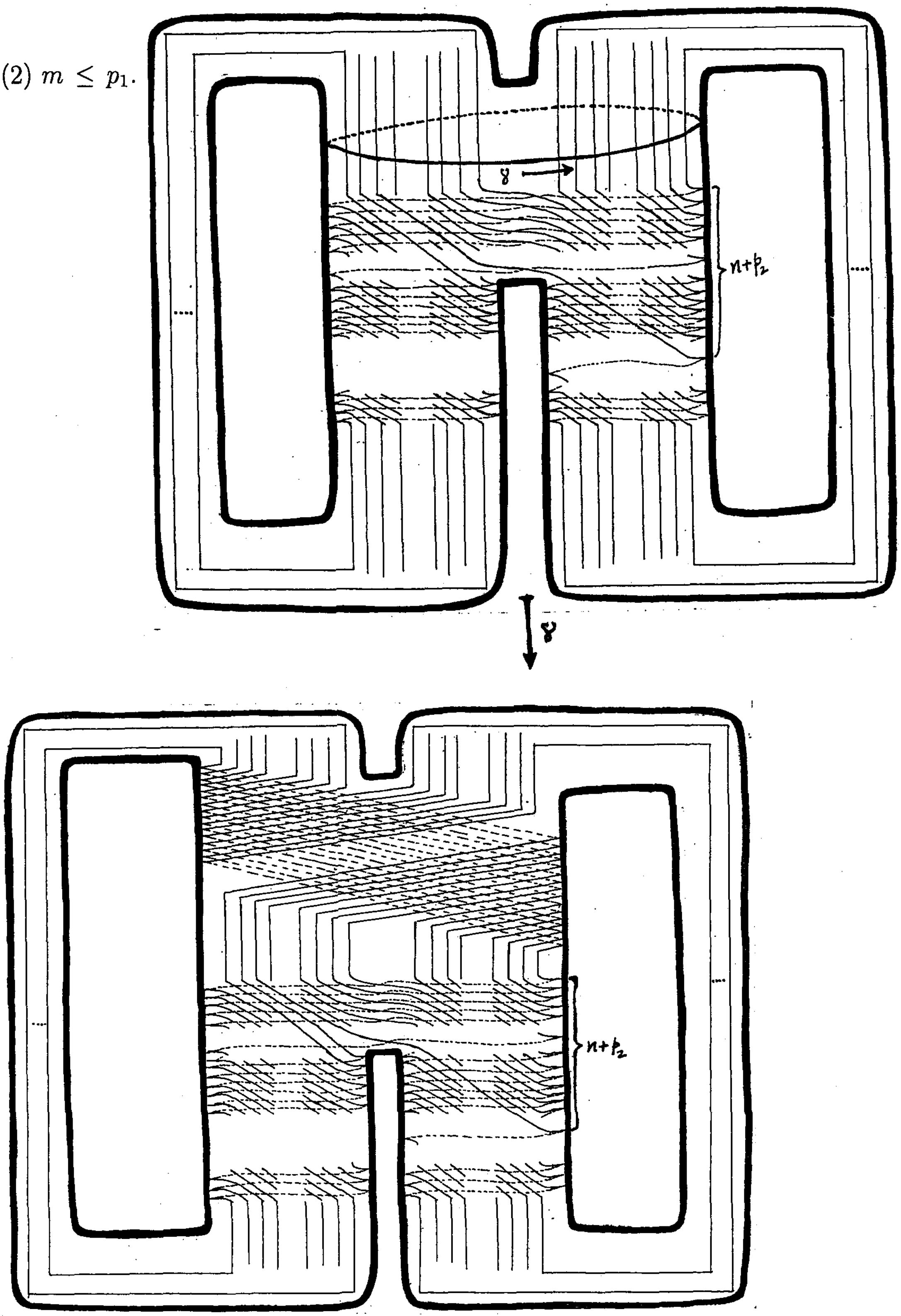


Figure 3.8(b) γ twist on $[(p_1, 0), (q_1, 0); (p_2, 0), (q_2, 0); m]$, $m \leq min\{p_1 + p_2, q_1, q_2\}$.

(See figure 3.8(b)) If $m \leq min\{p_1+p_2, q_1, q_2\}$ and $m \leq p_1$, apply the γ twist to the double torus embedding $[(p_1,0),(q_1,0);(p_2,0),(q_2,0);m]$ to arrive at $[(p_1,0),(q_1,p_1+p_2);(p_2,0),(q_2,p_1+p_2);m+p_1+p_2]$. Then $m+p_2$ strands are isotoped over the waist handle in the direction of orientation of the canonical curve γ of the double torus towards the inner side of the right hand torus to arrive at $[(p_1,0),(q_1,p_1+p_2);(p_2,0),(q_2,p_1+p_2);p_1+p_2]$. Finally, m strands of these $m+p_2$ strands are isotoped from below the waist handle of the torus again in the direction of orientation of the canonical curve γ of the double torus towards the inner side of the left hand torus to arrive at $[(p_1,0),(q_1-m,p_1+p_2-m);(p_2,0),(q_2-m,p_1+p_2-m);p_1+p_2-m]$.

□

18. $\gamma^{-1}([(p_1,0),(0,q_1);(p_2,0),(0,q_2);m])$

$$= [(p_1,0),(p_1+p_2-m,q_1-m);(p_2,0),(p_1+p_2-m,q_2-m);p_1+p_2-m]$$

$$\text{if } m \leq min\{p_1+p_2,q_1,q_2\}.$$

19. $\gamma([(p_1,0),(q_1,0);(p_2,0),(q_2,0);m])$

$$= [(p_1,0),(q_1-(p_1+p_2),0);(p_2,0),(q_2-(p_1+p_2),0);m-(p_1+p_2)]$$

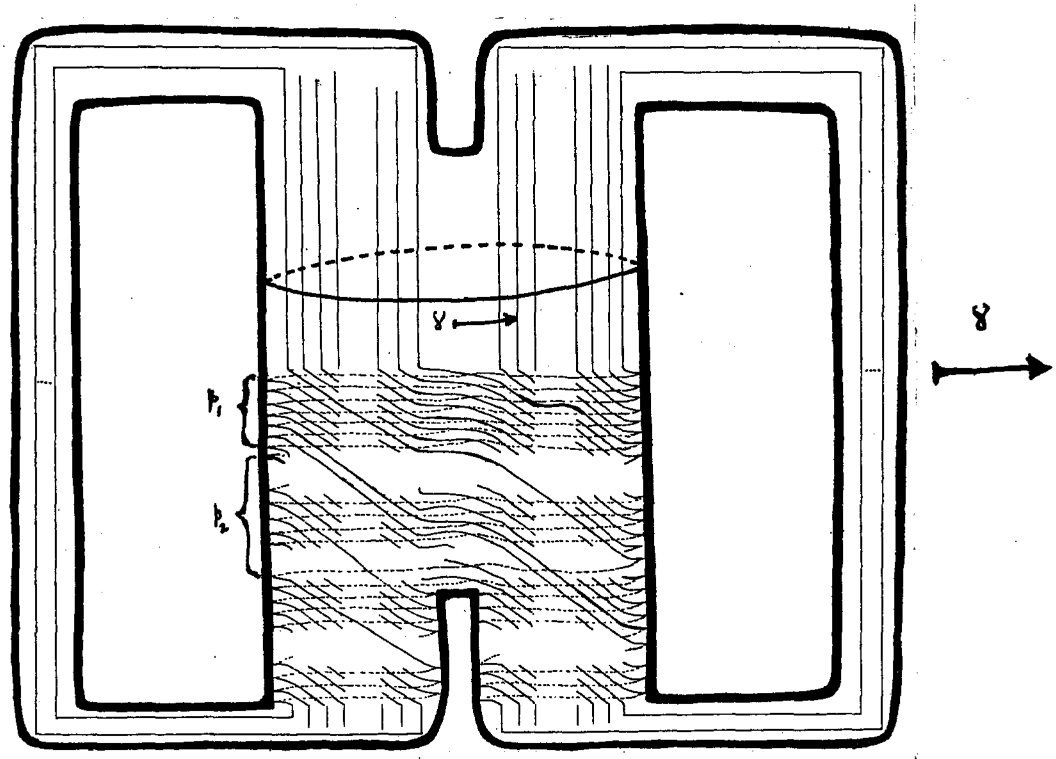
$$\text{if } p_1+p_2 \leq m \leq min\{q_1,q_2\}.$$

Proof: (See figure 3.9) If $p_1+p_2 \leq m \leq min\{q_1,q_2\}$, apply the γ twist to $[(p_1,0),(q_1,0);(p_2,0),(q_2,0);m]$ to arrive at $[(p_1,0),(q_1,(p_1+p_2));(p_2,0),(q_2,(p_1+p_2));m+p_1+p_2]$.

