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On a B-q bonacci Sequence

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ABSTRACT. In this paper q^{th} order linear recurrence relation is defined. This new sequence is an extension of Fibonacci sequence in such a way that the coefficients of the terms on the right hand side of its recurrence relation, are terms of the binomial expansion of $(a+b)^{q-1}$. Some properties of this extension like d'Ocagne, Catalan, Cassini identities are discussed by representing the recurrence relation in Matrix form.

1. Introduction

The well-known classical Fibonacci sequence [6], is defined by

$$F_{n+1} = F_n + F_{n-1} \text{ , for all } n \geq 1 \text{ with } F_0 = 0 \text{ and } F_1 = 1, \tag{1.1}$$

where F_n is the n^{th} Fibonacci number. This sequence has been extended in many ways [3, 4, 5, 6, 7]. One way of generalizing this sequence as given in [3] is

$$F_{n+1} = a\,F_n + b\,F_{n-1}, \text{ for all } n \geq 1 \text{ with } F_0 = 0 \text{ and } F_1 = 1, \tag{1.2}$$

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where a and b are any non zero real numbers. With $a = k,\ b = 1$, we get equation (1) of [5].
In [4], we had rewritten (1.2) as

$$({}^fB)_{n+1} = a\ ({}^fB)_n + b\ ({}^fB)_{n-1}, \forall\ n\ \in \mathbb{Z}, \tag{1.3}$$

with $({}^fB)_0 = 0$ and $({}^fB)_1 = 1$, where $({}^fB)_n$ is n^{th} term of B-Fibonacci sequence (1.3).

We called (1.3) as B-Fibonacci sequence. Here 'B' indicates that coefficients on the right hand side of the recurrence relation are terms of the binomial expansion of $(a+b)^1$. Similarly we defined B-Tribonacci sequence by

$$({}^tB)_{n+2} = a^2\ ({}^tB)_{n+1} + 2ab\ ({}^tB)_n + b^2\ ({}^tB)_{n-1}, \forall\ n\ \in \mathbb{Z}, \tag{1.4}$$

with $({}^tB)_0 = 0,\ ({}^tB)_1 = 0$, and $({}^tB)_2 = 1$, where $({}^tB)_n$ is n^{th} term of B-Tribonacci sequence (1.4). Note that the coefficients on the right hand side (1.4) are the terms of the binomial expansion of $(a+b)^2$.

As binomial coefficients play an important role in Combinatorics, we expect some applications of this sequence in combinatorial problems. In this paper we extend this idea to q^{th} order linear recurrence relation and call it B-q bonacci sequence. In [2], the author discusses m^{th} order linear recurrence relation and has obtained expressions for Generating functions and Binet type formula. We obtain these in new settings and also extend the results of B-Fibonacci sequence and B-Tribonacci sequence obtained in [4] for B-q bonacci sequence.

2. B-q bonacci sequence

In this section we define the B-q bonacci sequence as an extension of (1.3) and (1.4), that is as the q^{th} order linear recurrence relation and study some of its properties.

Definition 2.1: The *B-q* bonacci sequence is defined by

$$({}^qB)_{n+q-1} = \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\ a^{q-1-r}\ b^r\ ({}^qB)_{n+q-2-r}, \tag{2.1}$$

for all integers n and positive integer q ≥ 2 with $({}^qB)_i = 0,\ i = 0, 1, 2, 3, \cdots q-2$ and $({}^qB)_{q-1} = 1$,

where $({}^qB)_n$ is n^{th} term of B-q bonacci sequence (2.1) and $(q-1)^{\underline{r}}$ denote (q-1) falling factorial r.

We rearrange the terms of (2.1) as follows to obtain the terms for the negative integer values of n.

$$({}^qB)_{n-1} = \frac{1}{b^{q-1}}\Big[({}^qB)_{n+q-1} - \sum_{r=0}^{q-2} \frac{(q-1)^{\underline{r}}}{r!}\ a^{q-1-r}\ b^r\ ({}^qB)_{n+q-2-r}\Big]$$

for $n = \cdots, -2, -1, 0.$

Thus we have below few terms of (2.1) as:

$({}^qB)_{-2} = -(q-1)\frac{a}{b^q},\ ({}^qB)_{-1} = \frac{1}{b^{q-1}}, ({}^qB)_0 = ({}^qB)_1 = \cdots = ({}^qB)_{q-1} = 0,\ ({}^qB)_q = a^{q-1},$

