APPROXIMATION METHODS FOR ILL-POSED OPERATOR EQUATIONS

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DECLARATION

I do hereby declare that the thesis entitled **APPROXIMATION METHODS FOR ILL-POSED OPERATOR EQUATIONS** submitted to the Goa University for the award of the Degree of Doctor of Philosophy in Mathematics is a record of original and independent research work done by me under the supervision and guidance of Dr. M.T.Nair, Reader, Department of Mathematics, Goa University, and it has not previously formed the basis for the award of any Degree, Diploma, Associateship, Fellowship or other similar title to any candidate of any University.

Santhosh George

CERTIFICATE

This is to certify that the Thesis entitled **APPROXIMATION METHODS FOR ILL-POSED OPERATOR EQUATIONS** submitted to the Goa University by Shri. Santhosh George is a bonafide record of original and independent research work done by the candidate under my guidance. I further certify that this work has not formed the basis for the award of any Degree, Diploma, Associateship, Fellowship or other similar title to any candidate of any other University.

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CONTENTS

CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1. GENERAL INTRODUCTION

Many problems in mathematical physics and applied mathematics, particularly those involving remote sensing, indirect measurement, etc. have their mathematical formulation as an operator equation of the first kind,

$$Tx = y,$$

where $T : X \longmapsto Y$ is a bounded linear operator between Hilbert spaces X and Y. The above equation is, in general, 'ill-posed', i.e., the existence of a unique solution which depends continuously on the data y is not guaranteed. In case there is no solution in the usual sense, one seeks the 'least-square solution of minimal norm', which in general does not depend continuously on the data y. In fact, if $R(T)$, the range of T, is not closed, then the map which associates each $y \in R(T) + R(T)^{\perp}$ to the least-square solution of minimal norm is not continuous. In such situation one has to regularize $Tx = y$, often with an inexact data y^{δ} with $\|y-y^{\delta}\| \leq \delta$. The regularization which has been studied most extensively is the so called Tikhonov regularization, in which one considers x_{α}^{δ}, the solution of the equation

$$(T^*T + \alpha I)x_\alpha^\delta = T^*y^\delta, \quad \alpha > 0,$$

for obtaining approximations for $\hat{x}$, the least-square solution of minimal norm. The crucial problem here is to choose the regularization parameter α depending on δ and y^δ such that we must have $x_\alpha^\delta \to \hat{x}$ as $\delta \to 0$ and obtain the 'optimal' estimate for the error $\|\hat{x} - x_\alpha^\delta\|$. It is known [39] that if $\hat{x} \in R((T^*T)^\nu)$, $0 < \nu \le 1$, then the optimal rate for $\|\hat{x} - x_\alpha^\delta\|$ is $O(\delta^{2\nu/(2\nu+1)})$. Morozov [31] and Arcangeli [1] had considered 'discrepancy principles', namely,

$$\|Tx_\alpha^\delta - y^\delta\| = \delta \quad \text{and} \quad \|Tx_\alpha^\delta - y^\delta\| = \frac{\delta}{\sqrt{\alpha}}$$

respectively, for choosing the parameter α in Tikhonov regularization. For Morozov's method, the best possible rate for $\|\hat{x} - x_\alpha^\delta\|$ is $O(\delta^{1/2})$ ([14]) and for Arcangeli's, the known rate was $O(\delta^{1/3})$ which is attained for $\hat{x} \in R(T^*)$ ([18]). In an attempt to obtain optimal rate, i.e., $O(\delta^{2\nu/(2\nu+1)})$, Schock [38] considered the discrepancy principle

$$\|Tx_\alpha^\delta - y^\delta\| = \frac{\delta^p}{\alpha^q}, \quad p > 0, \quad q > 0$$

and proved that the rate is arbitrarily close to the optimal rate for large values of q. Later Nair [34], considered the above discrepancy principle and improved the result of Schock [38]. In fact, the result in [34], shows that, the Arcangeli's method does give the best rate $O(\delta^{2/3})$ for $\hat{x} \in R(T^*T)$.

In Chapter 2 we consider the discrepancy principle of Schock [38] and prove that if $\hat{x} \in R((T^*T)^{\nu})$, $1/2 \le \nu \le 1$, then the optimal rate $O(\delta^{2\nu/(2\nu+1)})$ is achieved. Our result improves the result of Nair [34] for $0 < \nu < 1$, and for $\nu = 1$ our result coincides with the result in [34]. In the final section of Chapter 2 we consider Schock's discrepancy principle for iterated Tikhonov regularization.

If $Y = X$ and the operator under consideration is 'positive and self-adjoint', then one can consider a simpler regularization method, namely, the Simplified regularization. In this case we use the notation A for the operator T, and consider the equation $Aw = g$. In Simplified regularization of the equation

$$Aw = g$$

one takes the solution w_α^δ of the equation

$$(A + \alpha I)w_\alpha^\delta = g^\delta$$

for obtaining approximations for $\hat{w}$, the minimal norm solution of the equation $Aw = g$. Here g^δ is such that $\|g-g^\delta\| \le \delta$. For choosing the regularization parameter α in Simplified regularization Groetsch and Guacaneme [16] considered Arcangeli's method and proved the convergence of w_α^δ to $\hat{w}$. But in [16], no attempt has been made for obtaining the estimate for the error $\|\hat{w} - w_\alpha^\delta\|$. In Section 3.1, we consider a generalized Arcangeli's method, namely,

$$\|Aw_\alpha^\delta - g^\delta\| = \frac{\delta^p}{\alpha^q}, \quad p > 0, \quad q > 0,$$

for obtaining the regularization parameter α. We obtain the optimal rate $O(\delta^{\nu/(\nu+1)})$ (see [39]) for the error $\|\hat{w} - w_\alpha^\delta\|$, whenever $\hat{w} \in R(A^\nu)$, $0 < \nu \le 1$. As a particular case we prove that the Arcangeli's method considered in [16] gives the rate $O(\delta^{1/3})$, and the best rate $O(\delta^{1/2})$ is obtained when $\nu = 1$ by taking $\frac{p}{q+1} = \frac{1}{2}$. The result for the case when $\nu = 1$ has also been considered by Guacaneme [19]. In Section 3.2, we consider the discrepancy principle, namely,

$$\alpha^{2(\rho+1)}\langle (A + \alpha I)^{-2(\rho+1)}Qg^\delta, Qg^\delta\rangle = c\delta^2, \quad \rho > 0,$$

where $c > 1$ is a constant and Q is the orthogonal projection onto the closure of the range of A. Result of this section includes a result of Guacaneme [21], which he proved when A is compact and $\nu = 1$. In the final section of Chapter 3 we consider the discrepancy principles considered in Sections 3.1 and 3.2 for iterated Simplified regularization.

In reality there are two occasions, where one has to consider perturbed operators instead of the original operator. One such occasion arises from the modeling error and the other when one considers numerical approximation. Many authors (e.g.,[31], [36], [37], [43]) considered the equation $Tx = y$ with a perturbed operator T_h instead of T with

$$\|T-T_h\| \le \varepsilon_h , \quad \varepsilon_h \to 0 \quad \text{as} \quad h \to 0.$$

In Chapter 4 we consider Tikhonov regularization and Simplified regularization with perturbed operators. Specifically, we modify the discrepancy principles of Chapter 2 and 3 so as to include the case of perturbed operators.

In Chapter 5 we consider projection method for the regularized equations

$$(T^*T + \alpha I)x_\alpha^\delta = T^*y^\delta \quad \text{and} \quad (A + \alpha I)w_\alpha^\delta = g^\delta.$$

The projection method for the equation

$$(T^*T + \alpha I)x_\alpha^\delta = T^*y^\delta$$

is a special case of the method considered in Section 4.2 and under certain conditions this method leads to a better error estimate than the one obtained in Section 4.2. In order to illustrate the theoretical results, some numerical experiments have been performed, and the results are reported in the last section of the thesis.

Now we formally define well-posed and ill-posed operator equations and discuss the peculiar problems associated with the solution of the ill-posed operator equations. Operator theoretic foundation for the sequel is laid by considering some preliminary results from Functional Analysis, which facilitates in discussing the concept of a generalized inverse and regularization methods.

WELL-POSED AND ILL-POSED PROBLEMS **Let** X and Y be Hilbert spaces(over real or complex field) and $T: X \longrightarrow Y$ be a linear operator. We consider the problem of solving the operator equation

$$(1.1) \qquad Tx = y.$$

A typical example of equation (1.1) is the Fredholm integral equation of the first kind

$$(1.2) \qquad \int_a^b k(s,t)x(t)dt = y(s), \quad a \le s \le b$$

with non-degenerate kernel $k(s,t)$. Here $X = Y = L^2[a,b]$.

An important fact concerning the equation (1.2) is that, the associated operator $T:L^2[a,b] \longrightarrow L^2[a,b]$ defined by

$$(Tx)(s) = \int_a^b k(s,t)x(t)dt, \quad a \leq s \leq b$$

is a 'compact operator' of infinite rank, and therefore T can not have a continuous inverse (See, [26], Theorem 17.2 and 17.4). This observation is very important in view of its application, for this amounts to large deviations in the solutions corresponding to 'nearby' data. Therefore equation (1.2) is a typical example of the so called 'ill-posed problems'. Many inverse problems in physical sciences lead to the solution of the equation of the above type.

In the begining of this century, Hadamad [22] specified the essential requirements for an equation to be well-posed. In our setting, the equation (1.1) is said to be *well-posed* if

(i) (1.1) has a solution x, for all $y \in Y$

(ii) (1.1) can not have more than one solution,

(iii) the unique solution x, if exists, depends continuously on the data y.

In operator theoretic language, (i), (ii), (iii) means that T is bijective and $T^{-1}: Y \longrightarrow X$ is a continuous operator. The equation (1.1) is said to be *ill-posed* if it is not well-posed. By the remark in the previous paragraph, if T is a compact operator of infinite rank, then the equation (1.1) is ill-posed.

We now mention a few examples of inverse problems in physical sciences which lead to solution of an integral equation of the type (1.2). Detailed discussions on these can be found in Groetch [15].

