Approximation Methods for Nonlinear Ill-posed Hammerstein Type Operator Equations

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Dedicated To Rt.Rev.Dr. George Valiamattam, Arch Bishop, Tellichery Dioces , Rev.Sr.Seraphia and Sisters of Carmelite Convent, Edoor.

DECLARATION

I do hereby declare that this Thesis entitled **"APPROXIMATION METHODS FOR ILL-POSED HAMMERSTEIN TYPE OPERATOR EQUATIONS"** submitted to **Goa University** in partial fulfillment of the requirements for the award of Degree of **Doctor of Philosophy** in **Mathematics** is a record of original and independent work done by me under the supervision and guidance of **Dr.Santhosh George,** Associate Professor, Department of MACS, National Institute of Technology Karnataka, Surathkal, with **Dr.Y.S.Valaulikar**, Associate Professor, Department of Mathematics, Goa University as co-guide, and it has not previously formed the basis for the award of any Degree, Diploma, Associteship, Fellowship or other similar title to any candidate of any University.

M.KUNHANANDAN

CERTIFICATE

This is to certify that the Thesis entitled "**APPROXIMATION METHODS FOR ILL-POSED HAMMERSTEIN TYPE OPERATOR EQUATIONS**" submitted to **Goa University** in partial fulfillment of the requirements for the award of Degree of **Doctor of Philosophy** in **Mathematics** by M.Kunhanandan is a bonafide record of original and independent research work done by the candidate under our guidance. We further certify that this work has not previously formed the basis for the award of any Degree, Diploma, Associteship, Fellowship or other similar title to any candidate of any University.

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Chapter 1

Introduction and Preliminaries

1.1 General Introduction

Driven by needs of application, the field of inverse problems has been one of the fastest growing area in applied mathematics in the last decades. It is well known that these problems typically lead to mathematical models that are ill-posed.

The notion of a well posed or correctly set problem makes its debut with the discussion in chapter 1 of J.Hadamard [29]. It represented a significant step forward in the classification of multitude of problems associated with differential equations, singling out those with sufficiently general properties of existence, uniqueness and stability of solutions. He expresses the opinion that only problems of physical interest are those that has a unique solution depending continuously on the given data. Such problems he called correctly set problem or well posed problems and problems that are not well posed are called incorrectly set problems or ill- posed problems. But Hadamard's notion of a mechanical or physical problem turns out to be too narrow. It applies when a problem is that of determining the effects(solutions) of a complete set of independent causes(data). But in many applied problems we have to get along without a precise knowledge of causes and in the others we are really trying to find causes that will produce the desired effect. We are then led to ill-posed problems. One might say that majority of applied problems are, and always have been ill-posed, particularly when they require numerical answers. Ill-posed problems include such classical problems of analysis and algebra as differentiation of functions known only approximately, solutions of integral equations of the first kind, summation of Fourier series with approximate coefficients, analytical continuation of functions, finding inverse Laplace transforms, the Cauchy problem for Laplace equations, solution of singular or ill-conditioned systems of linear algebraic equations and many others(cf.[59, 26]).

The next important question is in what sense ill-posed problems could have solutions that would be meaningful in applications. Often, existence and uniqueness can be forced by enlarging or reducing the solution space. For restoring stability, however, one has to change the topology of the space, which in many cases is impossible because of presence of measurement errors. At first glance it seems impossible to compute a solution of the problem numerically if the solution of the problem does not depend continuously on the data. If the initial data in such problems are known approximately and contain a random error, then the above mentioned instability of their solution leads to non uniqueness of the classically derived approximate solution and to serious difficulties in their physical interpretation. Under additional àpriori information about the solution such as smoothness and bounds on the derivatives, however, it is possible to restore stability and to construct efficient numerical algorithms for solving the ill-posed problems (cf.[59]). Ofcourse in solving such problems, one must first define the concept of an approximate solution that is stable to small changes in the initial data, and use special methods for deriving the solution. Tikhonov was one of the earliest workers in the field of ill-posed problems ([59]) who succeeded in giving a precise mathematical definition of approximate solution for general class of such problems and in constructing optimal solutions. Numerical methods that can cope with these problems are the so called regularization methods.

In the abstract setup, typically, ill-posed problems are classified as linear ill-posed problems or nonlinear ill-posed problems (cf. [48], [46]). A classical example of a linear ill-posed problem is the computerized tomography ([46]). Nonlinear ill-posed problems appear in a variety of natural models such as impendence tomography. The analysis of regularization methods for linear problems is relatively complete ([6], [9],

[10], [23], [30]). The theory of nonlinear problems is developed to a much lesser extend. Several results on the well known Tikhonov regularization are given in [11].

Due to rapidly evolving innovative processes in engineering and business, more and more nonlinear ill-posed problems arise and a deep understanding of the mathematical and physical aspects that would be necessary for deriving problem specific solution approaches can often not be gained for these new problems due to lack of time (see [35, 48]). Therefore one needs algorithms that can be used to solve inverse problems in their general formulations as nonlinear operator equations. In the last few years more emphasis was put on the investigation of iterative regularization methods. It turned out that they are an attractive alternative to Tikhonov regularization, especially for large scale inverse problems ([35, 48]). It is the topic of this thesis to propose such methods and algorithms for a special class of nonlinear ill-posed equations, namely, ill-posed Hammerstein type operator equations.

We will first set up the notations and introduce the formal notion and difficulties encountered with ill-posed problems.

1.2 Notations and Preliminaries

Throughout this thesis X and Y denote Hilbert spaces over real or complex field and $BL(X,Y)$ denote the space of all bounded linear transformations from X to Y. If $X = Y$, then we denote $BL(X,X)$ by $BL(X)$. We will use the symbol $\langle .,. \rangle$ to denote the inner product and $\|.\|$ denote the corresponding norm for the spaces under consideration.

For a subspace S of X, its closure is denoted by $\overline{S}$ and its annihilator is denoted by $S^{\perp}$ i.e.,

$$S^{\perp} = \{u \in X : \langle x, u \rangle = 0, \quad \forall x \in S\}.$$

If $T \in BL(X,Y)$, then its adjoint, denoted by T^*, is a bounded linear operator from Y to X defined by

$$\langle Tx, y \rangle = \langle x, T^*y \rangle, \quad \forall x \in X, y \in Y.$$

We shall denote the range and null space of T by $R(T)$ and $N(T)$ respectively.

The results quoted in this section with no reference can be found in any text book on functional analysis(for example, [43], [44]).

Theorem 1.2.1. *If $T \in BL(X,Y)$, then $R(T)^{\perp} = N(T^*)$, $N(T)^{\perp} = \overline{R(T^*)}$, $R(T^*)^{\perp} = N(T)$ and $N(T^*)^{\perp} = \overline{R(T)}$.*

The spectrum and spectral radius of an operator $T \in BL(X)$ are denoted by $\sigma(T)$ and $r_\sigma(T)$ respectively, i.e., $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ does not have bounded inverse$\}$ where I is the identity operator on X, and $r_\sigma(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$.

It is well known that $r_\sigma(T) \leq \|T\|$ and $\sigma(T)$ is a compact subset of the scalar field. If T is a non zero self adjoint operator, i.e.,$T^* = T$, then $\sigma(T)$ is a nonempty subset of real numbers and $r_\sigma(T) = \|T\|$.

If T is a positive self adjoint operator, i.e., $T = T^*$ and $\langle Tx, x\rangle \geq 0, \forall x \in X$, then $\sigma(T)$ is a subset of the set of non-negative reals. If $T \in BL(X)$ is compact, then $\sigma(T)$ is a countable set with zero as the only possible limit point. In fact the following result is well known:

Theorem 1.2.2. *Let $T \in BL(X)$ be a non-negative compact self adjoint operator . Then there is a finite or infinite sequence of non-zero real numbers (λ_n) with $|\lambda_1| \geq |\lambda_2| \geq \cdots$, and a corresponding sequence (u_n) of orthonormal vectors in X such that for all $x \in X$,*

$$Tx = \sum_n \lambda_n \langle x, u_n\rangle u_n$$

where $\lambda_n \to 0$ as $n \to \infty$, whenever the sequence (λ_n) is infinite. Here λ_n are eigenvalues of T with corresponding eigenvectors u_n.

If $T \in BL(X,Y)$ is a non-zero compact operator then T^*T is a positive compact self adjoint operator on X. Then by Theorem 1.2.2 and by the observation that $\sigma(T^*T)$ consists of non-negative reals, there exists a sequence (s_n) of positive reals with $s_1 \geq s_2 \geq \cdots$ and a corresponding sequence of orthonormal vectors (v_n) in X, satisfying

$$T^*Tx = \sum_n s_n \langle x, v_n \rangle v_n$$

for all $x \in X$ and $T^*Tv_n = s_n v_n, \;\; n = 1, 2, \cdots$.

Let $\lambda_n = \sqrt{s_n}, \quad \mu_n = \frac{1}{\lambda_n}, \quad u_n = \mu_n T v_n$ and $v_n = \mu_n T^* u_n$. The sequence $\{u_n, v_n, \mu_n\}$ is called a singular system for T.

In order to define functions of operators on a Hilbert space we require spectral theorem for self adjoint operators which is a generalization of Theorem 1.2.2.

Theorem 1.2.3. *Let $T \in BL(X)$ be self adjoint and let $a = \inf \sigma(T), \quad b = \sup \sigma(T)$. Then there exists a family $\{E_\lambda : a \leq \lambda \leq b\}$ of projection operators on X such that*

1. *$\lambda_1 \leq \lambda_2$ implies $\langle E_{\lambda_1} x, x \rangle \leq \langle E_{\lambda_2} x, x \rangle, \quad \forall x \in x$*

2. *$E_a = 0, \quad E_b = I$ where I is the identity operator on X*

3. *$T = \int_a^b \lambda dE_\lambda$.*

The above integral is in the sense of Riemann-Stieltje. The family $\{E_\lambda\}_{\lambda \in [a,b]}$ is called the spectral family of the operator T. If f is a continuous real valued function on $[a, b]$, then $f(T) \in BL(X)$ is defined by

$$f(T) = \int_a^b f(\lambda) dE_\lambda.$$

Then $\sigma(f(T)) = \{f(\lambda) : \lambda \in \sigma(T)\}$ and $\|f(T)\| = r_{\sigma(f(T))} = \sup\{|f(\lambda)| : \lambda \in \sigma(T)\}$.

For real valued function f and g we use the notation $f(x) = O(g(x))$ as $x \to 0$ to denote the relation

$$\frac{f(x)}{g(x)} \leq M$$

as $x \to 0$ where $M > 0$ is constant independent of x and $f(x) = o(g(x))$ as $x \to 0$ to denote

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 0.$$

We will be using the concept of Hilbert scales (cf.[47]) in Chapter3;

Definition 1.2.4. *(Hilbert Scales) Let L be a densely defined, self adjoint, strictly positive operator in a Hilbert space X that fulfills $\|Lx\| \geq \|x\|$ on its domain.For $s \geq 0$ let X_s be the completion of $\bigcap_{k=0}^{\infty} D(L^k)$ with respect to the Hilbert space norm induced by the inner product $\langle x, y\rangle_s := \langle L^s x, L^s y\rangle$ and for $s < 0$ let X_s be the dual space of X_-s.Then $(X)_{s\in\mathbb{R}}$ is called a Hilbert scale induced by the operator L.*

1.3 Basic Results from Nonlinear Functional Analysis

In this section we recall some definitions and basic results which will be used in this thesis.

Definition 1.3.1. *Let F be an operator mapping a Hilbert space X into a Hilbert space Y . If there exists a bounded linear operator L from X into Y such that*

$$\lim_{\|h\|\to o} \frac{\|F(x_0 + h) - F(x_0) - L(h)\|}{\|h\|} = 0,$$

then F is said to be Fréchet-differentiable at x_0, and the bounded linear operator

$$F'(x_0) = L$$

is called the first Fréchet derivative of F at x_0.

We assume that the Fréchet derivative F' of F satisfies the condition

$$\|F'(x) - F'(y)\| \leq k_0\|x - y\|, \quad \forall x, y \in B_{r_0}(x_0). \tag{1.3.1}$$

for some $r_0 > 0$.

We shall make use of the following lemma, extensively in our analysis.

Lemma 1.3.2. *Let $r_0 > 0$ and $x, y \in \overline{B_{r_0}(x_0)} \subset X$. Then*

$$\|F'(x_0)(x - x_0) - [F(x) - F(x_0)]\| \leq \frac{k_0 r_0}{2}\|x - x_0\|,$$

$$\|F'(x_0)(x - y) - [F(x) - F(y)]\| \leq k_0 r_0\|x - y\|.$$

Proof. By the Fundamental Theorem of Integral Calculus,

$$F(x) - F(y) = \int_0^1 F'(y + t(x - y))(x - y)dt,$$

and so

$$F'(x_0)(x - y) - (F(x) - F(y)) = \int_0^1 [F'(x_0) - F'(y + t(x - y))](x - y)dt. \quad (1.3.2)$$

Hence by (1.3.1)

$$\|F'(x_0)(x - y) - [F(x) - F(y)]\| \leq k_0\|x - y\| \int_0^1 \|x_0 - (y + t(x - y))\|dt.$$

Now since $y + t(x - y) \in B_{r_0}(x_0) \subset X$, then

$$\|x_0 - (y + t(x - y)\| \leq r_0$$

and

$$\|x_0 - (x_0 + t(x - x_0))\| \leq tr_0$$

hence

$$\|F'(x_0)(x - x_0) - [F(x) - F(x_0)]\| \leq \frac{k_0 r_0}{2}\|x - x_0\|$$

and

$$\|F'(x_0)(x - y) - [F(x) - F(y)]\| \leq k_0 r_0\|x - y\|.$$

This completes the proof. □

Definition 1.3.3. *Let X be a real Hilbert space and $F : D(F) \subseteq X \rightarrow X$ is an operator. Then F is said to be monotone if*

$$\langle F(x_1) - F(x_2), x_1 - x_2 \rangle \geq 0, \quad \forall x_1, x_2 \in D(F).$$

Remark 1.3.4. *1. If $F(x) = Ax$ where $A : X \rightarrow X$ is linear then F is monotone $\Leftrightarrow \langle Ax, x \rangle \geq 0, \quad \forall x \in X \Leftrightarrow A$ is positive semi definite.*

2. If F is continuously differentiable on X, then F is monotone $\Leftrightarrow F'(x)$ is positive semidefinite for all x.

In the analysis involving monotone operators we shall be using the concept of majorizing sequence.

Definition 1.3.5. *(see [2], Definition 1.3.11) A nonnegative sequence (t_n) is said to be a majorizing sequence of a sequence (x_n) in X if*

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n, \forall n \geq 0.$$

During the convergence analysis we will be using the following Lemma on majorization, which is a reformulation of Lemma 1.3.12 in [2]. For the sake of completeness, we supply its proof.

Lemma 1.3.6. *Let (t_n) be a majorizing sequence for (x_n) in X. If $\lim_{n \longrightarrow \infty} t_n = t^*$ then $x^* = \lim_{n \longrightarrow \infty} x_n$ exists and*

$$\|x^* - x_n\| \leq t^* - t_n, \forall n \geq 0. \tag{1.3.3}$$

Proof. Note that

$$\|x_{n+m} - x_n\| \leq \sum_{j=n}^{n+m-1} \|x_{j+1} - x_j\| \leq \sum_{j=n}^{n+m-1} (t_{j+1} - t_j) = t_{n+m} - t_n, \tag{1.3.4}$$

so (x_n) is a Cauchy sequence in X and hence (x_n) converges to some x^*. The error estimate in (1.3.3) follows from (1.3.4) as $m \longrightarrow \infty$. This completes the proof. □

Now we shall formally define the concept of ill-posedness.

1.4 Ill-posedness of Equations

Definition 1.4.1. *Let $F : X \to Y$ be an operator (linear or nonlinear) between Hilbert spaces X and Y. The equation*

$$F(x) = y \tag{1.4.1}$$

is said to be well-posed if the following three conditions hold.

1. *(1.4.1) has a solution*
2. *(1.4.1) cannot have more than one solution*
3. *the solution x of (1.4.1) depends continuously on the data y.*

In the operator theoretic language the above conditions together means that F is a bijection and F^{-1} is a continuous operator.

The equation (1.4.1) is said to be ill-posed if it is not well-posed.

An ill-posed operator equation is classified as linear or nonlinear as the operator F is linear or nonlinear. The subject matter of this thesis is nonlinear ill-posed operator equations.

Below we present some well-known examples for linear as well as nonlinear ill-posed problems.

Linear Ill-posed Problems

Example 1.4.2. *The Vibrating String (see [26]): The free vibration of a nonhomogeneous string of unit length and density distribution $\rho(x) > 0, 0 < x < 1$, is modeled by the partial defferential equation*

$$\rho(x)u_{tt} = u_{xx}; \tag{1.4.2}$$

where $u(x,t)$ is the position of the particle x at time t. Assume that the end of the string are fixed and $u(x,t)$ satisfies the boundary conditions

$$u(0,t) = 0, u(1,t) = 0.$$

Assuming the solution $u(x,t)$ is of the form

$$u(x,t) = y(x)r(t),$$

one observes that $y(x)$ satisfies the ordinary differential equation

$$y'' + \omega^2\rho(x)y = 0 \tag{1.4.3}$$

with boundary conditions

$$y(0) = 0, y(1) = 0.$$

Suppose the value of y at certain frequency ω is known, then by integrating equation (1.4.3) twice, first from zero to s and then from zero to one, we obtain

$$\int_0^1 y'(s;\omega)ds - y'(0;\omega) + \omega^2\int_0^1\int_0^s \rho(x)y(x;\omega)dxds = 0.$$

$$\int_0^1 (1-s)y(s;\omega)ds = \frac{y'(0;\omega)}{\omega^2}. \tag{1.4.4}$$

The inverse problem here is to determine the variable density ρ of the string, satisfying (1.4.4) for all allowable frequencies ω.

Example 1.4.3. *Simplified Tomography (see [26]): Consider a two dimensional object contained within a circle of radius R. The object is illuminated with a radiation of density I_0. As the radiation beams pass through the object it absorbs some radiation. Assume that the radiation absorption coefficient $f(x,y)$ of the object varies from point to point of the object.The absorption coefficient satisfies the law*

$$\frac{dI}{dy} = -fI$$

where I is the intensity of the radiation. By taking the above equation as the definition of the absorption coefficient, we have

$$I_x = I_0 \exp(-\int_{-y(x)}^{y(x)} f(x,y)dy)$$

where $y = \sqrt{R^2 - x^2}$. Let $p(x) = \ln(\frac{I_0}{I_x})$, i.e.,

$$p(x) = \int_{-y(x)}^{y(x)} f(x,y)dy.$$

Suppose that f is circularly symmetric,i.e.,$f(x,y) = f(r)$ with $r = \sqrt{x^2 + y^2}$, then

$$p(x) = \int_x^R \frac{2r}{\sqrt{r^2 - x^2}} f(r)dr. \tag{1.4.5}$$

The inverse problem is to find the absorption coefficient f satisfying the equation (1.4.5)

Nonlinear Ill-posed Problems

Example 1.4.4. *Nonlinear singular integral equation (see [8]):*

Consider the nonlinear singular integral equation in the form

$$\int_0^t (t-s)^{-\lambda} x(s)ds + F(x(t)) = f_0(t), \quad 0 < \lambda < 1, \tag{1.4.6}$$

where $f_0 \in L^2[0,1]$ and the nonlinear function $F(t)$ satisfies the following conditions:

- $|F(t)| \leq a_1 + a_2|t|, \quad a_1,\ a_2 > 0,$
- $F(t_1) \leq F(t_2) \Longleftrightarrow t_1 \leq t_2$, *and*
- F *is differentiable.*

Thus, F is a monotone operator from $X = L^2[0;1]$ into $X^* = L^2[0;1]$. In addition, assume that F is a compact operator. Then the equation (1.4.6) is an ill- posed problem, because the operator K defined by

$$Kx(t) = \int_0^t (t-s)^{-\lambda} x(s)ds,$$

also is compact.