Next, isotope p_1+p_2 strands over the waist handle in the direction of orientation of the canonical curve γ of the double torus towards the inner side of the right hand torus to arrive at $[(p_1,0),(q_1,(p_1+p_2));(p_2,0),(q_2,(p_1+p_2));m+p_2]$.

Further, isotope these p_1+p_2 strands from below the waist handle of the torus again in the direction of orientation of the canonical curve γ of the double torus towards the inner side of the left hand torus to arrive at $[(p_1,0),(q_1-(p_1+p_2),0);(p_2,0),(q_2-(p_1+p_2),0);m-p_1]$.

Finally, deform p_2 strands of these p_1+p_2 strands towards the right hand torus from over the waist handle in the direction of the orientation of the canonical curve γ of the double torus to arrive at $[(p_1,0),(q_1-(p_1+p_2),0);(p_2,0),(q_2-(p_1+p_2),0);m-(p_1+p_2)]$.



See the next page for the transformed link

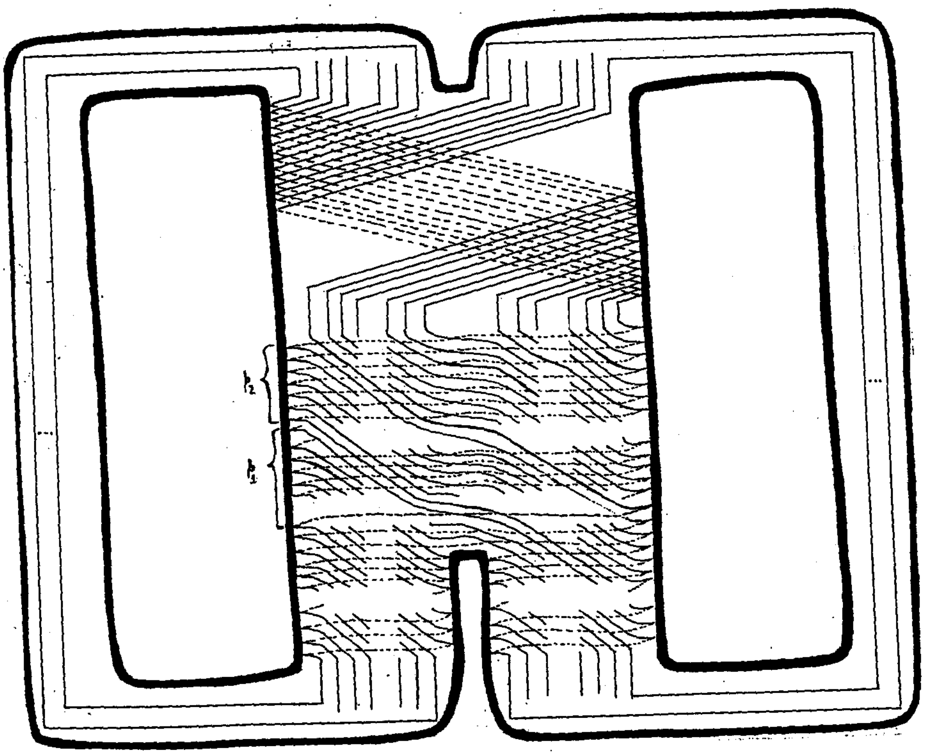


Figure 3.9 γ twist on $[(p_1,0),(q_1,0);(p_2,0),(q_2,0);m]$, $p_1+p_2 \leq m \leq min\{q_1,q_2\}$

□

20. $\gamma^{-1}([(p_1,0),(0,q_1);(p_2,0),(0,q_2);m])$

$$= [(p_1,0),(0,q_1-(p_1+p_2));(p_2,0),(0,q_2-(p_1+p_2));m-(p_1+p_2)]$$

$$\text{if } p_1+p_2 \leq m \leq min\{q_1,q_2\}.$$

Stated below are the transformations of the parametric representations of the multiple connected sums in which both torus links involved are m-cut longitudinally, when the twists $l_1^{\pm1}$, $l_2^{\pm1}$, $m_1^{\pm1}$, $m_2^{\pm1}$, and $\delta^{\pm1}$ are applied to them under specific conditions on the parameters. These transformations suffice for the purpose of developing an algorithm to generate the mapping class element associated with a multiple connected sum. To indicate the shifted longitudinal m-cuts in this case, wide hats are inserted over the longitudinal parameters in the parametric representation of the multiple connected sums. These are not proved since they are similar to the ones above and are not used in our algorithm. These are stated for the sake of those who may wish to explore for an algorithm using them, similar to the one given above. These come handy if one wants to work with multiple connected sums involving shifts of longitudinal cuts.

1. $l_1([\widehat{(p_1,0)},(q_1,0);\widehat{(p_2,0)},(q_1,0);m])$
   $= [(\widehat{p_1-q_1},0),(q_1,0);(\widehat{(p_2,0)},(q_2,0);m]$ if $p_1 - m \geq q_1$.

2. $l_1^{-1}([\widehat{(p_1,0)},(0,q_1),;\widehat{(p_2,0)},(0,q_2);m])$
   $= [(\widehat{p_1-q_1},0),(0,q_1);\widehat{(p_2,0)},(0,q_2);m]$ if $p_1 - m \geq q_1$.

3. $l_2([\widehat{(p_1,0)},(q_1,0);\widehat{(p_2,0)},(q_2,0);m])$
   $= [\widehat{(p_1,0)},(q_1,0);(\widehat{p_2-q_2},0),(q_2,0);m]$ if $p_2 - m \geq q_2$.

4. $l_2^{-1}([\widehat{(p_1,0)},(0,q_1);\widehat{(p_2,0)},(0,q_2);m])$
   $= [\widehat{(p_1,0)},(0,q_1);(\widehat{p_2-q_2},0),(0,q_2);m]$ if $p_2 - m \geq q_2$.

5. $l_1^{-1}([(m,\widehat{m-p_1}),(q_1,0);\widehat{(p_2,0)},(q_2,0);m])$
   $$= \begin{cases} [(m,\widehat{m-p_1}-q_1),(q_1,0);\widehat{(p_2,0)},(q_2,0);m] & \text{if } m-p_1 \geq q_1 \\ [(\widehat{p_1+q_1},0),(q_1,0);\widehat{(p_2,0)},(q_2,0);m] & \text{if } m-p_1 \leq q_1 \end{cases}$$

6. $l_1([\widehat{(m-p_1,m)},(q,0);\widehat{(p_2,0)},(q_2,0);m])$
   $$= \begin{cases} [(\widehat{m-p_1-q_1,m}),(q,0);\widehat{(p_2,0)},(q_2,0);m] & \text{if } m-p_1 \geq q_1 \\ [(0,\widehat{p_1+q_1}),(q_1,0);\widehat{(p_2,0)},(q_2,0);m] & \text{if } m-p_1 \leq q_1 \end{cases}$$

7. $l_1^{-1}([(m,\widehat{m+p_1}),(q_1,0);\widehat{(p_2,0)},(q_2,0);m])$
   $$= \begin{cases} [(m,\widehat{m+p_1}-q_1),(q_1,0);\widehat{(p_2,0)},(q_2,0);m] & \text{if } m+p_1 \geq q_1 \\ [(\widehat{q_1-p_1},0),(q_1,0);\widehat{(p_2,0)},(q_2,0);m] & \text{if } m+p_1 \leq q_1 \end{cases}$$

8. $l_1([\widehat{(m+p_1,m)},(q_1,0);\widehat{(p_2,0)},(q_2,0);m])$
$$= \begin{cases} [\widehat{(m+p_1-q_1,m)},(q_1,0);\widehat{(p_2,0)},(q_2,0);m] & \text{if } m+p_1 \geq q_1 \\ [\widehat{(0,q_1-p_1)},(q_1,0);\widehat{(p_2,0)},(q_2,0);m] & \text{if } m+p_1 \leq q_1 \end{cases}$$

9. $l_2^{-1}([\widehat{(p_1,0)},(q_1,0);\widehat{(m,m-p_2)},(q_2,0);m])$
$$= \begin{cases} [\widehat{(p_1,0)},(q_1,0);\widehat{(m,m-p_2-q_2)},(q_2,0);m] & \text{if } m-p_2 \geq q_2 \\ [\widehat{(p_1,0)},(q_1,0);\widehat{(p_2+q_2,0)},(q_2,0);m] & \text{if } m-p_2 \leq q_2 \end{cases}$$

10. $l_2([\widehat{(p_1,0)},(q_1,0);\widehat{(m-p_2,m)},(q_2,0);m])$
$$= \begin{cases} [\widehat{(p_1,0)},(q_1,0);\widehat{(m-p_2-q_2,m)},(q_2,0);m] & \text{if } m-p_2 \geq q_2 \\ [\widehat{(p_1,0)},(q_1,0);\widehat{(0,p_2+q_2)},(q_2,0);m] & \text{if } m-p_2 \leq q_2 \end{cases}$$