THE VIBRATING STRING. The free vibration of a nonhomogeneous string of unit length and density distribution $\rho(x) > 0$, $0 < x < 1$, is modeled by the partial differential equation

$$\rho(x)U_{tt} = U_{xx}; \tag{1.3}$$

where $U(x,t)$ is the position of the particle 'x' at time t. Assume that the end of the string are fixed and $U(x,t)$ satisfies the boundary conditions

$$U(0,t) = 0, \quad U(1,t) = 0.$$

Assuming the solution $U(x,t)$ is of the form

$$U(x,t) = y(x)r(t),$$

one observes that $y(x)$ satisfies the ordinary differential equation

$$(1.4) \qquad y'' + \omega^2 \rho(x) y = 0$$

with boundary conditions

$$y(0) = 0, \quad y(1) = 0.$$

Suppose the value of y at certain frequency ω is known, then by integrating equation (1.4) twice, first from zero to s and then from zero to one, we obtain

$$(1.5) \qquad \int_0^1 y'(s;\omega)ds - y'(0;\omega) + \omega^2 \int_0^1 \int_0^s \rho(x) y(x;\omega) dx ds = 0.$$

or

$$(1.6) \qquad \int_0^1 (1-s) y(s;\omega) \rho(s) ds = \frac{y'(0;\omega)}{\omega^2}.$$

The inverse problem here is to determine the variable density ρ of the string, satisfying (1.6) for all allowable frequencies ω.

THERMAL ARCHAEOLOGY. Consider a uniform bar of length π which is insulated on its lateral surface so that heat is constrained to flow in only one direction. With certain normalizations and scaling the temperature $U(x,t)$ satisfies the partial differential equation

$$U_t = U_{xx}, \quad 0 < x < \pi.$$

We assume that the ends of the bar are kept at temperature zero, i.e,

$$U(0,t) = 0 \quad \text{and} \quad U(\pi,t) = 0.$$

If $f(x) = U(x,0)$, $0 \le x \le \pi$, is the initial temperature distribution, then the temperature distribution at a later time, say at time $t = 1$, is given by

$$(1.7) \qquad g(x) = U(x,1) = \sum_{n=1}^{\infty} a_n \sin nx,$$

where

$$(1.8) \qquad a_n = (2/\pi)\int_0^{\pi} f(u)\sin nu\, du . e^{-n^2}$$

The inverse problem associated with the above consideration is to determine the initial temperature distribution $f(x)$, knowing a later temperature $g(x)$. From (1.7) and (1.8), the problem, then is to solve the integral equation of the first kind,

$$\int_0^{\pi} k(x,u) f(u)du = g(x)$$

where

$$k(x,u) = (2/\pi)\sum_{n=1}^{\infty} e^{-n^2} \sin nx . \sin nu .$$

GEOLOGICAL PROSPECTING. Here the problem is to determine the location, shape and constitution of subterranean bodies from measurements at the earth's surface. Consider a variable distribution of mass along a parallel line below one unit of the earth's surface. Suppose that a horizontal line measurement is made of the vertical component of the gravitational force due to the mass. If the variable mass density $x(t)$ is distributed along the horizontal axis for $0 \le t \le 1$ and one measures the vertical component of the force $y(s)$, then a small mass element $x(t)\Delta t$ at position t gives rise to a vertical force $\Delta y(s)$ at s, given by

$$\Delta y(s) = \gamma(x(t)\Delta t/((s-t)^2 + 1))\cos\theta$$

$$= \gamma\, x(t)\Delta t/((s-t)^2+1)^{3/2}$$

where γ is the gravitational constant. Now the Fredholm integral equation

$$\gamma \int_0^1 ((s-t)^2+1)^{-3/2} x(t)\,dt = y(s)$$

gives the relation between the vertical force $y(s)$ at s and the density distribution $x(t)$.

SIMPLIFIED TOMOGRAPHY. Consider a two dimensional object contained within a circle of radius **R**. The object is illuminated with a radiation of intensity I_0. As the radiation beams passes through the object it absorbs some radiation. Assume that the radiation absorption coefficient $f(x,y)$, of the object varies from point to point of the object. The absorption coefficient satisfies the law

$$\frac{dI}{dy} = -fI$$

where I is the intensity of the radiation. By taking the above equation as the definition of the absorption coefficient, we have

$$I_x = I_0 \exp\left(-\int_{-y(x)}^{y(x)} f(x,y)dy \right)$$

where $y = \sqrt{R^2-x^2}$. Let $p(x) = \ln(I_0/I_x)$, i.e.,

$$p(x) = \int_{-y(x)}^{y(x)} f(x,y)dy.$$

Suppose that f is circularly symmetric, i.e., $f(x,y) = f(r)$ with $r = \sqrt{x^2+y^2}$, then

(1.9) $$p(x) = \int_x^R (2r/\sqrt{r^2-x^2})f(r)dr.$$

The inverse problem is to find the absorption coefficient f satisfying the equation (1.9).

BLACK BODY RADIATION. When a black body is heated, it emits thermal radiation from its surface at various frequencies. The distribution of thermal power, per unit area of radiating surface, over the various frequencies is known as the power spectrum of the black body. The relation between the power radiation by a unit area of surface at a given frequency ν and absolute temperature T of the surface is given by the relation

$$P(\nu) = \frac{2h\nu^2}{c^2} \cdot \frac{1}{\exp(h\nu/kT-1)}$$

where c is the speed of light, h is Planck's constant and k is Boltzmann's constant.

Suppose that different patches of the surface of the black body are at different temperatures. Let $a(T)$ represents the area of the surface which is at temperature T, i.e, $a(.)$ is the area-temperature distribution of the radiating surface. Then the total radiated power at frequency ν, $W(\nu)$, is given by

$$(1.10) \qquad W(\nu) = (2h\nu^3/c^2)\int_0^{\infty}(\exp(h\nu/(kT-1)))^{-1}a(T)dT.$$

The inverse problem is to find the area-temperature distribution $a(.)$ that can account for an observed power spectrum $W(.)$, i.e, to solve the integral equation (1.10).

1.2. NOTATIONS AND SOME BASIC RESULTS FROM FUNCTIONAL ANALYSIS.

Throughout this thesis X and Y denote Hilbert spaces over real or complex field and $BL(X,Y)$ denotes the space of all bounded linear operators from X to Y. If $Y = X$, then we denote $BL(X,X)$ by $BL(X)$. We will use the symbol $\langle \cdot , \cdot \rangle$ to denote the innerproduct and $\| \cdot \|$ to denote the corresponding norm for the spaces under consideration. The results quoted in this section with no references can be found in any text book on functional analysis, for example, [26] or [13].

For a subspace S of X, its closure is denoted by $\bar{S}$, and its annihilator is denoted by $S^{\perp}$, i.e.,

$$S^{\perp} = \{u \in X : \langle x,u \rangle = 0 \text{ for all } x \in S\}.$$

If $T \in BL(X,Y)$, then its adjoint, denoted by T^*, is a bounded linear operator from Y to X defined by,

$$\langle Tx,y \rangle = \langle x,T^*y \rangle$$

for all $x \in X$ and $y \in Y$. Denoting the range and null space of T by $R(T)$ and $N(T)$ respectively, i.e.,

$$R(T) = \{Tx : x \in X\}$$

and

$$N(T) = \{x \in X : Tx = 0\},$$

we have the following.

Theorem 1.2.1. If $T \in BL(X,Y)$, then $R(T)^{\perp} = N(T^*)$, $N(T)^{\perp} = \overline{R(T^*)}$, $R(T^*)^{\perp} = N(T)$ and $N(T^*)^{\perp} = \overline{R(T)}$. ■

The spectrum and the spectral radius of an operator $T \in BL(X)$ are denoted by $\sigma(T)$ and $r_{\sigma}(T)$ respectively, i.e.,

$$\sigma(T) = \{\lambda \in C : T - \lambda I \text{ does not have bounded inverse}\},$$

where I is the identity operator on X, and

$$r_{\sigma}(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

It is known that

$$r_{\sigma}(T) \le \|T\|,$$

and $\sigma(T)$ is a compact subset of the scalar field. If T is a nonzero self-adjoint operator, i.e., $T = T^*$, then $\sigma(T)$ is a non-empty set of real numbers, and

$$r_{\sigma}(T) = \|T\|. \tag{1.11}$$

If T is a positive self-adjoint operator, i.e., $T = T^*$ and $\langle Tx,x\rangle \ge 0$, $x \in X$, then $\sigma(T)$ is a subset of the set of non-negative reals. If $T \in BL(X)$ is compact, i.e., closure of $\{Tx : x \in X, \|x\| \le 1\}$ is compact, then $\sigma(T)$ is a countable set with zero as the only possible limit point. In fact we have the following result.

Theorem 1.2.2. Let $T \in BL(X)$ be a non-zero compact self-adjoint operator. Then there is a finite or infinite sequence of non-zero real number's (λ_n) with $|\lambda_1| \geq |\lambda_2| \geq \ldots$, and a corresponding sequence (u_n) of orthonormal vectors in X such that for all $x \in X$,

$$Tx = \sum_n \lambda_n \langle x, u_n \rangle u_n,$$

where $\lambda_n \longrightarrow 0$ whenever the sequence (λ_n) is infinite. Here λ_n's are eigenvalues of T with corresponding eigenvectors u_n. ∎

If $T \in BL(X,Y)$ is a non zero compact operator, then T^*T is a positive, compact and self-adjoint operator on X. Then by Theorem 1.2.2, and by the observation that $\sigma(T^*T)$ consists of non-negative reals, there exist a sequence (s_n) of positive reals with $s_1 \geq s_2 \geq \ldots$ and a corresponding sequence of orthonormal vectors (v_n) in X satisfying.

$$T^*Tx = \sum_n s_n \langle x, v_n \rangle v_n, \quad \text{for all} \quad x \in X$$

and $T^*Tv_n = s_n v_n$, $n = 1, 2, \ldots$. Let λ_n be the positive square root of s_n, $\mu_n = 1/\lambda_n$ and $u_n = \mu_n T v_n$. Then (u_n) is a complete orthonormal sequence in Y and $\mu_n T^* u_n = v_n$. Using Theorem 1.2.2, it can be seen (See, [12]) that (u_n) is a complete orthonormal set for $\overline{R(T)} = N(T^*)^{\perp}$ and (v_n) is a complete orthonormal set for $\overline{R(T^*)} = N(T)^{\perp}$. The sequence $\{u_n ; v_n , \mu_n\}$ is called a singular system for T.

In order to define functions of operators on a Hilbert space, we require the spectral theorem for a self-adjoint operator which is a generalization of Theorem 1.2.2.

