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Hyers-Ulam and Hyers-Ulam-Aoki-Rassias Stability for Linear Ordinary Differential Equations

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Abstract

Here we prove the Hyers-Ulam stability and Hyers-Ulam-Aoki-Rassias stability of the n-th order ordinary linear differential equation with smooth coefficients on compact and semi-bounded intervals using successive integration by parts.

Keywords: Ordianry differential equations; Hyers-Ulam stability

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1. Introduction

Stanislaw Marcin Ulam, in 1940, posed a problem concerning the stability of functional equation to give conditions in order for a linear mapping near a approximate linear mapping to exist. Hyers solved the problem for a pair of Banach spaces, thus came the terminology **Hyers-Ulam stability** (in short **HU stability**). The result of Hyers was further generalised by (Aoki, 1950) and (Rassias, 1978), which is termed as **Hyers-Ulam-Aoki-Rassias stability** (in short **HUAR stability**) or simply **Hyers-Ulam-Rassias stability** or **generalised Hyers-Ulam stability**. Since then the stability problems for functional equations have been studied by many mathematicians.

The study of stability for linear ordinary differential equations was started with the investigation by (Obloza, 1993; Obloza, 1997) and soon after by (Alsina and Ger, 1998). They studied the stability of $y'(t) = y(t)$. This was further generalised by (Miura et al., a). They studied the Hyers-

Ulam stability of the differential equation $y'(t) = \lambda y(t)$ where λ is a complex number. After this many have investigated the Hyer-Ulam stability of various types of differential equations. In this note we prove the Hyers-Ulam stability and the HUAR stability for n-th order ordinary linear differential equation.

For more on Hyers-Ulam type stability of ordinary differential equations, we refer to (Jung, 2004; Jung, 2005; Jung, ; Miura et al., a; Qarawani, 2012; Rus, 2009; Miura et al., 2003a; Miura et al., 2003b; Miura et al., b; Cimpean and Popa, 2010).

Consider a linear differential equation of n-th order

$$L_n y(x) \equiv y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y + q = 0, \tag{.1}$$

on two types of intervals; compact interval and non-compact interval. Assume that the coefficient functions $p_1, \cdots, p_n$ are sufficiently smooth on the interval under consideration. For non compact intervals, the Hyers-Ulam stability is somewhat difficult to prove.

For a non-negative function $\epsilon(t)$ on an interval, we say that a n times continuously differentiable function y is an $\epsilon(t)$-**approximate solution** of (.1) if y satisfies

$$|y^{(n)}(t) + p_1(t) y^{(n-1)}(t) + \cdots + p_n(t) y(t) + q(t)| \leq \epsilon(t), \tag{.2}$$

for all t in the interval. Similarly we say a n times continuously differentiable function z is an **exact solution** of (.1) if

$$L_n z(t) = 0.$$

Definition 0.1: The differential equation (.1) on an interval is said to be **HU stable** on an interval if the following holds:

For any $\epsilon > 0$ there exists a constant $K > 0$ (independent of ϵ) such that whenever y is a n-times differentiable function satisfying $|L_n y(x)| \leq \epsilon$, there exists a solution z of (.1) such that $|y(x) - z(x)| \leq K\epsilon$ for all x.

Definition 0.2: The differential equation (.1) on an interval is said to be **HUAR stable** on if the following holds:

Let $\epsilon(t) \geq 0$ be a continuous function. Then there exists an nonnegative function $\epsilon_1(t)$, which depends only on $\epsilon(t)$ and the coefficients of the ODE (.1), such that whenever y is a n-times differentiable function satisfying (.2), there exists a solution z of (.1) such that $|y(t) - z(t)| \leq \epsilon_1(t)$.

Consider the first order linear differential equation

$$p_0 y' + p_1 y + q = 0, \tag{.3}$$

where $p_i' s$ and q are assumed to be continuous functions on $I = (a, b)$. In this case assuming (i) $p_0(t) \neq 0$ for all $t \in I$, (ii) $|p_1(t)| \geq \delta$ for some $\delta > 0$, and (iii) $\int_a^b \frac{p_1(t)}{p_0(t)} dt < \infty$, the HU stability was proved in (Wang et al., 2008).