Example 1.4.5. *Parameter identification problem (see [12]):*

A nonlinear ill-posed problem which arises frequently is applications is the inverse problem of identifying a parameter in a two point boundary value problem. Consider a two point boundary value problem given by

$$-u_{ss} + cu = f, \quad u(0) = u(1) = 0, \tag{1.4.7}$$

where $f \in L^2[0,1]$ is given and $c \in L^2[0,1]$ is such that $c \geq 0$ almost everywhere. The inverse problem here is to estimate the parameter c from noisy measurements $u_\delta \in L^2[0,1]$. It is assumed that the unperturbed data u is attainable, i.e., there exists $\hat{c} \in L^2[0,1]$, $\hat{c} \geq 0$ almost everywhere, with $u_{\hat{c}} = u$. Here $u_{\hat{c}}$ denotes the solution of the differential equation with $c = \hat{c}$. Under the assumption that $c \geq 0$ and $f \in L^2[0,1]$, *it is known that the above boundary value problem (1.4.7) has a unique solution. In the context of this problem, the operator* $F : D(F) \subseteq L^2[0,1] \mapsto L^2[0,1]$ *is given by:*

$$F(c) := u_c$$

with domain

$$D(F) := \{c \in L^2[0,1] : c \geq 0 \quad \textit{almost everywhere}\}$$

The problem of estimating c is ill-posed as can be seen from the following argument, as in [12]:-

Let f *be the constant function say* $f \equiv 16$. *Then, for the data*

$$u(s) := 8s(1-s), \quad u_n(s) := u(s) + e_n(s), \quad n \geq 2,$$

where

$$e_n(s) := \begin{cases} n^{-5/4}(2s)^{2n} - 4n^{-1/4}s & , s \leq 1/2 \\ n^{-5/4}(2-2s)^{2n} - 4n^{-1/4}(1-s) & , s > 1/2 \end{cases}$$

the unique solution in $D(F)$ *are given by*

$$c = 0 \quad \textit{and} \quad c_n = \frac{(e_n)_{ss}}{u + e_n}.$$

Here $\|u_n - u\| \to 0$ *and* $u_n \to u$ *in* $L^2[0,1]$, *but* $\|c_n\|_2 \simeq n^{1/4} \to \infty$, *and hence* c_n *does not converge to c in* $L^2[0,1]$.

Example 1.4.6. *Nonlinear Hammerstein integral equation (see [12]):*

$$F(x) = y$$

where $F : D = L^2[0,1] \to L^2[0,1]$ *defined by*

$$F(x)(t) := \int_0^1 k(s,t)u(s,x(s))ds,$$

is injective with a non-degenerate kernel $k(.,.) \in L^2([0,1]\times[0,1])$ *and,* $u:[0,1]\times\mathbb{R}\rightarrow\mathbb{R}$ *satisfies*

$$|u(t,s)| \leq a(t) + b|s|, \quad t \in [0,1], \quad s \in \mathbb{R}$$

for some $a \in L^2[0,1]$ *and* $b > 0$. *It can be seen that* F *is compact and continuous on* $L^2[0,1]$ *(see [34]). Further, since* $D(F)$ *is weakly closed and* F *is injective, it follows that the problem of solving* $F(x) = y$ *is ill posed (see [12], Proposition 10.1).*

Example 1.4.7. *Exponential growth model (see [26])*

For a given $c > 0$, *consider the problem of determining* $x(t), t \in (0,1)$, *in the initial value problem*

$$\frac{dy}{dt} = x(t)y(t), \quad y(0) = c, \tag{1.4.8}$$

where $y \in L^2[0,1]$. *This problem can be written as an operator equation of the form (1.4.1), where* $F: L^2[0,1] \rightarrow L^2[0,1]$ *is defined by*

$$F(x)(t) = c\exp(\int_0^t x(t)dt), \quad c \in L^2[0,1], \quad t \in (0,1).$$

It can be seen from the following argument that the problem is ill-posed. Suppose, in place of an exact data y, *we have a perturbed data*

$$y^\delta(t) := y(t)\exp(\delta\sin(\frac{t}{\delta^2})), \quad t \in (0,1).$$

Then, from (1.4.8), the solution corresponding to $y^\delta(t)$ *is given by*

$$x^\delta(t) := \frac{d}{dt}\log(y^\delta(t)), \quad t \in (0,1).$$

Note that,

$$\|y - y^\delta\|_2 \rightarrow 0 \quad as \quad \delta \rightarrow 0.$$

But

$$x^{\delta}(t) - x(t) = \frac{d}{dt}\log(\exp(\delta \sin\frac{t}{\delta^2})) = \frac{d}{dt}(\delta \sin\frac{t}{\delta^2}),$$

so that

$$\|x^{\delta} - x\|_2^2 = \frac{\sin(2/\delta^2)}{4} + \frac{1}{2\delta^2} \to \infty \quad as \quad \delta \to 0.$$

Hence, the solution dose not depend continuously on the given data and thus the problem is ill-posed.

1.5 Regularization of Ill-posed Operator Equations

Let us first consider the case when the operator F in (1.4.1) is a linear operator.

Generalized Inverse

If $y \notin R(F)$ then clearly (1.4.1) has no solution and hence the equation (1.4.1) is ill-posed. In such a case we may broaden the notion of a solution in a meaningful sense. For $F \in BL(X,Y)$ and $y \in Y$, an element $u \in X$ is said to be a least square solution of (1.4.1) if

$$\|F(u) - y\| = \inf\{\|F(x) - y\| : x \in X\}.$$

Observe that if F is not one-one, then the least square solution (cf.[23]) u, if exists , is not unique since $u + v$ is also a least square solution for every $v \in N(F)$. The following theorem provides a characterization of least square solutions.

Theorem 1.5.1. *([23], Theorem 1.3.1) For $F \in BL(X,Y)$ and $y \in Y$, the following are equivalent.*

(i) $\|F(u) - y\| = \inf\{\|F(x) - y\| : x \in X\}$

(ii) $F^*F(u) = F^*y$

(iii) $F(u) = Py$

where $P : Y \to Y$ *is the orthogonal projection onto* $\overline{R(F)}$.

From (iii) it is clear that (1.4.1) has a least square solution if and only if $Py \in R(F)$. i.e., if and only if y belongs to the dense subset $R(F) + R(F)^{\perp}$. By Theorem 1.5.1 it is clear that the set of all least square solutions is a closed convex set and hence by Theorem 1.1.4 in [24], there is a unique least square solution of smallest norm. For $y \in R(F) + R(F)^{\perp}$, the unique least square solution of minimal norm of (1.4.1) is called the generalized solution or the pseudo solution of (1.4.1). It can be easily seen that the generalized solution belongs to the subspace $N(F)^{\perp}$ of X. The map $F^{\dagger} : D(F^{\dagger}) := R(F) + R(F)^{\perp} \to X$ which assigns each $y \in D(F^{\dagger})$ with the unique least square solution of minimal norm is called the generalized inverse or Moore-Penrose inverse of F. Note that if $y \in R(F)$ and if F is injective the generalized solution of (1.4.1) is nothing but the solution of (1.4.1). If F is bijective then it follows that $F^{\dagger} = F^{-1}$.

Theorem 1.5.2. *([44],Theorem 4.4) Let* $F \in BL(X, Y)$. *Then* $F^{\dagger} : D(F^{\dagger}) := R(F) + R(F)^{\perp} \to X$ *is closed densely defined operator and* $F^{\dagger}$ *is bounded if and only if* $R(F)$ *is closed.*

If the equation (1.4.1) is ill-posed then one would like to obtain the generalized solution of (1.4.1). But by Theorem 1.5.2, the problem of finding the generalized solution of (1.4.1) is also ill-posed, i.e., $F^{\dagger}$ is discontinuous if $R(F)$ is not closed. This observation is important since a wide class of operators of practical importance, especially compact operators of infinite rank falls into this category ([26]). Further in application the data y may not be available exactly. So one has to work with an approximation $\tilde{y}$ of y. If $F^{\dagger}$ is discontinuous then for $\tilde{y}$ close to y, the generalized solution $F^{\dagger}\tilde{y}$, even when it is defined need not be close to $F^{\dagger}y$. To manage this situation the so called regularization procedures have to be employed and obtain approximations for $F^\dagger y$.

1.6 Regularization Principle and Tikhonov Regularization

Let us first consider the problem of finding the generalized solution of (1.4.1) with $F \in BL(X,Y)$ and $y \in D(F^\dagger)$. For $\delta > 0, y^\delta \in Y$ be an inexact data such that $\|y - y^\delta\| \leq \delta$. By a regularization of equation (1.4.1) with y^δ in place of y we mean a procedure of obtaining a family (x_α^δ) of vectors in X such that each $x_\alpha^\delta, \alpha > 0$ is a solution of a well posed equation and $x_\alpha^\delta \to F^\dagger y$ as $\alpha \to 0, \delta \to 0$.

A regularization method which has been studied most extensively is the so called Tikohonov regularization ([23]) introduced in the early sixties, where x_α^δ is taken as the minimizer of the functional $J_\alpha^\delta(x)$, where

$$J_\alpha^\delta(x) = \|F(x) - y^\delta\|^2 + \alpha\|x\|^2 \tag{1.6.1}$$

The fact that x_α^δ is the unique solution of the well-posed equation

$$(F^*F + \alpha I)x_\alpha^\delta = F^* y^\delta$$

is included in the following well known result (see [44]).

Theorem 1.6.1. *Let $F \in BL(X,Y)$. For each $\alpha > 0$ there exists unique $x_\alpha^\delta \in X$ which minimizes the functional $J_\alpha^\delta(x)$ in (1.6.1). Moreover the map $y^\delta \to x_\alpha^\delta$ is continuous for each $\alpha > 0$ and*

$$x_\alpha^\delta = (F^*F + \alpha I)^{-1} F^* y^\delta.$$

If $Y = X$ and F is a positive self adjoint operator on X, then one may consider ([3]) a simplcr regularization method to solve (1.6.1) where the vectors w_α^δ satisfying

$$(F + \alpha I)w_\alpha^\delta = y^\delta \tag{1.6.2}$$

are considered to obtain approximation for $F^{\dagger}y$. Note that for positive self adjoint operator F, the ordinary Tikhonov regularization applied to the equation (1.4.1) results in a more complicated equation $(F^2 + \alpha I)x_\alpha^\delta = Fy^\delta$ than (1.6.2). Moreover it is known that (see [56]) the approximation obtained by the regularization procedure (1.6.2) has better convergence property than the approximation obtained by Tikhonov regularization. As in [27] we call the above regularization procedure (1.6.2) the simplified regularization of (1.4.1).

One of the prime concerns of regularization methods is the convergence of x_α^δ (w_α^δ in the case of simplified regularization) to $F^{\dagger}y$, as $\alpha \to 0$ and $\delta \to 0$. It is known that ([23]) if $R(F)$ is not closed then there exist sequences (δ_n) and $\alpha_n = \alpha(\delta_n)$ such that $\delta_n \to 0$ and $\alpha_n \to 0$ as $n \to \infty$ but the sequence $(x_{\alpha_n}^{\delta_n})$ diverges as $\delta_n \to 0$.Therefore it is important to choose the regularization parameter α depending on the error level δ and also possibly on y^δ, say $\alpha := \alpha(\delta, y^\delta)$ such that $\alpha(\delta, y^\delta) \to 0$ and $x_\alpha^\delta \to F^{\dagger}y$ as $\delta \to 0$. Practical considerations suggest that it is desirable to choose the regularization parameter at the time of solving x_α^δ using a so called a posteriori method which depend on y^δ as well as on δ ([50]). For our work we have used the adaptive selection of parameter proposed by Pereverzeve and Schock ([50]) in 2005. Before explaining this procedure in detail we shall briefly refer to the topic of Tikhonov regularization for a nonlinear ill-posed operator equation.

For the equation (1.4.1) with F a nonlinear operator, the least square solution $\hat{x}$ is defined by the requirement

$$\|F(\hat{x}) - y\| = \inf_{x \in D(F)} \|F(x) - y\| \tag{1.6.3}$$

and an x_0 minimum norm solution should satisfy (1.6.3)([13]) and also

$$\|\hat{x} - x_0\| = \min\{\|x - x_0\| : F(x) = y, x \in D(F)\} \tag{1.6.4}$$

here x_0 is some initial guess. Such a solution:

- need not exist
- need not be unique, even when it exists.

Tikhonov regularization for nonlinear ill-posed problem (1.4.1) provides approximate solutions as solutions of the minimization problem $J_F^\delta(x)$, where

$$J_F^\delta(x) = \|F(x) - y^\delta\|^2 + \alpha\|x - x_0\|^2$$

$\alpha > 0$. If x_α^δ is an interior point of $D(F)$, then the regularized approximation x_α^δ satisfies the normal operator equation

$$F'^*(x)[F(x) - y^\delta] + \alpha(x - x_0) = 0$$

of the Tikhonov functional $J_F^\delta(x)$. Here $F'^*(.)$ is the adjoint of the Fréchet derivative $F'(.)$ of F. For the special case when F is a monotone operator the least squares minimization (and hence the use of adjoint) can be avoided and one can use the simpler regularized equation

$$F(x) + \alpha(x - x_0) = y^\delta. \tag{1.6.5}$$

The method in which the regularized approximation x_α^δ is obtained by solving the singularly perturbed operator equation (1.8.1) is called the method of Lavrentiev regularization ([39]), or sometimes the method of singular perturbation ([40]). In general a regularized solution x_α^δ can be written as $x_\alpha^\delta = R_\alpha y^\delta$, where R_α is a regularization function.

1.6.1 Iterative Methods

Iterative methods have the following form:

(1) Beginning with a starting value x_0,

(2) Successive approximates x_i, $i = 1, 2, \cdots$ to x_α^δ are computed with the aid of an iteration function $G : X \mapsto X$:

$$G(x_i) = x_{i+1} \quad i = 1, 2, \cdots,$$

(3) If x_α^δ is a fixed point of G i.e., $G(x_\alpha^\delta) = x_\alpha^\delta$, all fixed points of G are also zeros of F, and if G is continuous in a neighborhood of each of its fixed points, then each limit point of the sequence x_i, $i = 1, 2, \cdots$, is a fixed point of G, and hence a solution of the equation (1.4.1).

1.7 Selection of the Regularization Parameter

Making a right choice of a regularization parameter in a regularization method is as important as the method itself. A choice $\alpha = \alpha_\delta$ of the regularization parameter may be made in either an apriori (before computing, α_δ fixed) or a posteriori way (after computing we fix α_δ)(cf.[23]). The question of making an implicit (aposteriori) choice of a suitable value for the regularization parameter in ill-posed problems without the knowledge about the solution smoothness (which may not be accessible) has been discussed extensively in regularization theory (see [21], [42]). A first aposteriori rule of choice is described by Phillips in [51].

Suppose there exist a function φ on $[0, \infty)$ such that

$$x_0 - \hat{x} = \varphi(F'(\hat{x}))v \tag{1.7.1}$$

where x_0 is an initial guess, $\hat{x}$ is the solution of (1.4.1) and $F'(\hat{x})$ is the Fréchet derivative (see Definition 1.3.1) of F at $\hat{x}$ and

$$\|\hat{x} - R_\alpha y\| \leq \varphi(\alpha),$$

then φ is called a source function and the condition (1.7.1) is called source condition.

Note that (See [23]) the choice of the parameter α_δ depends on the unknown source conditions. In applications, it is desirable that α is chosen independent of the source function φ, but may depend on the data (δ, y^δ), and consequently on the regularized solutions. For linear ill-posed problems there exist many such a posteriori parameter choice strategies. These strategies include the ones proposed by Archangeli (see,[27]), [28], [16] , and [58].

In [50], Pereverzev and Schock considered an adaptive selection of the parameter which does not involve even the regularization method in an explicit manner. Let us briefly discuss this adaptive method in a general context of approximating an element $\hat{x} \in X$ by elements from a set $\{x_\alpha^\delta : \alpha > 0, \delta > 0\}$.

Suppose $\hat{x} \in X$ is to be approximated by using elements x_α^δ for $\alpha > 0, \delta > 0$. Assume that there exist increasing functions $\varphi(t)$ and $\psi(t)$ for $t > 0$ such that

$$\lim_{t \to o} \varphi(t) = 0 = \lim_{t \to o} \psi(t),$$

and

$$\|\hat{x} - x_\alpha^\delta\| \leq \varphi(t) + \frac{\delta}{\psi(t)}$$

for all $\alpha > 0, \delta > 0$. Here, the function φ may be associated with the unknown element $\hat{x}$, whereas the function ψ may be related to the method involved in obtaining x_α^δ. Note that the quantity $\varphi(\alpha) + \frac{\delta}{\psi(\alpha)}$ attains its minimum for the choice $\alpha := \alpha_\delta$ such that $\varphi(\alpha_\delta) = \frac{\delta}{\psi(\alpha_\delta)}$, that is for

$$\alpha_\delta = (\varphi\psi)^{-1}(\delta)$$

and in that case

$$\|\hat{x} - x_{\alpha_\delta}^\delta\| \leq 2\varphi(\alpha_\delta).$$

The above choice of the parameter is a priori in the sense that it depends on the unknown functions φ and ψ.

In an aposteriori choice, one finds a parameter α_δ without making use of the unknown source function φ such that one obtains an error estimate of the form

$$\|\hat{x} - x_{\alpha_\delta}^\delta\| \leq c\varphi(\alpha_\delta).$$

for some $c > 0$ with $\alpha_\delta = (\varphi\psi)^{-1}(\delta)$. The procedure considered by Pereverzev and Schock in [50] starts with a finite number of positive real numbers, α_0, α_1, α_2, . . . ,α_N, such that

$$\alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_N$$

The following theorem is essentially a reformulation of a theorem proved in [50].

Theorem 1.7.1. *([20] Theorem 4.3) Assume that there exists $i \in \{0, 1, 2, \cdots, N\}$ such that $\varphi(\alpha_i) \leq \frac{\delta}{\psi(\alpha_i)}$ and for some $\mu > 1$,*

$$\psi(\alpha_i) \leq \mu\psi(\alpha_{i-1}) \quad \forall i \in \{0, 1, 2, \cdots, N\}.$$

Let

$$l := max\{i : \varphi(\alpha_i) \leq \frac{\delta}{\psi(\alpha_i)}\} < N,$$

$$k := max\{i : \|x^\delta_{\alpha_i} - x^\delta_{\alpha_j}\| \leq 4\frac{\delta}{\psi(\alpha_j)}, \quad \forall j = 0; 1, \cdots, i.\}.$$

Then $l \leq k$ *and*

$$\|\hat{x} - x^\delta_{\alpha_k}\| \leq 6\mu\varphi(\alpha_\delta), \quad \alpha_\delta := (\varphi\psi)^{-1}(\delta)$$

1.8 Hammerstein Operators

Let a function $k(t, s, u)$ be defined for $t \in [a, b]$, $s \in [c, d]$ and $-\infty < u < \infty$. Then the nonlinear integral operator

$$Ax(t) = \int_c^d k(t, s, x(s))ds \tag{1.8.1}$$

is called an Uryson integral operator and the function $k(t, s, u)$ is called its kernel. If k has the special form $k(t, s, u) = k(t, s)f(s, u)$, then the operator A in (1.8.1) is called a Hammerstein integral operator.

Note that each Hammerstein integral operator A admits a representation of the form $A = KF$ where K is a linear integral operator with kernel $k(t, s)$:

$$Kx(t) = \int_c^d k(t, s)x(s)ds$$

and F is the nonlinear superposition operator (cf. [37])

$$Fx(s) = f(s, x(s)).$$

Hence the study of a Hammerstein operator can be reduced to the study of the linear operator K and the nonlinear operator F. An equation of of the form

$$KFx(t) = y(t) \tag{1.8.2}$$

is called a Hammerstein type operator equation ([14]).

Subject matter of this thesis is the ill-posed Hammerstein type operator equations.

1.9 Summary of the Thesis

Chapter 2: We consider an ill-posed Hammerstein type operator equation (1.8.2) with $R(K)$, the range of K not closed. For obtaining approximate solutions for the equation (1.8.2), for $n \in \mathbb{N}$ we consider $x_{n,\alpha}^{\delta}$, defined iteratively as

$$x_{n+1,\alpha}^{\delta} = x_{n,\alpha}^{\delta} - F'(x_{n,\alpha}^{\delta})^{-1}(F(x_{n,\alpha}^{\delta}) - z_{\alpha}^{\delta}), \tag{1.9.1}$$

with $x_{0,\alpha}^{\delta} = x_0$ and $z_{\alpha}^{\delta} = (K^*K + \alpha I)^{-1}K^*(y^{\delta} - KF(x_0)) + F(x_0)$.

We shall make use of the adaptive parameter selection procedure suggested by Pereverzev and Schock [50] for choosing the regularization parameter α, depending on the inexact data y^{δ} and the error δ satisfying

$$\|y - y^{\delta}\| \leq \delta. \tag{1.9.2}$$

It is shown that the method that we consider give quadratic convergence compared to the linear convergence obtained in [20].

Chapter 3: In this chapter we consider the Hilbert scale ([46]) variant of the method considered by George and Nair in [20] and obtained improved error estimate. Here we take $X = Y = Z = H$. Let $L : D(L) \subset H \to H$, be a linear, unbounded, self-adjoint, densely defined and strictly positive operator on H. We consider the Hilbert scale $(H_r)_{r\in\mathbb{R}}$ (see , [38]) generated by L for our analysis. Recall (c.f.[17])that the space H_t is the completion of $D := \cap_{k=0}^{\infty} D(L^k)$ with respect to the norm $\|x\|_t$, induced by the inner product

$$\langle u, v\rangle_t := \langle L^t u, L^t v\rangle, \quad u, v \in D. \tag{1.9.3}$$

In order to obtain stable approximate solution to (1.8.2), for $n \in \mathbf{N}$ we consider the n^{th} iterate;

$$x_{n+1,\alpha,s}^{\delta} = x_{n,\alpha,s}^{\delta} - F'(x_0)^{-1}[F(x_{n,\alpha,s}^{\delta}) - z_{\alpha,s}^{\delta}], \alpha > 0 \tag{1.9.4}$$

where $x_{0,\alpha,s}^{\delta} := x_0$ and $z_{\alpha,s}^{\delta} = F(x_0) + (K + \alpha L^s)^{-1}(y^{\delta} - KF(x_0))$, as an approximate solution for (1.8.2). Here α is the regularization parameter to be chosen appropriately depending on the inexact data y^{δ} and the error level δ satisfying (1.9.2), and for this we shall use the adaptive parameter selection procedure suggested by Pereverzev and Schock in [50].