Theorem 1.2.3. Let $T \in BL(X)$ be self-adjoint and let $a = \inf\sigma(T)$, $b = \sup\sigma(T)$. Then there exists a family $\{E_\lambda : a \le \lambda \le b\}$ of projection operators on X such that

(i) $\lambda_1 < \lambda_2$ implies $\langle E_{\lambda_1} x, x\rangle \le \langle E_{\lambda_2} x, x\rangle$ for all $x \in X$.

(ii) $E_a = 0$, $E_b = I$, where I is the identity operator on X

(iii) $T = \int_a^b \lambda dE_\lambda$. ∎

The integral in (iii) is understood in the Riemann-Stieltjes sense. The family $\{E_\lambda\}_{\lambda\in[a,b]}$ is called *the spectral family* of the operator T. If f is a continuous real valued function on $[a,b]$, then $f(T) \in BL(X)$ is defined by

$$f(T) = \int_a^b f(\lambda)dE_\lambda.$$

Then

$$\sigma(f(T)) = \{f(\lambda) : \lambda \in \sigma(T)\}.$$

Now by (1.11) we have

$$(1.12) \qquad \|f(T)\| = r_\sigma(f(T)) = \sup\{|f(\lambda)| : \lambda \in \sigma(T)\}.$$

For real valued functions f and g, we use the notation

$$f(\alpha) = O(g(\alpha)) \quad \text{as} \quad \alpha \to 0$$

to denote the relation

$$\left|\frac{f(\alpha)}{g(\alpha)}\right| \leq M \quad \text{as} \quad \alpha \to 0,$$

where $M > 0$ is a constant independent of α, and

$$f(\alpha) = o(g(\alpha)) \quad \text{as} \quad \alpha \to 0$$

to denote

$$\lim_{\alpha \to 0} \frac{f(\alpha)}{g(\alpha)} = 0.$$

1.3. GENERALIZED INVERSE.

If the operator equation (1.1) has no solution in the usual sense, i.e., if y does not belong to the range of T, then one may broaden the notion of a solution in a meaningful sense. This can be done using the concept of a least-square solution.

For $T \in BL(X,Y)$ and $y \in Y$, we say that $u \in X$ is a *least square solution* of the equation (1.1), $Tx = y$, if

$$\|Tu-y\| = \inf\{\|Tx-y\| : x \in X\}.$$

It is to be remarked that if T is not one-one, then a least-square solution u, if it exists, is not unique, since $u+v$ is also a least-square solution for every $v \in N(T)$. The following Theorem provides characterizations of least-square solutions.

Theorem 1.3.1. (Groetsch [12], Theorem 1.3.1). For $T \in BL(X,Y)$ and $y \in Y$, the following are equivalent.

(i) $\|Tu-y\| = \inf\{\|Tx-y\| : x \in X\}$

(ii) $T^*Tu = T^*y$

(iii) $Tu = Py$

where $P : Y \longmapsto Y$ is the orthogonal projection onto $\overline{R(T)}$. ∎

From (iii) it is clear that (1.1) has a least-square solution if and only if $Py \in R(T)$, i.e., if and only if y belongs to the dense subspace $R(T) + R(T)^{\perp}$ of Y. Any of (i)-(iii) in Theorem 1.3.1 shows that the set of all least-square solutions is a closed convex set, and therefore, by Theorem 1.1.4 in [11], there is a unique least-square solution of smallest norm. For $y \in R(T) + R(T)^{\perp}$, the unique least-square solution of minimal norm of (1.1) is called *the generalized solution* or *pseudo solution* of (1.1). It can be easily seen that the generalized solution belongs to the subspace $N(T)^{\perp}$ of X. For $T \in BL(X,Y)$, the map $T^{\dagger}$ which associates each $y \in D(T^{\dagger}) := R(T) + R(T)^{\perp}$, the generalized solution of (1.1) is called the *generalized inverse* of T. We note that if $y \in R(T)$ and T is injective, then the generalized solution of (1.1) is the solution of (1.1). If T is bijective, then it follows that $T^{\dagger} = T^{-1}$.

Theorem 1.3.2. (Groetch [11], [13]). Let $T \in BL(X,Y)$. Then $T^{\dagger}: D(T^{\dagger}) \longrightarrow X$ is a closed densely defined linear operator, and $T^{\dagger}$ is bounded if and only if $R(T)$ is closed. ∎

If equation (1.1) is ill-posed then one would like to obtain the generalized solution of (1.1). But Theorem 1.3.2 shows that the problem of finding the generalized solution of (1.1) is also ill-posed, i.e., $T^{\dagger}$ is discontinuous, if $R(T)$ is not a closed subspace of Y. Recall that if $T \in BL(X,Y)$ is a compact operator of infinite rank, then $R(T)$ is not closed. This observation is important since a wide class of operators of practical interest, as we have seen in Section 1.2, are compact operators of infinite rank. In application, the data y may not be available exactly. So, one has to work with an approximation, say $\tilde{y}$, of y. If $T^{\dagger}$ is discontinuous, then for $\tilde{y}$ close to y, the generalized solution $T^{\dagger}\tilde{y}$, even when it is defined, need not be close to $T^{\dagger}y$. Therefore some regularization procedures have to be employed, to obtain approximations for $T^{\dagger}y$, for $y \in D(T^{\dagger})$.

1.4. THE REGULARIZATION PRINCIPLE AND THE TIKHONOV REGULARIZATION.

Here onwards we are concerned with the problem of of finding the generalized solution of (1.1) where $T \in BL(X,Y)$ and $y \in D(T^{\dagger}) = R(T) + R(T)^{\perp}$. For $\delta > 0$, let $\tilde{y} \in Y$ be an inexact data such that $\|y-\tilde{y}\| \leq \delta$. By *regularization* of the equation (1.1) with $\tilde{y}$ in place of y, we mean a procedure of obtaining a family $(\tilde{x}_{\alpha})$ of vectors in X such that each $\tilde{x}_{\alpha}$, $\alpha > 0$, is a solution of a well-posed equation satisfying $\tilde{x}_{\alpha} \to T^{\dagger}y$ as $\alpha \to 0$ and $\delta \to 0$.

A regularization method which has been studied most extensively is the so called Tikhonov regularization ([43], [44]) introduced in the early sixties, where $\tilde{x}_{\alpha}$ is taken as the minimizer of the functional

$$x \longmapsto F_{\alpha}(x) = \|Tx - \tilde{y}\|^2 + \alpha\|x\|^2, \quad x \in X, \ \alpha > 0.$$

The fact that $\tilde{x}_{\alpha}$ is the unique solution of the well-posed equation, namely, (1.13) $$(T^*T + \alpha I)\tilde{x}_\alpha = T^*\tilde{y},$$

is included in the following well known result, the proof of which is included for the sake of completion.

Theorem 1.4.1. (See [35]) Let $T \in BL(X;Y)$ and $y \in Y$. For each $\alpha > 0$ there exists a unique $x_\alpha \in X$ which minimizes the function

(1.14) $$x \longmapsto F_\alpha(x) = \|Tx-y\|^2 + \alpha\|x\|^2, \quad x \in X.$$

More over, the map $y \longmapsto x_\alpha$ is continuous for each $\alpha > 0$, and

$$x_\alpha = (T^*T+\alpha I)^{-1}T^*y.$$

Proof: First we prove that there exists a unique x_α which minimizes the function (1.14). Consider the product space $X \times Y$ with the usual innerproduct defined by

$$\langle (x_1,y_1),(x_2,y_2)\rangle = \langle x_1,x_2\rangle + \langle y_1,y_2\rangle, \quad x_1, x_2 \in X \; ; \; y_1, y_2 \in Y.$$

It is seen that, with respect to this innerproduct, $X \times Y$ is a Hilbert space. For $\alpha > 0$, consider the function

$$F_\alpha : X \longrightarrow X \times Y, \quad F_\alpha(x) = (\sqrt{\alpha}x, Tx), \quad x \in X.$$

Since $T \in BL(X,Y)$, the graph of T,

$$G(T) = \{(x,Tx) : x \in X\},$$

is a closed subspace of $X \times Y$, so that the range $R(F_\alpha)$ of F_α is closed in $X \times Y$. Thus by Theorem 1.3.2, the generalized inverse $F_\alpha^\dagger$ is a bounded linear operator from $X \times Y$ into X. Let $x_\alpha = F_\alpha^\dagger(0,y)$. Since F_α is one-one, it is clear from the definition of the generalized inverse that x_α is the unique element in X satisfying

$$\|F_\alpha(x_\alpha) - (0,y)\| = \inf\{\|F_\alpha(x) - (0,y)\| : x \in X\}$$

i.e.,

$$\|Tx_\alpha - y\|^2 + \alpha\|x_\alpha\|^2 = \inf\{\|Tx-y\|^2 + \alpha\|x\|^2 : x \in X\}.$$

Now since the function $J : Y \longrightarrow X \times Y$ defined by $J(y) = (0,y)$, $y \in Y$, is continuous, the function $y \longmapsto x_\alpha := F_\alpha^\dagger(0,y)$ is also continuous.

Now to prove that x_α is given by $x_\alpha = (T^*T + \alpha I)^{-1}T^*y$, first we note that T^*T is a positive self adjoint operator and hence $-\alpha \notin \sigma(T^*T)$, if $\alpha > 0$. Thus for $\alpha > 0$, $(T^*T + \alpha I)^{-1}$ exist and is a bounded linear operator on X. Let $u_\alpha = (T^*T + \alpha I)^{-1}T^*y$, $\alpha > 0$, then

$$\|T(u_\alpha+v)-y\|^2 + \alpha\|u_\alpha+v\|^2 = \|Tu_\alpha-y\|^2 + \alpha\|u_\alpha\|^2 + \langle (T^*T+\alpha I)v \, , \, v\rangle,$$

for all $v \in X$. Now since $\langle (T^*T + \alpha I)v , v\rangle \geq 0$, for all $v \in X$, it follows that

$$\|Tu_\alpha - y\|^2 + \alpha\|u_\alpha\|^2 \leq \|Tx-y\|^2 + \alpha\|x\|^2, \text{ for all } x \in X.$$

This, together with the fact that $x_\alpha = F_\alpha^\dagger(0,y)$ is the unique element in X such that

$$\|Tx_\alpha - y\|^2 + \alpha\|x_\alpha\|^2 = \inf\{\|Tx-y\|^2 + \alpha\|x\|^2 : x \in X\}$$

shows that $x_\alpha = u_\alpha = (T^*T + \alpha I)^{-1}T^*y$. ∎

If $Y = X$ and T is a positive self-adjoint operator on X, then one may consider (See Bakushiniskii [2]) a simpler regularization method to solve equation (1.1), where the family of vectors $\tilde{w}_\alpha$, $\alpha > 0$, satisfying

(1.15) $$(T + \alpha I)\tilde{w}_\alpha = \tilde{y},$$

is considered to obtain approximations for $T^\dagger y$. Note that for positive self adjoint operator T, the ordinary Tikhonov regularization applied to (1.1) results in a more complicated equation $(T^2 + \alpha I)\tilde{x}_\alpha = T\tilde{y}$ than (1.15). Moreover it is known (See Schock [40]) that the approximation obtained by regularization procedure (1.15) has better convergence properties than the approximation obtained by Tikhonov regularization. As in Groetsch and Guacaneme, [16], we call the above regularization procedure which gives the family of vectors $\tilde{w}_\alpha$ in (1.15), the *Simplified regularization* of (1.1).