However, (Jung,) considered the equation (.3) in a complex Banach space X with complex valued continuous coefficients. He proved the HUAR stability of (.3):

Theorem 0.3: ((Jung,)) Let X be a complex Banach space, and let $q : I \to X$ strongly continuous function. Let p_1 be a complex valued continuous function and $\epsilon(t)$ be a non-negative function on I. Denote G_a by $G_a(t) = e^{-\int_a^t p_1(u)du}$. Assume that
(i) $p_1(t)$, $\left(\exp \int_a^t p_1(u)du\right) q(t)$ are integrable on (a, c) for each $c \in I$,
(ii) $\epsilon(t) \exp \int_a^t p_1(u)du$ is integrable on I.

Let $y : I \to X$ be an $\epsilon(t)$-approximate solution of (.3) with $p_0(t) \equiv 1$, where the derivative is understood to exist in the strong sense. Then there exists a unique $x_0 \in X$ given by

$$x_0 = \mathrm{s} - \lim_{t \to b^-} \left(\frac{1}{G_a(t)} y(t) + \int_a^t \frac{q(u)}{G_a(u)} du \right),$$

such that the function $y_1(t) = G_a(t) \left(x_0 - \int_a^t \frac{q(u)}{G_a(u)} du \right)$ is an unique exact solution of (.3) (with $p_0(t) \equiv 1$) and satisfies

$$||y(t) - y_1(t)|| \leq |G_a(t)| \int_t^b \frac{\epsilon(u)}{|G_a(u)|} du.$$

This result gave an impetus to study the stability (in terms of a unique solution) of higher order linear differential equations.

In the general case of n-th order, for constant and non constant coefficients, the HUAR stability was proved by (**?**), and (Popa and Rosa, 2012) respectively. In this case the argument for the n-th order linear equation was basically successive application of the Theorem 0.3, assuming that the linear part of the equation is factorised into a product of first order terms (although not mentioned explicitly)

$$\left(\frac{d}{dx} + a_1(x) \right) \left(\frac{d}{dx} + a_2(x) \right) \cdots \left(\frac{dy}{dx} + a_n(x) y \right) + q(x),$$

and on certain conditions on $a_i(x)$. For $n = 2$, the HU stability was proved in (Li and Shen, 2010) using the above factorisation.

The conditions on the $a_i(x)$'s can be replaced by some integrability conditions to prove the stability for these equations. Also there are other methods, such as reducing a second order linear non homogeneous equation to a first order equation using a known solution of the corresponding second order homogeneous equation (Javadian et al., 2011), or reducing the second order non homogeneous equation to a first order linear non homogeneous equation if the second order equation is exact (Ghaemi et al., 2012). For third order, the stability was studied explicitly using the above factorisation method in (Jung, 2012) and (Abdollahpour et al., 2012).

As it has been noted, the constraints on the coefficient functions for stability for higher order equation is fairly strong. However, if the underlying interval is compact, the conditions on the coefficients can be relaxed and hence the above techniques work under less number of conditions.

Here we assume that the interval under consideration is either compact or semibounded and we prove the HU and HUAR stability of a n-th order linear differential equation with smooth variable coefficients by successively integrating it and converting it to an integral equation, where certain initial or terminal conditions are satisfied. This method is simple and seems to have been either remained unnoticed so far or is considered too elementary to be discussed in a research article.

We note that on compact intervals the HU stability was studied for linear differential equations in (Li and Shen, 2009; Gavruta et al., 2011; Qarawani, 2012; Abdollahpour and Najati, 2011; Abdollahpour et al., 2012; Li and Shen, 2010), using different methods. The intervals on which we prove the stability are either compact or semi bounded.