11. $l_2^{-1}([\widehat{(p_1,0)},(q_1,0);\widehat{(m,m+p_2)},(q_2,0);m])$
$$= \begin{cases} [\widehat{(p_1,0)},(q_1,0);\widehat{(m,m+p_2-q_2)},(q_2,0);m] & \text{if } m+p_2 \geq q_2 \\ [\widehat{(p_1,0)},(q_1,0);\widehat{(q_2-p_2,0)},(q_2,0);m] & \text{if } m+p_2 \leq q_2 \end{cases}$$

12. $l_2([\widehat{(p_1,0)},(q_1,0);\widehat{(m+p_2,m)},(q_2,0);m])$
$$= \begin{cases} [\widehat{(p_1,0)},(q_1,0);\widehat{(m+p_2-q_2,m)},(q_2,0);m] & \text{if } m+p_2 \geq q_2 \\ [\widehat{(p_1,0)},(q_1,0);\widehat{(0,q_2-p_2)};m] & \text{if } m+p_2 \leq q_2 \end{cases}$$

13. $m_1([\widehat{(p_1,0)},(q_1,0);\widehat{(p_2,0)},(q_2,0);m])$
$$= \begin{cases} [\widehat{(p_1,0)},(q_1-p_1,0);\widehat{(p_2,0)},(q_2,0);m] & \text{if } q_1 \geq p_1 \\ [\widehat{(p_1,0)},(0,p_1-q_1);\widehat{(p_2,0)},(q_2,0);m] & \text{if } q_1 \leq p_1 \end{cases}$$

14. $m_1^{-1}([\widehat{(p_1,0)},(0,q_1);\widehat{(p_2,0)},(0,q_2);m])$
$$= \begin{cases} [(\widehat{(p_1,0)},(0,q_1-p_1);\widehat{(p_2,0)},(0,q_2);m] & \text{if } q_1 \geq p_1 \\ [\widehat{(p_1,0)},(p_1-q_1,0);\widehat{(p_2,0)},(0,q_2);m] & \text{if } q_1 \leq p_1 \end{cases}$$

15. $m_2([\widehat{(p_1,0)},(q_1,0);\widehat{(p_2,0)},(q_2,0);m])$
$$= \begin{cases} [\widehat{(p_1,0)},(q_1,0);\widehat{(p_2,0)},(q_2-p_2,0);m] & \text{if } q_2 \geq p_2 \\ [\widehat{(p_1,0)},(q_1,0);\widehat{(p_2,0)},(0,p_2-q_2);m] & \text{if } q_2 \leq p_2 \end{cases}$$

16. $m_2^{-1}([\widehat{(p_1,0)},(0,q_1);\widehat{(p_2,0)},(0,q_2);m])$
$$= \begin{cases} [\widehat{(p_1,0)},(0,q_1);\widehat{(p_2,0)},(0,q_2-p_2);m] & \text{if } q_2 \geq p_2 \\ [\widehat{(p_1,0)},(0,q_1);\widehat{(p_2,0)},(p_2-q_2,0);m] & \text{if } q_2 \leq p_2 \end{cases}$$

17. $\delta([\widehat{(p_1,0)},(q_1,0);\widehat{(p_2,0)},(q_2,0);m])$
$= [(p_1-\widehat{m,q_1+q_2}-m),(q_1,0);(p_2-\widehat{m,q_1+q_2}-m),(q_2,0);q_1+q_2-m]$

if $m \leq min\{q_1+q_2,p_1,p_2\}$.

18. $\delta^{-1}([\widehat{(p_1,0)},(0,q_1);\widehat{(p_2,0)},(0,q_2);m])$
$= [(p_1-\widehat{m,q_1+q_2}-m),(0,q_1);(p_2-\widehat{m,q_1+q_2}-m),(0,q_2);q_1+q_2-m]$

if $m \leq min\{q_1+q_2,p_1,p_2\}$.

19. $\delta([\widehat{(p_1,0)},(q_1,0);\widehat{(p_2,0)},(q_2,0);m])$
$= [(p_1-\widehat{(q_1+q_2)},0),(q_1,0);(p_2-\widehat{(q_1+q_2)},0),(q_2,0);m-(q_1+q_2)]$

if $q_1+q_2 \leq m \leq min\{p_1,p_2\}$.

20. $\delta^{-1}([\widehat{(p_1,0)},(0,q_1);\widehat{(p_2,0)},(0,q_2);m])$
$= [(p_1-\widehat{(q_1+q_2)},0),(0,q_1);(p_2-\widehat{(q_1+q_2)},0),(0,q_2);m-(q_1+q_2)]$ if $q_1+q_2 \leq m \leq min\{p_1,p_2\}$.

3.1.3 The Algorithm

An algorithm to generate a mapping class element f using the twists $l_1^{\pm 1}, l_2^{\pm 1}, m_1^{\pm 1}, m_2^{\pm 1}$ and $\gamma^{\pm 1}$ under specific conditions on the parameters governing $L_1\sharp_m L_2$ so that f^{-1} sends $L_1\sharp_m L_2$ to canonical curves. In the algorithm, the multiple connected sums have both its torus links cut meridionally. The algorithm consists of three phases. For $p_i, q_i, m \in \mathbb{N}$, let $p_i' = -p_i \, mod \, q_i$, $q_i' = [(q_i-m) \, mod \, p_i'] + m$, $p_i'' = -p_i' \, mod \, q_i'$, where $i = 1,2$. For $t \in \mathbb{R}$, $[t] \in \mathbb{N}$ is defined as the unique natural number such that $t-1 < [t] \leq t$. The following three lemmas serve as prerequisites.

Lemma 3.1.1. *If a standard multiple connected sum is one of the following type*

(1) $[(p_1,0),(q_1,0);(p_2,0),(q_2,0);m]$, *or (2)* $[(p_1,0),(0,q_1);(p_2,0),(0,q_2);m]$ *with* $m \leq min\{q_1,q_2\}$, $p_i < q_i$, $q_i - m < p_i$ *and* $p_i \leq p_i'$ *for* $i = 1,2$, *then* $p_1 + p_2 < 2m$.

Proof: Since $q_i - m < p_i$, $i = 1,2$, it follows that $p_1' + p_i' = (q_1+q_2)-(p_1+p_2) < 2m$. But, $p_1 + p_2 \leq p_1' + p_2'$ since $p_i \leq p_i'$ for $i = 1,2$. □

Lemma 3.1.2. *If a standard multiple connected sum is one of the following type (1)* $[(p_1,0),(q_1,0);(p_2,0),(q_2,0);m]$ *or (2)* $[(p_1,0),(0,q_1);(p_2,0),(0,q_2);m]$ *where* $m \leq min\{q_1,q_2\}$ *and its parameters satisfy the conditions* $p_i < q_i, q_i - m < p_i$, $i = 1,2$ *and* $2m \leq p_1 + p_2$, *then there exists* $j \in \{1,2\}$ *such that* $p_j' < p_j$ $q_j - m \geq . p_j'$ *and* $q_j > q_j'$. *Further, if* $p_j' < q_j'$, *then* $p_j > p_j''$.

Proof: Since $2m \leq p_1 + p_2$, there exists $j \in \{1,2\}$ such that $p_j \geq m$. Therefore $q_j = p_j + p_j' \geq m + p_j'$, and $p_j > q_j - m \geq p_j'$.
The inequality $q_j > q_j'$ is immediate.
When $p_j' < q_j'$, we have $p_j + p_j' = q_j > q_j' = p_j' + p_j''$ and therefore $p_j > p_j''$. □

Lemma 3.1.3. *Suppose a standard multiple connected sum is one of the following type* (1)$[(p_1,0),(q_1,0);(p_2,0),(q_2,0);m]$ *or* (2) $[(p_1,0),(0,q_1);(p_2,0),(0,q_2);m]$ *where* $m \leq min\{q_1,q_2\}$, *and its parameters satisfy the conditions* $p_i < q_i, q_i - m < p_i, i = 1,2$, *and* $2m \leq p_1 + p_2$. *Then each can be reduced using twists to another standard multiple connected sum again of one of the two types above such that either two of the original parameters* $\{p_1,q_1\}$ *or* $\{p_2,q_2\}$ *or all the four parameters* $\{p_1,q_1,p_2,q_2\}$ *decreased in the latter standard multiple connected sum, while the rest of the parameters remain unchanged.*

Proof: Since $2m \leq p_1 + p_2$, either (1) $p_1' < p_1$ and $p_2' \geq p_2$, or (2) $p_1' \geq p_1$ and $p_2' < p_2$, or (3) $p_1' < p_1$ and $p_2' < p_2$.

Without loss of generality, assume that the standard multiple connected sum with both torus links are positive.