Chapter 4: In this chapter we consider the special case of a Hammerstein type operator equation (1.8.2) when the nonlinear operator F is monotone. i.e., we take $Z = X$ and $F : D(F) \subseteq X \to X$ satisfies

$$\langle F(x_1) - F(x_2), x_1 - x_2 \rangle \geq 0, \qquad \forall x_1, x_2 \in D(F)$$

and $K : X \to Y$ is, as usual, bounded linear operator. We propose two iterative methods:

$$x^\delta_{n+1,\alpha} = x^\delta_{n,\alpha} - (F'(x^\delta_{n,\alpha}) + I)^{-1}(F(x^\delta_{n,\alpha}) - z^\delta_\alpha + (x^\delta_{n,\alpha} - x_0)),$$

and

$$\tilde{x}^\delta_{n+1} := \tilde{x}^\delta_n - (F'(x_0) + I)^{-1}(F(\tilde{x}^\delta_n) - z^\delta_\alpha + (\tilde{x}^\delta_n - x_0))$$

where x_0 is the starting point of the iterations and $z^\delta_\alpha = (K^*K + \alpha I)^{-1}K^*y^\delta$ in both cases. Note that in these methods we do not require invertibility of the Fréchet derivative $F'(.)$ as against the hypothesis in chapter 2 and chapter 3. The methods used in this chapter differ from the treatment in chapter 2 and chapter 3, in as much as, that the convergence analysis is carried out by means of suitably constructed majorizing sequences, thanks to the monotonicity of F. Further this approach enables us to get an apriori error estimate which can be used to determine the number of iterations needed to achieve a prescribed solution accuracy before actual computation takes place. Adaptive selection of the parameter in the linear part is, once again, done by the method of Pereverzev and Schock [50].

Chapter 5: We end the thesis with some concluding remarks in this chapter. □

Chapter 2

An Iterative Regularization Method for Ill-posed Hammerstein Type Operator Eqations

In this chapter we discuss in detail a combination of Newton's method and a regularization method for obtaining a stable approximate solution for ill-posed Hammerstein type operator equation. By choosing the regularization parameter according to an adaptive scheme considered by Pereverzev and Schock [50] an order optimal error estimate has been obtained. The method that we consider is shown to give quadratic convergence compared to the linear convergence obtained by George and Nair in [20].

2.1 Introduction

Regularization methods used for obtaining approximate solution of nonlinear ill-posed operator equation

$$Tx = y, \tag{2.1.1}$$

where T is a nonlinear operator with domain $D(T)$ in a Hilbert space X, and with its range $R(T)$ in a Hilbert space Y, include Tikhonov regularization (see [13, 23, 33, 53]) Landweber iteration [31], iteratively regularized Gauss-Newton method [4] and Marti's method [32]. Here the equation (2.1.1) is ill-posed in the sense that the solution of (2.1.1) does not depend continuously on the data y.

The optimality of these methods are usually obtained under a number of restrictive conditions on the operator T (see for example assumptions (10)-(14) and (93)-(98) in [54]). For the special case where T is a Hammerstein type operator, George [14], [15] and George and Nair [20] studied a new iterative regularization method and had obtained optimality under weaker conditions on T (that are more easy to verify in concrete problems).

Recall ([20]) that a Hammerstein type operator is an operator of the form $T = KF$, where $F : D(F) \subset X \mapsto Z$ is nonlinear and $K : Z \mapsto Y$ is a bounded linear operator where we take X, Y, Z to be Hilbert spaces.

So we consider an equation of form

$$KF(x) = y. \tag{2.1.2}$$

In [20], George and Nair, studied a modified form of Newton Lavrentiev Regularization (NLR) method for obtaining approximations for a solution $\hat{x} \in D(F)$ of (2.1.2), which satisfies

$$\|F(\hat{x}) - F(x_0)\| = min\{\|F(x) - F(x_0)\| : KF(x) = y, x \in D(F)\}. \tag{2.1.3}$$

In this chapter we assume that the solution $\hat{x}$ satisfies (2.1.3) and that $y^\delta \in Y$ are the available noisy data with

$$\|y - y^\delta\| \leq \delta. \tag{2.1.4}$$

The method considered in [20] gives only linear convergence. Here we attempt to obtain quadratic convergence.

Recall that a sequence (x_n) is X with $lim x_n = x^*$ is said to be convergent of order $p > 1$, if there exist positive reals β, γ, such that for all $n \in \mathbb{N}$

$$\|x_n - x^*\| \leq \beta e^{-\gamma p^n}. \tag{2.1.5}$$

If the sequence (x_n) has the property, that

$$\|x_n - x^*\| \leq \beta q^n, \qquad 0 < q < 1$$

then (x_n) is said to be linearly convergent. For an extensive discussion of convergence rate see Kelley [36].

This chapter is organized as follows. In section 2 we introduce the iterated regularization method. In section 3 we give error analysis and in section 4 we derive error bounds under general source conditions by choosing the regularization parameter by an a priori manner as well as by an adaptive scheme proposed by Pereverzev and Schock in [50]. In section 5 we consider the stopping rule and the algorithm for implementing the iterated regularization method.

2.2 Iterated Regularization Method

Assume that the function F in (2.1.2) satisfies the following:

1. F possesses a uniformly bounded Frèchet derivative $F'(.)$ in a ball $B_r(x_0)$ of radius $r > 0$ around $x_0 \in X$, where x_0 is an initial approximation for a solution $\hat{x}$ of (2.1.2).

2. There exist a constant $\kappa_0 > 0$ such that

$$||F'(x) - F'(y)|| \leq \kappa_0 ||x - y||, \qquad \forall x, y \in B_r(x_0) \tag{2.2.1}$$

3. $F'(x)^{-1}$ exist and is a bounded operator for all $x \in B_r(x_0)$.

Consider e.g.,(c.f.[54])the nonlinear Hammerstein operator equation

$$(KFx)(t) = \int_0^1 k(s,t)h(s,x(s))x(s)ds$$

with k continuous and h is differentiable with respect to the second variable. Here $F : D(F) = H^1(]0,1[) \mapsto L^2(]0,1[)$ is given by

$$F(x)(s) = h(s, x(s)), \quad s \in [0,1]$$

and $K : L^2(]0,1[) \mapsto L^2(]0,1[)$ is given by

$$Ku(t) = \int_0^1 k(s,t)u(s)ds, \quad t \in [0,1].$$

Then F is Frèchet differentiable and we have

$$[F'(x)]u(t) = \partial_2 h(t, x(t))u(t), \qquad t \in [0,1].$$

Assume that $N : H^1(]0,1[) \mapsto H^1(]0,1[)$ defined by $(Nx)(t) := \partial_2 h(t, x(t))$ is locally Lipschitz continuous, i.e., for all bounded subsets $U \subseteq H^1$ there exists $\kappa_0 := \kappa_0(U)$ such that

$$\|\partial_2 h(.,x(.)) - \partial_2 h(.,y(.))\|_{H^1} \leq \kappa_0 \|x - y\| \tag{2.2.2}$$

for all $x, y \in H^1$. Further if we assume that there exists κ_1 such that

$$\partial_2 h(t, x_0(t)) \geq \kappa_1 \qquad t \in [0,1], \tag{2.2.3}$$

then by (2.2.2) and (2.2.3), there exists a neighborhood $U(x_0)$ of x_0 in H^1 such that

$$\partial_2 h(t, x(t)) \geq \frac{\kappa_1}{2}$$

for all $t \in [0,1]$ and for all $x \in U(x_0)$. So $F'(x)^{-1}$ exists and is a bounded operator for all $x \in U(x_0)$.

Observe that (cf. [20]) equation (2.1.2) is equivalent to

$$K[F(x) - F(x_0)] = y - KF(x_0) \tag{2.2.4}$$

for a given x_0, so that the solution $\hat{x}$ of (2.1.2) is obtained by first solving

$$Kz = y - KF(x_0) \tag{2.2.5}$$

for z and then solving the nonlinear equation

$$F(x) = z + F(x_0). \tag{2.2.6}$$

For fixed $\alpha > 0$, $\delta > 0$ we consider the regularized solution of (2.2.5) with y^δ in place of y as

$$z_\alpha^\delta = (K + \alpha I)^{-1}(y^\delta - KF(x_0)) + F(x_0) \tag{2.2.7}$$

if the operator K in (2.2.5) is positive self adjoint and $Z = Y$, otherwise we consider

$$z_\alpha^\delta = (K^*K + \alpha I)^{-1} K^*(y^\delta - KF(x_0)) + F(x_0). \tag{2.2.8}$$

Note that (2.2.7) is the simplified or Lavrentiev regularized solution of equation (2.2.5) and (2.2.8) is the Tikhonov regularized solution of (2.2.5).

Now for obtaining approximate solutions for the equation (2.1.2), for $n \in \mathbb{N}$ we consider $x^\delta_{n,\alpha}$, defined iteratively as

$$x^\delta_{n+1,\alpha} = x^\delta_{n,\alpha} - F'(x^\delta_{n,\alpha})^{-1}(F(x^\delta_{n,\alpha}) - z^\delta_\alpha), \tag{2.2.9}$$

with $x^\delta_{0,\alpha} = x_0$.

Note that the iteration (2.2.9) is the Newton's method for the nonlinear problem

$$F(x) - z^\delta_\alpha = 0.$$

We shall make use of the adaptive parameter selection procedure suggested by Pereverzev and Schock [50] for choosing the regularization parameter α, depending on the inexact data y^δ and the error δ satisfying (2.1.4).

2.3 Error Analysis

For investigating the convergence of the iterate $(x^\delta_{n,\alpha})$ defined in (2.2.9) to an element $x^\delta_\alpha \in B_r(x_0)$ we introduce the following notations: Let for $n = 1, 2, 3, \cdots$,

$$\begin{aligned} \beta_n &:= \|F'(x^\delta_{n,\alpha})^{-1}\|, \\ e_n &:= \|x^\delta_{n+1,\alpha} - x^\delta_{n,\alpha}\|, \\ \gamma_n &:= \kappa_0 \beta_n e_n, \\ d_n &:= 3\gamma_n(1-\gamma_n)^{-1}, \\ \omega &:= \|F(\hat{x}) - F(x_0)\|. \end{aligned} \tag{2.3.1}$$

Further we assume that

$$\gamma_0 := \kappa_0 e_0 \beta_0 < \frac{1}{4} \tag{2.3.2}$$

and

$$\eta := 2e_0 < r. \tag{2.3.3}$$

THEOREM 2.3.1. *Suppose (2.2.1), (2.3.2) and (2.3.3) hold. Then* $x^\delta_{n,\alpha}$ *defined in (2.2.9) belong to* $B_\eta(x_0)$ *and is a Cauchy sequence with* $\lim_{n\to\infty} x^\delta_{n,\alpha} = x^\delta_\alpha \in \overline{B_\eta(x_0)} \subset B_r(x_0)$. *Further we have the following:*

$$\|x^\delta_{n,\alpha} - x^\delta_\alpha\| \le \frac{\eta d_0^{2^n-1}}{2^n} = \beta e^{-\gamma 2^n} \tag{2.3.4}$$

where $\beta = \frac{\eta}{d_0}$ *and* $\gamma = -log d_0$.

Proof. First we shall prove that

$$\|x^\delta_{n+1,\alpha} - x^\delta_{n,\alpha}\| \le \frac{3}{2}\beta_n\kappa_0\|x^\delta_{n,\alpha} - x^\delta_{n-1,\alpha}\|^2, \tag{2.3.5}$$

and then by induction we prove, $x^\delta_{n,\alpha} \in B_\eta(x_0)$.

Let $G(x) = x - F'(x)^{-1}[F(x) - z^\delta_\alpha]$. Then

$$\begin{aligned}
G(x) - G(y) &= x - y - F'(x)^{-1}[F(x) - z^\delta_\alpha] + F'(y)^{-1}[F(y) - z^\delta_\alpha] \\
&= x - y + [F'(x)^{-1} - F'(y)^{-1}]z^\delta_\alpha - F'(x)^{-1}F(x) + F'(y)^{-1}F(y) \\
&= x - y + [F'(x)^{-1} - F'(y)^{-1}](z^\delta_\alpha - F(y)) \\
&\quad -F'(x)^{-1}[F(x) - F(y)] \\
&= F'(x)^{-1}[F'(x)(x-y) - (F(x) - F(y))] \\
&\quad +F'(x)^{-1}[F'(y) - F'(x)]F'(y)^{-1}(z^\delta_\alpha - F(y)) \\
&= F'(x)^{-1}[F'(x)(x-y) - (F(x) - F(y))] \\
&\quad +F'(x)^{-1}[F'(y) - F'(x)](G(y) - y).
\end{aligned} \tag{2.3.6}$$

Now observe that $G(x^\delta_{n,\alpha}) = x^\delta_{n+1,\alpha}$, so by putting $x = x^\delta_{n,\alpha}$ and $y = x^\delta_{n-1,\alpha}$ in (2.3.6), we obtain

$$\begin{aligned}
x^\delta_{n+1,\alpha} - x^\delta_{n,\alpha} &= F'(x^\delta_{n,\alpha})^{-1}\left[F'(x^\delta_{n,\alpha})(x^\delta_{n,\alpha} - x^\delta_{n-1,\alpha}) - (F(x^\delta_{n,\alpha}) - F(x^\delta_{n-1,\alpha}))\right] \\
&\quad +F'(x^\delta_{n,\alpha})^{-1}\left[F'(x^\delta_{n-1,\alpha}) - F'(x^\delta_{n,\alpha})\right](x^\delta_{n,\alpha} - x^\delta_{n-1,\alpha}).
\end{aligned} \tag{2.3.7}$$

Thus by Lemma 1.3.2 and (2.2.1),

$$\|x^\delta_{n+1,\alpha} - x^\delta_{n,\alpha}\| \le \frac{\beta_n\kappa_0}{2}\|x^\delta_{n,\alpha} - x^\delta_{n-1,\alpha}\|^2 + \beta_n\kappa_0\|x^\delta_{n,\alpha} - x^\delta_{n-1,\alpha}\|^2. \tag{2.3.8}$$

This proves (2.3.5). Again, since

$$\begin{aligned} F'(x^\delta_{n,\alpha}) &= F'(x^\delta_{n-1,\alpha}) + F'(x^\delta_{n,\alpha}) - F'(x^\delta_{n-1,\alpha}) \\ &= F'(x^\delta_{n-1,\alpha})[I + F'(x^\delta_{n-1,\alpha})^{-1}(F'(x^\delta_{n,\alpha}) - F'(x^\delta_{n-1,\alpha}))], \end{aligned} \tag{2.3.9}$$

$$F'(x^\delta_{n,\alpha})^{-1} = [I + F'(x^\delta_{n-1,\alpha})^{-1}(F'(x^\delta_{n,\alpha}) - F'(x^\delta_{n-1,\alpha}))]^{-1}F'(x^\delta_{n-1,\alpha})^{-1}. \tag{2.3.10}$$

So if

$$\|F'(x^\delta_{n-1,\alpha})^{-1}(F'(x^\delta_{n,\alpha}) - F'(x^\delta_{n-1,\alpha}))\| \leq \beta_{n-1}\kappa_0 e_{n-1} = \gamma_{n-1} < 1,$$

then

$$\beta_n \leq \beta_{n-1}(1-\gamma_{n-1})^{-1} \tag{2.3.11}$$

and by (2.3.5)

$$\begin{aligned} e_n &\leq \frac{3}{2}\kappa_0\beta_{n-1}(1-\gamma_{n-1})^{-1}e^2_{n-1} && (2.3.12) \\ &= \frac{3}{2}\gamma_{n-1}(1-\gamma_{n-1})^{-1}e_{n-1} && (2.3.13) \\ &= \frac{1}{2}d_{n-1}e_{n-1}. && (2.3.14) \end{aligned}$$

Again by (2.3.11) and (2.3.13),

$$\begin{aligned} \gamma_n = \kappa_0 e_n \beta_n &\leq \frac{3}{2}\kappa_0\gamma_{n-1}(1-\gamma_{n-1})^{-1}e_{n-1}.\beta_{n-1}(1-\gamma_{n-1})^{-1} \\ &= \frac{3}{2}\gamma^2_{n-1}(1-\gamma_{n-1})^{-2}. \end{aligned} \tag{2.3.15}$$

The above relation together with $\gamma_0 = \kappa_0 e_0 \beta_0 < \frac{1}{4}$ implies $\gamma_n < \frac{1}{4}$. Consequently by (2.3.13),

$$e_n < \frac{1}{2}e_{n-1}, \tag{2.3.16}$$

for all $n \geq 1$. So $e_n \leq 2^{-n}e_0$, and hence

$$\begin{aligned} \|x^\delta_{n+1,\alpha} - x_0\| &\leq \sum_{j=0}^{n} \|x^\delta_{j+1,\alpha} - x^\delta_{j,\alpha}\| \\ &\leq \sum_{j=0}^{n} 2^{-j}e_0 \\ &\leq 2e_0 < r. \end{aligned}$$

Thus $(x^\delta_{n,\alpha})$ is well defined and is a Cauchy sequence with $x^\delta_\alpha = \lim_{n\to\infty} x^\delta_{n,\alpha} \in \overline{B_\eta(x_0)} \subset B_r(x_0)$. So from (2.2.9), it follows that $F(x^\delta_\alpha) = z^\delta_\alpha$.

Further note that since $\gamma_n \leq 1/4$, and by (2.3.15) we have

$$d_n = 3\gamma_n(1-\gamma_n)^{-1} < 4\gamma_n < 4.\frac{3}{2}\gamma_{n-1}^2(1-\gamma_{n-1})^{-2} < d_{n-1}^2.$$

Hence

$$d_n \leq d_0^{2^n}, \tag{2.3.17}$$

consequently, by (2.3.14), (2.3.16) and (2.3.17)

$$e_n \leq \frac{1}{2}d_{n-1}e_{n-1} \;\leq\; 2^{-n}d_0^{2^{n-1}}e_0.$$

Therefore

$$\begin{aligned}
\|x_{n,\alpha}^\delta - x_\alpha^\delta\| &= \lim_i \|x_{n,\alpha}^\delta - x_{n+i,\alpha}^\delta\| \leq \sum_{j=n}^{\infty} e_j \\
&\leq \sum_{j=n}^{\infty} 2^{-j}d_0^{2^{j-1}}e_0 \leq 2.2^{-n}d_0^{2^{n-1}}e_0 = \frac{2e_0 d_0^{2^{n-1}}}{2^n} \qquad (2.3.18)\\
&\leq \frac{\eta d_0^{2^{n-1}}}{2^n} = \frac{\eta}{d_0 2^n}e^{-\gamma 2^n} \\
&= \frac{\eta}{d_0}e^{-\gamma 2^n} = \beta e^{-\gamma 2^n}.
\end{aligned}$$

This completes the proof.

REMARK 2.3.2. *Note that $\gamma > 0$ because $\gamma_0 < 1/4 \Longrightarrow d_0 < 1$. So by (2.1.5), sequence $(x_{n,\alpha}^\delta)$ converges quadratically to x_α^δ.*

THEOREM 2.3.3. *Suppose (2.2.1), (2.3.2) and (2.3.3) hold. If, in addition, $\|x_0 - \hat{x}\| \leq \eta < r < \frac{1}{\beta_0\kappa_0}$, then*

$$\|\hat{x} - x_\alpha^\delta\| \leq \frac{\beta_0}{1-\beta_0\kappa_0 r}\|F(\hat{x}) - z_\alpha^\delta\|.$$

Proof. Observe that

$$\begin{aligned}
\|\hat{x} - x_\alpha^\delta\| &= \|\hat{x} - x_\alpha^\delta + F'(x_0)^{-1}[F(x_\alpha^\delta) - F(\hat{x}) + F(\hat{x}) - z_\alpha^\delta]\| \\
&\leq \|F'(x_0)^{-1}[F'(x_0)(\hat{x} - x_\alpha^\delta) - (F(\hat{x}) - F(x_\alpha^\delta))]\| + \|F'(x_0)^{-1}[F(\hat{x}) - z_\alpha^\delta]\| \\
&\leq \beta_0\kappa_0 r\|\hat{x} - x_\alpha^\delta\| + \beta_0\|F(\hat{x}) - z_\alpha^\delta\|.
\end{aligned}$$

Thus

$$(1-\beta_0\kappa_0 r)\|\hat{x}-x_\alpha^\delta\| \leq \beta_0\|F(\hat{x})-z_\alpha^\delta\|.$$

This completes the proof.