2. Hyers-Ulam stability of linear ODE on compact and semibounded intervals

For the remaining part of our discussion we will denote

$$\begin{aligned} I_1 &= [a,b], \ \ -\infty < a < b < \infty, \\ I_2 &= [a,b), \ \ -\infty < a < b \leq \infty, \\ I_3 &= (a,b], -\infty \leq a < b < \infty. \end{aligned}$$

Here we need a few lemmas which are required for the main result:

Lemma 0.4: **(a)** Let f be a continuous function on an interval I, where $I = I_1$ or $I = I_2$. Then n-successive integrations near the end point a yield

$$\text{(i)} \qquad \int_a^{t_n} dt_{n-1} \int_a^{t_{n-1}} dt_{n-2} \cdots \int_a^{t_1} f(t)\ dt = \int_a^{t_n} f(t) \frac{(t_n - t)^{n-1}}{(n-1)!}\ dt. \tag{.4}$$

$$\text{(ii)} \quad \int_a^{t_n} dt_{n-1} \int_a^{t_{n-1}} dt_{n-2} \cdots \int_a^{t_1} dt = \frac{(t_n - a)^n}{(n)!}, \tag{.5}$$

where $t_n \in I$.

(b) Let f be a continuous function on I, where $I = I_1$ or $I = I_3$. Then n-successive integrations near the end point b give

$$\text{(iii)} \qquad \int_{t_n}^{b} dt_{n-1} \int_{t_{n-1}}^{b} dt_{n-2} \cdots \int_{t_1}^{b} f(t)\ dt = \int_{t_n}^{b} f(t) \frac{(t - t_n)^{n-1}}{(n-1)!}\ dt. \tag{.6}$$

$$\text{(iv)} \quad \int_{t_n}^{b} dt_{n-1} \int_{t_{n-1}}^{b} dt_{n-2} \cdots \int_{t_1}^{b} dt = \frac{(b - t_n)^n}{(n)!}, \tag{.7}$$

where $t_n \in I$.

Proof: we will prove (i) by induction. For $n = 1$ the identity is trivially true. Assume it to hold for $n-1$, i.e.

$$\int_a^{t_{n-1}} dt_{n-2} \cdots \int_a^{t_1} f(t)\ dt = \int_a^{t_{n-1}} f(t) \frac{(t_{n-1} - t)^{n-2}}{(n-2)!}\ dt.$$

Then for n,

$$\int_a^{t_n} dt_{n-1} \int_a^{t_{n-1}} dt_{n-2} \cdots \int_a^{t_1} f(t)\, dt = \int_a^{t_n} dt_{n-1} \int_a^{t_{n-1}} f(t) \frac{(t_{n-1}-t)^{n-2}}{(n-2)!}\, dt.$$

With a change of region using $a \leq t \leq t_{n-1} \leq t_n$, one has the t-integral from a to t_n and t_{n-1} integral from t to t_n. So the right hand side of the integral in the above becomes

$$\int_a^{t_n} dt\ f(t) \int_t^{t_n} dt_{n-1} \frac{(t_{n-1}-t)^{n-2}}{(n-2)!} = \int_a^{t_n} f(t) \frac{(t_n-t)^{n-1}}{(n-1)!}\, dt.$$

This proves part (i). Now part (ii) follows from part (i) by setting $f(t) \equiv 1$.

The proofs of parts (iii) and (iv) are similar to that of parts (i) and (ii) respectively. So we omit the proof. □

Lemma 0.5: **(a)** Let ξ be an k times continuously differentiable on I, where $I = I_1$ or $I = I_2$, such that $\xi(a) = \xi'(a) = \cdots = \xi^{(k-1)}(a) = 0$. Then for any k times continuously differentiable function f on I and for any $t, t_k \in I$,

$$(i) \int_a^t f(u)\xi^{(k)}(u)du = \sum_{j=0}^{k-1} (-1)^j f^{(j)}(t)\xi^{(k-j)}(t) + (-1)^k \int_a^t f^{(k)}(t)\xi(t)dt, \tag{.8}$$

$$\begin{aligned} (ii) \qquad & \int_a^{t_k} dt_{k-1} \int_a^{t_{k-1}} dt_{k-2} \cdots \int_a^{t_1} f(u)\xi^{(k)}(u)du \\ = \ & \sum_{m=0}^{k} (-1)^m \binom{k}{m} \int_a^{t_k} dt_{k-1} \int_a^{t_{k-1}} dt_{k-2} \cdots \int_a^{t_{k-m+1}} f^{(m)}(u)\xi(u)du, \end{aligned} \tag{.9}$$

where the term for $m = 0$ is understood to be $f(t_k)\xi(t_k)$.