One of the prime concerns of regularization methods is the convergence of $\tilde{x}_\alpha$ ($\tilde{w}_\alpha$ in the case of Simplified regularization) to $T^\dagger y$, as $\alpha \to 0$ and $\delta \to 0$. It is known ([12], Theorem 2.3.5) that, if $R(T)$ is not closed, then there exist sequences (δ_n) and $(\alpha_n) := (\alpha(\delta_n))$ such that $\delta_n \to 0$ and $\alpha_n \to 0$ as $n \to \infty$ but the sequence $(\tilde{x}_{\alpha_n})$ is divergent as $\delta_n \to 0$. Therefore it is important to choose the regularization parameter α depending on the error level δ and also possibly on $\tilde{y}$, say $\alpha = \alpha(\delta, \tilde{y})$, such that $\alpha(\delta, \tilde{y}) \to 0$ and $\tilde{x}_\alpha \to T^\dagger y$ as $\delta \to 0$. We shall see later (Section 2.1) that in the case of Tikhonov regularization, if we take $\alpha = \delta$ à priorily then $\tilde{x}_\alpha \to T^\dagger y$ as $\delta \to 0$. Practical considerations suggest that, it is desirable to choose the regularization parameter α at the time of solving $\tilde{x}_\alpha$, using a so-called à posteriori method which depends on $\tilde{y}$ as well as δ. (See, [3]).

1.5. THE CHOICE OF REGULARIZATION PARAMETER BY DISCREPANCY PRINCIPLES.

For choosing the regularization parameter a posteriorly, 'discrepancy principles' have been used extensively in the literature (e.g., [4], [6], [7], [10], [32], [38]). This idea was first enunciated by Morozov[31]. The method is based on the reasonable view that the quality of the results of a computation must be comparable to the quality of the input data. To be more precise the magnitude of the error must be in agreement with the accuracy of the assignment of the input data (See, Morozov [31] or Groetsch [12]). The practical difficulty here is that even an asymptotic bound for the quantity $\|\tilde{x}_\alpha - T^\dagger y\|$ usually requires information on the data y. Therefore one has to consider an 'optimal order' (optimal in the sense that, in general, the order can not be improved) of the quantity $\|\tilde{x}_\alpha - T^\dagger y\|$, based on the available information of the data. Now the crucial problem is to find the value of the regularization parameter α which gives the optimal order of the quantity $\|\tilde{x}_\alpha - T^\dagger y\|$.

The subject matter of this thesis is to provide optimal error bounds for the existing discrepancy principles for Tikhonov regularization and simplified regularization, and also to generalize a discrepancy principle for simplified regularization considered by Guacaneme [21]. Computational results are given in the last section of the thesis which confirm the theoretical results.

1.6 SUMMARY OF THE THESIS

In Chapter 2 we consider Tikhonov regularization for approximately solving the ill-posed operator equation $Tx = y$, where $T : X \longrightarrow Y$ is a bounded linear operator between Hilbert spaces X and Y and $y \in R(T) + R(T)^{\perp}$, i.e., the problem of minimizing the functional

$$x \longmapsto \|Tx-y\|^2 + \alpha\|x\|^2, \quad \alpha > 0.$$

When only an approximation of the data y is known, say y^{δ}, with $\|y-y^{\delta}\| \le \delta$, then the problem of choosing the regularization parameter α depending on δ and y^{δ} is important. For this purpose many discrepancy principles are known in the literature (e.g., [4], [10], [38]). In Section 2.2 we consider the discrepancy principle

$$\|Tx_{\alpha}^{\delta} - y^{\delta}\| = \frac{\delta^p}{\alpha^q}, \quad p > 0, \quad q > 0,$$

considered by Schock [38] and later by Nair [34] and prove that this discrepancy principle gives the optimal estimate $O(\delta^{2\nu/(2\nu+1)})$, $1/2 \le \nu \le 1$, for the error $\|\hat{x} - x_{\alpha}^{\delta}\|$ whenever $\hat{x}$ belongs to $R((T^*T)^{\nu})$. The result of this section improves the result of Schock [38], and also it improves the result of Nair [34], except for the case $\nu = 1$. Particular case of the result, as proved in [34], shows that the Arcangeli's method does give the optimal rate $O(\delta^{2/3})$. In Section 2.3 we show that one can use the discrepancy principle considered above for iterated Tikhonov regularization also.

Chapter 3 is concerned with the problem of approximately solving ill-posed operator equation $Aw = g$, where $A : X \longrightarrow X$ is a positive self-adjoint operator on a Hilbert space X and $g \in R(A)$, the range of A. Here we consider the Simplified regularization, where the solution w_α of the equation

$$(A + \alpha I)w_\alpha = g$$

is taken as an approximation for the minimal norm solution $\hat{w}$ of the equation $Aw = g$. If the data g is known only approximately, say g^δ, with $\|g-g^\delta\| \le \delta$, then we consider the solution w_α^δ of the equation

$$(A + \alpha I)w_\alpha^\delta = g^\delta$$

for obtaining approximations for $\hat{w}$. In this case, for choosing the parameter α, Groetsch and Guacaneme [16] considered the discrepancy principle

$$\|Aw_\alpha^\delta - g^\delta\| = \frac{\delta}{\sqrt{\alpha}}$$

and proved that $w_\alpha^\delta \to \hat{w}$ as $\delta \to 0$, but no attempt has been made for obtaining estimate for the error $\|\hat{w} - w_\alpha^\delta\|$. In Section 3.1 we consider a general class of discrepancy principle, namely,

$$\|Aw_\alpha^\delta - g^\delta\| = \frac{\delta^p}{\alpha^q}, \quad p > 0, \quad q > 0,$$

which includes the one considered by Groetsch and Guacaneme [16], and obtain the optimal estimate for the error $\|\hat{w} - w_\alpha^\delta\|$. In Section 3.2 we consider a generalized form of a discrepancy principle considered by Guacaneme [21], namely,

$$\alpha^{2(\rho+1)}\langle (A + \alpha I)^{-2(\rho+1)} Qg^\delta, Qg^\delta \rangle = c\delta^2, \quad \rho > 0$$

where $c > 1$ is a constant and Q is the orthogonal projection onto $\overline{R(T)}$, the closure of the range of A. Results of this section includes a result of Guacaneme [21], which he proved when A is, in addition, compact and $\hat{w} \in R(A)$. In the last section of Chapter 3, we consider the discrepancy principles considered in Sections 3.1 and 3.2 for iterated Simplified regularization.

Chapter 4 is devoted to the study of Tikhonov regularization and Simplified regularization in the presence of modelling and data error, i.e., both the operator and the data are known only approximately. Knowing a family of operators T_h, $h > 0$, with

$$\|T - T_h\| \le \varepsilon_h, \quad \varepsilon_h \to 0 \quad \text{as} \quad h \to 0,$$

we consider the solution $x_{\alpha,h}^\delta$ of the equation

$$(T_h^* T_h + \alpha I) x_{\alpha,h}^\delta = T_h^* y^\delta,$$

as an approximation for $\hat{x}$, the minimal norm solution of the equation $Tx = y$. In this case we consider the discrepancy principle

$$\|T_h x_\alpha^\delta - y^\delta\| = \frac{(\delta+\varepsilon_h)^p}{\alpha^q}, \qquad p > 0, \quad q > 0,$$

and obtain the optimal rate $O((\delta+\varepsilon_h)^{2\nu/(2\nu+1)})$, $1/2 \le \nu \le 1$ for $\|\hat{x} - x_{\alpha,h}^\delta\|$ under the assumption $\hat{x} \in R((T^*T)^\nu)$. In Sections 4.3 and 4.4 we consider a family of self-adjoint operators A_h with

$$\|A-A_h\| \le \varepsilon_h, \quad \varepsilon_h \to 0 \text{ as } h \to 0.$$

For choosing the parameter in the case of Simplified regularization of $Aw = g$, we consider the discrepancy principles

$$\|A_h w_\alpha^\delta - g^\delta\| = \frac{(\delta+\varepsilon_h)^p}{\alpha^q}, \quad p > 0, \ q > 0$$

and

$$\alpha^{2(\rho+1)}\langle (A_h + \alpha I)^{-2(\rho+1)} Q_h g^\delta, Q_h g^\delta\rangle = (c\delta + d\varepsilon_h)^2, \quad \rho > 0,$$

where c and d are properly chosen constants and Q_h is the orthogonal projection onto $\overline{R(A_h)}$.

In Chapter 5 we consider a projection method for the regularized equations

$$(T^*T + \alpha I)x_\alpha^\delta = T^*y^\delta \text{ and } (A + \alpha I)w_\alpha^\delta = g^\delta.$$

For the first equation, the method is a special case of the procedure considered in Section 4.2 and is a generalization and modification of the Marti's method. Also in this case the regularized projection method improves the result of Section 4.2 under certain conditions. In order to illustrate the theoretical results, some numerical experiments have been performed, and the results are reported in the last section of the thesis.