REMARK 2.3.4. *If z_α^δ is as in (2.2.8)and if $\|F(x_0)-F(\hat{x})\|+\frac{\delta}{\sqrt{\alpha}}<\frac{r}{2\beta_0}<\frac{1}{2\beta_0^2\kappa_0}$ then $\|x_0-\hat{x}\|\leq\eta<r<\frac{1}{\beta_0\kappa_0}$, holds (see section 2.5).*

The following Theorem is a consequence of Theorem 3.3.4 and Theorem 2.3.3

THEOREM 2.3.5. *Suppose (2.2.1), (2.3.2) and (2.3.3) hold. If, in addition, $\beta_0\kappa_0 r<1$, then*

$$\|\hat{x}-x_{n,\alpha}^\delta\| \leq \frac{\beta_0}{1-\beta_0\kappa_0 r}\|F(\hat{x})-z_\alpha^\delta\|+\frac{\eta d_0^{2^n-1}}{2^n}.$$

REMARK 2.3.6. *Hereafter we consider z_α^δ as the Tikhonov regularization of (2.2.5) given in (2.2.8). All results in the forthcoming sections are valid for the simplified regularization of (2.2.5).*

In view of the estimate in the Thcorcm 2.3.5, the next task is to find an estimate $\|F(\hat{x})-z_\alpha^\delta\|$. For this, let us introduce the notation;

$$z_\alpha := F(x_0)+(K^*K+\alpha I)^{-1}K^*(y-KF(x_0)).$$

We may observe that

$$\begin{aligned}\|F(\hat{x})-z_\alpha^\delta\| &\leq \|F(\hat{x})-z_\alpha\|+\|z_\alpha-z_\alpha^\delta\| \\ &\leq \|F(\hat{x})-z_\alpha\|+\frac{\delta}{\sqrt{\alpha}}, \qquad (2.3.19)\end{aligned}$$

and

$$\begin{aligned}F(\hat{x})-z_\alpha &= F(\hat{x})-F(x_0)-(K^*K+\alpha I)^{-1}K^*K[F(\hat{x})-F(x_0)] \\ &= [I-(K^*K+\alpha I)^{-1}K^*K][F(\hat{x})-F(x_0)] \\ &= \alpha(K^*K+\alpha I)^{-1}[F(\hat{x})-F(x_0)]. \qquad (2.3.20)\end{aligned}$$

Note that for $u \in R(K^*K)$ with $u = K^*Kz$ for some $z \in Z$,

$$\|\alpha(K^*K+\alpha I)^{-1}u\| = \|\alpha(K^*K+\alpha I)^{-1}K^*Kz\| \le \alpha\|z\| \to 0$$

as $\alpha \to 0$. Now since $\|\alpha(K^*K+\alpha I)^{-1}\| \le 1$ for all $\alpha > 0$, it follows that for every $u \in \overline{R(K^*K)}$, $\|\alpha(K^*K+\alpha I)^{-1}u\| \to 0$ as $\alpha \to 0$. Thus we have the following theorem.

THEOREM 2.3.7. *If $F(\hat{x}) - F(x_0) \in \overline{R(K^*K)}$, then $\|F(\hat{x}) - z_\alpha\| \to 0$ as $\alpha \to 0$.*

2.4 Error Bounds Under Source Conditions

In view of the above theorem, we assume that

$$\|F(\hat{x}) - z_\alpha\| \le \varphi(\alpha) \tag{2.4.1}$$

for some positive monotonic increasing function φ defined on $(0, \|K\|^2]$ such that

$$\lim_{\lambda \to 0} \varphi(\lambda) = 0.$$

Suppose φ is a source function in the sense that $\hat{x}$ satisfies a source condition of the form

$$F(\hat{x}) - F(x_0) = \varphi(K^*K)w, \qquad \|w\| \le 1,$$

such that

$$\sup_{0<\lambda<\|K\|^2} \frac{\alpha\varphi(\lambda)}{\lambda+\alpha} \le \varphi(\alpha), \tag{2.4.2}$$

then the assumption (2.4.1) is satisfied. For example if $F(\hat{x}) - F(x_0) \in R((K^*K)^\nu)$, for some ν with, $0 < \nu \le 1$, then by (2.3.20)

$$\begin{aligned}\|F(\hat{x}) - z_\alpha\| &\le \|\alpha(K^*K+\alpha I)^{-1}(K^*K)^\nu\omega\| \\ &\le \sup_{0<\lambda\le\|K\|^2} \frac{\alpha\lambda^\nu}{\lambda+\alpha}\|\omega\| \le \alpha^\nu\|\omega\|.\end{aligned}$$

Thus in this case $\varphi(\lambda) = \lambda^\nu\|\omega\|$ satisfies the assumption (2.4.1). Therefore by (2.3.19) and by the assumption (2.4.1), we have

$$\|F(\hat{x}) - z_\alpha^\delta\| \le \varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}. \tag{2.4.3}$$

So, we have the following theorem.

THEOREM 2.4.1. *Under the assumptions of Theorem 2.3.5 and (2.4.3),*

$$\|\hat{x} - x_{n,\alpha}^{\delta}\| \leq \frac{\beta_0}{1-\beta_0\kappa_0 r}(\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}) + \frac{\eta d_0^{2^{n-1}}}{2^n}.$$

2.4.1 Apriori Choice of the Parameter

Note that the estimate $\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}$ in (2.4.2) attains minimum for the choice $\alpha := \alpha_\delta$ which satisfies $\varphi(\alpha_\delta) = \frac{\delta}{\sqrt{\alpha_\delta}}$. Let $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \leq \|K\|^2$. Then we have $\delta = \sqrt{\alpha_\delta}\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$, and

$$\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta)). \tag{2.4.4}$$

So the relation (2.4.3) leads to

$$\|F(\hat{x}) - z_\alpha^\delta\| \leq 2\psi^{-1}(\delta).$$

Theorem 2.4.1 and the above observation leads to the following.

THEOREM 2.4.2. *Let $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \leq \|K\|^2$ and the assumptions of Theorem 2.3.5 and (2.4.1) are satisfied. For $\delta > 0$, let $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$. If*

$$n_\delta := min\{n : \frac{r d_0^{2^{n-1}}}{2^n} < \frac{\delta}{\sqrt{\alpha_\delta}}\},$$

then

$$\|\hat{x} - x_{\alpha_\delta,n_\delta}^\delta\| = O(\psi^{-1}(\delta)).$$

2.4.2 An Adaptive Choice of the Parameter

The error estimate in the above Theorem has optimal order with respect to δ. Unfortunately, an a priori parameter choice (2.4.4) cannot be used in practice since the smoothness properties of the unknown solution $\hat{x}$ reflected in the function φ are generally unknown. There exist many parameter choice strategies in the literature, for example see [5], [27], [28], [16], [18], [52] and [58].

In [50], Pereverzev and Schock considered an adaptive selection of the parameter which does not involve even the regularization method in an explicit manner. In this method the regularization parameter α_i are selected from some finite set $\{\alpha_i : 0 < \alpha_0 < \alpha_1 < \cdots < \alpha_N\}$ and the corresponding regularized solution, say $u^\delta_{\alpha_i}$ are studied on-line. Later George and Nair [20] considered this adaptive selection of the parameter for choosing the regularization parameter in Newton-Lavrentiev regularization method for solving Hammerstein type operator equation. We too follow the same adaptive method for selecting the parameter α in $x^\delta_{\alpha,n}$. Rest of this section is essentially a reformulation of the adaptive method considered in [50] in this special context.

Let $i \in \{0, 1, 2, \cdots, N\}$ and $\alpha_i = \mu^{2i}\alpha_0$ where $\mu > 1$ and $\alpha_0 = \delta^2$. Let

$$l := max\{i : \varphi(\alpha_i) < N \leq \frac{\delta}{\sqrt{\alpha_i}}\} \tag{2.4.5}$$

and

$$k := max\{i : \|z^\delta_{\alpha_i} - z^\delta_{\alpha_j}\| \leq \frac{4\delta}{\sqrt{\alpha_j}}, j = 0, 1, 2, \cdots, i\}. \tag{2.4.6}$$

The proof of the next theorem is analogous to the proof of Theorem 1.2 in [50], but for the sake of completeness, we supply its proof as well.

THEOREM 2.4.3. *Let l be as in (2.4.5), k be as in (2.4.6) and $z^\delta_{\alpha_k}$ be as in (2.2.8) with $\alpha = \alpha_k$. Then $l \leq k$ and*

$$\|F(\hat{x}) - z^\delta_{\alpha_k}\| \leq (2 + \frac{4\mu}{\mu - 1})\mu\psi^{-1}(\delta).$$

Proof. Note that, to prove $l \leq k$, it is enough to prove that, for $i = 1, 2, \cdots, N$

$$\varphi(\alpha_i) \leq \frac{\delta}{\sqrt{\alpha_i}} \implies \|z^\delta_{\alpha_i} - z^\delta_{\alpha_j}\| \leq \frac{4\delta}{\sqrt{\alpha_j}}, \forall j = 0, 1, 2, \cdots, i.$$

For $j \leq i$,

$$\begin{aligned}
\|z_{\alpha_i}^\delta - z_{\alpha_j}^\delta\| &\leq \|z_{\alpha_i}^\delta - F(\hat{x})\| + \|F(\hat{x}) - z_{\alpha_j}^\delta\| \\
&\leq \varphi(\alpha_i) + \frac{\delta}{\sqrt{\alpha_i}} + \varphi(\alpha_j) + \frac{\delta}{\sqrt{\alpha_j}} \\
&\leq \frac{2\delta}{\sqrt{\alpha_i}} + \frac{2\delta}{\sqrt{\alpha_j}} \\
&\leq \frac{4\delta}{\sqrt{\alpha_j}}.
\end{aligned}$$

This proves the relation $l \leq k$. Now since $\sqrt{\alpha_{l+m}} = \mu^m \sqrt{\alpha_l}$, by using triangle inequality successively, we obtain for $l < k$,

$$\begin{aligned}
\|F(\hat{x}) - z_{\alpha_k}^\delta\| &\leq \|F(\hat{x}) - z_{\alpha_l}^\delta\| + \sum_{j=l+1}^{k} \frac{4\delta}{\sqrt{\alpha_{j-1}}} \\
&\leq \|F(\hat{x}) - z_{\alpha_l}^\delta\| + \sum_{m=0}^{k-l-1} \frac{4\delta}{\sqrt{\alpha_l}\mu^m} \\
&\leq \|F(\hat{x}) - z_{\alpha_l}^\delta\| + (\frac{\mu}{\mu-1})\frac{4\delta}{\sqrt{\alpha_l}}
\end{aligned}$$

Therefore by (2.4.2) and (2.4.5) we have

$$\begin{aligned}
\|F(\hat{x}) - z_{\alpha_k}^\delta\| &\leq \varphi(\alpha_l) + \frac{\delta}{\sqrt{\alpha_l}} + (\frac{\mu}{\mu-1})\frac{4\delta}{\sqrt{\alpha_l}} \\
&\leq (2 + \frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta). \qquad (2.4.7)
\end{aligned}$$

The last step follows from the inequality $\sqrt{\alpha_\delta} \leq \sqrt{\alpha_{l+1}} \leq \mu\sqrt{\alpha_l}$ and $\frac{\delta}{\sqrt{\alpha_\delta}} = \psi^{-1}(\delta)$. Note that (2.4.7) holds for the case $k = l$ as well. This completes the proof.

2.5 Stopping Rule

Note that

$$\begin{aligned}
e_0 = \|x_{1,\alpha}^\delta - x_0\| &= \|F'(x_0)^{-1}(K^*K + \alpha I)^{-1}K^*(y^\delta - KF(x_0))\| \\
&= \|F'(x_0)^{-1}(K^*K + \alpha I)^{-1}K^*(y^\delta - y + y - KF(x_0))\| \\
&\leq \beta_0(\|(K^*K + \alpha I)^{-1}K^*(y^\delta - y)\| + \\
&\quad \|(K^*K + \alpha I)^{-1}K^*K(F(\hat{x}) - F(x_0))\|) \\
&\leq \beta_0(\omega + \frac{\delta}{\sqrt{\alpha}}),
\end{aligned}$$

so if

$$\omega + \frac{\delta}{\sqrt{\alpha}} < \frac{1}{2\beta_0} min\{r, \frac{1}{2\beta_0\kappa_0}\}, \tag{2.5.1}$$

then $2e_0 \leq 2\beta_0(\omega + \frac{\delta}{\sqrt{\alpha}}) < r$, and

$$\gamma_0 = e_0\beta_0\kappa_0 < \frac{1}{4}.$$

Again since $\alpha_j = \mu^{2j}\delta^2$, $\frac{\delta}{\sqrt{\alpha_k}} = \mu^{-k}$; the condition (2.5.1) with $\alpha = \alpha_k$ takes the form

$$\omega + \frac{1}{\mu^k} < \frac{1}{2\beta_0} min\{r, \frac{1}{2\beta_0\kappa_0}\}. \tag{2.5.2}$$

Note that if we assume that $2\beta_0\kappa_0 r < 1$, then condition (2.5.2) takes the form $\omega + \frac{1}{\mu^k} < \frac{r}{2\beta_0}$. So if we assume

$$r < 2\beta_0(1+\omega), \qquad \frac{1}{\mu} + \omega < \frac{r}{2\beta_0}$$

then $\mu > 1$ and (2.3.2) and (2.3.3) hold. The above discussion leads to the following theorem.

THEOREM 2.5.1. *Assume that* $\mu > \frac{2\beta_0}{r - 2\beta_0\omega}$, $2\beta_0\omega < r < min\{2\beta_0(1+\omega), \frac{1}{2\beta_0\kappa_0}\}$. *Let* $\alpha_0 = \delta^2$, $\alpha_j = \mu^{2j}\delta^2$ *for* $j = 1, 2, \cdots, N$ *and* $k := max\{i : \|z^\delta_{\alpha_i} - z^\delta_{\alpha_j}\| \leq 4\mu^{-j}, j = 0, 1, 2, \cdots, i\}$. *Then*

$$\|F(\hat{x}) - z^\delta_{\alpha_k}\| \leq (2 + \frac{4\mu}{\mu - 1})\mu\psi^{-1}(\delta)$$

where $\psi(t) = t\sqrt{\varphi^{-1}(t)}$ *for* $0 < t < \|K\|^2$. *Further* $\gamma_k := \beta_k e_k \kappa_0 < \frac{1}{4}$ *and if*

$$n_k := min\{n : \frac{r d_0^{2^n - 1}}{2^n} < \frac{1}{\mu^k}\};$$

then

$$\|\hat{x} - x^\delta_{n_k, \alpha_k}\| = O(\psi^{-1}(\delta)).$$

2.5.1 Algorithm:

Note that for $i, j \in \{0, 1, 2, \cdots, n\}$

$$\|z^\delta_{\alpha_i} - z^\delta_{\alpha_j}\| = (\alpha_j - \alpha_i)(K^*K + \alpha_j I)^{-1}(K^*K + \alpha_i I)^{-1}K^*(y^\delta - KF(x_0)).$$

Therefore the adaptive algorithm associated with the choice of the parameter specified in the above theorem is as follows.

```
begin
   i=0
    repeat
      i=i+1
```

Solve for $w_i : (K^*K + \alpha_i I)w_i = K^*(y^\delta - KF(x_0))$

```
        j=-1
        repeat
          j=j+1
```

Solve for $z_{i,j} : (K^*K + \alpha_j I)z_{i,j} = (\alpha_j - \alpha_i)w_i$

until($\|z_{i,j}\| \leq 4\mu^{-j}$AND $j < i$)

until($\|z_{i,j}\| \leq 4\mu^{-j}$)

k = i-1.

m=0

```
repeat
  m=m+1
```

until$(\frac{rd_0^{2^{m-1}}}{2^m} > \frac{1}{\mu^k})$

$n_k = m$

for l=1 to n_k

Solve for $u_{l-1} : F'(x^\delta_{l-1,\alpha_k})u_{l-1} = F(x^\delta_{l-1,\alpha_k}) - z^\delta_{\alpha_k}$

$x^\delta_{l,\alpha_k} := x^\delta_{l-1,\alpha_k} - u_{l-1}$

end □

Chapter 3

Iterative Regularization Methods for Ill-posed Hammerstein Type Operator Equations in Hilbert Scales

In this chapter we report on a method for regularizing a nonlinear Hammerstein type operator equation in Hilbert scales. The proposed method is a combination of Lavrentieve regularization methods in Hilbert scales and a modified Newton's iteration. Under the assumptions that the operator F is continuously Fréchet differentiable with a Lipschitz-continuous first derivative and that the solution of (3.1.1) fulfills a general source condition, we give an optimal order convergence rate result with respect to the general source function.

3.1 Introduction

Let us take $X = Y = Z = H$ and consider an ill-posed Hammertein type operator equation

$$KF(x) = y, \tag{3.1.1}$$

where $K : H \mapsto H$ is a positive self-adjoint operator with its range $R(K)$ not closed in H and $F : D(F) \subseteq H \mapsto H$ is a nonlinear operator. The equation (3.1.1) is ill-posed, in the sense that a unique solution that depends continuously on the data does not exist.

In [20], George and Nair studied a modified NLR method for obtaining an approximation for the x_0-minimum norm solution (x_0-MNS) of the equation (3.1.1). Recall that a solution $\hat{x} \in D(F)$ of (3.1.1) is called an x_0-MNS of (3.1.1), if

$$\|F(\hat{x}) - F(x_0)\| = min\{\|F(x) - F(x_0)\| : KF(x) = y,\, x \in D(F)\}. \tag{3.1.2}$$

As in chapter 2, we assume the existence of an x_0-MNS for exact data y, i.e.,

$$KF(\hat{x}) = y.$$

Not that, due to the nonlinearity of F, the above solution need not be unique. The element $x_0 \in X$ in (4.1.2) plays the role of a selection criterion.

Further we assume that $y^\delta \in H$ are the available noisy data with

$$\|y - y^\delta\| \leq \delta. \tag{3.1.3}$$

Since (3.1.1) is ill-posed, regularization methods are to be employed for obtaining a stable approximate solution for (3.1.1). See, for example [7], [10], [11], [58], [45] for various regularization methods for ill-posed operator equations.

In [20], George and Nair considered the n^{th} iterate

$$x_{n,\alpha}^\delta = x_{n-1,\alpha}^\delta - F'(x_0)^{-1}[F(x_{n-1,\alpha}^\delta) - F(x_0) - (K + \alpha I)^{-1}(y^\delta - KF(x_0))] \tag{3.1.4}$$

as an approximation for the x_0-minimum norm solution of (3.1.1). In order to improve the error estimate available in [20], in this chapter we consider the Hilbert scale variant of (3.1.4).

Let $L : D(L) \subset H \to H$, be a linear, unbounded, self-adjoint, densely defined and strictly positive operator on H. We consider the Hilbert scale $(H_r)_{r\in\mathbb{R}}$ (see ,[17], [19], [38] and [45]) generated by L for our analysis. Recall (c.f.[17]) that the space H_t is the completion of $D := \cap_{k=0}^{\infty} D(L^k)$ with respect to the norm $\|x\|_t$, induced by the inner product

$$\langle u, v\rangle_t := \langle L^t u, L^t v\rangle, \quad u, v \in D. \tag{3.1.5}$$

In order to obtain stable approximate solution to (3.1.1), for $n \in \mathbf{N}$ we consider the n^{th} iterate;

$$x^{\delta}_{n+1,\alpha,s} = x^{\delta}_{n,\alpha,s} - F'(x_0)^{-1}[F(x^{\delta}_{n,\alpha,s}) - z^{\delta}_{\alpha,s}], \alpha > 0 \tag{3.1.6}$$

where $x^{\delta}_{0,\alpha,s} := x_0$ and $z^{\delta}_{\alpha,s} = F(x_0) + (K + \alpha L^s)^{-1}(y^{\delta} - KF(x_0))$, as an approximate solution for (3.1.1). Here α is the regularization parameter to be chosen appropriately depending on the inexact data y^{δ} and the error level δ satisfying (3.1.3).

Note that, if $D(L) = H$ and $L = I$, then the above procedure is the modified Newton-Lavrentieve regularization method considered in [20]. Further note that under the assumptions on L, the iterates in (3.1.6) is well defined. We observe that, regularization methods for nonlinear ill-posed problems in Hilbert scales, an assumption of the form;

$$m\|x\|_{-a} \leq \|F'(x_0)x\| \tag{3.1.7}$$

on the smoothness of $F'(x_0)$ is used (cf.[57]). Another feature of the proposed method is that no assumption of the form (3.1.7) on $F'(x_0)$ is used in our analysis.

Again in many cases one is not interested in completely knowing $\hat{x}$, but some derived quantities of $\hat{x}$ (see [42], [22]). Often such derived quantities correspond to bounded linear functionals of the solution. Then the problem is to estimate $\langle f, \hat{x}\rangle$, where f is any given functional. A straight forward approach to find an approximation to $\langle f, \hat{x}\rangle$, is to find some approximate solution of (3.1.1) and then apply the given functional to this. This approach is referred to as the solution-functional strategy (cf.[1]).

Note that, if $f \in H_u$ for some u, then

$$|\langle f, x\rangle| \leq \|f\|_u \|x\|_{-u}$$

for all $x \in H$. Thus to obtain an estimate for $|\langle f, \hat{x}\rangle - \langle f, x^{\delta}_{n,\alpha,s}\rangle|$, it is enough to find an estimate for $\|\hat{x} - x^{\delta}_{n,\alpha,s}\|_{-u}$. So our main aim in this chapter is to obtain an optimal order error estimate for $\|\hat{x} - x^{\delta}_{n,\alpha,s}\|_{-u}$ under an apriori and an aposteriori parameter choice strategy.