(b) Let ξ be an n times continuously differentiable on I where $I = I_1$ or $I = I_3$ such that $\xi(b) = \xi'(b) = \cdots = \xi^{(k-1)}(b) = 0$. Then for any k times continuously differentiable function f on I and for any $t_k \in I$,

$$\begin{aligned} & \int_{t_k}^{b} dt_{k-1} \int_{t_{k-1}}^{b} dt_{k-2} \cdots \int_{t_1}^{b} f(u)\xi^{(k)}(u)du \\ = \ & (-1)^k \sum_{m=0}^{k} \binom{k}{m} \int_{t_k}^{b} dt_{k-1} \int_{t_{k-1}}^{b} dt_{k-2} \cdots \int_{t_{k-m+1}}^{b} f^{(m)}(u)\xi(u)du, \end{aligned} \tag{.10}$$

where the term for $m = 0$ is understood to be $f(t_k)\xi(t_k)$.

Proof: We will prove part (a)(i) by induction. Note that for $k = 1$, the conclusion holds trivially by integration by parts. Assuming that it is true for k for any $1 \leq k \leq n-1$, we will prove it for $k+1$. Now, the hypothesis that it is true for k and an integration by parts yield

$$\begin{aligned}
&\int_a^t f(u)\xi^{(k+1)}(u)du \\
= \;& f(t)\xi^{(k)}(t) - \int_a^t f'(u)\xi^{(k)}(u)du \\
= \;& f(t)\xi^{(k)}(t) - \left\{\sum_{j=0}^{k-1}(-1)^j f^{(j+1)}(t)\xi^{k-j}(t) + (-1)^k \int_a^t f^{(k+1)}(u)\xi(u)du\right\} \\
= \;& f(t)\xi^{(k)}(t) + \sum_{j=0}^{k-1}(-1)^{j+1} f^{(j+1)}(t)\xi^{(k-j)}(t) + (-1)^{k+1}\int_a^t f^{(k+1)}(u)\xi(u)du \\
= \;& \sum_{l=0}^{k}(-1)^l f^{(l)}(t)\xi^{(k+1-j)}(t) + (-1)^{k+1}\int_a^t f^{(k+1)}(u)\xi(u)du.
\end{aligned}$$

Part (a)(ii) can also be proved using induction. This identity is satisfied for $k=1$. Assume that part (ii) holds for $k=m$. We will prove it for $k=m+1$. For $k=m+1$, an integration by part, the conditions on ξ at a and the hypothesis that (.9) holds for $m=k$ yield

$$\begin{aligned}
&\int_a^{t_{k+1}} dt_k \int_a^{t_k} dt_{k-1}\cdots\int_a^{t_1} f(u)\xi^{(k+1)}(u)du \\
&= \int_a^{t_{k+1}} dt_k \int_a^{t_k} dt_{k-1}\cdots\int_a^{t_2} dt_1 \left(\int_a^{t_1} f(u)\xi^{(k+1)}(u)du\right) \\
&= \int_a^{t_{k+1}} dt_k \cdots \int_a^{t_2} dt_1 \left(f(t_1)\xi^{(k)}(t_1) - \int_a^{t_1} f'(u)\xi^{(k)}(u)du\right) \\
&= \int_a^{t_{k+1}} dt_k \cdots \int_a^{t_2} dt_1 f(t_1)\xi^{(k)}(t_1) - \int_a^{t_{k+1}} dt_k \cdots \int_a^{t_1} f'(u)\xi^{(k)}(u)du \\
&= \int_a^{t_{k+1}} dt_k \cdots \int_a^{t_2} f(t)\xi^{(k)}(t)dt - \int_a^{t_{k+1}} dt_k \left(\int_a^{t_k} dt_{k-1}\cdots\int_a^{t_1} f'(u)\xi^{(k)}(u)du\right) \\
&= \sum_{j=0}^{k}(-1)^j\binom{k}{j}\int_a^{t_{k+1}} dt_k \int_a^{t_k} dt_{k-1}\cdots\int_a^{t_{k-j+2}} f^{(j)}(u)\xi(u)du \\
&- \int_a^{t_{k+1}} dt_k \left[\sum_{i=0}^{k}(-1)^i\binom{k}{i}\int_a^{t_k} dt_{k-1}\int_a^{t_{k-1}} dt_{k-2}\cdots\int_a^{t_{k-i+1}} f^{(i+1)}(u)\xi(u)du\right].
\end{aligned}$$

