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# Congruence properties of some partition functions

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**Abstract:** Let $M(n) = (N_3(n)/48)$, where $N_3(n)$ denotes the number of ways in which $n$ can be written as sum of three squares. We study the congruence properties of some partition functions in relation to $M(16n + 14)$.
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## 1 Introduction

In [2], Andrew, Dyson and Hickerson used arithmetic of $Q(\sqrt{6})$ to study some partition function. This is the first time the interaction between the partition function and some indefinite quadratic form was established. In [3], [4] and [5], more interaction between some partitions functions and number of solutions of some Diophantine equations was shown. In this paper we study congruence properties of some partition functions modulo 2 and 3 using $M(n) = (N_3(n)/48)$, where $N_3(n)$ denotes the number of ways in which $n$ can be written as sum of three squares.

## 2 Main theorem

Denote by $P(n)$ the partition function defined by

$$P(n) = \Sigma_{\pi}(2^{\rho(\pi)}(-1)^{\sigma(\pi)}), \tag{2.1}$$

where sum on the right is over all partitions π of $n$ into parts which are repeated exactly 1, $4j$ or $4j + 1$ times with $1 \leq j \leq 5$, and $\rho(\pi)$ is the number of parts in π which are repeated exactly $4j$ or $4j + 1$ times, where $j = 2$ or 3, while $\sigma(\pi)$ is the number of parts in $\pi$, which are repeated exactly $4j$ or $4j + 1$ times, where $j = 1$, 2 or 5.

Let

$$M(n) = (N_3(n)/48),$$

where $N_3(n)$ denotes the number of ways in which $n$ can be written as sum of three squares.

**Theorem 1.** *For* $n \geq 0$,

$$P(n) = M(16n + 14).$$

*Proof.* Using the Jacobi's triple product identity ([1], p. 21) and proceeding as in the proof of the Theorem in [4] we get:

$$\sum_{x(i)\in\mathbb{Z}} q^{\sum_{i=1}^{3}\left(4x(i)^2+ix(i)\right)} = \prod_{n\geq 1}(1-q^{8n})^2(1-q^{4n})(1+q^n)$$
$$= \prod_{n\geq 1}(1-2q^{8n}+q^{16n})(1+q^n-q^{4n}-q^{5n})$$
$$= \prod_{n\geq 1}(1+q^n-q^{4n}-q^{5n}-2q^{8n}-2q^{9n}+2q^{12n}+2q^{13n}+q^{16n}+q^{17n}-q^{20n}-q^{21n})$$
$$= \sum_{n\geq 0} P(n)q^n. \tag{2.2}$$

Finally,

$$\sum_{n\geq 0} P(n)q^{16n+14} = q^{14}\sum_{x(i)\in\mathbb{Z}} q^{\sum_{i=1}^{3}\left(16(4x(i)^2+ix(i))\right)}$$
$$= \sum_{x(i)\in\mathbb{Z}} q^{\sum_{i=1}^{3}(8x(i)+i)^2}$$
$$= \sum_{n\geq 0} M(16n+14)q^{16n+14}.$$

This proves the theorem. □

# 3 Congruence properties of some partition functions

Looking at equations (2.1) and (2.2) modulo 3, and Theorem 1, we get the following:

**Proposition1.** Let $A(n)$ denote the number of partitions of $n$ into parts which are repeated exactly 1, 4, 5, 8, 9, 12, 13, 16, 17, 20, or 21 times with parts repeated exactly 4, 5, 12, 13, 20, or 21 times being even number of times minus the number of parts repeated exactly 4, 5, 12, 13, 20, or 21 times being odd in number. Then $A(n) \equiv M(16n + 14) \pmod 3$.

While looking at equations (2.1) and (2.2) modulo 2, and Theorem 1, we get the following:

**Proposition 2.** Let $B(n)$ denote the number of partitions of $n$ into parts which are repeated exactly 1, 4, 5, 16, 17, 20, or 21 times with parts repeated exactly 4, 5, 20, or 21 times being even number of times minus the number of parts repeated exactly 4, 5, 20, or 21 times being odd in number. Then $B(n)$ and $M(16n + 14)$ have the same parity.

## References

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