In section 2 we give some preliminary results which are required in the remaining sections of the chapter. In section 3 we derived error bounds for $\|x^{\delta}_{n,\alpha,s} - \hat{x}\|_{-u}$. In section 4 we derived optimal order error bounds for $\|x^\delta_{n,\alpha,s} - \hat{x}\|_{-u}$, under general source condition, provided α and n are chosen apriorily. In section 5 we considered an adaptive scheme for choosing the regularization parameter α and in section 5.1 we considered a stopping rule for the iterative index n.

3.2 Preliminaries

Let $K \in L(H)$ be a bounded, positive self-adjoint operator on H (i.e., $\langle Kx, x\rangle \geq 0$ for every $x \in H$) with its range $R(K)$ not closed in H. Let us introduce the operator

$$K_s := L^{-s/2}KL^{-s/2}. \tag{3.2.1}$$

Note that the operator K_s is a positive and self-adjoint bounded operator on H. We shall make use of the relation

$$\|(K_s + \alpha I)^{-1}K_s^\tau\| \leq \alpha^{\tau-1}, \quad \alpha > 0, \quad 0 < \tau \leq 1, \tag{3.2.2}$$

which follows from the spectral properties of the positive self adjoint operator K_s, $s > 0$.

We need the following assumptions for our analysis.

ASSUMPTION (A1) There exist constants $c_1 > 0, c_2 > 0$ and $a > 0$ such that

$$c_1\|x\|_{-a} \leq \|Kx\| \leq c_2\|x\|_{-a}. \tag{3.2.3}$$

ASSUMPTION (A2)$\|F(\hat{x}) - F(x_0)\|_t \leq E$ for some $t \geq 0$.

Further we need the function f and g; defined by

$$f(v) = min\{c_1^v, c_2^v\}, \qquad g(v) = max\{c_1^v, c_2^v\}, \quad v \in \mathbb{R}, |v| \leq 1, \tag{3.2.4}$$

respectively. One of the crucial results for proving the results in this chapter is the following Proposition.

PROPOSITION 3.2.1. *(See [17], Proposition3.1) For $s \geq 0$ and $|\nu| \leq 1$,*

$$f(\nu/2)\|x\|_{-\nu(s+a)/2} \leq \|K_s^{\nu/2}x\| \leq g(\nu/2)\|x\|_{-\nu(s+a)/2}, \qquad x \in H. \tag{3.2.5}$$

Let

$$z^\delta_{\alpha,s} := F(x_0) + (K + \alpha L^s)^{-1}(y^\delta - KF(x_0)) \tag{3.2.6}$$

and

$$z_{\alpha,s} := F(x_0) + (K + \alpha L^s)^{-1}(y - KF(x_0)). \tag{3.2.7}$$

THEOREM 3.2.2. *Suppose that Assumption A2 holds for, $0 < u + t \leq s + a$, $0 < u \leq a$ and $\alpha > 0$. Then*

$$\|z^\delta_{\alpha,s} - z_{\alpha,s}\|_{-u} \leq \psi(s)\alpha^{(u-a)/(s+a)}\delta, \tag{3.2.8}$$

$$\|F(x_0) - z_{\alpha,s}\|_{-u} \leq \psi_1(s)\|F(\hat{x}) - F(x_0)\|_{-u}, \tag{3.2.9}$$

$$\|F(\hat{x}) - z_{\alpha,s}\|_{-u} \leq \phi(s,t)\alpha^{(u+t)/(s+a)}E, \tag{3.2.10}$$

where $\psi(s) = \frac{g(-s/(2s+2a))}{f((2u+s)/(2s+2a))}$, $\psi_1(s) = \frac{g((2u+s)/(2s+2a))}{f((2u+s)/(2s+2a))}$ *and* $\phi(s,t) = \frac{g((s-2t)/(2s+2a))}{f((2u+s)/(2s+2a))}$.

Proof. Not that

$$\begin{aligned}\|z^\delta_{\alpha,s} - z_{\alpha,s}\|_{-u} &= \|(K + \alpha L^s)^{-1}(y^\delta - y)\|_{-u} \\ &= \|L^{-(u+s/2)}(K_s + \alpha I)^{-1}L^{-s/2}(y^\delta - y)\|\end{aligned}$$

now by taking $\nu = (2u+s)/(s+a)$ and $x = (K_s + \alpha I)^{-1}L^{-s/2}(y^\delta - y)$ in proposition 3.2.1, we have

$$\begin{aligned}\|z^\delta_{\alpha,s} - z_{\alpha,s}\|_{-u} &\leq \frac{1}{f(\frac{2u+s}{2s+2a})}\|K_s^{(2u+s)/(2s+2a)}(K_s + \alpha I)^{-1}L^{-s/2}(y^\delta - y)\| \\ &= \frac{1}{f(\frac{2u+s}{2s+2a})}\|(K_s + \alpha I)^{-1}K_s^{(2u+s)/(2s+2a)}L^{-s/2}(y^\delta - y)\| \\ &\leq \frac{1}{f(\frac{2u+s}{2s+2a})}\|(K_s + \alpha I)^{-1}K_s^{(u+s)/(s+a)}\| \\ &\quad \times\|K_s^{-s/(2s+2a)}L^{-s/2}(y^\delta - y)\|\end{aligned} \tag{3.2.11}$$

We note that the relation (3.2.2) with $\tau = (u+s)/(s+a)$ gives

$$\|(K_s + \alpha I)^{-1}K_s^{s/(s+a)}\| \leq \alpha^{(u-a)/(s+a)} \tag{3.2.12}$$

and Proposition 3.2.1, with $\nu = -s/(s+a)$ and $x = L^{-s/2}(y^\delta - y)$, gives

$$\begin{aligned} \|K_s^{-s/(2s+2a)} L^{-s/2}(y^\delta - y)\| &\leq g(\frac{-s}{2s+2a})\|L^{-s/2}(y^\delta - y)\|_{s/2} \\ &\leq g(\frac{-s}{2s+2a})\|y^\delta - y\|. \end{aligned} \tag{3.2.13}$$

Now (3.2.8) follows from (3.2.11), (3.2.12), (3.2.13) and (3.1.3). Again

$$\begin{aligned} \|z_{\alpha,s} - F(x_0)\|_{-u} &= \|(K+\alpha L^s)^{-1}K(F(\hat{x}) - F(x_0))\|_{-u} \\ &= \|L^{-(u+s/2)}(K_s+\alpha I)^{-1}L^{-s/2}K(F(\hat{x}) - F(x_0))\| \\ &= \|L^{-(u+s/2)}(K_s+\alpha I)^{-1}K_s L^{s/2}(F(\hat{x}) - F(x_0))\|. \end{aligned} \tag{3.2.14}$$

So by taking $\nu = (2u+s)/(s+a)$ and $x = (K_s+\alpha I)^{-1}K_s L^{s/2}(F(\hat{x}) - F(x_0))$ in Proposition 3.2.1, we obtain

$$\begin{aligned} &\|L^{-((2u+s)/2)}(K_s+\alpha I)^{-1}K_s L^{s/2}(F(\hat{x}) - F(x_0))\| \\ \leq\ & \frac{1}{f(\frac{2u+s}{2s+2a})}\|K_s^{(2u+s)/(2s+2a)}(K_s+\alpha I)^{-1}K_s L^{s/2}(F(\hat{x}) - F(x_0))\| \\ =\ & \frac{1}{f(\frac{2u+s}{2s+2a})}\|(K_s+\alpha I)^{-1}K_s K_s^{(2u+s)/(2s+2a)} L^{s/2}(F(\hat{x}) - F(x_0))\|. \end{aligned} \tag{3.2.15}$$

Now by taking $\nu = (2u+s)/(s+a)$ and $x = L^{s/2}(F(\hat{x}) - F(x_0))$ in Proposition 3.2.1, we have

$$\begin{aligned} \|K_s^{(2u+s)/(2s+2a)} L^{s/2}(F(\hat{x}) - F(x_0))\| &\leq g(\frac{2u+s}{2s+2a})\|L^{s/2}(F(\hat{x}) - F(x_0))\|_{-(u+s/2)} \\ &\leq g(\frac{u+s}{2s+2a})\|F(\hat{x}) - F(x_0)\|_{-u}. \end{aligned} \tag{3.2.16}$$

Thus by (3.2.14), (3.2.15), (3.2.16) and the relation $\|(K_s+\alpha I)^{-1}K_s\| \leq 1$;

$$\|z_{\alpha,s} - F(x_0)\|_{-u} \leq \psi_1(s)\|F(\hat{x}) - F(x_0)\|_{-u}. \tag{3.2.17}$$

Further we observe that

$$\begin{aligned}
\|z_{\alpha,s} - F(\hat{x})\|_{-u} &= \|((K+\alpha L^s)^{-1}K - I)(F(\hat{x}) - F(x_0))\|_{-u} \\
&= \|\alpha L^{-(u+s/2)}(K_s+\alpha I)^{-1}L^{s/2}(F(\hat{x}) - F(x_0))\| \\
&\leq \frac{1}{f(\frac{2u+s}{2s+2a})}\|K_s^{(2u+s)/(2s+2a)}\alpha(K_s+\alpha I)^{-1}L^{s/2}(F(\hat{x}) - F(x_0))\| \\
&= \frac{1}{f(\frac{2u+s}{2s+2a})}\|\alpha(K_s+\alpha I)^{-1}K_s^{(u+t)/(s+a)}K_s^{(s-2t)/(2s+2a)} \\
&\quad \times L^{s/2}(F(\hat{x}) - F(x_0))\| \\
&\leq \frac{1}{f(\frac{2u+s}{2s+2a})}\|\alpha(K_s+\alpha I)^{-1}K_s^{(u+t)/(s+a)}\|\|K_s^{(s-2t)/(2s+2a)} \\
&\quad \times L^{s/2}(F(\hat{x}) - F(x_0))\| \\
&\leq \frac{g(\frac{s-2t}{2s+2a})}{f(\frac{2u+s}{2s+2a})}\alpha^{(u+t)/(s+a)}\|L^{s/2}(F(\hat{x}) - F(x_0))\|_{t-s/2} \\
&\leq \varphi(s,t)\alpha^{(u+t)/(s+a)}E. \qquad (3.2.18)
\end{aligned}$$

3.3 Error Analysis

In addition to the assumptions on K, we assume that, F possess a uniformly bounded Fréchet derivative $F'(.)$ in a ball $B_r(x_0) \subset H_{-u}, 0 < u \leq a$ of radius $r > 0$ such that

$$\|F'(x) - F'(y)\|_{-u} \leq \kappa_0\|x-y\|_{-u}, \qquad x, y \in B_r(x_0), \qquad (3.3.1)$$

and that $F'(x_0)^{-1}$ exists and is a bounded operator. Further we assume that,

$$\|L^{-u}F'(x_0)^{-1}L^u\| = \beta < \infty. \qquad (3.3.2)$$

Now we shall give examples that satisfies the assumptions on F, L and K.

EXAMPLE 3.3.1. *Consider the nonlinear Hammerstein equation $Tx = y$, where the operator $T : H^u[0,1] \to L^2[0,1]$ given by*

$$T(x)(s) = \int_0^1 k(s,t)(x(t) + f(t))dt, \quad 0 \leq s \leq 1, \qquad (3.3.3)$$

with $k(s,t) := \sum_{n=0}^{\infty}(n+1)^{-2}u_n(s)u_n(t)$, $f(t) \in L^2[0,1]$; *where* $u_n(s) = \sqrt{2}cos(2n\pi s)$. *Then (3.3.3) can be written as*

$$KF(x)(s) = y(s)$$

where $K : L^2[0,1] \to L^2[0,1]$ *defined by*

$$Ku(s) = \int_0^1 k(s,t)u(t)dt \tag{3.3.4}$$

and $F : H^u[0,1] \to L^2[0,1]$ *is given by*

$$F(x)(s) = x(s) + f(s). \tag{3.3.5}$$

Note that K *in (3.3.4) is compact, positive self adjoint with positive eigenvalues* $(n+1)^{-2}$ *and corresponding eigenvectors* $u_n(.)$ *for* $n = 0, 1, 2, \ldots$.

Further note that $F'(x)h(s) = h(s)$ *and hence*

$$\|F'(x) - F'(y)\|_{-u} \leq \|x - y\|_{-u}.$$

Thus F *satisfies (3.3.1) and* $F'(x_0)^{-1} = I$ *exist and is bounded.*

Let

$$Lx := \sum_{j=0}^{\infty}(j+1)^2\langle x, u_j\rangle u_j, \quad u_j(s) = \sqrt{2}cos(2\pi jt)$$

with

$$D(L) := \{x \in L^2[0,1] : \sum_{j=0}^{\infty}(j+1)^4|\langle x, u_j\rangle|^2 < \infty\}.$$

Then since $F'(x_0)^{-1} = I$, $L^{-u}F'(x_0)^{-1}L^u = I$ *so that (3.3.2) holds. Note that*

$$H_t = \{x \in L^2[0,1] : \sum_{j=0}^{\infty}(j+1)^{4t}|\langle x, u_j\rangle|^2 < \infty\},$$

and a, c_1, c_2 *in* $Assumption(A_1)$ *are given by* $a = 1, c_1 = c_2 = \frac{\pi^2}{6}$.

Next example is based on an orthogonal linear splines.

3.3.1 Orthogonal Linear Splines

Let $x_k, \quad k = 0, 1, 2, \ldots n$ be a set of ordered knots and $x_{-1} < x_0 = a, \quad b = x_n < x_{n+1}$ are the exterior knots. Then the conventional family $\{L_k\}, \quad k = 0, 1, 2, \ldots n$ of linear B-splines are defined as

$$L_k(x) = \begin{cases} \frac{x - x_{k-1}}{x_k - x_{k-1}}, & x \in [x_{k-1}, x_k], \\ \frac{x_{k+1} - x}{x_{k+1} - x_k}, & x \in [x_k, x_{k+1}]. \end{cases}$$

Note that L_k is continuous with support $[x_{k-1}, x_{k+1}]$. Now we shall convert the linear splines $\{L_k\}$ to a basis of orthogonal splines $\{Q_k\} \quad k = 0, 1, 2, \ldots n$ by the relation

$$Q_k(x) = \begin{cases} 0 & x \in [a, x_{k-1}], \\ |L_k - L_{k-1}|, & x \in [x_{k-1}, x_k], \\ 0, & x \in [x_k, b]. \end{cases}$$

Here Q_k is a linear spline with support $[x_{k-1}, x_k]$, so $\{Q_k\} \quad k = 0, 1, 2, \ldots, n$ are orthogonal splines.

Let

$$P_k = \frac{Q_k}{\|Q_k\|}.$$

Then $\{P_k\}, \quad k = 0, 1, 2, \ldots, n$ are orthonormal splines. We now approximate any continuous function f on $[a, b]$ by $\{P_k\}$ in the form

$$f \approx \sum_{k=0}^{n} \langle f, P_k \rangle P_k.$$

Since $C[a, b]$ is dense in $L^2[a, b]$, we approximate every $f \in L^2[a, b]$ in the form $\sum_{k=0}^{n} \langle f, P_k \rangle P_k$.

EXAMPLE 3.3.2. *Consider the nonlinear Hammerstein equation $Tx = y$, where the operator $T : H^u[0, 1] \to L^2[0, 1]$ given by*

$$T(x)(s) = \int_0^1 k(s, t) x^2(t) dt, \quad 0 \leq s \leq 1, \tag{3.3.6}$$

with $k(s, t) := \lim_{n \to \infty} \sum_{k=0}^{n} (k + 1)^{-2} P_k(s) P_k(t)$; where $P_k(s)$ is the orthonormal spline defined in section 3.1 with $x_i = \frac{i}{n} (i = 0, 1, 2, \ldots n)$. Then (3.3.6) can be written as

$$KF(x)(s) = y(s)$$

where $K : L^2[0,1] \to L^2[0,1]$ *defined by*

$$Ku(s) = \int_0^1 k(s,t)u(t)dt \tag{3.3.7}$$

and $F : H^u[0,1] \to L^2[0,1]$ *is given by*

$$F(x)(s) = x^2(s). \tag{3.3.8}$$

Note that K *in (3.3.4) is compact, positive self adjoint with positive eigenvalues* $(k+1)^{-2}$ *and corresponding eigenvectors* $P_k(.)$ *for* $k = 0,1,2,...n.$

Further note that $F'(x)h(s) = 2x(s)h(s)$ *and hence*

$$\|F'(x) - F'(y)\|_{-u} \leq 2\|x-y\|_{-u}.$$

Thus F *satisfies (3.3.1) and* $F'(x_0)^{-1} = \frac{1}{2x_0(.)}$ *exist and is bounded, if* $x_0(s) \geq A > 0, \forall s \in [0,1]$*. So we assume that* $x_0(s) \geq A > 0, \forall s \in [0,1]$*. Let*

$$Lx := \lim_{n\to\infty} \sum_{j=0}^{n} (j+1)^2 \langle x, P_j\rangle P_j,$$

with

$$D(L) := \{x \in L^2[0,1] : \lim_{n\to\infty} \sum_{j=0}^{n} (j+1)^4 |\langle x, P_j\rangle|^2 < \infty\}.$$

Then since support of P_k *is* $[x_{k-1}, x_k]$ *and* $\|F'(x_0)^{-1}\| = \|\frac{1}{2x_0(.)}\| \leq \frac{1}{2K}$*, we have,*

$$\begin{aligned}(L^{-u}F'(x_0)^{-1}L^u)(x) &= \lim_{n\to\infty} \sum_{k=0}^{n} (k+1)^{-2u} \lim_{n\to\infty} \sum_{j=0}^{n} (j+1)^{2u} \\ &\quad \times \langle x, P_j\rangle \langle \frac{P_j}{2x_0}, P_k\rangle P_k \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} (k+1)^{2u}(k+1)^{-2u} \\ &\quad \times \langle x, P_k\rangle \langle \frac{P_k}{2x_0}, P_k\rangle P_k,\end{aligned}$$

so that

$$\|(L^{-u}F'(x_0)^{-1}L^u)(x)\|^2 \leq \frac{1}{4K^2} \lim_{n\to\infty} \sum_{k=0}^{n} |\langle x, P_k\rangle|^2 \tag{3.3.9}$$

$$\leq \frac{1}{4A^2}\|x\|^2. \tag{3.3.10}$$

Thus (3.3.2) holds. Note that

$$H_t = \{x \in L^2[0,1] : \lim_{n\to\infty} \sum_{j=0}^{n} (j+1)^{4t} |\langle x, P_j\rangle|^2 < \infty\},$$

and a, c_1, c_2 *in Assumption*(A_1) *are given by* $a = c_1 = c_2 = 1$.

We shall make use of the following lemma, extensively in our analysis.

LEMMA 3.3.3. *Let* $0 < r_0 < r$ *and* $x, y \in \overline{B_{r_0}(x_0)} \subset H_{-u}$. *Then*

$$\|F'(x_0)(x - x_0) - [F(x) - F(x_0)]\|_{-u} \leq \frac{\kappa_0 r_0}{2}\|x - x_0\|_{-u},$$

$$\|F'(x_0)(x - y) - [F(x) - F(y)]\|_{-u} \leq \kappa_0 r_0 \|x - y\|_{-u}.$$

Proof. By fundamental Theorem of Integral Calculus,

$$F(x) - F(y) = \int_0^1 F'(y + t(x - y))(x - y)dt,$$

so

$$F'(x_0)(x - y) - (F(x) - F(y)) = \int_0^1 [F'(x_0) - F'(y + t(x - y))](x - y)dt.$$

Hence by (3.3.1)

$$\|F'(x_0)(x - y) - [F(x) - F(y)]\|_{-u} \leq \kappa_0 \|x - y\|_{-u} \int_0^1 \|x_0 - (u + t(x - y))\|_{-u} dt.$$

Now since $y + t(x - y) \in B_{r_0}(x_0) \subset H_{-u}$, $\|x_0 - (y + t(x - y))\|_{-u} \leq r_0$ and $\|x_0 - (x_0 + t(x - x_0))\|_{-u} \leq t r_0$ and hence

$$\|F'(x_0)(x - x_0) - [F(x) - F(x_0)]\|_{-u} \leq \frac{\kappa_0 r_0}{2}\|x - x_0\|_{-u},$$

$$\|F'(x_0)(x-y) - [F(x) - F(y)]\|_{-u} \leq \kappa_0 r_0 \|x-y\|_{-u}.$$

This completes the proof.