CHAPTER 2

PARAMETER CHOICE STRATEGY FOR TIKHONOV REGULARIZATION

In this Chapter we consider one of the important points to be taken into account while using Tikhonov regularization method for ill-posed operator equations, namely, choosing the regularization parameter depending on the inexact data as well as the error level in the data. In Section 2.1 we present some known results which motivated our investigations in the later sections. These results are presented in suitable forms required for later references and their proofs are included for the sake of completion. In Section 2.2, a discrepancy principle suggested by Schock [38] is considered for ordinary Tikhonov regularization. We show that the 'optimal' rate is achieved under certain smoothness assumption on 'the solution'. In the final section, above discrepancy principle is applied to the iterated Tikhonov regularization and it is compared with a procedure adopted by Engl [4].

2.1 PRELIMINARIES

We are concerned with the problem of approximately solving the operator equation

$$Tx = y, \tag{2.1}$$

where $T \in BL(X,Y)$, with non-closed range $R(T)$, and $y \in D(T^\dagger) := R(T) + R(T)^\perp$. The idea is to look for approximations for the generalized solution $\hat{x} := T^\dagger y$ of (2.1) with the help of well-posed equations. Here $T^\dagger$ is the generalized inverse of T (see Section 1.3). We consider Tikhonov regularization for solving the equation (2.1). In practice the data y may not be known exactly, instead we may have an approximation, say y^δ of y within error level $\delta > 0$, i.e., $y^\delta \in D^\delta := \{u \in Y : \|u-y\| \le \delta\}$. In Tikhonov regularization, as we have seen in Section 1.4, one solves the equation

$$(2.2) \qquad (T^*T + \alpha I)x_\alpha^\delta = T^*y^\delta, \quad \alpha > 0.$$

We recall (Section 1.4) that $(T^*T + \alpha I)^{-1}$ exists for each $\alpha > 0$, and is a bounded linear operator. Also we note that for each $\alpha > 0$,

$$(2.3) \qquad T(T^*T + \alpha I)^{-1} = (TT^* + \alpha I)^{-1}T.$$

We have the following result which gives certain bounds for the error $\|\hat{x} - x_\alpha^\delta\|$.

Theorem 2.1.1. (Schock [41]). Let x_α^δ is as in (2.2) with $y^\delta \in D^\delta$ and $x_\alpha := x_\alpha^0$. Then we have the following.

a) $x_\alpha \to \hat{x}$ as $\alpha \to 0$.

b) If $\hat{x} \in R((T^*T)^\nu)$, $0 < \nu \le 1$, then

(i) $\|\hat{x} - x_\alpha\| \le c_1\alpha^\nu$,

(ii) $\|\hat{x} - x_\alpha^\delta\| \le c_1\alpha^\nu + \frac{\delta}{\sqrt{\alpha}}$,

where $c_1 > 0$ is a positive constant.

In particular we have the following,

(iii) if $\alpha = \alpha(\delta)$ is such that $\alpha(\delta) \to 0$ and $\frac{\delta}{\sqrt{\alpha}} \to 0$ as $\delta \to 0$ then $x_\alpha^\delta \to \hat{x}$ as $\delta \to 0$.

(iv) If $\alpha = c\delta^{2/(2\nu+1)}$ for some constant $c > 0$, then

$$\|\hat{x} - x_\alpha^\delta\| = O(\delta^{2\nu/(2\nu+1)}).$$

Proof: To prove the convergence of x_α to $\hat{x}$, we let $R_\alpha = \alpha(T^*T + \alpha I)^{-1}$, then $\hat{x} - x_\alpha = R_\alpha\hat{x}$. Thus it is enough to prove that $R_\alpha\hat{x} \to 0$ as $\delta \to 0$. But $\|R_\alpha\| \le 1$ for every $\alpha > 0$, and for any $u \in R(T^*T)$ let $u = T^*Tv$, so that $\|R_\alpha u\| = \|R_\alpha(T^*T)v\| \le \alpha\|v\|$. Thus $R_\alpha u \to 0$ as $\alpha \to 0$ for every $u \in R(T^*T)$. Therefore by using the fact that $R(T^*T)$ is dense subset of the orthogonal compliment of the null space of T and $\|R_\alpha\| \le 1$, it follows that $R_\alpha\hat{x} \to 0$ as $\alpha \to 0$.

Now using the definition of x_α and the relation $T^*T\hat{x} = T^*y$,

(ii) on p. 19

we have

$$\|\hat{x} - x_\alpha\| = \|\hat{x} - (T^*T + \alpha I)^{-1}T^*y\|$$

$$= \|\alpha(T^*T + \alpha I)^{-1}\hat{x}\|$$

$$= \|\alpha(T^*T + \alpha I)^{-1}(T^*T)^{\nu}z\|$$

where $\hat{x} = (T^*T)^{\nu}z$ for some $z \in X$, since $\hat{x} \in R((T^*T)^{\nu})$. Since

$$\|\alpha(T^*T + \alpha I)^{-1}(T^*T)^{\nu}z\| \le \|\alpha(T^*T + \alpha I)^{-1}(T^*T)^{\nu}\| \; \|z\|,$$

by (1.12), we have

$$\|\alpha(T^*T + \alpha I)^{-1}(T^*T)^{\nu}z\| \le \sup_{0\le\lambda\le\|T\|} \frac{\alpha\lambda^{\nu}}{\lambda + \alpha} \quad \|z\|$$

$$= \alpha^{\nu} \sup_{0\le\lambda\le\|T\|} \frac{(\lambda/\alpha)^{\nu}}{1+(\lambda/\alpha)} \|z\|.$$

Now the result (i) follows from the fact that $(\lambda/\alpha)^{\nu} < 1 + (\lambda/\alpha)$ for $0 < \nu \le 1$ and (ii) follows from (i) and the following inequality,

$$(2.4) \qquad \|\hat{x} - x_\alpha^{\delta}\| \le \|\hat{x} - x_\alpha\| + \|x_\alpha - x_\alpha^{\delta}\|$$

where

$$(2.5) \qquad \|x_\alpha - x_\alpha^{\delta}\| = \|(T^*T + \alpha I)^{-1}T^*(y-y^{\delta})\|$$

$$\leq \|(T^*T + \alpha I)^{-1}(T^*T)^{1/2}\| \|y - y^\delta\|$$

not clear

$$\leq \frac{\delta}{\sqrt{\alpha}}.$$

Now (iii) follows from (ii), and (iv) follows from (ii) by noting that $\alpha^\nu = \frac{\delta}{\sqrt{\alpha}} = O(\delta^{2\nu/(2\nu+1)})$ if $\alpha = c\delta^{2/(2\nu+1)}$. ∎

The following Theorem shows that the rate

$$\|\hat{x} - x_\alpha^\delta\| = O(\delta^{2\nu/(2\nu+1)}) \tag{2.6}$$

in Theorem 2.1.1 (iv) is optimal in the sense that in general it cannot be improved.

Theorem 2.1.2. (Groetsch [12], Schock [39]). Let $T \in BL(X,Y)$ be a compact linear operator. Assume that $0 \neq x \in R((T^*T)^\nu)$, $0 < \nu \leq 1$, $\alpha(\delta) = c\delta^{2/(2\nu+1)}$ for some constant $c > 0$, and that, for each $y^\delta \in D^\delta$, we have $\|\hat{x} - x_\alpha^\delta\| = o(\delta^{2\nu/(2\nu+1)})$. Then, range of T^*T is of finite dimension.

Proof: Let (u_n, v_n, μ_n) be a singular system for T. Suppose that T^*T does not have a finite rank. Then by the remarks that follow Theorem 1.2.2, $\mu_n \to \infty$ as $n \to \infty$. Let $\delta_n = \mu_n^{-(2\nu+1)}$ and $y^{\delta_n} = y + \delta_n u_n$. For simplicity we replace δ_n by δ and $\alpha(\delta_n)$ by α. Then

$$x_\alpha^\delta - \hat{x} = x_\alpha - \hat{x} + x_\alpha^\delta - x_\alpha$$

$$= x_\alpha - \hat{x} + \delta(T^*T + \alpha I)^{-1}T^*u_n$$

$$= x_\alpha - \hat{x} + \frac{\delta\mu_n^{-1}v_n}{\mu_n^{-2}+\alpha}.$$

Therefore

$$\|x_\alpha^\delta - \hat{x}\|^2 = \|x_\alpha - \hat{x}\|^2 + \left(\frac{2\delta\mu_n}{1+\alpha\mu_n^2}\right)\langle x_\alpha - \hat{x} , v_n\rangle + \left(\frac{\delta\mu_n}{1+\alpha\mu_n^2}\right)^2 \|v_n\|^2,$$

and hence using $\alpha = c\delta^{2/(2\nu+1)}$ and $\delta = \mu_n^{-(2\nu+1)}$, we obtain

$$\delta^{\frac{-4\nu}{2\nu+1}}\|x_\alpha^\delta - \hat{x}\|^2 \geq \frac{\delta^{\frac{-2\nu}{2\nu+1}}}{1+c}\langle x_\alpha - \hat{x} , v_n\rangle + (1+c)^{-2}\|v_n\|^2.$$

Now by hypothesis, we have

$$0 \geq 2\limsup_{\delta \to 0} \frac{\delta^{-2\nu/(2\nu+1)}}{1+c} \langle x_\alpha - \hat{x} , v_n\rangle + (1+c)^{-2}\|v_n\|^2,$$

so that

$$(1+c)^{-2}\|v_n\| \leq 2\limsup_{\delta \to 0} \frac{\delta^{-2\nu/(2\nu+1)}}{1 + c} \langle \hat{x} - x_\alpha , v_n\rangle$$

$$\leq 2\limsup_{\delta \to 0} \delta^{-2\nu/(2\nu+1)}\|\hat{x} - x_\alpha\|.$$

However, by hypothesis we also have $\|x_{\alpha(\delta)} - \hat{x}\| = o(\delta^{2\nu/(2\nu+1)})$ and hence $(1 + c)^{-2}\|v_n\| \leq 0$, a contradiction. This completes the proof of the Theorem. ∎

Remark 2.1.3. Theorem 2.1.2 is proved in Groetsch [12] when $\nu = 1$ and the proof for the case $0 < \nu \leq 1$ is given in Schock [39]. The proof given above is a modification of the one given in [12].