Case (1) $p_1' < p_1$ and $p_2' \geq p_2$

Let the multiple connected sum $[(p_1,0),(q_1,0);(p_2,0),(q_2,0);m]$ be such that the condition (1) holds. Using formula (13), we get $l_1([(p_1,0),(q_1,0);(p_2,0),(q_2,0);m])$ $= [(0,p_1'),(q_1,0);(p_2,0),(q_2,0);m]$. From lemma 3.1.2, $q_1 - m \geq p_1'$. Applying m_1^{-1} repeatedly using formula (2), we get $m_1^{-s}([(0,p_1'),(q_1,0);(p_2,0),(q_2,0);m]) =$ $[(0,p_1'),(q_1',0);(p_2,0),(q_2,0);m]$ where $s = [(q_1 - m)/p_1']$. Repeatedly using formula (14), we get $l_1^{-t}([(0,p_1'),(q_1',0);(p_2,0),(q_2,0);m]) = [(p_1'',0),(q_1',0);(p_2,0),(q_2,0);m]$ where $t = [p_1'/q_1'] + 1$. Here, we know that the initial parameter q_1 has decreased to q_1' and all initial parameters other than p_1 and q_1 are unchanged. Now, if $p_1'' \geq p_1$, then it follows that $p_1'' + p_2 \geq p_1 + p_2$. Hence, parameters of $[(p_1'',0),(q_1',0);(p_2,0),(q_2,0);m]$ satisfy all the conditions in the hypothesis of the lemma. Therefore, we can similarly reduce this standard multiple connected sum further until the parameter p_1 also decreases after a finite stage.

Case (2) $p_1' \geq p_1$ and $p_2' < p_2$.

The proof in this case is exactly similar to the case (1) and hence is omitted.

Case (3) $p_1' < p_1$ and $p_2' < p_2$.

Let the multiple connected sum $[(p_1,0),(q_1,0);(p_2,0),(q_2,0);m]$ be such that the condition (3) holds. Applying l_1 and l_2 using the formulae (13) and (15) respectively, we get $l_1([(p_1,0),(q_1,0);(p_2,0),(q_2,0);m]) = [(0,p_1'),(q_1,0);(p_2,0),(q_2,0);m]$ and $l_2([(0,p_1'),(q_1,0);(p_2,0),(q_2,0);m]) = [(0,p_1'),(q_1,0);(0,p_2'),(q_2,0);m]$.

From lemma 3.1.2, the inequalities $q_1 - m \geq p_1'$ and $q_2 - m \geq p_2'$ follow. Hence, by applying m_1^{-1} and m_2^{-1} repeatedly using the formulae (2) and (4) respectively, we get

$m_1^{-s}([(0,p_1'),(q_1,0);(0,p_2'),(q_2,0);m]) = [(0,p_1'),(q_1',0);(0,p_2'),(q_2,0);m]$ where $s = [(q_1-m)/p_1']$. $m_2^{-t}([(0,p_1'),(q_1',0);(0,p_2'),(q_2,0);m]) = [(0,p_1'),(q_1',0);(0,p_2'),(q_2',0);m]$, where $t = [(q_2-m)/p_2']$. Here, the initial parameters p_1, q_1, p_2 and q_2 have decreased to p_1', q_1', p_2' and q_2' respectively while the parameter m remains unaltered. This completes the proof of case (3). □

The algorithm is described below in three phases.

Phase (1) In this phase we reduce the link $L_1\sharp_m L_2$ to one of the two standard forms (1) $[(p_1,0),(q_1,0);(p_2,0),(q_2,0);m]$, or (2) $[(p_1,0),(0,q_1);(p_2,0),(0,q_2);m]$.

This can always be done in the following manner. If there are more than five non-zero terms in the parametric representation, then the m-cut on one or both the torus links $L_i(p_i,q_i)$; $i = 1,2$ must be of one of the type (1) $[(p_i,0),(m,m-q_i);m]$, or (2) $[(p_i,0),(m+q_i,m);m]$, or (3) $[(p_i,0),(m-q_i,m);m]$, or (4) $[(p_i,0),(m,m+q_i);m]$.

We describe the method in one of the cases now. Suppose the multiple connected sum is of the form $[(p_1,0),(m,m-q_1);(p_2,0),(m+q_2,m);m]$. Then, the parameters $m-q_1$ and $m+q_2$ can be got rid of by repeatedly applying the twists m_1^{-1} and m_2 using the formulae (5) and (12) respectively to arrive at $[(p_1,0),(m+s,0);(p_2,0),(0,m+t);m]$ where $s = (q_1-m) \, mod \, p_1$ and $t = (-m-q_2) \, mod \, p_2$. Next, apply the twist l_2^{-1} repeatedly using the formula (16) until we arrive at $[(p_1,0),(m+s,0);(p_2,0),(r,0);m]$ where $r = (-m-t) \, mod \, p_2$. Similarly, all other cases can be reduced to one of the standard multiple connected sums mentioned above. This concludes phase (1) of the algorithm.

Phase (2) This phase consists of five steps to be iterated in the order of their occurrence until both of the longitudinal parameters $p_i, i = 1,2$ vanish or the waist parameter m representing the cuts vanishes. If a standard multiple connected sum is of the type $[(p_1, 0), (0, q_1); (p_2, 0), (0, q_2); m]$, then it can be converted into the type $[(p_1, 0), (q_1, 0); (p_2, 0), (q_2, 0); m]$ by repeatedly applying the twists l_1^{-1} and l_2^{-1} as in the formulae (14) and (16) respectively. Hence, without loss of generality, assume that the multiple connected sum is of the type $[(p_1, 0), (q_1, 0); (p_2, 0), (q_2, 0); m]$ where $m \leq min\{q_1, q_2\}$.

Step (1) If $p_j \geq q_j$ for some $j \in \{1, 2\}$, then apply the l_j twist repeatedly until the longitudinal parameter p_j reduces to $p_j\,(mod\, q_j)$, while all other parameters remain unaltered. Here the range of $mod\, q_j$ is considered as $\{0, 1, ..., q_j - 1\}$. For instance, if $p_1 \geq q_1$, then applying the l_1 twist repeatedly using the formula (13) we get $l_1^s([(p_1, 0), (q_1, 0); (p_2, 0), (q_2, 0); m]) = [(p_1\,(mod\, q_1), 0), (q_1, 0); (p_2, 0), (q_2, 0); m]$ where $s = [p_1/q_1]$.

Step (2) If $p_i < q_i$ for $i = 1, 2$, and $q_j - m \geq p_j$ for some $j \in \{1, 2\}$, then apply the twist m_j repeatedly to reduce q_j to $[\{(q_j - m)\, mod\, p_j\} + m]$ while all other parameters remain unaltered. Here the range of $mod\, p_j$ is considered as $\{0, 1, ..., p_j - 1\}$. For instance if $p_i < q_i$, $i = 1, 2$ and $q_1 - m \geq p_1$, then applying the twist m_1 repeatedly using the formula (1) we get $m_1^t([(p_1, 0), (q_1, 0); (p_2, 0), (q_2, 0); m]) = [(p_1, 0), ((q_1 - m)\, mod\, p_1 + m, 0); (p_2, 0), (q_2, 0); m]$ where $t = [(q_1 - m)/p_1]$.

Step (3) If $p_i < q_i$, $q_i - m < p_i$, $i = 1, 2$ and $m \geq p_1 + p_2$, then applying the twist γ to the link using the formula (19), we get $\gamma([(p_1, 0), (q_1, 0); (p_2, 0), (q_2, 0); m]) = [(p_1, 0), (q_1 - (p_1 + p_2), 0); (p_2, 0), (q_2 - (p_1 + p_2), 0); m - (p_1 + p_2)]$. This reduces the waist handle parameter m to $[m - (p_1 + p_2)]$ and both the meridional parameters q_i to $[q_i - (p_1 + p_2)]$ while the two longitudinal parameters p_i, $i = 1, 2$ remain unaltered.

Step (4) If $p_i < q_i$, $q_i - m < p_i$, $i = 1, 2$ and $m < p_1 + p_2 < 2m$, then applying the twist γ to the link using the formula (17), we get $\gamma([(p_1, 0), (q_1, 0); (p_2, 0), (q_2, 0); m]) =$

$[(p_1, 0), (q_1 - m, p_1 + p_2 - m); (p_2, 0), (q_2 - m, p_1 + p_2 - m); p_1 + p_2 - m]$. This reduces the waist handle parameter m to $[(p_1 + p_2) - m]$. The meridional parameters change from $(q_i, 0)$ to $(q_i - m, p_1 + p_2 - m)$ thereby making the resultant multiple connected sum non-standard. To convert this non-standard multiple connected sum to a standard one, we apply the twists m_i, $i = 1, 2$ using the formulae (6) and (10) respectively. Then we get $m_1([(p_1, 0), (q_1 - m, p_1 + p_2 - m); (p_2, 0), (q_2 - m, p_1 + p_2 - m); p_1 + p_2 - m]) = [(p_1, 0), (0, 2p_1 + p_2 - q_1); (p_2, 0), (q_2 - m, p_1 + p_2 - m); (p_1 + p_2) - m]$. $m_2([(p_1, 0), (0, 2p_1 + p_2 - q_1); (p_2, 0), (q_2 - m, p_1 + p_2 - m); (p_1 + p_2) - m]) = [(p_1, 0), (0, 2p_1 + p_2 - q_1); (p_2, 0), (0, p_1 + 2p_2 - q_2); (p_1 + p_2) - m]$. Though the waist handle parameter m reduces as already mentioned above and the longitudinal parameters p_i $i = 1, 2$ remain unaltered, the meridional parameters q_1 and q_2 changed to $2p_1 + p_2 - q_1$ and $p_1 + 2p_2 - q_2$ respectively can be greater than the original parameters.