Now that the l-th term of the first sum adds up to the $(l-1)$-th term of the second sum in the above to give

$$(-1)^l\binom{n+1}{l}\int_a^{t_{k+1}} dt_k \int_a^{t_k} dt_{k-1}\cdots\int_a^{t_{k-l+2}} f^{(l)}(u)\xi(u)du.$$

After summing up there are $k+1$ terms which are the terms of the expansion for the case $k+1$.

The proof of part (b) is same as that of part (a)(ii). So we omit its proof.

□

Remark 0.6: We may call the identities (.9), (.10) as *Leibnitz formulae for successive integration*.

Lemma 0.7: **(a)** Let $I = I_1$ or I_2, and assume that $p_i \in C^{n-i}(I)$ for $1 \leq i \leq n$. Suppose that ξ is a solution of the differential equation

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x), \tag{.11}$$

on I with $\xi^{(k)}(a) = 0$ for $0 \leq k \leq n-1$, where g is a given continuous function on I. Then for any $t_n \in I$

$$\int_a^{t_n} g(t)\frac{(t_n - t)^{n-1}}{(n-1)!}\, dt = \xi(t_n) + \sum_{j=1}^{n}\sum_{m=0}^{n-j}(-1)^m \binom{n-j}{m} \int_a^{t_n} \left[p_j^{(m)}(t)\frac{(t_n - t)^{j+m-1}}{(j+m-1)!}\right] \xi(t)\, dt. \tag{.12}$$

(b) Let $I = I_1$ or $I = I_3$. Let ζ be a solution of the differential equation (.11) on I, with $\zeta^{(k)}(b) = 0$ for $0 \leq k \leq n-1$, where g and p_i be as in the part(a) of this lemma. Then for any $t_n \in I$

$$\begin{aligned} &\int_{t_n}^{b} g(t)\frac{(t - t_n)^{n-1}}{(n-1)!}\, dt \\ = \quad & (-1)^n\zeta(t_n) + \sum_{j=1}^{n}(-1)^{n-j}\sum_{m=0}^{n-j}\binom{n-j}{m}\int_{t_n}^{b}\left[p_j^{(m)}(t)\frac{(t_n - t)^{j+m-1}}{(j+m-1)!}\right]\zeta(t)\, dt. \end{aligned} \tag{.13}$$

Proof: We will omit the proof of part (b) since it is similar to that of part (a). To prove part(a), using Lemmas 0.4(i) and 0.5(ii) we have