In the above Theorem, if $\nu = 1$ then the condition $\alpha = c\delta^{2/(2\nu+1)}$ in Theorem 2.1.2 is redundant, because in this case we have

$$\alpha = O(\delta) + O(\|\hat{x} - x_\alpha^\delta\|)$$

(Groetsch [12], Theorem 3.2.3). In fact, the above relation together with the condition $\|\hat{x} - x_\alpha^\delta\| = o(\delta^{2/3})$ implies that

$$\alpha = o(\delta^{2/3}).$$

As we mentioned in Section 1.4, in a posteriori parameter choice strategies the regularization parameter $\alpha = \alpha(\delta)$ (depending on y^δ and the error level δ) is determined during the course of computation of x_α^δ. Well-known methods in this regard are the discrepancy principles

$$\|Tx_\alpha^\delta - y^\delta\| = \delta \quad \text{and} \quad \|Tx_\alpha^\delta - y^\delta\| = \frac{\delta}{\sqrt{\alpha}}$$

of Morozov [31] and Arcangeli's [1] respectively. Groetsch [14] has shown that Morozov's method does not yield a better rate than $O(\delta^{1/2})$. In the case of Arcangeli's method Groetsch and Schock [18] have shown that if $\hat{x} \in R(T^*)$, then the rate is $O(\delta^{1/3})$ instead of $O(\delta^{1/2})$.

In an attempt to achieve the best rate $O(\delta^{2/3})$, Schock [38] considered a generalized form of Arcangeli's method, namely,

$$\|Tx_\alpha^\delta - y^\delta\| = \frac{\delta^p}{\alpha^q}, \quad p > 0 , q > 0. \tag{2.7}$$

for choosing the regularization parameter α. The following proposition shows the existence and the order of α (with respect to δ) satisfying (2.7).

Proposition 2.1.4. (Schock [38]) For $\delta > 0$, there exist a unique $\alpha := \alpha(\delta)$ satisfying (2.7), and if $y \neq 0$, then

$$\alpha(\delta) \to 0 \text{ as } \delta \to 0. \text{ Moreover } \alpha(\delta) = O(\delta^{\frac{p}{q+1}}), \quad 0 < \delta \leq \frac{\|y\|}{2}.$$

Proof: We observe that

$$\|Tx_\alpha^\delta - y^\delta\| = \|\alpha(TT^* + \alpha I)^{-1}y^\delta\| \tag{2.8}$$

and hence

$$\frac{\alpha\|y^\delta\|}{\alpha+\|T\|^2} \leq \|Tx_\alpha^\delta - y^\delta\| \tag{2.9}$$

$$\leq \|y^\delta\|.$$

For fixed δ, y^δ, let $\phi(\alpha) = \alpha^{2q}\|Tx_\alpha^\delta - y^\delta\|^2$. Then from (2.9) it follows that

$$\lim_{\alpha \to 0} \phi(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha \to \infty} \phi(\alpha) = \infty.$$

Therefore by Intermediate Value Theorem, there exist an $\alpha := \alpha(\delta)$ satisfying (2.7). The uniqueness follows from the fact that the derivative of $\phi(\alpha)$ is strictly positive; i.e., $\phi(\alpha)$ is strictly increasing.

Now suppose $\alpha(\delta)$ does not converges to zero as $\delta \to 0$. Then there exists a sequence (δ_n) such that $\delta_n \to 0$ and $\alpha_n := \alpha(\delta_n) \to c > 0$ as $n \to \infty$. Then by (2.7) we have

$$0 = \lim_{n \to \infty} \alpha_n^{q} \|Tx_{\alpha_n}^{\delta_n} - y^{\delta_n}\| = c^{q} \|T(T^*T + cI)^{-1}T^*y - y\|$$

and hence

$$TT^*y = TT^*y + cy$$

i.e., $y = 0$, a contradiction. Thus $\alpha(\delta) \to 0$ as $\delta \to 0$.
Note that

$$\|y^{\delta}\| - \frac{\delta^{p}}{\alpha^{q}} = \|y^{\delta}\| - \|Tx_{\alpha}^{\delta} - y^{\delta}\|$$

$$\leq \|Tx_{\alpha}^{\delta}\|$$

$$= \frac{1}{\alpha} \|T(\alpha x_{\alpha}^{\delta})\|$$

$$\leq \frac{1}{\alpha} \|T\| \, \|\alpha x_{\alpha}^{\delta}\|$$

$$= \frac{\|T\|\delta^p}{\alpha^{q+1}}.$$

Therefore

$$\alpha(\delta)^{q+1} \le \frac{\|T\|+\alpha(\delta)}{\|y^\delta\|}\delta^p$$

$$\le \frac{2(\|T\|+\alpha(\delta))}{\|y\|}\delta^p.$$

Hence

$$\alpha(\delta) = O(\delta^{\frac{p}{q+1}}).$$

This completes the proof. ∎

Schock [38] proved that if $\hat{x} \in R((T^*T)^\nu)$, $0 < \nu \le 1$, $\frac{p}{q+1} = \frac{2}{2\nu+1+(1/2q)}$ and $\alpha = \alpha(\delta)$ is chosen according to (2.7), then

$$\|\hat{x} - x_\alpha^\delta\| = O(\delta^t) \quad \text{with} \quad t = \frac{2\nu}{2\nu+1+(1/2q)}.$$

Latter Nair [34] improved the above result of Schock by showing that if $\hat{x} \in R((T^*T)^\nu)$, $0 < \nu \le 1$, $\frac{p}{q+1} = \frac{2}{2\nu+1+((1-\nu)/2q)}$ and $\alpha = \alpha(\delta)$ is chosen according to (2.7), then

(2.10) $$\|\hat{x} - x_\alpha^\delta\| = O(\delta^s) \quad \text{with} \quad s = \frac{2\nu}{2\nu+1+((1-\nu)/2q)}.$$

The above result (2.10), in particular, gives the best rate $O(\delta^{2/3})$ for $\hat{x} \in R(T^*T)$ for the choice of $\frac{p}{q+1} = \frac{2}{3}$, showing there by that the Arcangeli's method, i.e., for $p = 1$, $q = 1/2$, gives the optimal rate for $\nu = 1$. This result also includes the main Theorem of Guacaneme [20] which he proved for $p = 2$, $q = 2$ and $\nu = 1$.

In order to obtain the optimal rate (2.6), Engl [4] had considered a variant of (2.7), namely,

$$(2.11) \qquad \| T^*Tx_\alpha^\delta - T^*y^\delta \|^2 = \frac{\delta^p}{\alpha^q}, \quad p > 0, \quad q > 0,$$

and proved that if $\hat{x} \in R((T^*T)^\nu)$, $0 < \nu \le 1$, $\frac{p}{q+1} = \frac{2}{2\nu+1}$, and $\alpha = \alpha(\delta)$ is chosen according to (2.11), then the optimal rate in (2.6) is achieved. It is to be recalled that Engl and his collaborators stated in many papers (e.g., [4],[6],[7],[8]) that the Arcangeli's method can not have the optimal rate $O(\delta^{2/3})$ and therefore the introduction of a new discrepancy principle such as (2.11). This remark was based on a wrong observation on a result in [18]. What in fact, proved in [18] was that the rate $o(\delta^{2/3})$ is not possible for Arcangeli's method unless $\hat{x} = 0$, and the rate $O(\delta^{2/3})$ is attained if T is of finite rank.

Next section is an attempt to show that Engl's modification (2.11) is not necessary to obtain the rate $O(\delta^{\frac{2\nu}{2\nu+1}})$. We achieve this goal for $1/2 \le \nu \le 1$. Also for $0 < \nu < 1$, the result of the forthcoming section is an improvement over the result (2.10) of Nair [34].

2.2. GENERALIZED ARCANGELI'S METHOD FOR TIKHONOV REGULARIZATION.

Here onwards we assume that $y \in R(T)$, so that least square solution of (2.1) is its solution and the generalized solution of (2.1) is the solution of minimal norm, i.e., the unique element $\hat{x} \in N(T)^{\perp}$ such that $T\hat{x} = y$. In order to choose the regularization parameter α in (2.2) we consider the generalized Arcangeli's method (2.7). Now Theorem 2.1.1 (ii) shows that, estimates for $\alpha(\delta)$ and $\frac{\delta}{\sqrt{\alpha(\delta)}}$ in terms of powers of δ, will lead to the estimates for the error $\|\hat{x} - x_{\alpha}^{\delta}\|$. Thus, in view of Proposition 2.1.4, the aim is to obtain estimate for $\frac{\delta}{\sqrt{\alpha}}$. Before that we show the convergence of the method.

Theorem 2.2.1. If $\alpha = \alpha(\delta)$ is chosen according to (2.7) with $p < \frac{4q(q+1)}{2q+1}$, then

$$x_{\alpha}^{\delta} \to \hat{x} \quad \text{as} \quad \delta \to 0.$$

Proof: In view of Theorem 2.1.1 (a), (2.4) and (2.5) it is enough to prove that $\frac{\delta}{\sqrt{\alpha}} \to 0$ as $\delta \to 0$. But

$$\|Tx_{\alpha}^{\delta} - y^{\delta}\| = \|T(T^*T + \alpha I)^{-1}T^*y^{\delta} - y^{\delta}\|$$

$$= \|\alpha(TT^* + \alpha I)^{-1}y^{\delta}\|$$

$$\leq \|\alpha(TT^* + \alpha I)^{-1}(y^\delta - y)\| + \|\alpha(TT^* + \alpha I)^{-1}y\|$$

(2.12) $$\leq \delta + \|\alpha(TT^* + \alpha I)^{-1}T\hat{x}\|$$

where $\|\alpha(TT^* + \alpha I)^{-1}(y^\delta - y)\| \leq \delta$ and $y = T\hat{x}$. Thus we have

(2.13) $$\|Tx_\alpha^\delta - y^\delta\| \leq \delta + \|\alpha T(T^*T + \alpha I)^{-1}\hat{x}\|.$$

Let $T = U(T^*T)^{1/2}$ be the polar decomposition of T where U is the unitary operators on X. Then we have

$$\|\alpha T(T^*T + \alpha I)^{-1}\hat{x}\| \leq \|\alpha U(T^*T)^{1/2}(T^*T + \alpha I)^{-1}\hat{x}\|$$

$$\leq \sup_{\lambda > 0} \frac{\alpha\lambda^{1/2}}{\lambda+\alpha} \|\hat{x}\|$$

$$\leq \alpha^{1/2} \sup_{\lambda > 0} \frac{(\lambda/\alpha)^{1/2}}{1+\lambda/\alpha} \|\hat{x}\|$$

$$\leq \alpha^{1/2}\|\hat{x}\|.$$

Therefore by (2.13) and Proposition 2.1.4 we have

$$\frac{\delta^p}{\alpha^q} = \|Tx_\alpha^\delta - y^\delta\|$$

$$\leq \delta + c\delta^{\frac{p}{2(q+1)}}.$$

Thus

$$\frac{\delta}{\sqrt{\alpha}} = \delta^{1-\frac{p}{2q}}\left(\frac{\delta^p}{\alpha^q}\right)^{\frac{1}{2q}}$$

$$\le \delta^{1-\frac{p}{2q}}\left(\delta + c\delta^{\frac{p}{2(q+1)}}\right)^{\frac{1}{2q}}$$

$$\le O\left(\delta^{1-\frac{p}{2q}+\frac{p}{4q(q+1)}}\right)$$

$$\le O\left(\delta^{1-\frac{p(2q+1)}{4q(q+1)}}\right).$$

Now by the assumption on (p, q) we have $\frac{\delta}{\sqrt{\alpha}} \to 0$ as $\delta \to 0$. This completes the proof of the Theorem. ■

In order to obtain the main result (Theorem 2.2.4) of this Section, we require the following two Lemmas.