$({}^qB)_{q+1} = a^{2(q-1)} + (q-1)a^{q-2}\, b,$

$({}^qB)_{q+2} = a^{3(q-1)} + \frac{(2(q-1))^{\underline{1}}}{1!}\, a^{2q-3}\, b + \frac{(q-1)^{\underline{2}}}{2!}\, a^{q-3} b^2,$

$({}^qB)_{q+3} = a^{4(q-1)} + \frac{(3(q-1))^{\underline{1}}}{1!}\, a^{3q-4}\, b + \frac{(2(q-1))^{\underline{2}}}{2!}\, a^{2q-4}\, b^2 + \frac{(q-1)^{\underline{3}}}{3!} a^{q-4}\, b^3,$

$({}^qB)_{q+4} = a^{5(q-1)} + \frac{(4(q-1))^{\underline{1}}}{1!}\, a^{4q-5}\, b + \frac{(3(q-1))^{\underline{2}}}{2!}\, a^{3q-5}\, b^2 + \frac{(2(q-1))^{\underline{3}}}{3!} a^{2q-5}\, b^3 + \frac{(q-1)^{\underline{4}}}{4!} a^{q-5}\, b^4.$

Following results can be derived from equation (2.1).

(1) The n^{th} term of (2.1) is given by

$$({}^qB)_n = \frac{\sum_{k=1}^{q}(-1)^{k+1}\prod_{1\le i<j\le q, i,j\ne k}(\phi_i-\phi_j)\phi_k^n}{\prod_{1\le i<j\le q}(\phi_i-\phi_j)}, \tag{2.2}$$

where ϕ_p, $p = 1, 2, \cdots, q$ are q distinct roots of characteristic equation corresponding to (2.1). Equation (2.2) is a Binet type formula for (2.1).

(2) The generating function for B-q bonacci sequence (2.1) is given by

$${}^qG(x) = \sum_{r=-\infty}^{\infty} x^r (a+bx)^{(q-1)r}.$$

(3) (a) For all $n \ge q-1$ and $q \ge 2$, the n^{th} term of (2.1) is given by

$$({}^qB)_n = \sum_{r=0}^{\left[\frac{(q-1)(n-(q-1))}{q}\right]} \frac{\Big((q-1)(n-(q-1)-r)\Big)^{\underline{r}}}{r!}\; a^{(q-1)\left(n-(q-1)-r\right)-r}\, b^r. \tag{2.3}$$

(b) For all $n \le -1$ and $q \ge 2$, the n^{th} term of (2.1) is given by

$$({}^qB)_n = \sum_{r=\left[\frac{(q-1)(n-(q-1))}{q}\right]}^{n} \frac{\Big((q-1)(n-(q-1)-r)\Big)^{\underline{r}}}{r!}\; a^{(q-1)\left(n-(q-1)-r\right)-r}\, b^r. \tag{2.4}$$

(4) (a) For any integer $n \ge 0$, we have

$$\sum_{r=0}^{n}({}^qB)_r = \frac{({}^qB)_{n+1} + \sum_{i=0}^{q-2}\sum_{r=1+i}^{q-1}\frac{(q-1)^{\underline{r}}}{r!}\, a^{q-1-r}\, b^r\, ({}^qB)_{n-i} - 1}{(a+b)^{q-1}-1}, \tag{2.5}$$

provided $\begin{cases} a+b \ne 1, \text{if q is even,} \\ a+b \ne \pm 1, \text{if q is odd.} \end{cases}$

(b) For any integer $n < 0$, we have

$$\sum_{r=-1}^{-n}({}^qB)_r = -\frac{({}^qB)_{-n} + \sum_{i=0}^{q-2}\sum_{r=1+i}^{q-1}\frac{(q-1)^{\underline{r}}}{r!}\, a^{q-1-r}\, b^r\, ({}^qB)_{-n-1-i} - 1}{(a+b)^{q-1}-1}, \tag{2.6}$$

provided $\begin{cases} a+b \ne 1, \text{if q is even,} \\ a+b \ne \pm 1, \text{if q is odd.} \end{cases}$

Combining (2.5) and (2.6) we have the following.