$$
\begin{aligned}
&\int_a^{t_n} g(t)\frac{(t_n-t)^{n-1}}{(n-1)!}\,dt \\
&= \int_a^{t_n} dt_{n-1}\int_a^{t_{n-1}} dt_{n-2}\cdots\int_a^{t_1} g(u)du \\
&= \int_a^{t_n} dt_{n-1}\int_a^{t_{n-1}} dt_{n-2}\cdots\int_a^{t_1} dt\left[\xi^{(n)}(t)+p_1(t)\xi^{(n-1)}(t)+\cdots+p_n(t)\xi(t)\right] \\
&= \xi(t_n)+\int_a^{t_n} dt_{n-1}\left(\int_a^{t_{n-1}} dt_{n-2}\int_a^{t_{n-2}} dt_{n-3}\cdots\int_a^{t_1} p_1(t)\xi^{(n-1)}(t)dt\right)+ \\
&\vdots \\
&+\int_a^{t_n} dt_{n-1}\int_a^{t_{n-1}} dt_{n-2}\cdots\left(\int_a^{t_{n-j}} dt_{n-j-1}\cdots\int_a^{t_1} p_j(t)\xi^{(n-j)}(t)\,dt\right)+ \\
&\vdots \\
&+\int_a^{t_n} dt_{n-1}\int_a^{t_{n-1}} dt_{n-2}\cdots\int_a^{t_1} p_n(u)\xi(u)du \\
&= \xi(t_n)+\sum_{j=1}^{n}\int_a^{t_n} dt_{n-1}\cdots\int_a^{t_{n-j+1}} dt_{n-j}\left(\int_a^{t_{n-j}} dt_{n-j-1}\cdots\int_a^{t_1} p_j(t)\xi^{(n-j)}(t)\,dt\right) \\
&= \xi(t_n)+\sum_{j=1}^{n}\int_a^{t_n} dt_{n-1}\cdots \\
&\cdots\int_a^{t_{n-j+1}} dt_{n-j}\left(\sum_{m=0}^{n-j}(-1)^m\binom{n-j}{m}\int_a^{t_{n-j}} dt_{n-j-1}\cdots\int_a^{t_{n-j-m+1}} p_j^{(m)}(t)\xi(t)\,dt\right) \\
&= \xi(t_n)+\sum_{j=1}^{n}\sum_{m=0}^{n-j}(-1)^m\binom{n-j}{m}\int_a^{t_n} dt_{n-1}\cdots\int_a^{t_{n-j-m+1}} p_j^{(m)}(t)\xi(t)\,dt \\
&= \xi(t_n)+\sum_{j=1}^{n}\sum_{m=0}^{n-j}(-1)^m\binom{n-j}{m}\int_a^{t_n}\left[p_j^{(m)}(t)\frac{(t_n-t)^{j+m-1}}{(j+m-1)!}\right]\xi(t)\,dt. \qquad (.14)
\end{aligned}
$$

□

Our main result is as follows:

Theorem 0.8: Consider the differential equation (.1) on an interval I. Assume that the coefficients p_k are $n-k$ times continuously differentiable on I for $1 \le k \le n$. Assume that q is a complex valued continuous function on I. Let $\epsilon(t)$ be an arbitrary nonnegative continuous function on I.

(a) Assume that the above hypotheses hold on $I = I_1$ or $I = I_2$. Then there exists a nonnegative function $\epsilon_1(x)$ (depending on $\epsilon(x)$ and the coefficient functions p_i only) such that if an n-times continuously differentiable function y satisfies the inequality (.2), then there exists a nonzero solution z_1 of (.1) such that

$$|y(x) - z_1(x)| \le \epsilon_1(x), \qquad (.15)$$

where $\epsilon_1(x)$ is given by (.18).

(b) Assume that the above hypotheses hold on I, where $I = I_1$ or $I = I_3$. Then there exists a nonnegative function $\epsilon_2(x)$ (depending on $\epsilon(x)$ and the coefficient functions p_i only) such that if an n-times continuously differentiable function y satisfies the inequality (.2), then there exists a nonzero solution z_2 of (.1) satisfying

$$|y(x) - z_2(x)| \le \epsilon_2(x), \tag{.16}$$

where

$$\epsilon_2(x) \equiv \left[\int_x^b \epsilon(t)\frac{(t-x)^{n-1}}{(n-1)!}dt\right] \exp\left(\int_x^b \left|\sum_{j=1}^{n}(-1)^{n-j}\sum_{m=0}^{n-j}\binom{n-j}{m} p_j^{(m)}(t)\frac{(t-x)^{j+m-1}}{(j+m-1)!}\right| dt\right). \tag{.17}$$