Step (5) If $p_i < q_i$ and $q_i - m < p_i$ for $i = 1, 2$, and $2m \leq p_1 + p_2$, then (1) $p'_1 < p_1$ and $p_2 \leq p'_2$, or (2) $p_1 \leq p'_1$ and $p'_2 < p_2$, or (3) $p'_1 < p_1$ and $p'_2 < p_2$ (by lemma 3.1.1.).

Case (1) $p'_1 < p_1$ and $p_2 \leq p'_2$.

$l_1([(p_1, 0), (q_1, 0); (p_2, 0), (q_2, 0); m]) = [(0, p'_1), (q_1, 0); (p_2, 0), (q_2, 0); m]$ by the formula (13) reducing p_1 to $p'_1 = q_1 - p_1$. From lemma 3.1.2, it follows that $(q_1 - m) \geq p'_1$. ${m_1}^{-t}([(0, p'_1), (q_1, 0); (p_2, 0), (q_2, 0); m]) = [(0, p'_1), (q'_1, 0); (p_2, 0), (q_2, 0); m]$ where $t = [(q_1 - m)/p'_1]$ by repeated use of the formula (2) reducing q_1 to $(q_1 - m)\ (mod\, p'_1) + m$. Finally, ${l_1}^{-1}([(0, p'_1), (q'_1, 0); (p_2, 0), (q_2, 0); m]) = [(p''_1, 0), (q'_1, 0); (p_2, 0), (q_2, 0); m]$ by the formula (14) to change p'_1 to $-p'_1$ mod q'_1. It is clear that q_1 decreases to q'_1 as asserted in lemma 3.1.2. However, if the inequality $p_1 > p'_1{\prime}$ does not hold, then we have the inequalities $p_1 \leq p''_1$, and $2m \leq p_1 + p_2 \leq p''_1 + p_2$. In this case, continue this process to further reduce q_1. By lemma 3.1.3, this process must terminate after a finite number of iterations and result in a multiple connected sum of the initial type. Note that the original parameters p_1 and q_1 will both be reduced while all the other parameters will remain unaltered in this step.

Case (2) $p_1 \leq p'_1$ and $p'_2 < p_2$

This case is analogous to case (1) and can be dealt in a similar way.

Case (3) $p'_1 < p_1$ and $p'_2 < p_2$.

Note that $l_1([(p_1,0),(q_1,0);(p_2,0),(q_2,0);m]) = [(0,p'_1),(q_1,0);(p_2,0),(q_2,0);m]$ and $l_2([(0,p'_1),(q_1,0);(p_2,0),(q_2,0);m]) = [(0,p'_1),(q_1,0);(0,p'_2),(q_2,0);m]$ by the formulae (13) and (15). This reduces p_i to $p'_i = q_i - p_i$ for $i = 1,2$. By applying the formula (2) to the above, we get ${m_1}^{-s}([(0,p'_1),(q_1,0);(0,p'_2),(q_2,0);m]) = [(0,p'_1),(q'_1,0);(0,p'_2),(q_2,0);m]$ where $s = [(q_1 - m)/p'_1]$ and then applying formula (4), we get ${m_2}^{-t}([(0,p'_1),(q'_1,0);(0,p'_2),(q_2,0);m]) = [(0,p'_1),(q'_1,0);(0,p'_2),(q'_2,0);m]$ where $t = [(q_2 - m)/p'_2]$. This reduces q_i to $q'_i = [(q_i - m) \, mod \, p'_i + m]$ for $i = 1,2$.

$q_i > q'_i$ for $i = 1,2$ by lemma 3.1.2, and $p_i > p'_i$ for $i = 1,2$ by hypothesis. This brings us to the end of phase (2) of the algorithm.

Note that in the phase (2), the parameter m is never allowed to increase and whenever it does not decrease at any step, no other parameter is allowed to increase and moreover at least one other parameter must decrease. These facts guarantee that after a finite number of iterations of the steps in phase (2) in the order given, either the parameter m or both the parameters p_i, $i = 1,2$ must vanish.

Phase (3) If we arrive at a multiple connected sum with $p_i = 0$ for $i = 1,2$ at the end of the phase (2), then this multiple connected sum will be of the type

$[(0,0),(q_1,0);(0,0),(q_2,0);m]$, and will consist of only canonical curves. More precisely, there will be m canonical curves around the waist handle and q_1-m and q_2-m meridional curves around the two handles of the double torus.

If we arrive at a multiple connected sum with $m = 0$ at the end of phase (2), then this multiple connected sum will be of the type (1)$[(p_1,0),(q_1,0);(p_2,0),(q_2,0);0]$, or (2)$[(p_1,0),(0,q_1);(p_2,0),(0,q_2);0]$. In this case, we have two torus links embedded in a double torus. These can both be reduced using $l_i^{\pm 1}$ and $m_i^{\pm 1}$, $i = 1,2$ to arrive at a multiple connected sum of one of the following type.

1. $[(p_1,0),(0,0);(p_2,0),(0,0);0] = [(0,p_1),(0,0);(p_2,0),(0,0);0]$. In this case, there are p_1+p_2 longitudinal curves.

2. $[(p_1,0),(0,0);(0,0),(0,q_2);0] = [(p_1,0),(0,0);(0,0),(q_2,0);0]$. In this case, there are p_1 longitudinal and q_2 meridional curves.

3. $[(0,0),(q_1,0);(p_2,0),(0,0);0] = [(0,0),(q_1,0);(0,p_2),(0,0);0]$. In this case, there are q_1 meridional and p_2 longitudinal curves.

4. $[(0,0),(0,q_1);(0,0),(0,q_2);0] = [(0,0),(0,q_1);(0,0),(q_2,0);0]$. In this case, there are q_1+q_2 meridional strands.

Following is an example that illustrates the application of the algorithm to generate a mapping class element of the double torus associated with a multiple connected sum using the Dehn twists. Consider the reverse multiple connected sum $[(5,0),(78,76);(7,0),(75,78);78]$ of the torus links $L_1((5,0),(2,0))$ and $L_2((7,0),(0,3))$. The Dehn twists can be applied successively to each resulting multiple connected sum until a multiple connected sum having components isotopic to the canonical curves is arrived at as follows.

$m_1^{-16} \circ m_2^{11}([(5,0),(78,76);(7,0),(75,78);78]) = [(5,0),(82,0);(7,0),(0,80);78].$

$l_2^{-1}([(5,0),(82,0);(7,0),(0,80);78]) = [(5,0),(82,0);(73,0),(80,0);78].$

$\gamma([(5,0),(82,0);(73,0),(80,0);78]) = [(5,0),(4,0);(73,0),(2,0);0].$

$m_1 \circ m_2^{36}([(5,0),(4,0);(73,0),(2,0);0]) = [(1,0),(4,0);(1,0),(2,0);0].$

$l_1^4 \circ l_2^2([(1,0),(4,0);(1,0),(2,0);0]) = [(1,0),(0,0);(1,0),(0,0);0].$

$l_1^{-1} \circ m_1 \circ l_2^{-1} \circ m_2([(1,0),(0,0);(1,0),(0,0);0]) = [(0,0),(1,0);(0,0),(1,0);0].$

Hence, it follows from the last expression that $n(L_1\sharp_{78}L_2) = 2$ and the number of distinct isotopy classes is also two. The latter fact indicates that the original multiple connected sum $L_1\sharp_{78}L_2$ is a genus two link. Hence a mapping class element f associated with the multiple connected sum $[(5,0),(78,76);(7,0),(75,78);78]$ is given by the inverse of the composition of these Dehn twists, that is $f^{-1} = l_1^{-1} \circ m_1 \circ l_2^{-1} \circ m_2 \circ l_1^4 \circ l_2^2 \circ m_1 \circ m_2^{36} \circ \gamma \circ l_2^{-1} \circ m_1^{-16} \circ m_2^{11}$.