We start our error analysis by introducing the following notations: Let

$$\omega := \|F(\hat{x}) - F(x_0)\|_{-u}$$

and for $\alpha > 0$, $\delta > 0$, let

$$\gamma_{s,a} := 2\beta^2\kappa_0(\psi(s)\delta\alpha^{\frac{u-a}{s+a}} + \psi_1(s)\omega).$$

Note that if

$$\psi(s)\delta\alpha^{\frac{u-a}{s+a}} + \psi_1(s)\omega < \frac{1}{2\beta}\min\Big\{r,\ \frac{1}{\beta\kappa_0}\Big\}, \tag{3.3.11}$$

then

$$\gamma_{s,a} < 1 \quad \text{and} \quad \eta_{s,a} := \frac{1-\sqrt{1-\gamma_{s,a}}}{\beta\kappa_0} < r.$$

THEOREM 3.3.4. *Suppose (3.3.1),(3.3.2) and (3.3.11) hold. Then the sequence $(x^\delta_{n,\alpha,s})$ defined in (3.1.6) converges, and its limit $x^\delta_{\alpha,s} := \lim_{n\to\infty} x^\delta_{n,\alpha,s}$ belong to $\overline{B_{\eta_{s,a}}(x_0)} \subset B_r(x_0) \subset H_{-u}$. Further,*

$$\|x^\delta_{\alpha,s} - x^\delta_{n,\alpha,s}\|_{-u} \leq \frac{\eta_{s,a}q^n}{1-q}, \tag{3.3.12}$$

where $q := \eta_{s,a}\beta\kappa_0 = 1 - \sqrt{1-\gamma_{s,a}}$.

Proof. First we prove that $x^\delta_{n,\alpha,s} \in B_{\eta_{s,a}}(x_0)$. Suppose $x^\delta_{m,\alpha,s} \in B_{\eta_{s,a}}(x_0)$. Then

$$\begin{aligned}
\|x^\delta_{m+1,\alpha,s} - x_0\|_{-u} &= \|L^{-u}(x^\delta_{m,\alpha,s} - x_0) - L^{-u}F'(x_0)^{-1}(F(x^\delta_{m,\alpha,s}) - z^\delta_{\alpha,s})\| \\
&= \|L^{-u}F'(x_0)^{-1}L^uL^{-u}[F'(x_0)(x^\delta_{m,\alpha,s} - x_0) - (F(x^\delta_{m,\alpha,s}) \\
&\quad -F(x_0)) + (z^\delta_{\alpha,s} - F(x_0))]\|.
\end{aligned}$$

Thus by Lemma 3.3.3 and (3.3.2),

$$\begin{aligned}
\|x^\delta_{m+1,\alpha,s} - x_0\|_{-u} &\leq \beta\frac{\kappa_0}{2}\eta_{s,a}\|x^\delta_{m,\alpha,s} - x_0\|_{-u} + \beta\|z^\delta_{\alpha,s} - F(x_0)\|_{-u} && (3.3.13)\\
&\leq \beta(\frac{\kappa_0}{2}\eta^2_{s,a} + \psi(s)\alpha^{(u-a)/(s+a)}\delta + \psi_1(s)\omega) && (3.3.14)\\
&\leq \eta_{s,a}. && (3.3.15)
\end{aligned}$$

The last but one step follows from Theorem 3.2.2. Since $x_0 \in B_{\eta_{s,a}}(x_0)$, by induction $x^\delta_{n,\alpha,s} \in B_{\eta_{s,a}}(x_0)$ for all $n = 1, 2, 3....$

Now we prove that $x^\delta_{n,\alpha,s}$ is a Cauchy sequence in $B_{\eta_{s,a}}(x_0)$. Observe that

$$\begin{aligned}
\|x^\delta_{n+1,\alpha,s} - x^\delta_{n,\alpha,s}\|_{-u} &= \|L^{-u}(x^\delta_{n,\alpha,s} - x^\delta_{n-1,\alpha,s}) - L^{-u}F'(x_0)^{-1} \\
&\quad \times (F(x^\delta_{n,\alpha,s}) - F(x^\delta_{n-1,\alpha,s}))\| \\
&\leq \|L^{-u}F'(x_0)^{-1}L^uL^{-u}[F'(x_0)(x^\delta_{n,\alpha,s} - x^\delta_{n-1,\alpha,s}) \\
&\quad -(F(x^\delta_{n,\alpha,s}) - F(x^\delta_{n-1,\alpha,s}))]\| \\
&\leq \beta\|F'(x_0)(x^\delta_{n,\alpha,s} - x^\delta_{n-1,\alpha,s}) \\
&\quad -(F(x^\delta_{n,\alpha,s}) - F(x^\delta_{n-1,\alpha,s}))\|_{-u} \\
&\leq \beta\kappa_0\eta_{s,a}\|x^\delta_{n,\alpha,s} - x^\delta_{n-1,\alpha,s}\|_{-u} \\
&= q\|x^\delta_{n,\alpha,s} - x^\delta_{n-1,\alpha,s}\|_{-u} \qquad (3.3.16)
\end{aligned}$$

where $q = \beta\kappa_0\eta_{s,a} < 1 - \sqrt{1-\gamma_{s,a}} < 1$. Thus $x^\delta_{n,\alpha,s}$ is a Cauchy sequence in $B_{\eta_{s,a}}(x_0)$ and hence converges, and its limit $x^\delta_{\alpha,s} := \lim_{n\to\infty} x^\delta_{n,\alpha,s} \in \overline{B_{\eta_{s,a}}(x_0)} \subset B_r(x_0) \subset H_{-u}$. Now by (3.3.16), we have

$$\begin{aligned}
\|x^\delta_{\alpha,s} - x^\delta_{n,\alpha,s}\|_{-u} &\leq \lim_{i\to\infty} \|x^\delta_{i,\alpha,s} - x^\delta_{n,\alpha,s}\|_{-u} \qquad (3.3.17) \\
&\leq \sum_{j=n}^{\infty} \eta_{s,a}\, q^j \qquad (3.3.18) \\
&\leq \frac{\eta_{s,a}\, q^n}{1-q}. \qquad (3.3.19)
\end{aligned}$$

This completes the proof.

THEOREM 3.3.5. *Suppose (3.3.1),(3.3.2) and (3.3.11) hold. If, in addition, $\beta\kappa_0 r < 1$, then*

$$\|\hat{x} - x^\delta_{\alpha,s}\|_{-u} \leq \frac{\beta}{1-\beta\kappa_0 r}\|F(\hat{x}) - z^\delta_{\alpha,s}\|_{-u}.$$

Proof. Observe that

$$\|\hat{x} - x^\delta_{\alpha,s}\|_{-u} = \lim_{n\to\infty} \|\hat{x} - x^\delta_{n,\alpha,s}\|_{-u}.$$

Now since,

$$\begin{aligned}
\|x^{\delta}_{n+1,\alpha,s} - \hat{x}\|_{-u} &= \|x^{\delta}_{n,\alpha,s} - \hat{x} - F'(x_0)^{-1}(F(x^{\delta}_{n,\alpha,s}) - z^{\delta}_{\alpha,s})\|_{-u} \\
&= \|L^{-u}F'(x_0)^{-1}L^{u}L^{-u}\{F'(x_0)(x^{\delta}_{n,\alpha,s} - \hat{x}) \\
&\quad -(F(x^{\delta}_{n,\alpha,s}) - z^{\delta}_{\alpha,s})\}\| \\
&= \|L^{-u}F'(x_0)^{-1}L^{u}L^{-u}\{F'(x_0)(x^{\delta}_{n,\alpha,s} - \hat{x}) \\
&\quad -(F(x^{\delta}_{n,\alpha,s}) - F(\hat{x})) - (F(\hat{x}) - z^{\delta}_{\alpha,s})\}\|.
\end{aligned}$$

Thus by Lemma 3.3.3 and (3.3.2),

$$\|x^{\delta}_{n+1,\alpha,s} - \hat{x}\|_{-u} \leq \beta\kappa_0 r\|x^{\delta}_{n,\alpha,s} - \hat{x}\|_{-u} + \beta\|F(\hat{x}) - z^{\delta}_{\alpha,s}\|_{-u}.$$

In particular,

$$\|x^{\delta}_{a,s} - \hat{x}\|_{-u} \leq \beta\kappa_0 r\|x^{\delta}_{\alpha,s} - \hat{x}\|_{-u} + \beta\|F(\hat{x}) - z^{\delta}_{\alpha,s}\|_{-u}.$$

so that the result follows.

Combining the estimates in Theorem 3.3.4 and Theorem 3.3.5 we obtain the following.

THEOREM 3.3.6. *Suppose (3.3.1),(3.3.2) and (3.3.11) hold. Assume, in addition, that $\beta\kappa_0 r < 1$. Then*

$$\|\hat{x} - x^{\delta}_{n,\alpha,s}\|_{-u} \leq \frac{\beta}{1-\beta\kappa_0 r}\|F(\hat{x}) - z^{\delta}_{\alpha,s}\|_{-u} + \frac{\eta_{s,a}\, q^n}{1-q}.$$

In view of the estimate in the above theorem, it is desirable to find out the nature of the quantity $\|F(\hat{x}) - z^{\delta}_{\alpha,s}\|_{-u}$. But by (3.2.8), and triangle inequality we have

$$\|F(\hat{x}) - z^{\delta}_{\alpha,s}\|_{-u} \leq \|F(\hat{x}) - z_{\alpha,s}\|_{-u} + \psi(s)\alpha^{(u-a)/(s+a)}\delta. \tag{3.3.20}$$

Further by (3.2.10) for $0 < u+t \leq s+a$, if $\|F(\hat{x}) - F(x_0)\|_t \leq E$, for some constant $E > 0$, we have the following.

THEOREM 3.3.7. *If $\|F(\hat{x}) - F(x_0)\|_t \leq E$ for $0 < u+t \leq s+a$, then $\|F(\hat{x}) - z_{\alpha,s}\|_{-u} \to 0$ as $\alpha \to 0$.*

3.4 Error Bounds and Parameter Choice in Hilbert Scales

We start our study with the following observation: by Theorem 3.3.7 if $\|F(\hat{x}) - F(x_0)\|_t \leq E$ for some $0 < u + t \leq s + a$, then $\|F(\hat{x}) - z_{\alpha,s}\|_{-u} \to 0$ as $\alpha \to 0$. So we assume that

$$\|F(\hat{x}) - z_{\alpha,s}\|_{-u} \leq \varphi_{s,a}(\alpha) \tag{3.4.1}$$

for some positive function $\varphi_{s,a}$ defined on $(0, \|K_s\|]$ such that $\lim_{\lambda\to 0}\varphi_{s,a}(\lambda) = 0$. We further assume that $\varphi_{s,a}$ is monotonically increasing. Note that $\varphi_{s,a}(\lambda) := \varphi(s,t)E\lambda^{\frac{u+t}{s+a}}$ satisfies the above assumptions. Again by (3.4.1), (3.2.8) and triangle inequality, we have

$$\|F(\hat{x}) - z^{\delta}_{\alpha,s}\|_{-u} \leq \varphi_{s,a}(\alpha) + \psi(s)\alpha^{(u-a)/(s+a)}\delta. \tag{3.4.2}$$

Thus we have the following theorem.

THEOREM 3.4.1. *Under the assumptions of Theorem 3.3.4 and (3.4.2)*

$$\|\hat{x} - x^{\delta}_{n,\alpha,s}\|_{-u} \leq \frac{\beta}{1-\beta\kappa_0 r}(\varphi_{s,a}(\alpha) + \psi(s)\alpha^{(u-a)/(s+a)}\delta) + \frac{\eta_{s,a}q^n}{1-q}. \tag{3.4.3}$$

Again the error estimate $\varphi_{s,a}(\alpha) + \psi(s)\alpha^{(u-a)/(s+a)}\delta$ in (3.4.2) attains minimum for the choice $\alpha := \alpha_\delta$ which satisfies $\varphi_{s,a}(\alpha) = \psi(s)\alpha^{(u-a)/(s+a)}\delta$. Clearly $\alpha_\delta = \varphi_{s,a}\lambda_{s,a})^{-1}(\delta)$, where

$$\lambda_{s,a}(\lambda) = \frac{\lambda^{(a-u)/(s+a)}}{\psi(s)}, \quad 0 < \lambda \leq \|A_s\| \tag{3.4.4}$$

and in this case

$$\|F(\hat{x}) - z^{\delta}_{\alpha,s}\|_{-u} \leq 2\varphi_{s,a}((\varphi_{s,a}\lambda_{s,a})^{-1}(\delta)),$$

which has at least optimal order with respect to δ (See [50]).

In view of the above observation, Theorem 3.4.1 leads to the following.

THEOREM 3.4.2. *Let* $\lambda_{s,a}(\lambda) = \frac{\lambda^{(a-u)/(s+a)}}{\psi(s)}$, *for* $0 < \lambda \leq \|K_s\|$, *assumptions in Theorem 3.4.1 and (3.4.1) are satisfied. For* $\delta > 0$, *let* $\alpha_\delta = (\varphi_{s,a}\lambda_{s,a})^{-1}(\delta)$. *If*

$\varphi_{s,a}(\alpha_\delta) + \psi_1(s)\omega < \frac{1}{2\beta}min\{r, \frac{1}{\beta\kappa_0}\}$, *and* $n_\delta := min\{n : q^n \leq \frac{\delta}{\lambda_{s,a}(\alpha_\delta)}\}$, *then*

$$\|\hat{x} - x^\delta_{\alpha_\delta,n_\delta}\|_{-u} = O(\varphi_{s,a}(\varphi_{s,a}\lambda_{s,a})^{-1}(\delta))).$$

3.5 Adaptive Scheme and Stopping Rule

As in any regularization strategy the next important point under consideration is the choice of the regularization parameter $\alpha := \alpha_\delta$ and stopping rule for the iteration in (3.3.1), independent of the source function $\varphi_{s,a}$, but may depend on the data (δ, y^δ). For linear ill-posed problems in Hilbert scales, there exist many such a posteriori parameter choice strategies (See [16], [17]).

In this chapter we shall modify the adaptive scheme considered by Pereverzev and Schock in [50], to suit the Hilbert scale set up.

Let us introduce the following notations:

$$\alpha_\delta := (\varphi_{s,a}\lambda_{s,a})^{-1}(\delta). \tag{3.5.1}$$

$i \in \{0, 1, 2, \cdots, N\}$ and $\alpha_i = \mu^i\alpha_0$ where $\mu = \rho^{(s+a)/(a-u)}$, $\rho > 1$ and $\alpha_0 = (\psi(s)\delta)^{(s+a)/(a-u)}$. Let

$$l := max\{i : \varphi_{s,a}(\alpha_i) \leq \frac{\delta}{\lambda_{s,a}(\alpha_i)}\}. \tag{3.5.2}$$

and

$$k := max\{i : \|z^\delta_{\alpha_i,s} - z^\delta_{\alpha_j,s}\|_{-u} \leq \frac{4\delta}{\lambda_{s,a}(\alpha_j)}, j = 0, 1, 2, \cdots, i\}. \tag{3.5.3}$$

Now we have the following.

THEOREM 3.5.1. *Let* l *be as in (3.5.2),* k *be as in (3.5.3),* $\lambda_{s,a}$ *be as in (3.4.4) and* $z^\delta_{\alpha_k,s}$ *be as in (3.2.6) with* $\alpha = \alpha_k$*. Then* $l \leq k$*; and*

$$\|F(\hat{x}) - z^\delta_{\alpha_k,s}\|_{-u} \leq (2 + \frac{4\rho}{\rho - 1})\rho\varphi_{s,a}((\varphi_{s,a}\lambda_{s,a})^{-1}(\delta)). \tag{3.5.4}$$

Proof To see that $l \leq k$, it is enough to show that, for $i = 1, 2, \cdots, N$,

$$\varphi_{s,a}(\alpha_i) \leq \frac{\delta}{\lambda_{s,a}(\alpha_i)} \implies \|z^\delta_{\alpha_i,s} - z^\delta_{\alpha_j,s}\|_{-u} \leq \frac{4\delta}{\lambda_{s,a}(\alpha_j)}, \qquad \forall j = 0, 1, \cdots, i.$$

For $j \leq i$, by (3.4.2)

$$\begin{aligned}
\|z^\delta_{\alpha_i,s} - z^\delta_{\alpha_j,s}\|_{-u} &\leq \|z^\delta_{\alpha_i,s} - F(\hat{x})\|_{-u} + \|F(\hat{x}) - z^\delta_{\alpha_j,s}\|_{-u} \\
&\leq \varphi_{s,a}(\alpha_i) + \frac{\delta}{\lambda_{s,a}(\alpha_i)} + \varphi_{s,a}(\alpha_j) + \frac{\delta}{\lambda_{s,a}(\alpha_j)} \\
&\leq \frac{2\delta}{\lambda_{s,a}(\alpha_i)} + \frac{2\delta}{\lambda_{s,a}(\alpha_j)} \\
&\leq \frac{4\delta}{\lambda_{s,a}(\alpha_j)}.
\end{aligned}$$

This proves the relation $l \leq k$. Now by the relation $\lambda_{s,a}(\alpha_{l+m}) = \rho^m \lambda_{s,a}(\alpha_l)$ and by using triangle inequality successively, we obtain

$$\begin{aligned}
\|F(\hat{x}) - z^\delta_{\alpha_k,s}\|_{-u} &\leq \|F(\hat{x}) - z^\delta_{\alpha_l,s}\|_{-u} + \sum_{i=l+1}^{k} \|z^\delta_{\alpha_i,s} - z^\delta_{\alpha_{i-1},s}\|_{-u} \\
&\leq \|F(\hat{x}) - z^\delta_{\alpha_l,s}\|_{-u} + \sum_{i=l+1}^{k} \frac{4\delta}{\lambda_{s,a}(\alpha_{i-1})} \\
&\leq \|F(\hat{x}) - z^\delta_{\alpha_l,s}\|_{-u} + \sum_{m=0}^{k-l-1} \frac{4\delta}{\lambda_{s,a}(\alpha_l)\rho^m} \\
&\leq \|F(\hat{x}) - z^\delta_{\alpha_l,s}\|_{-u} + \frac{4\rho}{\rho - 1} \frac{\delta}{\lambda_{s,a}(\alpha_l)}.
\end{aligned}$$

Therefore by (3.4.2) we have

$$\begin{aligned}
\|F(\hat{x}) - z^\delta_{\alpha_k,s}\|_{-u} &\leq \varphi_{s,a}(\alpha_l) + \frac{\delta}{\lambda_{s,a}(\alpha_l)} + \frac{4\rho}{\rho - 1} \frac{\delta}{\lambda_{s,a}(\alpha_l)} \\
&\leq (2 + \frac{4\rho}{\rho - 1}) \frac{\delta}{\lambda_{s,a}(\alpha_l)} \\
&\leq (2 + \frac{4\rho}{\rho - 1}) \rho \varphi_{s,a}((\varphi_{s,a}\lambda_{s,a})^{-1}(\delta)).
\end{aligned}$$

The last step follows from the inequality $\alpha_\delta \leq \alpha_{l+1}$ and $\lambda_{s,a}(\alpha_\delta) \leq \lambda_{s,a}(\alpha_{l+1}) = \rho\lambda_{s,a}(\alpha_l)$.

THEOREM 3.5.2. *Let $x^\delta_{\alpha_k,s}$ be as in Theorem 3.3.4 with $\alpha = \alpha_k$, $\lambda_{s,a}$ and $z^\delta_{\alpha_k,s}$ be as in Theorem 3.5.1 and the assumptions (3.4.1) hold. Let k be as in (3.5.3). Then*

$$\|\hat{x} - x^\delta_{\alpha_k,s}\|_{-u} \leq \frac{\beta}{1 - \beta\kappa_0 r}(2 + \frac{4\rho}{\rho - 1})\rho\varphi_{s,a}((\varphi_{s,a}\lambda_{s,a})^{-1}(\delta)). \tag{3.5.5}$$

Proof. The result follows from Theorem 3.3.5, Theorem 3.5.1.

3.5.1 Stopping Rule

Note that if $\alpha_0 = (\psi(s)\delta)^{(s+a)/(a-u)}$, $\alpha_j = \rho^{((s+a)/(a-u))j}\alpha_0$ for $\rho > 1$ *and* $j = 1, 2, 3, \cdots, N$ and $k := max\{i : \|z^\delta_{\alpha_i,s} - z^\delta_{\alpha_j,s}\|_{-u} \leq 4\rho^{-j}, j = 0, 1, 2, \cdots, i\}$, then

$$\psi(s)\frac{\delta}{\alpha_k^{(a-u)/(s+a)}} = \rho^{-k}.$$

Thus the condition (3.3.11) takes the form

$$\frac{1}{\rho^k} + \psi_1(s)\omega < \frac{r}{2\beta}.$$

Further if we assume that

$$r < 2\beta(1 + \psi_1(s)\omega), \quad \frac{1}{\rho} + \psi_1(s)\omega < \frac{r}{2\beta}$$

then $\rho > 1$ and (3.3.11) holds.