Proof: For part (a), for simplicity, we will denote $L_n y$ to be the left hand side of (.1). Suppose that $|L_n y(t)| \le \epsilon(t)$ for all $t \in I$. Let z_1 satisfies $L_n z(t) = 0$ and that $z_1^{(k)}(a) = y^{(k)}(a)$ for $0 \le k \le n-1$. Then

$$|L_n y(t) - L_n z_1(t)| \le \epsilon(t).$$

Setting $g(t) = L_n y(t) - L_n z_1(t)$, and $\xi(t) = y(t) - z_1(t)$, note that $|g(t)| \le \epsilon(t)$, and that g and ξ satisfy the hypotheses of Lemma 0.7. So

$$\xi^{(n)}(t) + p_1(t)\xi^{(n-1)}(t) + \cdots + p_n(t)\xi(t) = g(t).$$

Upon integrating successively n times near a we obtain (.12) using Lemma 0.7(a). Using triangle inequality of the absolute value, and that $|g(t)| \le \epsilon(t)$, we have

$$\begin{aligned}
&|\xi(t_n)| - \left|\sum_{j=1}^{n}\sum_{m=0}^{n-j}(-1)^m\binom{n-j}{m}\int_a^{t_n}\left[p_j^{(m)}(t)\frac{(t_n-t)^{j+m-1}}{(j+m-1)!}\right]\xi(t)\,dt\right| \\
&\le \left|\xi(t_n) + \sum_{j=1}^{n}\sum_{m=0}^{n-j}(-1)^m\binom{n-j}{m}\int_a^{t_n}\left[p_j^{(m)}(t)\frac{(t_n-t)^{j+m-1}}{(j+m-1)!}\right]\xi(t)\,dt\right| \\
&= \left|\int_a^{t_n} g(t)\frac{(t_n-t)^{n-1}}{(n-1)!}\,dt\right| \le \int_a^{t_n}\epsilon(t)\frac{(t_n-t)^{n-1}}{(n-1)!}\,dt.
\end{aligned}$$

Setting $t_n = x$ in the above inequality, we have

$$|\xi(x)| \le \int_a^x \epsilon(t)\frac{(x-t)^{n-1}}{(n-1)!}\,dt + \int_a^x \left|\left[\sum_{j=1}^{n}\sum_{m=0}^{n-j}(-1)^m\binom{n-j}{m}p_j^{(m)}(t)\frac{(x-t)^{j+m-1}}{(j+m-1)!}\right]\right| |\xi(t)|\,dt.$$

So by Gronwall's inequality (see Theorem 1.3.1 of (Pachpatte, 1998) and recalling that $\xi(x) =$

$y(x) - z_1(x)$, one has

$$\begin{aligned}
&|y(x) - z_1(x)| \\
&\leq \left[\int_a^x \epsilon(t)\frac{(x-t)^{n-1}}{(n-1)!}dt\right] \\
&\times \exp\left(\int_a^x \left|\sum_{j=1}^{n}\sum_{m=0}^{n-j}(-1)^m \binom{n-j}{m} p_j^{(m)}(t)\frac{(x-t)^{j+m-1}}{(j+m-1)!}\right| dt\right) \\
&\equiv \epsilon_1(x),
\end{aligned} \tag{.18}$$

for all x.

The proof of part (b) is similar to that of part (a), where z_2 is a solution of (.1), $z_2^{(k)}(b) = y^{(k)}(b)$ for $0 \leq k \leq n-1$, and uses Lemmas 0.4(b), 0.5(b) and 0.7(b).

□

Remark 0.9: If the interval under consideration is I_1 and $\epsilon(x) \equiv \epsilon$, then it follows that the function $\epsilon_1(x)$ in (.18) and $\epsilon_2(x)$ in (.17) are bounded by $K\epsilon$ for some $K > 0$. Hence in this case the linear ODE is HU stable.

Remark 0.10: It is interesting to compare the error estimates in the above Theorem 0.3 with that of Theorem 0.8. Note that for $n = 1$, $z_2(x) = y_1(x)$, where z_2 and y_1 are obtained in Theorems 0.8(b) and 0.3 respectively. So z_2 in Theorem 0.8 is unique. For $n > 1$, the function z_1 satisfying (.18) is not unique, as can be seen from the next example.