Lemma 2.2.2. If $\hat{x} \in R((T^*T)^\nu)$, $0 < \nu \le 1$, then

$$\|\alpha(TT^* + \alpha I)^{-1}T\hat{x}\| = O(\alpha^\omega) \quad \text{with} \quad \omega = \min\{1, \nu+1/2\}.$$

proof: Let $T = U(T^*T)^{1/2}$ be the polar decomposition of T where U is the unitary operator, and let $u \in X$ be such that $\hat{x} = (T^*T)^\nu u$. Then we have

$$\|\alpha(TT^* + \alpha I)^{-1}T\hat{x}\| = \|\alpha T(T^*T + \alpha I)^{-1}\hat{x}\|$$

$$= \|\alpha U(T^*T)^{1/2}(T^*T + \alpha I)^{-1}(T^*T)^{\nu}u\|$$

$$\le \sup_{\lambda > 0} \frac{\alpha\lambda^{\nu+1/2}}{\lambda + \alpha} \|u\|$$

$$\le \alpha^{\nu+1/2}\|u\| \sup_{\lambda > 0} \frac{(\lambda/\alpha)^{\nu+1/2}}{1 + \lambda/\alpha}$$

Now the result follows using the fact that $\hat{x} \in R((T^*T)^{1/2})$ whenever $\nu \ge 1/2$ and $(\lambda/\alpha)^{\nu+1/2} \le 1 + \lambda/\alpha$. ■

Lemma 2.2.3. Let $\hat{x} \in R((T^*T)^{\nu})$, $0 < \nu \le 1$. Suppose that $\omega = \min\{1, \nu+1/2\}$, $\frac{p}{q+1} \le \min\{1/\omega, \frac{2}{1+(1-\omega)/q}\}$ and $\alpha := \alpha(\delta)$ is chosen according to (2.7). Then

$$\frac{\delta}{\sqrt{\alpha}} = O(\delta^{\mu}) \text{ with } \mu = 1 - \frac{p}{2(q+1)}(1+(1-\omega)/q).$$

proof: From (2.12), by using Lemma 2.2.2 and Proposition 2.1.4 we have,

$$\frac{\delta^p}{\alpha^q} = \|Tx_\alpha^\delta - y^\delta\| \le \delta + c\delta^{\frac{p\omega}{q+1}}.$$

Hence

$$\frac{\delta}{\sqrt{\alpha}} = \delta^{1 - p/2q}\left(\frac{\delta^p}{\alpha^q}\right)^{1/2q}$$

$$\le \left(\delta^{2q - p + 1} + c\delta^{2q-p+\frac{p\omega}{q+1}}\right)^{1/2q}.$$

From this the lemma follows. ■

Theorem 2.2.4. Let $\hat{x} \in R((T^*T)^\nu)$, $0 < \nu \le 1$, and p, q, μ are as in Lemma 2.2.2 and $\alpha := \alpha(\delta)$ be chosen according to (2.7). Then

(i) $\|\hat{x} - x_\alpha^\delta\| = O(\delta^r)$ with $r = \min\{\mu, \frac{p\nu}{q+1}\}$.

In particular, if $\frac{p}{q+1} = \frac{2}{2\nu+1+(1-\omega)/q}$, then

(ii) $\|\hat{x} - x_\alpha^\delta\| = O(\delta^t)$ with $t = \frac{2\nu}{2\nu+1+(1-\omega)/q}$ $= \frac{p\nu}{q+1}$ ← better in agreement with (iii) 4.2.2

where $\omega = \min\{1, \nu+1/2\}$.

Proof: The proof of the first part is a consequence of Theorem 2.1.1 (ii), using Lemma 2.2.3 and Proposition 2.1.4. The second part follows by noting that $\mu = \frac{p\nu}{q+1}$ if and only if $\frac{p}{q+1} = \frac{2}{2\nu+1+(1-\omega)/q}$.

Corollary 2.2.5. Let p, q be positive reals satisfying $\frac{p}{q+1} \le 1$ and let $\hat{x} \in R((T^*T)^\nu)$, $0 < \nu \le 1$, $\omega = \min\{1, \nu+1/2\}$, $l = \min\{1/\omega, \frac{2}{2\nu+1+(1-\omega)/q}\}$. If $\alpha := \alpha(\delta)$ is chosen according to (2.7), then

$$\|\hat{x} - x_\alpha^\delta\| \le c\delta^\tau$$

where $\tau = \begin{cases} \frac{p\nu}{q+1}, & \frac{p}{q+1} \le l \\ 1-\frac{p}{2(q+1)}, & \frac{p}{q+1} \ge l \end{cases}$

Proof: With μ as defined in Lemma 2.2.3, we note that $\frac{p\nu}{q+1} = \mu$ if and only if $\frac{p}{q+1} = \frac{2}{2\nu+1+(1-\omega)/q}$.

Also

$$0 < \nu \le 1/2 \quad \text{implies} \quad \frac{2}{2\nu+1+(1-\omega)/q} \ge 1$$

and

$$1/2 < \nu \le 1 \quad \text{implies} \quad \frac{2}{2\nu+1+(1-\omega)/q} \le 1.$$

Now the result follows from Theorem 2.2.4 (i). ∎

Remark 2.2.6. We observe that if $0 < \nu < 1$, then Theorem 2.2.4 improves upon the result (2.10) of Nair [34], and the optimal rate $O(\delta^{\frac{2\nu}{2\nu+1}})$ is attained for $1/2 \le \nu \le 1$ by choosing $\frac{p}{q+1} = \frac{2}{2\nu+1}$. This result agrees with the result of Nair [34] for $\nu = 1$ and $\frac{p}{q+1} = \frac{2}{3}$ giving the best rate $O(\delta^{2/3})$. For a general ν, i.e; $0 < \nu \le 1$, and $\frac{p}{q+1} = \frac{2}{3}$ with $q \ge 1/2$, Theorem 2.2.4 gives the rate $O(\delta^{2\nu/3})$ which agrees with the result in Groetsch and Schock [18] for $\hat{x} \in R(T^*)$, i.e., $\hat{x} \in R((T^*T)^{1/2})$. In particular, this result includes the Arcangeli's method, i.e., $p = 1, q = 1/2$, and the result of Guacaneme [20] which he proved for $p = 2$, $q = 2$ and $\nu = 1$.

2.3. ITERATED TIKHONOV REGULARIZATION

In order to obtain approximations which give better rates than the one given in (2.6), many authors (e.g., [8], [9], [10]) considered the iterated version of Tikhonov regularization, in which the approximation $x_\alpha^{\delta,j}$ is obtained by solving

$$(2.14) \qquad (T^*T + \alpha I)x_\alpha^{\delta,i} = T^*y^\delta + \alpha\, x_\alpha^{\delta,i-1}, \quad i = 1, \ldots, j$$

iteratively with $x_\alpha^{\delta,0} = 0$. This is motivated from the identity

$$(2.15) \qquad (T^*T + \alpha I)\hat{x} = T^*y + \alpha\hat{x}.$$

We note that the case for $j = 1$ is the ordinary Tikhonov regularization (2.2).

If $\hat{x} \in R((T^*T)^\nu)$, $0 < \nu \le j$, then, analogous to the results in Theorem 2.1.1, we have (See. [8], [9])

$$(2.16) \qquad \|\hat{x} - x_\alpha^{\delta,j}\| \le c_1\alpha^\nu$$

and

$$(2.17) \qquad \|\hat{x} - x_\alpha^{\delta,j}\| \le c_1\alpha^\nu + j\frac{\delta}{\sqrt{\alpha}},$$

where c_1 is a constant (independent of j). In particular if $\alpha = c\delta^{\frac{2}{2\nu+1}}$ for some constant c, then

$$(2.18) \qquad \|\hat{x} - x_\alpha^{\delta,j}\| \le c_j\delta^{\frac{2\nu}{2\nu+1}}, \quad 0 < \nu \le j.$$

for some constant $c_j > 0$.

The following is a companion result to Theorem 2.2.1.

Theorem 2.3.1. If $\alpha := \alpha(\delta)$ is chosen according to (2.7) with $p < \frac{4q(q+1)}{2q+1}$, then for each $j = 1,2,\ldots,$

$$x_\alpha^{\delta,j} \to \hat{x} \quad \text{as} \quad \delta \to 0.$$

Proof: We note that

$$\|\hat{x} - x_\alpha^{\delta,j}\| \le \|\hat{x} - x_\alpha^{0,j}\| + \|x_\alpha^{0,j} - x_\alpha^{\delta,j}\|,$$

here

$$\|x_\alpha^{0,j} - x_\alpha^{\delta,j}\| = \|\sum_{i=1}^{j} \alpha^{i-1}(T^*T + \alpha I)^{-i}T^*(y^\delta - y)\|$$

$$\le j\frac{\delta}{\sqrt{\alpha}}$$

and

$$\|\hat{x} - x_\alpha^{0,j}\| = \|\hat{x} - \sum_{i=1}^{j} \alpha^{i-1}(T^*T + \alpha I)^{-i}T^*y\|.$$

We note that

$$\hat{x} - (T^*T + \alpha I)^{-1}T^*y = \alpha(T^*T + \alpha I)^{-1}\hat{x}.$$

Therefore, by induction, it follows that

$$\|\hat{x} - x_\alpha^{0,j}\| = \| \alpha^{j}(T^*T + \alpha I)^{-j}\hat{x}\|.$$

Now the Theorem follows as in the proof of Theorem 2.1.1 (a) with $R_\alpha = \alpha^j(T^*T + \alpha I)^{-j}$. ∎

Next result is a companion result to Theorem 2.2.4.