THEOREM 3.5.3. *Assume that* $2\beta\psi_1(s)\omega < r < min\{2\beta(1+\psi_1(s)\omega), 1/\beta\kappa_0\}$, $\rho > 2\beta/(r-2\beta\psi_1(s)\omega)$. *Let* $\alpha_0 = (\psi(s)\delta)^{(s+a)/(a-u)}$, $\alpha_j = \rho^{((s+a)/(a-u))j}\alpha_0$ *for* $\rho > 1$ *and* $j = 1, 2, 3, \cdots, N$ *and* $k := max\{i : \|z^\delta_{\alpha_i,s} - z^\delta_{\alpha_j,s}\|_{-u} \leq 4\rho^{-j}, j = 0, 1, 2, \cdots, i\}$. *Then*

$$\|F(\hat{x}) - z^\delta_{\alpha_k,s}\|_{-u} \leq \left[\frac{\beta}{1-\beta\kappa_0 r}(2 + \frac{4\rho}{\rho-1})\right]\rho\varphi_{s,a}((\varphi_{s,a}\lambda_{s,a})^{-1}(\delta)). \tag{3.5.6}$$

Further

$$\gamma_{s,a,k} := 2\beta^2\kappa_0(\psi_1(s)\omega + 1/\rho^k) < 1,$$

and if

$$n_k := min\{n : q_k^n \leq \frac{1}{\rho^k}\}$$

with $q_k := 1 - \sqrt{1-\gamma_{s,a,k}}$, *then*

$$\|\hat{x} - x^\delta_{n_k,\alpha_k,s}\|_{-u} = O(\varphi_{s,a}((\varphi_{s,a}\lambda_{s,a})^{-1}(\delta))).$$

Proof. The result follows from Theorem 3.4.1, Theorem 3.5.2 and the triangle inequality,

$$\|\hat{x} - x^\delta_{n_k,\alpha_k,s}\|_{-u} \leq \|\hat{x} - x^\delta_{\alpha_k,s}\|_{-u} + \|x^\delta_{\alpha_k,s} - x^\delta_{n_k,\alpha_k,s}\|_{-u}. \tag{3.5.7}$$

□

Chapter 4

Iterative Regularization Methods for Ill-posed Hammerstein Type Operator Equation with Monotone Nonlinear Part

In this chapter we consider a procedure for solving an ill-posed Hammerstein type operator equation $KF(x) = y$, where F is a nonlinear monotone operator, by solving the linear equation $Kz = y$ first for z and then solving the nonlinear equation $F(x) = z$. Convergence analysis is carried out by means of suitably constructed majorizing sequences. The derived error estimate using an adaptive method proposed by Perverzev and Schock [50] in relation to the noise level and a stopping rule based on the majorizing sequences are shown to be of optimal order with respect to certain assumptions on $F(\hat{x})$, where $\hat{x}$ is the solution of $KF(x) = y$.

4.1 Introduction

In this chapter we consider the problem of approximately solving a nonlinear ill-posed operator equation of the Hammerstein type with a monotone nonlinear part. Recall that a Hammerstein type operator (see [13, 14, 15, 20]) is an operator of the form KF, where $F : D(F) \subset X \mapsto Z$ is nonlinear and $K : Z \mapsto Y$ is a bounded linear operator and X, Y, Z are taken to be real Hilbert spaces in this chapter. We are interested in the case when $Z = X$ and F is a monotone operator (cf. [58]).

i.e.,$F : D(F) \subset X \mapsto X$ satisfies

$$\langle F(x_1) - F(x_2), x_1 - x_2 \rangle \geq 0, \qquad \forall x_1, x_2 \in D(F).$$

So we consider an equation of form

$$KF(x) = y \tag{4.1.1}$$

where $F : D(F) \subset X \mapsto X$ is monotone and $K : X \mapsto Y$ is linear. It is assumed that (4.1.1) has a solution $\hat{x} \in D(F)$ satisfying

$$\|\hat{x} - x_0\| = min\{\|x - x_0\| : KF(x) = y,\ x \in D(F)\}. \tag{4.1.2}$$

We assume throughout that $y^\delta \in Y$ are the available noisy data with

$$\|y - y^\delta\| \leq \delta \tag{4.1.3}$$

and Observe that (cf. [20]) the solution $\hat{x}$ of (4.1.1) can be obtained by first solving the linear equation

$$Kz = y \tag{4.1.4}$$

for z and then solving the nonlinear equation

$$F(x) = z. \tag{4.1.5}$$

For the treatment of nonlinear ill-posed problems the standard regularization method is the method of Tikhonov regularization. But if the nonlinear operator is monotone then a simpler regularization strategy available is the Lavrentiev regularization. Note that KF need not be monotone even if F is monotone. So in the straight forward approach one has to consider Tihkonòv regularization method for approximately solving (4.1.1).

What we show in this chapter is that for the special case when K is linear and F is monotone, by splitting the equation (4.1.1) into (4.1.4) and (4.1.5), one can simplify the procedure by specifying a regularization strategy for linear part (4.1.4) and an iterative method for nonlinear part (4.1.5). More precisely , for fixed $\alpha > 0$, $\delta > 0$ we consider the regularized solution of (4.1.4) with y^δ in place of y as

$$z_\alpha^\delta = (K + \alpha I)^{-1} y^\delta \tag{4.1.6}$$

if the operator K in (4.1.4) is positive self adjoint and $X = Y$, otherwise we consider

$$z_\alpha^\delta = (K^*K + \alpha I)^{-1}K^*y^\delta. \tag{4.1.7}$$

Note that (4.1.6) is the simplified or Lavrentiev regularization (see [28]) of the equation (4.1.4) and (4.1.7) is the Tikhonov regularization (see [10, 13, 23, 18, 54, 55]) of (4.1.4). The regularization parameter is chosen according to an adaptive method proposed by Pereverzev and Schock in [50]. Also one can see that the iterative method we considered in section 3 and section 4 for the nonlinear equation (4.1.5) do not involve any regularization parameter explicitly.

In [20], it is assumed that the bounded inverse of $F'(x_0)$ exist and considered the sequence

$$x_{n+1,\alpha}^\delta = x_{n,\alpha}^\delta - F'(x_0)^{-1}(F(x_{n,\alpha}^\delta) - z_\alpha^\delta), \tag{4.1.8}$$

with $x_{0,\alpha}^\delta = x_0$ and proved that $(x_{n,\alpha}^\delta)$ converges linearly to the solution x_α^δ of

$$F(x) = z_\alpha^\delta. \tag{4.1.9}$$

In chapter 2, we considered the sequence $(x_{n,\alpha}^\delta)$ defined iteratively as

$$x_{n+1,\alpha}^\delta = x_{n,\alpha}^\delta - F'(x_{n,\alpha}^\delta)^{-1}(F(x_{n,\alpha}^\delta) - z_\alpha^\delta), \tag{4.1.10}$$

with $x_{0,\alpha}^\delta = x_0$ and proved that $(x_{n,\alpha}^\delta)$ converges quadratically to the solution x_α^δ of (4.1.9) under the assumption that the bounded inverse of $F'(x)$ exist in a neighborhood of x_0. For the special case when F is monotone we can do away with the above requirement of invertibility of F' even at x_0.

Recall that a sequence (x_n) is X with $\lim x_n = x^*$ is said to converge quadratically, if there exists positive number M, not necessarily less than 1, such that for all n sufficiently large

$$\|x_{n+1} - x^*\| \leq M\|x_n - x^*\|^2. \tag{4.1.11}$$

If the sequence (x_n) has the property that

$$\|x_{n+1} - x^*\| \leq q\|x_n - x^*\|, \qquad 0 < q < 1$$

then (x_n) is said to be linearly convergent. For an extensive discussion of convergence rate, see Ortega and Rheinboldt [49].

Note that the ill-posedness of equation (4.1.1) in [20] and in chapter 2 is due to the ill-posedness of the linear equation (4.1.4). In the present chapter we assume that (4.1.1) is ill-posed in both the linear part (4.1.4) and the nonlinear part (4.1.5). Using the monotonicity of F, we carry out the convergence analysis by means of suitably constructed majorizing sequences, deviating from the methods used in [20] and chapter 2. An advantage of this approach is that the majorizing sequence gives an a priori error estimate which can be used to determine the number of iterations needed to achieve a prescribed solution accuracy before actual computation takes place.

Organization of this chapter is as follows. We collected some preparatory results in section 2. Convergence analysis of an iterated sequence converging quadratically is given in section 3 and in section 4 we consider another sequence which converges linearly. In section 5 we give error analysis and derive optimal order error bounds. Finally in section 6 we consider an algorithm for implementing method considered in this chapter.

4.2 Preparatory Results

Throughout this chapter we assume that the operator F satisfies the following assumptions.

Assumption 4.2.1. *There exists $r > 0$ such that $B_r(x_0) \subseteq D(F)$ and F is Fréchet differentiable at all $x \in B_r(x_0)$.*

Assumption 4.2.2. *There exists a constant $k_0 > 0$ such that for every $x, u \in B_r(x_0)$ and $v \in X$, there exists an element $\Phi(x, u, v) \in X$ satisfying*

$$[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v), \|\Phi(x, u, v)\| \leq k_0\|v\|\|x - u\|.$$

The next assumption on source condition is based on a source function φ and a property of the source function φ. We will be using this assumption to obtain an error estimate for $\|F(\hat{x}) - z_\alpha^\delta\|$.

Assumption 4.2.3. *There exists a continuous, strictly monotonically increasing function* $\varphi : (0, a] \to (0, \infty)$ *with* $a \geq \|K^*K\|$ *satisfying;*

- $\lim_{\lambda \to 0} \varphi(\lambda) = 0$
-
$$\sup_{\lambda \geq 0} \frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \leq c_\varphi \varphi(\alpha), \qquad \forall \alpha \in (0, a].$$
- *there exists* $v \in X$ *such that*

$$F(\hat{x}) = \varphi(K^*K)v \tag{4.2.1}$$

Let

$$z_\alpha := (K^*K + \alpha I)^{-1} K^* y.$$

Hereafter we consider z_α^δ as in (4.1.7). We observe that

$$\begin{aligned} \|F(\hat{x}) - z_\alpha^\delta\| &\leq \|F(\hat{x}) - z_\alpha\| + \|z_\alpha - z_\alpha^\delta\| \\ &\leq \|F(\hat{x}) - z_\alpha\| + \frac{\delta}{\sqrt{\alpha}}, \end{aligned} \tag{4.2.2}$$

and

$$\begin{aligned} F(\hat{x}) - z_\alpha &= F(\hat{x}) - (K^*K + \alpha I)^{-1} K^*KF(\hat{x}) \\ &= [I - (K^*K + \alpha I)^{-1} K^*K]F(\hat{x}) \\ &= \alpha(K^*K + \alpha I)^{-1} F(\hat{x}). \end{aligned} \tag{4.2.3}$$

So by Assumption 4.2.3,

$$\|F(\hat{x}) - z_\alpha\| \leq \|v\| c_\varphi \varphi(\alpha). \tag{4.2.4}$$

Thus we have the following theorem.

THEOREM 4.2.1. *Let z_α^δ be as in (4.1.7) and the Assumption 4.2.3 holds. Then*

$$\|F(\hat{x}) - z_\alpha^\delta\| \leq max\{\|v\|c_\varphi, 1\}(\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}). \tag{4.2.5}$$

4.2.1 Apriori Choice of the Parameter

Note that the estimate $\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}$ in (4.2.5) attains minimum for the choice $\alpha := \alpha_\delta$ which satisfies $\varphi(\alpha_\delta) = \frac{\delta}{\sqrt{\alpha_\delta}}$. Let $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, \quad 0 < \lambda \leq \|K\|^2$. Then we have $\delta = \sqrt{\alpha_\delta}\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$, and

$$\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta)). \tag{4.2.6}$$

So Theorem 4.2.1 and the above observation lead to the following.

THEOREM 4.2.2. *Let $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, \quad 0 < \lambda \leq \|K\|^2$ and the assumptions of Theorem 4.2.1 are satisfied. For $\delta > 0$, let $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$. Then*

$$\|F(\hat{x}) - z_\alpha^\delta\| \leq O(\psi^{-1}(\delta)).$$

4.2.2 An Adaptive Choice of the Parameter

The error estimate in the above Theorem has optimal order with respect to δ. As we have stated in the previous chapters, an a priori parameter choice (4.2.6) cannot be used in practice since the smoothness properties of the unknown solution $\hat{x}$ reflected in the function φ are generally unknown.

In this chapter also, we consider the adaptive method for selecting the parameter α in z_α^δ.

Let $i \in \{0, 1, 2, \cdots, N\}$ and $\alpha_i = \mu^{2i}\alpha_0$ where $\mu > 1$ and $\alpha_0 = \delta^2$. Let

$$l := max\{i : \varphi(\alpha_i) \leq \frac{\delta}{\sqrt{\alpha_i}}\} \tag{4.2.7}$$

and

$$k := max\{i : \|z_{\alpha_i}^\delta - z_{\alpha_j}^\delta\| \leq \frac{4\delta}{\sqrt{\alpha_j}}, j = 0, 1, 2, \cdots, i\}. \tag{4.2.8}$$

Then analogous to the proof of Theorem 2.4.3 one can prove the following Theorem.

THEOREM 4.2.3. *Let l be as in (4.2.7), k be as in (4.2.8) and $z^\delta_{\alpha_k}$ be as in (4.1.7) with $\alpha = \alpha_k$. Then $l \leq k$ and*

$$\|F(\hat{x}) - z^\delta_{\alpha_k}\| \leq (2 + \frac{4\mu}{\mu - 1})\mu\psi^{-1}(\delta).$$

4.3 Quadratic Convergence

Now consider the nonlinear equation (4.1.5) with $z^\delta_{\alpha_k}$ in place of z. It can be seen as in [58], Theorem 1.1, that for monotone operator F, the equation

$$F(x) + (x - x_0) = z^\delta_{\alpha_k}. \tag{4.3.1}$$

has a unique solution $x^\delta_{\alpha_k}$. It is interesting to note that the camouflaged presence of regularization parameter in α_k, in (4.3.1) relieves us of the labour of Lavrentiev regularization in the nonlinear part.

We propose the following iterative method for computing the solution $x^\delta_{\alpha_k}$. For $n \geq 0$, let

$$x^\delta_{n+1,\alpha_k} = x^\delta_{n,\alpha_k} - (F'(x^\delta_{n,\alpha_k}) + I)^{-1}(F(x^\delta_{n,\alpha_k}) - z^\delta_{\alpha_k} + (x^\delta_{n,\alpha_k} - x_0)), \tag{4.3.2}$$

where x_0 is a starting point of the iteration. The main goal of this section is to provide sufficient conditions for the quadratic convergence of method (4.3.2) to $x^\delta_{\alpha_k}$ and obtain an error estimate for $\|x^\delta_{\alpha_k} - x^\delta_{n,\alpha_k}\|$. We use a majorizing sequence for proving our results. Recall (see [2], Definition 1.3.11) that a nonnegative sequence (t_n) is said to be a majorizing sequence of a sequence (x_n) in X if

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n, \forall n \geq 0.$$

During the convergence analysis we will be using the following Lemma on majorization, which is a reformulation of Lemma 1.3.12 in [2].

LEMMA 4.3.1. *Let (t_n) be a majorizing sequence for (x_n) in X. If $\lim_{n\to\infty} t_n = t^*$, then $x^* = \lim_{n\to\infty} x_n$ exists and*

$$\|x^* - x_n\| \leq t^* - t_n, \forall n \geq 0. \tag{4.3.3}$$

The next Lemma on majorizing sequence is used to prove the convergence of the method (4.3.2).

LEMMA 4.3.2. *Assume there exist nonnegative numbers $q \in [0,1)$ and κ_0, η non-negative such that for all $n \geq 0$,*

$$\frac{3\kappa_0}{2} q^n \eta \leq q. \tag{4.3.4}$$

Then the iteration $(t_n), n \geq 0$, given by $t_0 = 0, t_1 = \eta$,

$$t_{n+1} = t_n + \frac{3\kappa_0}{2}(t_n - t_{n-1})^2 \tag{4.3.5}$$

is increasing, bounded above by $t^{**} := \frac{\eta}{1-q}$, and converges to some t^* such that $0 < t^* \leq \frac{\eta}{1-q}$. Moreover, for $n \geq 0$;

$$0 \leq t_{n+1} - t_n \leq q(t_n - t_{n-1}) \leq q^n \eta, \tag{4.3.6}$$

and

$$t^* - t_n \leq \frac{q^n}{1-q}\eta. \tag{4.3.7}$$

Proof. Since the result holds for $\eta = 0, \kappa_0 = 0$ or $q = 0$, we assume that $\kappa_0 \neq 0, \eta \neq 0$ and $q \neq 0$. Observe that $t_{i+1} - t_i \geq 0$ for all $i \geq 0$. If

$$\frac{3\kappa_0}{2}(t_{i+1} - t_i) \leq q, \tag{4.3.8}$$

then the estimate (4.3.7) follows from (4.3.5). So we shall prove (4.3.8) by induction on $i \geq 0$.

For $i = 0$, (4.3.8) holds by (4.3.4). Suppose (4.3.8) holds for all $i \leq k$ for some k. Then by (4.3.5) we have

$$\frac{3\kappa_0}{2}(t_{k+2} - t_{k+1}) \leq (\frac{3\kappa_0}{2}(t_{k+1} - t_k))^2 \leq q^2 < q.$$

Thus by induction (4.3.8) holds for all $i \geq 0$. Also, for $k \geq 0$,

$$\begin{aligned} t_{k+1} \leq t_k + q(t_k - t_{k-1}) &\leq \cdots \leq \eta + q\eta + \cdots + q^k\eta \\ &= \frac{1-q^{k+1}}{1-q}\eta < \frac{\eta}{1-q}. \end{aligned}$$

Hence the sequence $(t_n), n \geq 0$ is bounded above by $\frac{\eta}{1-q}$ and is nondecreasing. So it converges to some $t^* \leq \frac{\eta}{1-q}$. Further,

$$t^* - t_n = \lim_{i \to \infty} t_{n+i} - t_n \leq \lim_{i \to \infty} \sum_{j=0}^{i-1}(t_{n+1+j} - t_{n+j}) \leq \frac{q^n}{1-q}\eta.$$

This completes the proof of the lemma.

To prove the convergence of the sequence (x^δ_{n,α_k}) defined in (4.3.2) we introduce the following notations:

Let $R(x) := F'(x) + I$ and

$$G(x) := x - R(x)^{-1}[F(x) - z^\delta_{\alpha_k} + (x - x_0)]. \tag{4.3.9}$$

Note that with the above notation $G(x^\delta_{n,\alpha_k}) = x^\delta_{n+1,\alpha_k}$. Hereafter we assume that $\|x_0 - \hat{x}\| \leq \rho$ and

$$\begin{aligned} \frac{k_0}{2}\rho^2 + \rho + (2 + \frac{4\mu}{\mu - 1})\mu\psi^{-1}(\delta) &\leq \eta \\ &\leq min\{r(1-q), \frac{2q}{3k_0}\}. \end{aligned} \tag{4.3.10}$$

THEOREM 4.3.3. *Let η be as in (4.3.10). Under the assumption 4.2.2 and the assumptions in the Lemma 4.3.2 the sequence (x^δ_{n,α_k}) defined in (4.3.2) is well defined and $x^\delta_{n,\alpha_k} \in B_{t^*}(x_0)$ for all $n \geq 0$. Further (x^δ_{n,α_k}) is a Cauchy sequence in $B_{t^*}(x_0)$ and hence converges to $x^\delta_{\alpha_k} \in \overline{B_{t^*}(x_0)} \subset B_{t^{**}}(x_0)$ and $F(x^\delta_{\alpha_k}) = z^\delta_{\alpha_k} + (x_0 - x^\delta_{\alpha_k})$.*

Moreover, the following estimates hold for all $n \geq 0$,

$$\|x^\delta_{n+1,\alpha_k} - x^\delta_{n,\alpha_k}\| \leq t_{n+1} - t_n, \tag{4.3.11}$$

$$\|x^\delta_{n,\alpha_k} - x^\delta_{\alpha_k}\| \leq t^* - t_n \leq \frac{q^n\eta}{1-q}, \tag{4.3.12}$$

and

$$\|x^{\delta}_{n+1,\alpha_k} - x^{\delta}_{\alpha_k}\| \leq \frac{k_0}{2}\|x^{\delta}_{n,\alpha_k} - x^{\delta}_{\alpha_k}\|^2. \qquad (4.3.13)$$

Proof. First we shall prove that

$$\|x^{\delta}_{n+1,\alpha_k} - x^{\delta}_{n,\alpha_k}\| \leq \frac{3\kappa_0}{2}\|x^{\delta}_{n,\alpha_k} - x^{\delta}_{n-1,\alpha_k}\|^2. \qquad (4.3.14)$$

With G as in (4.3.9), we have for $u, v \in B_{t^*}(x_0)$,

$$\begin{aligned}
G(u) - G(v) &= u - v - R(u)^{-1}[F(u) - z^{\delta}_{\alpha_k} + (u - x_0)] \\
&\quad + R(v)^{-1}[F(v) - z^{\delta}_{\alpha_k} + (v - x_0)] \\
&= u - v - [R(u)^{-1} - R(v)^{-1}](F(v) - z^{\delta}_{\alpha_k} + (v - x_0)) \\
&\quad - R(u)^{-1}(F(u) - F(v) + (u - v)) \\
&= R(u)^{-1}[F'(u)(u - v) - (F(u) - F(v)] \\
&\quad - R(u)^{-1}[F'(v) - F'(u)]R(v)^{-1}(F(v) - z^{\delta}_{\alpha_k} + (v - x_0)) \\
&= R(u)^{-1}[F'(u)(u - v) - (F(u) - F(v))] \\
&\quad + R(u)^{-1}[F'(v) - F'(u)](v - G(v)) \\
&= R(u)^{-1}[F'(u)(u - v) + \int_0^1 (F'(u + t(v - u))(v - u)dt] \\
&\quad + R(u)^{-1}[F'(v) - F'(u)](v - G(v)) \\
&= \int_0^1 R(u)^{-1}[(F'(u + t(v - u)) - F'(u))(v - u)dt] \\
&\quad + R(u)^{-1}[F'(v) - F'(u)](v - G(v))
\end{aligned}$$

The last but one step follows from the Fundamental Theorem of Integral Calculus. So by the Assumption 4.2.2 and the estimate $\|R(u)^{-1}F'(u)\| \leq 1$, we have

$$\|G(u) - G(v)\| \leq \frac{\kappa_0}{2}\|u - v\|^2 + \kappa_0\|u - v\|\|v - G(v)\|. \qquad (4.3.15)$$

Now taking $u = x^{\delta}_{n,\alpha_k}$ and $v = x^{\delta}_{n-1,\alpha_k}$ in (4.3.15), we obtain (4.3.14).