(c) For any integer n,

$$\sum_{r=-n}^{n} ({}^{q}B)_r = \frac{\left(({}^{q}B)_{n+1} - ({}^{q}B)_{-n}\right) + \sum_{i=0}^{q-2}\sum_{r=1+i}^{q-1} \frac{(q-1)^{r}}{r!}\, a^{q-1-r}\, b^{r} \left(({}^{q}B)_{n-i} - ({}^{q}B)_{-n-1-i}\right)}{(a+b)^{q-1}-1}, \tag{2.7}$$

provided $\begin{cases} a+b \neq 1, \text{if q is even,} \\ a+b \neq \pm 1, \text{if q is odd.} \end{cases}$

3. Matrix representation of B-q bonacci sequence

In Matrix Form (2.1) is represented by

$$\begin{bmatrix} ({}^{q}B)_n \\ ({}^{q}B)_{n+1} \\ \cdots \\ ({}^{q}B)_{n+q-2} \\ ({}^{q}B)_{n+q-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ b^{q-1} & \frac{(q-1)^{1}}{1!} ab^{q-2} & \frac{(q-1)^{2}}{2!} a^{2}b^{q-3} & \frac{(q-1)^{3}}{3!} a^{3}b^{q-4} & \cdots & a^{q-1} \end{bmatrix} \begin{bmatrix} ({}^{q}B)_{n-1} \\ ({}^{q}B)_{n} \\ \cdots \\ ({}^{q}B)_{n+q-3} \\ ({}^{q}B)_{n+q-2} \end{bmatrix}.$$

Let $A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ b^{q-1} & \frac{(q-1)^{1}}{1!} ab^{q-2} & \frac{(q-1)^{2}}{2!} a^{2}b^{q-3} & \frac{(q-1)^{3}}{3!} a^{3}b^{q-4} & \cdots & a^{q-1} \end{bmatrix}$

$$= \begin{bmatrix} b^{q-1}({}^{q}B)_0 & \cdots & \sum_{r=q-j}^{q-1} \frac{(q-1)^{r}}{r!} a^{(q-1)-r}\, b^{r} ({}^{q}B)_{q-r-j} & \cdots & ({}^{q}B)_1 \\ \cdots & & & & \\ b^{q-1}({}^{q}B)_{i-1} & \cdots & \sum_{r=q-j}^{q-1} \frac{(q-1)^{r}}{r!} a^{(q-1)-r}\, b^{r} ({}^{q}B)_{q-1-r-j+i} & \cdots & ({}^{q}B)_i \\ \cdots & & & & \\ b^{q-1}({}^{q}B)_{q-1} & \cdots & \sum_{r=q-j}^{q-1} \frac{(q-1)^{r}}{r!} a^{(q-1)-r}\, b^{r} ({}^{q}B)_{2q-1-r-j} & \cdots & ({}^{q}B)_q \end{bmatrix},$$

where $1 \leq i, j \leq q$. Note that A is a matrix of order qxq and

$$A^n = \begin{bmatrix} b^{q-1}({}^qB)_{n-1}\cdots & \sum_{r=q-j}^{q-1}\frac{(q-1)^{\underline{r}}}{r!}a^{(q-1)-r}\ b^r({}^qB)_{n+q-1-r-j} & \cdots({}^qB)_n \\ \cdots & & \\ b^{q-1}({}^qB)_{n+(i-2)} & \sum_{r=q-j}^{q-1}\frac{(q-1)^{\underline{r}}}{r!}a^{(q-1)-r}\ b^r({}^qB)_{n+q-2-r-j+i} & \cdots({}^qB)_{n+(i-1)} \\ \cdots & & \\ b^{q-1}({}^qB)_{n+q-2}\cdots & \sum_{r=q-j}^{q-1}\frac{(q-1)^{\underline{r}}}{r!}a^{(q-1)-r}\ b^r({}^qB)_{n+2q-2-r-j} & \cdots({}^qB)_{n+q-1} \end{bmatrix}.$$

Following results can be obtained from the matrix representation.