3.2 General multiple connected sum

A multiple connected sum of torus links is a generalization of the concept of connected sum of torus links. Since multiple connected sums of torus links are accessible to combinatorial techniques, it is advantageous to study links from this perspective. To perform a multiple connected sum $L_1\sharp_{m_1}L_2\sharp_{m_2}L_3\sharp_{m_3}\cdots\sharp_{m_{n-1}}L_n$ of n torus links $L_i(p_i,q_i), i = 1,2,...,n$ we proceed as follows. Make an m_i-cut on each of the torus links L_i and $L_{i+1}, i = 1,2,...,n-1$ and then multiply connect the torus links L_i and L_{i+1} along the m_i-cuts. Note that each torus link $L_j, j = 2,3,...,n-1$ will be multiply connected to the torus links L_{j-1} along the m_{j-1}-cut and to the torus links L_{j+1} along the m_j-cut. The torus links L_1 and L_n will be multiply connected to the torus links

L_2 and L_{n-1} along the m_1-cut and the m_{n-1}-cut respectively. The multiple connected sum $L_1\sharp_{m_1}L_2\sharp_{m_2}L_3\sharp_{m_3}...\sharp_{m_{n-1}}L_n$ can be viewed as a chain of $(n-2)$ simple reverse multiple connections of $(n-1)$ submultiple connected sums $L_i''\sharp_{m_i}L_{i+1}'$, $i = 1, 2, ..., (n-1)$ where $L_1'' = L_1$ and $L_n' = L_n$ as shown in figure (3.11) below and is also denoted by the notation $(L_1\sharp_{m_1}L_2') \bigoplus (L_2''\sharp_{m_2}L_3') \bigoplus ... \bigoplus (L_{n-1}''\sharp_{m_{n-1}}L_n)$ where $\bigoplus$ indicates the simple reverse multiple connection between any two consecutive submultiple connected sums $L_i''\sharp_{m_i}L_{i+1}'$ and $L_{i+1}''\sharp_{m_{i+1}}L_{i+2}'$ as shown in the figure (3.10) below. The term simple indicates that the reverse m-cuts along which any two consecutive submultiple connected sums $L_i''\sharp_{m_i}L_{i+1}'$ and $L_{i+1}''\sharp_{m_{i+1}}L_{i+2}'$ are connected are such that m equals the number of longitudinal strands of the torus link L_{i+1}.

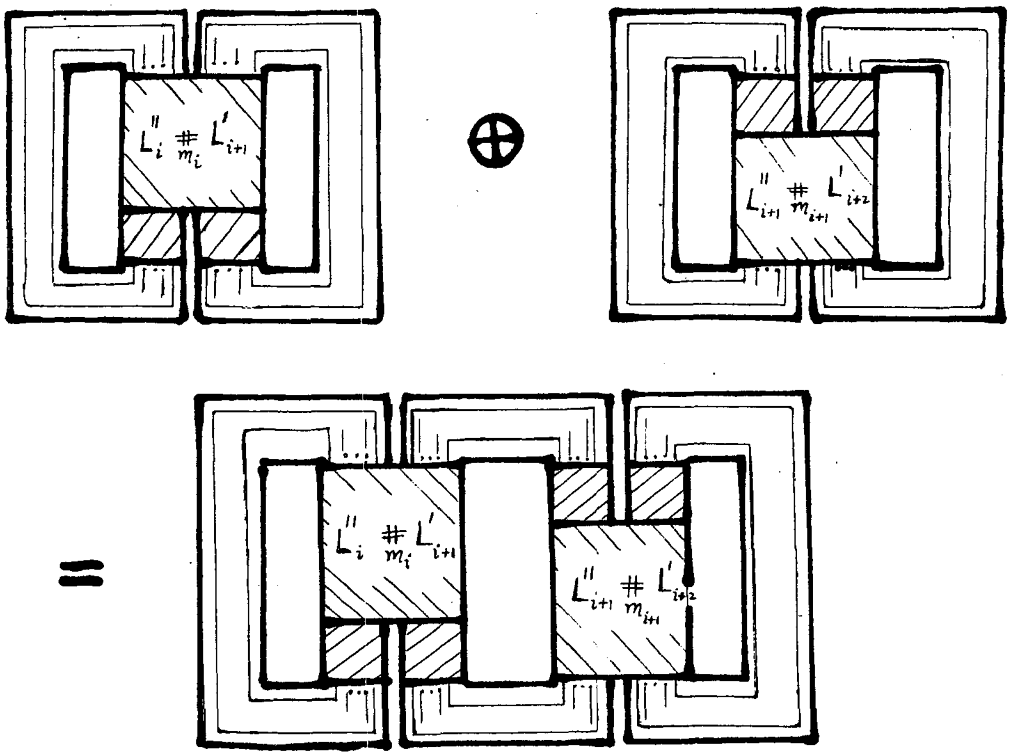


Figure 3.10 $L_i''\sharp_{m_i}L_{i+1}' \bigoplus L_{i+1}''\sharp_{m_{i+1}}L_{i+2}'$

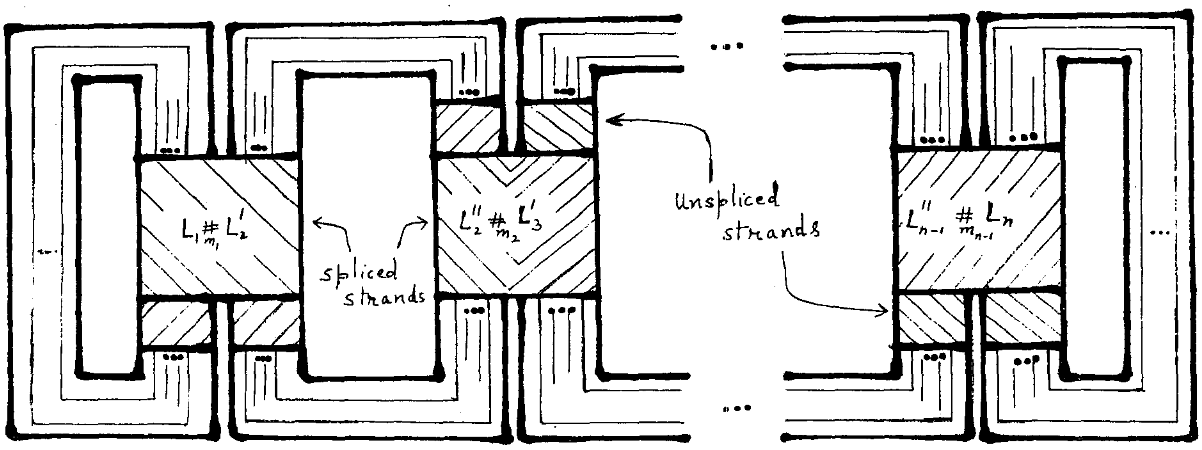


Figure 3.11 $(L_1\sharp_{m_1}L_2')\bigoplus(L_2''\sharp_{m_2}L_3')\bigoplus\cdots\bigoplus(L_{n-1}''\sharp_{m_{n-1}}L_n)$

The unspliced $|m_i - q_{i,2}^{(k_i)}|$ and $|m_i - q_{i+1,1}^{(j_{i+1})}|$ meridional strands of the $(n-1)$ submultiple connected sums $L_i''\sharp_{m_i}L_{i+1}'$ of $(L_1\sharp_{m_1}L_2')\bigoplus(L_2''\sharp_{m_2}L_3')\bigoplus\cdots\bigoplus(L_{n-1}''\sharp_{m_{n-1}}L_n)$ are arranged alternately below and above the spliced strands as shown in figure 3.10 for $i = 1, 2, ..., n-1$ and $k_i, j_{i+1} \in \{1, 2\}$. Arranging the unspliced meridional strands in $L_1\sharp_{m_1}L_2\sharp_{m_2}L_3\sharp_{m_3}\cdots\sharp_{m_{n-1}}L_n$ as mentioned above enables us to represent it in an unambiguous parametric form as follows.

$[(p_1,0)(q_1^{(1)},q_1^{(2)}),(p_2,0)(q_{2,1}^{(1)},q_{2,1}^{(2)});m_1]\bigoplus[(p_2,0)(q_{2,2}^{(1)},q_{2,2}^{(2)}),(p_3,0)(q_{3,1}^{(1)},q_{3,1}^{(2)});m_2]\bigoplus\cdots$

$\bigoplus\ [(p_{n-1},0)(q_{n-1,2}^{(1)},q_{n-1,2}^{(2)}),(p_n,0)(q_n^{(1)},q_n^{(2)});m_{n-1}]$ ...(3.1)

where $q_{i,1}^{(j_i)} + q_{i,2}^{(j_i)} = q_i^{(j_i)}$ and $q_i = |(q_{i,1}^{(1)} + q_{i,2}^{(1)}) - (q_{i,1}^{(2)} + q_{i,2}^{(2)})|, i = 2, 3, ..., n-1, j_i = 1, 2$. The above parametric representation (3.1) of $L_1\sharp_{m_1}L_2\sharp_{m_2}L_3\sharp_{m_3}\cdots\sharp_{m_{n-1}}L_n$ makes it possible to associate a permutation with it. Each cycle in this permutation will represent a component of the link generated by the multiple connected sum. To arrive at such a permutation, it is essential to label sequentially the longitudinal strands of the multiple connected sum under consideration in the following way.