Next we shall prove that the sequence (t_n) defined in Lemma 4.3.2 is a majorizing sequence of the sequence $(x^{\delta}_{n,\alpha_k})$.

Note that

$$\begin{aligned}
\|x^{\delta}_{1,\alpha_k} - x_0\| &= \|R(x_0)^{-1}(F(x_0) - z^{\delta}_{\alpha_k})\| \\
&\leq \|R(x_0)^{-1}(F(x_0) - F(\hat{x}) + F(\hat{x}) - z^{\delta}_{\alpha_k})\| \\
&\leq \|R(x_0)^{-1}(F(x_0) - F(\hat{x}) - F(x_0)(x_0 - \hat{x}) \\
&\quad + F(x_0)(x_0 - \hat{x}) + F(\hat{x}) - z^{\delta}_{\alpha_k})\| \\
&\leq \|R(x_0)^{-1}(F(x_0) - F'(\hat{x}) - F(x_0))(x_0 - \hat{x})\| \\
&\quad + \|R(x_0)^{-1}F'(x_0)(x_0 - \hat{x})\| + \|R(x_0)^{-1}[F(\hat{x}) - z^{\delta}_{\alpha_k}]\| \\
&\leq \|R(x_0)^{-1}\int_0^1 F'(\hat{x} + t(x_0 - \hat{x})) - F'(x_0)](x_0 - \hat{x})dt\| \\
&\quad + \|(x_0 - \hat{x})\| + (2 + \frac{4\mu}{\mu - 1})\mu\psi^{-1}(\delta) \\
&\leq \|R(x_0)^{-1}F'(x_0)\int_0^1 \Phi(\hat{x} + t(x_0 - \hat{x}), x_0, x_0 - \hat{x})dt\| \\
&\quad + \|x_0 - \hat{x}\| + (2 + \frac{4\mu}{\mu - 1})\mu\psi^{-1}(\delta) \\
&\leq \frac{k_0}{2}\|x_0 - \hat{x}\|^2 + \|x_0 - \hat{x}\| + (2 + \frac{4\mu}{\mu - 1})\mu\psi^{-1}(\delta) \\
&\leq \eta = t_1 - t_0. \qquad (4.3.16)
\end{aligned}$$

Assume that $\|x^{\delta}_{i+1,\alpha_k} - x^{\delta}_{i,\alpha_k}\| \leq t_{i+1} - t_i$ for all $i \leq k$ for some k. Then by (4.3.14),

$$\|x^{\delta}_{k+2,\alpha_k} - x^{\delta}_{k+1,\alpha_k}\| \leq \frac{3\kappa_0}{2}\|x^{\delta}_{k+1,\alpha_k} - x^{\delta}_{k,\alpha_k}\|^2 \leq \frac{3\kappa_0}{2}(t_{k+1} - t_k)^2 = t_{k+2} - t_{k+1}.$$

Thus by induction $\|x^{\delta}_{n+1,\alpha_k} - x^{\delta}_{n,\alpha_k}\| \leq t_{n+1} - t_n$ for all $n \geq 0$ and hence $(t_n), n \geq 0$ is a majorizing sequence of the sequence $(x^{\delta}_{n,\alpha_k})$. So by Lemma 4.3.1 $(x^{\delta}_{n,\alpha_k}), n \geq 0$ is a Cauchy sequence and converges to some $x^{\delta}_{\alpha_k} \in \overline{B_{t^*}(x_0)} \subset B_{t^{**}}(x_0)$ and

$$\|x^{\delta}_{\alpha_k} - x^{\delta}_{n,\alpha_k}\| \leq t^* - t_n \leq \frac{q^n\eta}{1-q}.$$

To prove (4.3.13), we observe that $G(x^{\delta}_{\alpha_k}) = x^{\delta}_{\alpha_k}$, so (4.3.13) follows from (4.3.15), by taking $u = x^{\delta}_{n,\alpha_k}$ and $v = x^{\delta}_{\alpha_k}$ in (4.3.15). Now by letting $n \to \infty$ in (4.3.1) we obtain $F(x^{\delta}_{\alpha_k}) = z^{\delta}_{\alpha_k} + (x_0 - x^{\delta}_{\alpha_k})$. This completes the proof of the Theorem.

REMARK 4.3.4. *Note that (4.3.13) implies $(x^{\delta}_{n,\alpha_k})$ converges quadratically to $x^{\delta}_{\alpha_k}$.*

4.4 Linear Convergence

In this section, we consider the sequence $(\tilde{x}_n^\delta)$ defined iteratively by

$$\tilde{x}_{n+1}^\delta := \tilde{x}_n^\delta - (F'(x_0)+I)^{-1}(F(\tilde{x}_n^\delta) - z_{\alpha_k}^\delta + (\tilde{x}_n^\delta - x_0)), \tag{4.4.1}$$

where x_0 is a starting point of the iteration. We prove that the sequence $(\tilde{x}_n^\delta)$ converge to the unique solution $x_{\alpha_k}^\delta$ of (4.3.1) and obtain an error estimate for $\|x_{\alpha_k}^\delta - \tilde{x}_n^\delta\|$. The proof of the following lemma is analogous to the proof of Lemma 4.3.2.

LEMMA 4.4.1. *Assume there exist $\tilde{r} \in [0,1)$ and nonnegative numbers κ_0, η, α such that*

$$\frac{\kappa_0}{(1-\tilde{r})}\eta \leq \tilde{r}. \tag{4.4.2}$$

Then the sequence $(\tilde{t}_n)$ defined by

$$\tilde{t}_{n+1} = \tilde{t}_n + \frac{\kappa_0}{(1-\tilde{r})}\eta(\tilde{t}_n - \tilde{t}_{n-1}) \tag{4.4.3}$$

is increasing, bounded above by $\tilde{t}^{**} := \frac{\eta}{1-\tilde{r}}$, and converges to some $\tilde{t}^*$ such that $0 < \tilde{t}^* \leq \frac{\eta}{1-\tilde{r}}$. Moreover, for $n \geq 0$;

$$0 \leq \tilde{t}_{n+1} - \tilde{t}_n \leq \tilde{r}(\tilde{t}_n - \tilde{t}_{n-1}) \leq \tilde{r}^n\eta, \tag{4.4.4}$$

and

$$\tilde{t}^* - \tilde{t}_n \leq \frac{\tilde{r}^n}{1-\tilde{r}}\eta. \tag{4.4.5}$$

We shall assume that

$$\begin{aligned}\frac{k_0}{2}\rho^2 + \rho + (2 + \frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta) &\leq \eta \\ &\leq min\{r(1-\tilde{r}), \frac{\tilde{r}(1-\tilde{r})}{k_0}\}.\end{aligned} \tag{4.4.6}$$

Let

$$\tilde{R}(x_0) := F'(x_0) + I$$

and

$$\tilde{G}(x) := x - \tilde{R}(x_0)^{-1}[F(x) - z_{\alpha_k}^\delta + (x - x_0)]. \tag{4.4.7}$$

Note that with the above notation, $\tilde{G}(\tilde{x}_n^\delta) = \tilde{x}_{n+1}^\delta$ and $\|\tilde{R}(x_0)^{-1}\| \leq 1$.

THEOREM 4.4.2. *Suppose Assumptions 4.2.1 and 4.2.2 hold. Let the assumptions in Lemma 4.4.1 are satisfied with η as in (4.4.6). Then the sequence $(\tilde{x}_n^\delta)$ defined in (4.4.1) is well defined and $\tilde{x}_n^\delta \in B_{\tilde{t}^*}(x_0)$ for all $n \geq 0$. Further $(\tilde{x}_n^\delta)$ is a Cauchy sequence in $B_{\tilde{t}^*}(x_0)$ and hence converges to $x_{\alpha_k}^\delta \in \overline{B_{\tilde{t}^*}(x_0)} \subset B_{\tilde{t}^{**}}(x_0)$ and $F(x_{\alpha_k}^\delta) + (x_{\alpha_k}^\delta - x_0) = z_{\alpha_k}^\delta$.*

Moreover, the following estimates hold for all $n \geq 0$,

$$\|\tilde{x}_{n+1}^\delta - \tilde{x}_n^\delta\| \leq \tilde{t}_{n+1} - \tilde{t}_n, \tag{4.4.8}$$

and

$$\|\tilde{x}_n^\delta - x_{\alpha_k}^\delta\| \leq \tilde{t}^* - \tilde{t}_n \leq \frac{\tilde{r}^n \eta}{1 - \tilde{r}}. \tag{4.4.9}$$

Proof.

Let G be as in (4.4.7). Then for $u, v \in B_{\tilde{t}^*}(x_0)$,

$$\begin{aligned}
\tilde{G}(u) - \tilde{G}(v) &= u - v - \tilde{R}(x_0)^{-1}[F(u) - z_{\alpha_k}^\delta + (u - x_0)] \\
&\quad + \tilde{R}(x_0)^{-1}[F(v) - z_{\alpha_k}^\delta + (v - x_0)] \\
&= \tilde{R}(x_0)^{-1}[\tilde{R}(x_0)(u - v) - (F(u) - F(v))] + \tilde{R}(x_0)^{-1}(v - u) \\
&= \tilde{R}(x_0)^{-1}[F'(x_0)(u - v) - (F(u) - F(v)) + (u - v)] \\
&\quad + \tilde{R}(x_0)^{-1}(v - u) \\
&= \tilde{R}(x_0)^{-1}[F'(x_0)(u - v) - (F(u) - F(v))]
\end{aligned}$$

Thus by Assumption 4.2.2 we have

$$\|\tilde{G}(u) - \tilde{G}(v)\| \leq \kappa_0 \tilde{t}^* \|u - v\|. \tag{4.4.10}$$

The rest of the proof is analogous to the proof of Theorem 4.3.3.

REMARK 4.4.3. *Now by taking $u = x^\delta_{\alpha_k}$ and $v = \tilde{x}_{n-1}$ in (4.4.10), we obtain linear convergence of $\tilde{x}_{n-1}$ to $x^\delta_{\alpha_k}$.*

REMARK 4.4.4. *For the remainder of the chapter we shall consider only the quadratically convergent sequence (x^δ_{n,α_k}) defined in (4.3.2) for detailed analysis. The results verbatim hold good in the case of linearly convergent sequence $(\tilde{x}^\delta_n)$ defined in (4.4.1).*

4.5 Error Bounds Under Source Conditions

The main objective of this section is to obtain an error estimate for $\|x^\delta_{n,\alpha_k} - \hat{x}\|$ under the assumption

$$\|x_0 - \hat{x}\| \leq c\frac{1}{\mu^k} \tag{4.5.1}$$

for some constant c and source condition (4.2.1) on $F(\hat{x})$. Note that $F(x^\delta_{\alpha_k}) + (x^\delta_{\alpha_k} - x_0) = z^\delta_{\alpha_k}$. So that $F(x^\delta_{\alpha_k}) - F(\hat{x}) + (x^\delta_{\alpha_k} - x_0) = z^\delta_{\alpha_k} - F(\hat{x})$. Therefore by monotonicity of F, by taking inner product with $x^\delta_{\alpha_k} - \hat{x}$ we obtain the following:

THEOREM 4.5.1. *Under the assumption 4.2.2,*

$$\|x^\delta_{\alpha_k} - \hat{x}\| \leq \|F(\hat{x}) - z^\delta_{\alpha_k}\| + \|x_0 - \hat{x}\|.$$

Combining the estimates in Theorem 4.2.1 Theorem 4.3.3, Theorem 4.5.1, (4.5.1) and the relations $\frac{1}{\mu^k} = \frac{1}{\mu^{k-l}\mu^l} = \frac{1}{\mu^{k-l}}\frac{\delta}{\sqrt{\alpha_l}}$ and $\sqrt{\alpha_\delta} \leq \sqrt{\alpha_{l+1}} = \mu\sqrt{\alpha_l}$ we have $\frac{1}{\mu^k} \leq \frac{1}{\mu^{k-l-1}}\frac{\delta}{\sqrt{\alpha_\delta}} = \frac{1}{\mu^{k-l-1}}\psi^{-1}(\delta)$, so we obtain the following.

THEOREM 4.5.2. *Let $x^\delta_{\alpha_k}$ be the unique solution of (4.3.1) and x^δ_{n,α_k} be as in (4.3.2). Let the assumptions in Theorem 4.2.3, Theorem 4.3.3, and Theorem 4.5.1 be satisfied. Then we have*

$$\|x^\delta_{n,\alpha_k} - \hat{x}\| \leq \frac{q^n\eta}{1-q} + O(\psi^{-1}(\delta)). \tag{4.5.2}$$

4.5.1 Stopping Index

Let

$$n_k = min\{n : q^n \leq \frac{1}{\mu^k}\}. \tag{4.5.3}$$

Then we have the following

THEOREM 4.5.3. *Let $x^{\delta}_{\alpha_k}$ be the unique solution of (4.3.1) and x^{δ}_{n,α_k} be as in (4.3.2). Let the assumptions in Theorem 4.2.1, Assumption 4.2.1, Assumption 4.2.2 and Assumption 4.2.3 be satisfied. Let n_k be as in (4.5.3). Then we have*

$$\|x^{\delta}_{n_k,\alpha_k} - \hat{x}\| = O(\psi^{-1}(\delta)). \tag{4.5.4}$$

4.6 Implementation of Adaptive Choice Rule

The main goal of this section is to provide a starting point for the iteration approximating the unique solution x^{δ}_{α} of (4.3.1) and then to provide an algorithm for the determination of a parameter fulfilling the balancing principle (4.2.8). Hereafter we assume without loss of generality that $k_0 \leq \frac{1}{4\eta}$ (if not, replace F by cF where $c \leq \frac{1}{4k_0\eta}$).

For $i, j \in \{0, 1, 2, \cdots, N\}$, we have

$$z^{\delta}_{\alpha_i} - z^{\delta}_{\alpha_j} = (\alpha_j - \alpha_i)(K^*K + \alpha_i I)^{-1}(K^*K + \alpha_j I)^{-1}K^*y^{\delta}.$$

The implementation of our method involves the following steps:

Step I

- i=1
- Solve for $w_i : (K^*K + \alpha_i I)w_i = K^*y^{\delta}$
- Solve for $z_{i,j} : (K^*K + \alpha_i I)^{-1}z_{i,j} = (\alpha_j - \alpha_i)w_i, j \leq i$
- If $\|z_{i,j}\| > \frac{4}{\mu^j}$, then take $k = i - 1$.
- Otherwise, repeat with $i + 1$ in place of i.

Step II

- Choose $q < 1$.
- Choose $x_0 \in D(F)$ such that $\|x_0 - \hat{x}\| < \frac{c}{\mu^k}$ for some constant c such that $\frac{k_0}{2}\frac{c^2}{\mu^{2k}} + \frac{c}{\mu^k} + (2 + \frac{4\mu}{\mu-1})\mu\psi^{-1}(\delta) \leq \eta$
$\leq min\{r(1-q), \frac{2q}{3k_0}\}$

Step III

- $n = 1$
- If $q^n \leq \frac{1}{\mu^k}$, then take $n_k := n$
- Otherwise , repeat with $n+1$ in place of n

Step IV

- Solve $x_{j,\alpha_k}^\delta : (F'(x_{j-1,\alpha_k}^\delta) + I)(x_{j,\alpha_k}^\delta - x_{j-1,\alpha_k}^\delta) = F(x_{j-1,\alpha_k}^\delta) - w_k + x_{j-1,\alpha_k}^\delta - x_0$ for $j = 1, 2, \cdots, n_k$.

□

Chapter 5

Concluding Remarks

In this thesis we focussed our attention exclusively on some iterative regularization methods for solving nonlinear ill-posed Hammerstein-type operator equation

$$KF(x) = y, \tag{5.0.1}$$

where $F : D(F) \subseteq X \to Z$ is a nonlinear operator and $K : Z \to Y$ is a bounded linear operator where X, Y, Z are Hilbert spaces. These type of methods for abstract Hammerstein operator equations were first considered by George in [14] , [15] for obtaining approximations for an x_0-minimal norm solution of (5.0.1). Recall that the solution $\hat{x}$ of (5.0.1) is called an x_0-minimal norm solution of (5.0.1) if

$$\|\hat{x} - x_0\| = min\{\|x - x_0\| : KF(x) = y,\, x \in D(F)\}. \tag{5.0.2}$$

Later in [20], George and Nair considered an iterative method for obtaining a stable approximate solution for a modified x_0-minimal norm solution of (5.0.1) i.e., solution $\hat{x}$ of (5.0.1) satisfies

$$\|F(\hat{x}) - F(x_0)\| = min\{\|F(x) - F(x_0)\| : KF(x) = y,\, x \in D(F)\}. \tag{5.0.3}$$

In chapter 2 we proposed an iterative method for obtaining stable approximate solution for (5.0.1) which guarantees quadratic convergence compared to the linear convergence obtained in [20]. We assume that solution $\hat{x}$ of (5.0.1) satisfies (5.0.3). The procedure involves, with the available data y^δ in place of the exact data y, solving the equation

$$(K^*K + \alpha I)z_\alpha^\delta = K^*(y^\delta - KF(x_0))$$

and finding the fixed point of the function

$$G(x) = x - F'(x)^{-1}(F(x) - F(x_0) - z_\alpha^\delta)$$

in an iterative manner. It is assumed, here, that the Fréchet derivative $F'(.)$ of F has continuous inverse in a neighborhood of some initial guess x_0 of the actual solution. For choosing the regularization parameter α and the stopping index for the iteration, we made use of the adaptive method suggested in [50].

In chapter 3, we take $X = Y = Z = H$ and study the procedure considered in [20] in the setting of Hilbert scales and obtained improved error estimates. Here it is assumed that the bounded linear operator K is positive self adjoint, $F'(x_0)$ is boundedly invertible. We consider the Hilbert scale $(H_r)_{r\in\mathbb{R}}$ generated by a densely defined, linear, unbounded, strictly positive self adjoint operator $L : D(L) \subseteq H \to H$.

Note that by the assumption of continuous invertibility of the Fréchet derivative $F'(.)$ the ill-posedness of the problems in chapter 2 and chapter 3 is due to the nonclosedness of the range of K. In chapter 4, we consider special case of the equation (5.0.1) when the nonlinear operator F is a monotone operator. Here we take $Z = X$ and $D(F) \subset X$. In this case we can do away with the assumption of invertibility of F' even at the initial guess x_0, and hence the problem may be ill-posed in the nonlinear part as well. We propose two different iterative methods with y^δ in place of y, for solving the equation (5.0.1). In the first method we find the fixed points of the function

$$G(x) = x - (F'(x) + I)^{-1}[F(x) - z_\alpha^\delta + x - x_0]$$

and in the second we find the fixed point of the function

$$\widetilde{G}(x) = x - (F'(x_0) + I)^{-1}[F(x) - z_\alpha^\delta + x - x_0]$$

where

$$z_\alpha^\delta = (K^*K + \alpha I)^{-1}K^*y^\delta.$$

The convergence analysis is carried out by means of suitably constructed majorizing sequences. The regularization parameter α for the linear part is chosen by the adaptive method of Pereverzev and Schock [50] and stopping index is prescribed using the majorizing sequence.

The methods considered in this thesis for solving ill-posed Hammerstien type operator equations, by no means, is exhaustive. From the perspective of this thesis itself, in future works, we would like to analyze the methods in chapter 2 and chapter 4 in the Hilbert scale set up and look for finite dimensional realizations of all these methods.

It is envisaged to show the effectiveness of our methods by computational verification for some well known examples. □

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Publications

List of Publications/Communications Based on the Thesis

1. S.George and M.Kunhanandan, An iterative regularization method for ill-posed Hammerstein type operator equation, J.Inv.Ill-Posed Problems, 17 , pp.831-844,(2009).

2. S.George and M.Kunhanandan, Iterative regularization methods for ill-posed Hammerstein type operator equation with monotone nonlinear part, Int.Journal of Math. Analysis, Vol. 4, no.34, pp.1673-1685, (2010).

3. S.George and M.Kunhanandan, Iterative Regularization Methods for Ill-posed Hammerstein Type Operator Equations in Hilbert Scales (communicated).