(1)(a) $({}^qB)_{n+m-1} = \sum_{r=0}^{q-1}\left(\sum_{s=0}^{r}\frac{(q-1)^{\underline{s}}}{s!}\ b^{q-1-s}\ a^s({}^qB)_{n-1+s-r}\right)({}^qB)_{m-1+r}.$

In particular, if $m = n$, then we have

(b) $({}^qB)_{2n-1} = \sum_{r=0}^{q-1}\left(\sum_{s=0}^{r}\frac{(q-1)^{\underline{s}}}{s!}\ b^{q-1-s}\ a^s({}^qB)_{n-1+s-r}\right)({}^qB)_{n-1+r}.$

and if $m = n+1$, we have

(c) $({}^qB)_{2n} = \sum_{r=0}^{q-1}\left(\sum_{s=0}^{r}\frac{(q-1)^{\underline{s}}}{s!}\ b^{q-1-s}\ a^s({}^qB)_{n-1+s-r}\right)({}^qB)_{n+r}.$

(2) **General q-linear formula**

For any integer $a_{i_m j}, 1 \le i_m, j, m \le q$ with distinct i_m and the following $(\frac{q^2}{2!})^2$ equations,
$a_{i_1 1} + a_{i_2 2} = a_{i_2 1} + a_{i_1 2}, \cdots, a_{i_1 1} + a_{i_q q} = a_{i_q 1} + a_{i_1 q}$
$a_{i_2 2} + a_{i_3 3} = a_{i_3 2} + a_{i_2 3}, \cdots, a_{i_2 2} + a_{i_q q} = a_{i_q 2} + a_{i_2 q}$
$\cdots$
$a_{i_{q-1}(q-1)} + a_{i_q q} = a_{i_q (q-1)} + a_{i_{q-1} q}$, we have

$$\begin{vmatrix} ({}^qB)_{a_{11}} & \cdots & ({}^qB)_{a_{1j}} & \cdots & ({}^qB)_{a_{1q}} \\ \cdots & & & & \\ ({}^qB)_{a_{i1}} & \cdots & ({}^qB)_{a_{ij}} & \cdots & ({}^qB)_{a_{iq}} \\ \cdots & & & & \\ ({}^qB)_{a_{q1}} & \cdots & ({}^qB)_{a_{qj}} & \cdots & ({}^qB)_{a_{qq}} \end{vmatrix}$$

$$= [(-b)^{q-1}]^s \begin{vmatrix} ({}^qB)_{a_{11}-s} & \cdots & ({}^qB)_{a_{1j}-s} & \cdots & ({}^qB)_{a_{1q}-s} \\ \cdots & & & & \\ ({}^qB)_{a_{i1}-s} & \cdots & ({}^qB)_{a_{ij}-s} & \cdots & ({}^qB)_{a_{iq}-s} \\ \cdots & & & & \\ ({}^qB)_{a_{q1}-s} & \cdots & ({}^qB)_{a_{qj}-s} & \cdots & ({}^qB)_{a_{qq}-s} \end{vmatrix}$$

$$=[(-b)^{q-1}]^s \sum_{\sigma\epsilon S_q} \prod_{i=1}^{q} \quad sgn(\sigma)({}^qB)_{a_{i\sigma(i)-s}},$$

ranging over the symmetric group S_q, where

$$\text{sgn}(\sigma)= \begin{cases} +1, & if\ \sigma\ is\ an\ even\ permutation, \\ -1 & if\ \sigma\ is\ an\ odd\ permutation. \end{cases}$$

(3) **d′Ocagne type identity**

For any integer $s_k, k = 1, 2, ..., q, 0 \leq i,\ j \leq q-1$,

$$\begin{vmatrix} ({}^qB)_{s_1} & \cdots & ({}^qB)_{s_j} & \cdots & ({}^qB)_{s_q} \\ ({}^qB)_{s_1+1} & \cdots & ({}^qB)_{s_j+1} & \cdots & ({}^qB)_{s_q+1} \\ \cdots & & & & \\ ({}^qB)_{s_1+i} & \cdots & ({}^qB)_{s_j+i} & \cdots & ({}^qB)_{s_q+i} \\ \cdots & & & & \\ ({}^qB)_{s_1+(q-1)} & \cdots & ({}^qB)_{s_j+(q-1)} & \cdots & ({}^qB)_{s_q+(q-1)} \end{vmatrix}$$

$$= [(-b)^{q-1}]^{s_q} \sum_{\sigma\epsilon S_{q-1}} \prod_{i=1}^{q-1} \quad sgn(\sigma)({}^qB)_{s_i-s_q+\sigma(i)-1}.$$