Case(1) Let the submultiple connected sum $L_i''\sharp_{m_i}L_{i+1}'$ of $L_1\sharp_{m_1}L_2\sharp_{m_2}L_3\sharp_{m_3}\cdots\sharp_{m_{n-1}}L_n$ be a direct multiple connected sum and suppose further that the labels $\{1, 2, ..., s\}$ are already assigned sequentially to the longitudinal strands of the submultiple connected sum $(L_1\sharp_{m_1}L_2')\bigoplus(L_2''\sharp_{m_2}L_3')\bigoplus\ldots\bigoplus(L_{i-1}''\sharp_{m_{i-1}}L_i')$ of $L_1\sharp_{m_1}L_2\sharp_{m_2}L_3\sharp_{m_3}\ldots\sharp_{m_{n-1}}L_n$. We label the p_i+p_{i+1} longitudinal strands of $L_i''\sharp_{m_i}L_{i+1}'$ sequentially by the labels $\{s-p_i+1,\ldots,s+p_i\}$ as shown in the figure (3.12) below.

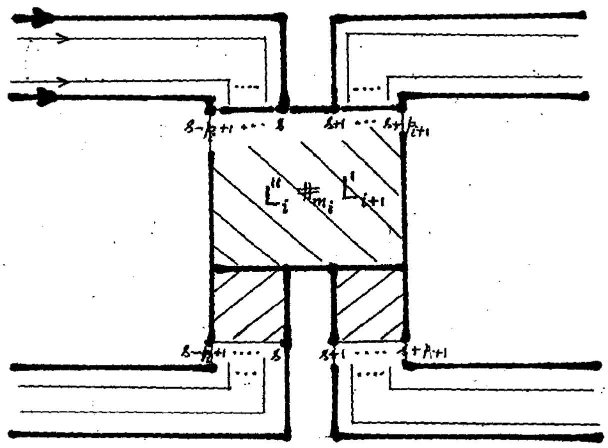


Figure 3.12 Labelling of strands of a direct submultiple connected sum $L_i''\sharp_{m_i}L_{i+1}'$

The permutation associated with a direct submultiple connected sum $L_i''\sharp_{m_i}L_{i+1}'$ is given by $p(m_i)(x)=[(x+m_i-s+p_i)\text{mod }(p_i+p_{i+1})-m_i+q_{i,2}]\text{mod }(p_i)+s-p_i$ if $1\leq(x+m_i-s+p_i)\text{mod }(p_i+p_{i+1})\leq p_i$ and $p(m_i)(x)=[(x+m_i-s+p_i)$ mod $(p_i+p_{i+1})-p_i-m_i+q_{i+1,1}]\text{mod }(p_{i+1})+s$, if $p_i+1\leq(x+m_i-s+p_i)$ mod $(p_i+p_{i+1})\leq p_i+p_{i+1}$ Here $q_{i,k}=|q_{i,k}^{(1)}-q_{i,k}^{(2)}|$ and $k=1,2$. Note that the labels $\{s-p_i+1,\ldots,s\}$ are used for the longitudinal strands of both the submultiple connected sums $L_{i-1}''\sharp_{m_{i-1}}L_i'$ and $L_i''\sharp_{m_i}L_{i+1}'$. These common labels between the two submultiple connected sum $L_{i-1}''\sharp_{m_{i-1}}L_i'$ and $L_i''\sharp_{m_i}L_{i+1}'$ are related by one of the two permutations given below.

(a) If the submultiple connected sum $L''_{i-1}\sharp_{m_{i-1}}L'_i$ is either a direct submultiple connected sum or a reverse submultiple connected sum with $p_i \geq p_{i-1}$, then the relation t_i between the common labels is a permutation given by the action $t_i(x) = 2s - p_i + 1 - x$ on the set of common labels $\{s - p_i + 1, ..., s\}$.

(b) If the submultiple connected sum $L''_{i-1}\sharp_{m_{i-1}}L'_i$ is a reverse submultiple connected sum with $p_i < p_{i-1}$, then the relation t_i between the common labels is the identity permutation.

Case(2) Let the submultiple connected sum $L''_i\sharp_{m_i}L'_{m_{i+1}}$ of $L_1\sharp_{m_1}L_2\sharp_{m_2}L_3\sharp_{m_3}...\sharp_{m_{n-1}}L_n$ be a reverse multiple connected sum and suppose further that the labels $\{1, 2, ..., s\}$ are already assigned sequentially to the longitudinal strands of the submultiple connected sum $(L_1\sharp_{m_1}L'_2)\bigoplus(L''_2\sharp_{m_2}L'_3)\bigoplus...\bigoplus(L''_{i-1}\sharp_{m_{i-1}}L'_i)$ of $L_1\sharp_{m_1}L_2\sharp_{m_2}L_3\sharp_{m_3}...\sharp_{m_{n-1}}L_n$. We label the r longitudinal strands of $L''_i\sharp_{m_i}L'_{i+1}$ sequentially by the labels $\{s - p_i + 1, ..., s + r - p_i\}$ where $r = max\{p_i, p_{i+1}\}$ as shown in the figure (3.13)(a) and (b).

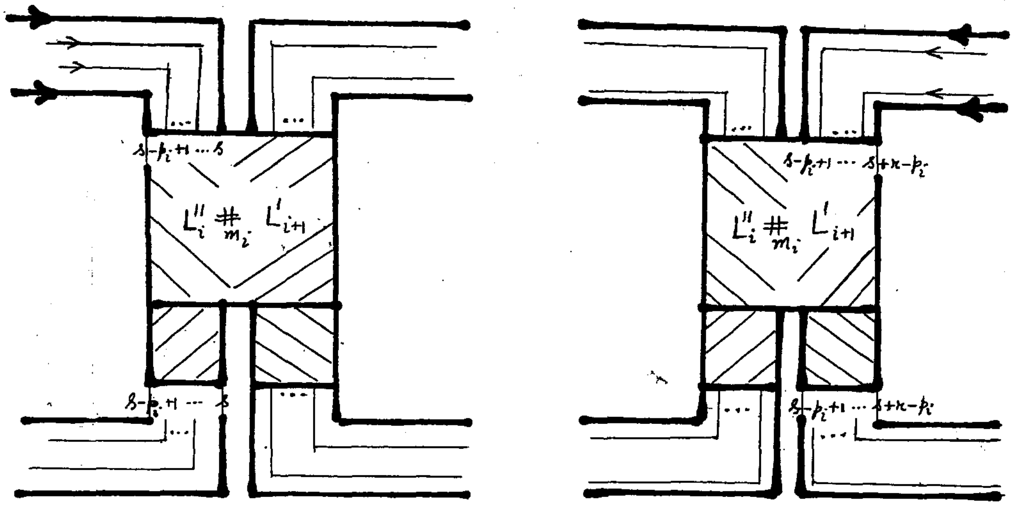


Figure3.13 Labelling of strands of a reverse submultiple connected sum $L''_i\sharp_{m_i}L'_{i+1}$

Subcase(i). $r = p_i$

The permutation $p(m_i)$ associated with a reverse submultiple connected sum $L''_i \sharp_{m_i} L'_{i+1}$ with $r = p_i$ is given by $p(m_i) = (\sigma(m_i))^{-1}$ where $\sigma(m_i)(x) = s+1-[p_{i+1}+1-(x-s+p_i+m_i-q_{i,2})\text{mod }(p_i)+m_i-q_{i+1,1}]\text{mod }(p_{i+1})$, if $1 \leq (x-s+p_i+m_i-q_{i,2})\text{mod }(p_i) \leq p_{i+1}$, and $\sigma(m_i)(x) = [(x-s+p_i+m_i-q_{i,2})\text{mod }(p_i)]\text{mod }(p_i-p_{i+1})+s-p_i$, if $p_{i+1}+1 \leq (x-s+p_i+m_i-q_{i,2})\text{mod }(p_i) \leq p_i$, where $q_{i,k} = |q^{(1)}_{i,k} - q^{(2)}_{i,k}|$ and $k = 1, 2$.

The labels $\{s-p_i+1, ..., s\}$ are used for the longitudinal strands of both the submultiple connected sums $L''_{i-1} \sharp_{m_{i-1}} L'_i$ and $L''_i \sharp_{m_i} L'_{i+1}$. These common labels between the two submultiple connected sum $L''_{i-1} \sharp_{m_{i-1}} L'_i$ and $L''_i \sharp_{m_i} L'_{i+1}$ are related by one of the two permutations given below:

(a) If the submultiple connected sum $L''_{i-1} \sharp_{m_{i-1}} L'_i$ is either a direct submultiple connected sum or a reverse submultiple connected sum with $p_i \geq p_{i-1}$, then the relation t_i between the common labels is a permutation given by the action $t_i(x) = 2s-p_i+1-x$ on the set of common labels $\{s-p_i+1, ..., s\}$.