Example 0.11: Consider the differential equation

$$u'' - \frac{x^2}{16} = 0.$$

on the interval $I = [0, 1]$. Let $\epsilon = 1/4$ and $y(x) = \frac{x^2}{16} + \frac{1}{16}$. Here $K = \sup_{x\in[0,1]}\frac{x^2}{2} = \frac{1}{2}$.

Then

$$\left|y'' - \frac{x^2}{16}\right| = \left|\frac{1}{8} - \frac{x^2}{16}\right| = \frac{(2-x^2)}{16} \leq \frac{1}{8} \leq \frac{1}{4} \equiv \epsilon.$$

Since $y(0) = 1/16$ and $y'(0) = 0$, according to Theorem 0.8, $z_1(x) = \frac{x^4}{192} + \frac{1}{16}$ and

$$|y(x) - z_1(x)| \leq \epsilon\frac{x^2}{2} \leq \frac{\epsilon}{2} = 1/8.$$

Let $z_2(x) = \frac{x^4}{192}$. Then z_2 satisfies $z_2'' - \frac{x^2}{16} = 0$ and, since $\frac{x^2}{16} \geq \frac{x^4}{192}$ for all $x \in [0, 1]$,

$$
\begin{aligned}
|y(x) - z_2(x)| &= \left|\frac{x^2}{16} - \frac{x^4}{192} + \frac{1}{16}\right| \\
&= \frac{x^2}{16} - \frac{x^4}{192} + \frac{1}{16} \\
&= \frac{x^2(12 - x^2)}{192} + \frac{1}{16} \leq \frac{12}{192} + \frac{1}{16} = \frac{1}{8} = \frac{\epsilon}{2}.
\end{aligned}
$$

Hence, z is not unique.

However, if we insist on z_j $(j = 1, 2)$ and its derivatives upto $(n-1)$-th order to have the same initial or terminal value as that of the derivatives of y, then z_j is unique (which follows from the uniqueness of solutions of the initial value problem) and we have $|y(t) - z_j(t)| \leq \epsilon_j(t)$ for $j = 1, 2$, appearing in (.15) and (.16) respectively.

Remark 0.12: All the results in this section can easily be generalised to a linear differential equation in a complex Banach space X, where the differentiability is considered in the strong sense. More precisely,

Theorem 0.13: Let $J = J_j$, $j = 1, 2$, where $J_1 = I_1$ or $J_1 = I_2$, and $J_2 = I_1$ or $J_2 = I_3$ respectively. Let X be a complex Banach space. Let $\epsilon : J \to [0, \infty)$ be a continuous function. If $y : J \to X$ is a strongly n-times continuously differentiable function satisfying (.11), where p_i are in $C^{(n-i)}(J, \mathbb{C})$ functions such that, whenever $||g(x)|| \leq \epsilon(x)$, there exist non-negative functions $\epsilon_j(x)$, $j = 1, 2$ (independent of y), and Banach space valued n-times strongly differential functions z_j, $j = 1, 2$, satisfying $L_n z_j(x) = 0$, $z_j^{(k)}(s_j) = y^{(k)}(s_j)$ (for $j = 1, 2$), $0 \leq k \leq n - 1$, with $s_1 = a,\ s_2 = b$, such that $||y(x) - z_j(x)|| \leq \epsilon_j(x)$ for $j = 1, 2$.

The proof of it goes almost in verbatim with that of the above theorem using results similar to Lemmas 0.5, 0.7 for Banach space valued functions.

3. Conclusion

Here we prove the Hyers-Ulam stability and Hyers-Ulam-Aoki-Rassias stability of n-th order linear ordinary differential equation with smooth coefficients on compact and semi-bounded intervals using successive integration by parts. The idea here is as follows: if y satisfies (.2) on I, where I is one of the form I_1 or I_2 or I_3, then choose a solution z of (.1) which alongwith all upto its $n-1$ derivatives agree with those of y at the finite end point of the interval. This solution z is used to prove that the differential equation (.1) is HUAR stable. This is achieved by applying the corresponding differential operator on $y - z$ and integrating successively n times near this end point (at which y and z alongwith their first $n-1$ derivatives agree) and making use of Gronwall's inequality.

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