(4) **Catalan type identity**

For any integers n, r,

$$\begin{vmatrix} ({}^qB)_n & \cdots & ({}^qB)_{n+(j-1)r} & \cdots & ({}^qB)_{n+(q-1)r} \\ \cdots & & & & \\ ({}^qB)_{n+(1-i)r} & \cdots & ({}^qB)_{n+(j-i)r} & \cdots & ({}^qB)_{n+(q-i)r} \\ \cdots & & & & \\ ({}^qB)_{n+(1-q)r} & \cdots & ({}^qB)_{n+(j-q)r} & \cdots & ({}^qB)_n \end{vmatrix}$$

$$= [(-b)^{q-1}]^n \sum_{\sigma\epsilon S_q} \prod_{i=1}^{q} \ sgn(\sigma)(^qB)_{(i-\sigma(i))r}.$$

Remark: 1. When q is odd, it is seen that the contribution of antidiagonal elements to the determinant value is zero. Hence the R.H.S. of the above identity takes the simpler form $[(-b)^{q-1}]^n \sum_{j=1}^{q-1} (^qB)_{jr}^{q-j} (^qB)_{-(q-j)r}^{j}$.

2. For $q = 2$ and $q = 3$, all the above results reduces to those of B-Fibonacci sequence and B-Tribonacci sequence obtained in [4].

Putting $r = 1$ in Catalan type identity, we get Cassini type identity.

(5) **Cassini type identity**

For any integer $n, 0 \leq i, j \leq q-1$

$$\begin{vmatrix} (^qB)_n & \cdots & (^qB)_{n+(j-1)} & \cdots & (^qB)_{n+(q-1)} \\ \cdots & & & & \\ (^qB)_{n+(1-i)} & \cdots & (^qB)_{n+(j-i)} & \cdots & (^qB)_{n+(q-i)} \\ \cdots & & & & \\ (^qB)_{n+(1-q)} & \cdots & (^qB)_{n+(j-q)} & \cdots & (^qB)_n \end{vmatrix} = [(-b)^{q-1}]^{n-(q-1)}.$$

(6) **Extended form of Cassini type identity**

For any integer $n, r,\ 0 \leq j \leq q-1$ and $0 \leq i \leq q-2$,

$$\begin{vmatrix} (^qB)_n & \cdots & (^qB)_{n-j} & \cdots & (^qB)_{n-(q-1)} \\ (^qB)_{n+1} & \cdots & (^qB)_{n+1-j} & \cdots & (^qB)_{n+1-(q-1)} \\ \cdots & & & & \\ (^qB)_{n+i} & \cdots & (^qB)_{n+i-j} & \cdots & (^qB)_{n+i-(q-1)} \\ \cdots & & & & \\ (^qB)_{n+q-2} & \cdots & (^qB)_{n+q-2-j} & \cdots & (^qB)_{n+q-2-(q-1)} \\ (^qB)_{n+r} & \cdots & (^qB)_{n+r-j} & \cdots & (^qB)_{n+r-(q-1)} \end{vmatrix} = \ [(-b)^{q-1}]^{n-(q-1)}(^qB)_r.$$

Following [1], we have pythagorean result for B-q bonacci sequence which can be proved using equation (2.1).

Theorem 3.1. *For all integers n, we have*

$$\Big[b^{q-1}(^qB)_{n-1}(2\ (^qB)_{n+q-1} - b^{q-1}(^qB)_{n-1})\Big]^2 + \Big[2(^qB)_{n+q-1}\big((^qB)_{n+q-1} - b^{q-1}(^qB)_{n-1}\big)\Big]^2 \qquad (3.1)$$

$$= \left[b^{2(q-1)}\,({}^{q}B)_{n-1}^{2} + 2\,({}^{q}B)_{n+q-1}\big(({}^{q}B)_{n+q-1} - b^{q-1}({}^{q}B)_{n-1}\big)\right]^{2}.$$

Next two theorems are related to the recurrence properties of B-q bonacci sequence which can be proved by induction on s.

Theorem 3.2. *For all* $s \geq 0$,

$$\sum_{i=0}^{(q-1)s} \frac{((q-1)s)^{\underline{i}}}{i!}\,({}^{q}B)_{n+i}\,a^{i}\,b^{(q-1)s-i} = ({}^{q}B)_{n+qs}. \tag{2}$$

Theorem 3.3. *For all* $s \geq 0$,

$$\sum_{i=0}^{s-1}\sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\,a^{(q-1)s-(q-1)i\,-r}\,b^{r}\,({}^{q}B)_{n+(q-1)+i-r} = ({}^{q}B)_{n+(q-1)+s} - a^{(q-1)s}\,({}^{q}B)_{n+q-1}. \tag{3}$$

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