(b) If the submultiple connected sum $L''_{i-1} \sharp_{m_{i-1}} L'_i$ a reverse submultiple connected sum with $p_i < p_{i-1}$, then the relation t_i between the common labels is the identity permutation.

Subcase(ii) $r = p_{i+1}$

The permutation $p(m_i)$ associated with a reverse submultiple connected sum $L''_i \sharp_{m_i} L'_{m_{i+1}}$ with $r = p_{i+1}$ is given by $p(m_i) = (\sigma(m_i))^{-1}$ where $\sigma(m_i) = \sigma_1 \circ \sigma_0 \circ \sigma_1$. Here σ_1 is the permutation given by the action $\sigma_1(m_i)(x) = 2s-2p_i+r+1-x$ on the set of labels $\{s-p_i+1, ..., s+r-p_i\}$ and σ_0 is the permutation defined by

$\sigma_0(m_i)(x) = s+1-[p_i+1-(x-s+p_i+m_i-q_{i+1,1})\text{mod }(p_{i+1})-m_i+q_{i,2}]\text{mod }(p_i)$ if $1 \leq (x-s+p_i-m_i+q_{i+1,1})\text{mod }(p_{i+1}) \leq p_i$, and $\sigma_0(m_i)(x) = [(x-s+p_i-m_i+q_{i+1,1})$ mod $(p_{i+1})]\text{mod }(p_{i+1}-p_i)+s-p_i$, if $p_i+1 \leq (x-s+p_i-m_i+q_{i+1,1})\text{mod }(p_{i+1}) \leq p_{i+1}$ where $q_{i,k} = |q_{i,k}^{(1)} - q_{i,k}^{(2)}|$ and $k = 1, 2$.

Note that the labels $\{s-p_i+1, ..., s\}$ are used for the longitudinal strands of both $L''_{i-1}\sharp_{m_{i-1}}L'_i$ and $L''_i\sharp_{m_i}L'_{i+1}$. These common labels between $L''_{i-1}\sharp_{m_{i-1}}L'_i$ and $L''_i\sharp_{m_i}L'_{i+1}$ are related by one of the two permutations given below:

(a) If $L''_{i-1}\sharp_{m_{i-1}}L'_i$ is either a direct submultiple connected sum or a reverse submultiple connected sum with $p_i \geq p_{i-1}$, then the relation t_i between the common labels is the identity permutation.

(b) If $L''_{i-1}\sharp_{m_{i-1}}L'_i$ a reverse submultiple connected sum with $p_i < p_{i-1}$, then the relation t_i between the common labels is a permutation given by the action $t_i(x) = 2s - p_i + 1 - x$ on the set of common labels $\{s - p_i + 1, ..., s\}$. Once the longitudinal strands of $L_1\sharp_{m_1}L_2\sharp_{m_2}L_3\sharp_{m_3}\cdots\sharp_{m_{n-1}}L_n$ are labelled in accordance with the above scheme and an orientation is assigned to the multiple connected sum it is possible to derive a permutation $p(m)$ associated with the multiple connected sum. This can be achieved by taking the product of the permutations $p(m_i)$ associated with the submultiple connected sums $L''_i\sharp_{m_i}L'_{i+1}$, $i = 1, 2, ..., n-1$ and the permutations $t(j)$ relating the common labels between the submultiple connected sums $L''_{i-1}\sharp_{m_{i-1}}L'_i$ and $L''_i\sharp_{m_i}L'_{i+1}, j = 1, 2, ..., n-2$ appropriately. The orientation assigned to the multiple connected sum $L_1\sharp_{m_1}L_2\sharp_{m_2}L_3\sharp_{m_3}\cdots\sharp_{m_{n-1}}L_n$ must ensure that the induced orientation of each component torus link $L_i, i = 1, 2, ..., n$ is such that all the longitudinal strands are compatibly oriented.

We illustrate below the method to compute the permutation $p(m)$ associated with an oriented multiple connected sum $L_1\sharp_{m_1}L_2\sharp_{m_2}L_3\sharp_{m_3}\cdots\sharp_{m_{n-1}}L_n$ for the case $n = 3$ with two examples .

Example 1 Consider the multiple connected sum $L_1\sharp_{m_1}L_2\sharp_{m_2}L_3$ having the parametric representation $[((4,0)(4,1)),((4,0)(4,3));4]\bigoplus[((4,0)(2,4)),((3,0)(2,4));4]$. Without loss of generality, we assign a positive orientation to the longitudes of the torus link $L_1((4,0)(3,0))$ that fixes the orientation of $L_1\sharp_{m_1}L_2\sharp_{m_2}L_3$. Therefore, the permutations associated with the submultiple connected sums $L_1\sharp_{m_1}L_2'$ and $L_2''\sharp_{m_2}L_3$ with respect to the assigned orientation are $p(m_1 = 4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 5 & 4 & 1 & 2 & 3 \end{pmatrix}$ and $p(m_2 = 4) = \begin{pmatrix} 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 10 & 11 & 9 & 7 & 8 & 5 & 6 \end{pmatrix}$ respectively. The permutation $t(1)$ relating the common labels $\{5,6,7,8\}$ of the two submultiple connected sums $L_1\sharp_{m_1}L_2'$ and $L_2''\sharp_{m_2}L_3$ is $t(1) = \begin{pmatrix} 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 \end{pmatrix}$. Hence, the permutation $p(m = 11)$ associated with the above torus link $L_1\sharp_{m_1}L_2\sharp_{m_2}L_3$ is given by the product $p(m_1) \circ t(1) \circ p(m_2) \circ t(1) = (1,9,5,4,6)(2,11,7)(3,10,8)$, and therefore the number of components $n(L_1\sharp_{m_1}L_2\sharp_{m_2}L_3) = 3$.

Example 2 Consider the multiple connected sum $L_1\sharp_{m_1}L_2\sharp_{m_2}L_3$ having the parametric representation $[((4,0)(5,2)),((5,0)(5,2));5]\bigoplus[((5,0)(3,1)),((4,0)(2,3));3]$. Without loss of generality, we assign a positive orientation to the longitudes of the torus link $L_1((4,0)(3,0))$, this fixes the orientation of $L_1\sharp_{m_1}L_2\sharp_{m_2}L_3$. Therefore, the permutations associated with the submultiple connected sums $L_1\sharp_{m_1}L_2'$ and $L_2''\sharp_{m_2}L_3$ with respect to the assigned orientation are $p(m_1 = 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 5 & 6 & 7 & 3 & 4 & 1 & 2 & 8 \end{pmatrix}$ and

$p(m_2 = 3) = \begin{pmatrix} 5 & 6 & 7 & 8 & 9 \\ 9 & 5 & 8 & 6 & 7 \end{pmatrix}$ respectively. The permutation $t(1)$ relating the common labels $\{5, 6, 7, 8, 9\}$ of the two submultiple connected sums $L_1 \sharp_{m_1} L_2'$ and $L_2'' \sharp_{m_2} L_3$ is $t(1) = \begin{pmatrix} 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 \end{pmatrix}$. Hence, the permutation $p(m = 9)$ associated with the above torus link $L_1 \sharp_{m_1} L_2 \sharp_{m_2} L_3$ is given by the product $p(m_1) \circ t(1) \circ p(m_2) \circ t(1) = (1, 5, 3, 8, 2, 7)(4, 6)(9)$, and hence the number of components $n(L_1 \sharp_{m_1} L_2 \sharp_{m_2} L_3) = 3$.

3.3 Some Open Questions

During the period of this work we were fascinated with many interesting problems connected with the materials presented here. Some of them we could not study in detail due to time constraints. We thought it apt to state some those problems here.

A. One of the most important question that comes to our mind is whether "*all links can be obtained as generalized multiple connected sums*". It may be true that the genus of the handle body on which a link can be embedded as a multiple connected sum may be higher than the genus of the link. Even if this is the case, the combinatorial advantage certainly makes it very useful. If this is not the case, then "*can one characterize in a reasonable way the links that appear as multiple connected sums*"?

B. Another problem is regarding the kind of ("*reduced*) permutations that appear as the permutations associated with a multiple connected sum". *Even for the multiple connected sum of two torus links this is not being satisfactorily answered. A reasonably good answer to this would probably have some bearing on the fundamental group of three manifolds obtained from them as is seen the thesis above. The general one would have to be studied further to say more about them.*

C. *Can we* "generalize the concept of multiple connected sum" *further by splicing the links along more arcs and meaningfully apply some combinatorial or other techniques to them to obtain interesting properties of them?*

D. *Can we use the concept of multiple connected sum to* "compute the various well known link invariants" *in a simpler way for such links?*

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