Theorem 2.3.2. For a fixed positive integer j, let $\hat{x} \in R((T^*T)^\nu)$, $0 < \nu \leq j$, p, q, μ, ω are as in Theorem 2.2.4 and $\alpha := \alpha(\delta)$ be chosen according to (2.7). Then

(i) $\|\hat{x} - x_\alpha^{\delta,j}\| = O(\delta^m)$ with $m = \min\{\mu, \frac{p\nu}{q+1}\}$.

In particular if $\frac{p}{q+1} = \frac{2}{2\nu+1+(1-\omega)/q}$, then

(ii) $\|\hat{x} - x_\alpha^{\delta,j}\| = O(\delta^s)$ with $s = \frac{2\nu}{2\nu+1+(1-\omega)/q}$.

(iii) If $\hat{x} \in R((T^*T)^\nu)$, $1/2 \leq \nu \leq j$, then $\frac{p}{q+1} = \frac{2}{2\nu+1}$ and

$$\|\hat{x} - x_\alpha^{\delta,j}\| = O(\delta^{\frac{2\nu}{2\nu+1}}).$$

Proof: In view of (2.17), the proof of (i) and (ii) follows as in Theorem 2.2.4. Proof of (iii) is a consequence of (ii) by noting that $\omega = 1$ for $\nu \geq 1/2$. ∎

Remark 2.3.3. We note that if $\hat{x} \in R((T^*T)^j)$ and $\frac{p}{q+1} = \frac{2}{2j+1}$ then, by Theorem 2.3.1 (iii), we have

$$\|\hat{x} - x_\alpha^{\delta,j}\| = O(\delta^{\frac{2j}{2j+1}}).$$

In [5], Engl considered the 'discrepancy principle'

(2.19) $$\|T^*Tx_\alpha^{\delta,j} - T^*y^\delta\|^2 = \frac{\delta^p}{\alpha^q}, \quad p > 0,\ q > 0.$$

for choosing the parameter α in (2.15) and obtained error bounds under certain conditions on p and q in terms of j. Later Engl and Neubauer [7] improved the results in [5] and showed that if $\hat{x} \in R((T^*T)^j)$ and $\frac{p}{2}(1+2j) - 2j = q \geq 2j^3 - 3j - 1$, then

$$\|\hat{x} - x_\alpha^{\delta,j}\| = O(\delta^{\frac{2j}{2j+1}}).$$

Analogous to (2.19) if we consider the discrepancy principle

(2.20) $$\|Tx_\alpha^{\delta,j} - y^\delta\| = \frac{\delta^p}{\alpha^q}, \quad p > 0,\ q > 0,$$

for choosing α in (2.15), then following the arguments in Section 2.2, we obtain

$$\|\hat{x} - x_\alpha^{\delta,j}\| = O(\delta^{\frac{2\nu}{2\nu+1}})$$

for $\hat{x} \in R((T^*T)^\nu)$ with $j-1/2 \leq \nu \leq j$, and $\frac{p}{q+j} = \frac{2}{2\nu+1}$.

Note that while using the discrepancy principle (2.19) (resp. (2.20)) for choosing the regularization parameter $\alpha := \alpha(\delta,j)$, one has to solve the linear equation (2.14) and the nonlinear equations (2.19) (resp. (2.20)), j times. But if one considers the discrepancy principle (2.7), then, one need to solve the linear equation (2.2) and the nonlinear equation (2.7) only once.

Comparison of the above results with Theorem 2.3.2, specifically the condition on ν in terms of j, shows the advantage of the discrepancy principle (2.7) over (2.19) or (2.20).

CHAPTER 3

PARAMETER CHOICE STRATEGIES FOR SIMPLIFIED REGULARIZATION

In this Chapter we are concerned with a special case of the operator equation (2.1) in which the operator T is a positive self-adjoint operator, and simplified regularization is used instead of the Tikhonov regularization.

For the purpose of relating the procedure of this Chapter with that of Chapter 2, we use different notations for the operator and data. In Section 3.1 we consider a class of discrepancy principles for determining the regularization parameter, in the line of the one considered in Section 2.2. This procedure generalizes the method adopted by Groetsch and Guacaneme [16] and Guacaneme [19]. A modified form of the discrepancy principle of Guacaneme [21] has been considered in Section 3.2, which facilitates handling of lesser smooth data. Iterated versions of both the above procedures have been considered in Section 3.3 and obtained results analogous to that of Sections 3.1 and 3.2.

3.1. GENERALIZED ARCANGELIES METHOD FOR SIMPLIFIED REGULARIZATION.

Let $A \in BL(X)$ be a positive self-adjoint operator and $g \in R(A)$. For regularization of the equation

(3.1) $$Aw = g$$

with an inexact data g^δ, we consider the simplified regularization procedure, namely,

(3.2) $$(A + \alpha I)w_\alpha^\delta = g^\delta, \quad \alpha > 0.$$

With $\|g-g^\delta\| \le \delta$ and A compact, Bakushinski [2] studied the above procedure, and showed that a sufficient condition for convergence of w_α^δ to $\hat{w}$, the minimal norm solution of (3.1), is $\delta = o(\alpha)$ (See also Ivanov [23], Khudok [24]). In [40] Schock considered the simplified regularization of (3.1) with positive self-adjoint operator (not necessarily compact) and proved that $w_\alpha := w_\alpha^0 \to \hat{w}$ as $\alpha \to 0$ and (w_α) has better convergence properties than the approximation obtained by Tikhonov regularization. It is also known (Schock [39]) that if $\hat{w} \in R(A^\nu)$, $0 < \nu \le 1$, then

(3.3) $$\|\hat{w} - w_\alpha\| = O(\alpha^\nu),$$

and if $\alpha(\delta) = c\delta^{1/(\nu+1)}$, then

(3.4) $$\|\hat{w} - w_\alpha^\delta\| = O(\delta^{\nu/(\nu+1)}).$$

This rate is optimal in the sense that $\|\hat{w} - w_\alpha^\delta\| = o(\delta^{\nu/(\nu+1)})$ implies $R(A)$ is finite dimensional (See [39]). For choosing the regularization parameter α in (3.2), Groetsch and Guacaneme [16] considered the Arcangeli's method, namely,

$$\|Aw_\alpha^\delta - g^\delta\| = \frac{\delta}{\sqrt{\alpha}}, \tag{3.5}$$

and proved that if $\alpha := \alpha(\delta)$ is chosen according to (3.5) and A is in addition, a compact operator, then $w_\alpha^\delta \to \hat{w}$ as $\delta \to 0$. But no attempt has been made for obtaining estimate for the error $\|\hat{w} - w_\alpha^\delta\|$. In this section we prove the convergence and also obtain error estimate under a general class of discrepancy principles,

$$\|Aw_\alpha^\delta - g^\delta\| = \frac{\delta^p}{\alpha^q}, \quad p > 0, \quad q > 0, \tag{3.6}$$

which is valid for $0 < p < q+1$. We do not require A to be compact. Also note that (3.6) includes (3.5) by taking $p = 1$, $q = 1/2$.

The proof of the following Lemma is analogous to the proof of Proposition 2.1.4.

Lemma 3.1.1. For each $\delta > 0$, there exist a unique $\alpha := \alpha(\delta)$ satisfying (3.6). Further $\alpha(\delta) \to 0$ as $\delta \to 0$. ■

Here onwards we assume that g^δ satisfies

$$\|g-g^\delta\| \le \delta \le \frac{\|g\|}{2}. \tag{3.7}$$

Theorem 3.1.2. If $\alpha := \alpha(\delta)$ is chosen according to (3.6), with g^δ satisfying (3.7), then

(i) $\alpha(\delta) = \mathcal{O}(\delta^{\frac{p}{q+1}})$.

If in addition, $p < q+1$, then

(ii) $\dfrac{\delta}{\alpha(\delta)} = \mathcal{O}(\delta^m)$, $\quad m = \dfrac{q+1-p}{q+1}$,

and

(iii) $w_\alpha^\delta \to \hat{w}$ as $\delta \to 0$.

Proof: First we note that

$$\|g^\delta\| - \frac{\delta^p}{\alpha^q} = \|g^\delta\| - \|Aw_\alpha^\delta - g^\delta\|$$

$$\leq \|Aw_\alpha^\delta\|$$

$$= \|A(Aw_\alpha^\delta - g^\delta)\|/\alpha$$

$$\leq \|A\| \frac{\delta^p}{\alpha^{q+1}}.$$

Therefore,

$$\alpha^{q+1} \leq \frac{\delta^p(\|A\| + \alpha)}{\|g^\delta\|}$$

$$\leq \frac{2\delta^p(\|A\| + \alpha)}{\|g\|}$$

so that

(3.8) $$\alpha(\delta) = O(\delta^{\frac{p}{q+1}}).$$

Note that

(3.9) $$\frac{\delta^p}{\alpha^q} = \|Aw_\alpha^\delta - g^\delta\|$$

$$= \|\alpha w_\alpha^\delta\|$$

$$\le \alpha(\|w_\alpha^\delta - w_\alpha\| + \|w_\alpha\|).$$

But

$$w_\alpha^\delta - w_\alpha = (A + \alpha I)^{-1}(g^\delta - g)$$

so that

(3.10) $$\|w_\alpha^\delta - w_\alpha\| \le \frac{\delta}{\alpha}.$$

Also

$$\|w_\alpha\| = \|(A + \alpha I)^{-1}A\hat{w}\| \le \|\hat{w}\|.$$

Therefore we obtain

$$\frac{\delta^p}{\alpha^q} \le \alpha(\frac{\delta}{\alpha} + \|\hat{w}\|)$$

$$= \delta + \alpha\|\hat{w}\|.$$

Now using the estimate (3.8), we get

$$\frac{\delta}{\alpha} = \delta^{1-\frac{p}{q}}\left(\frac{\delta^p}{\alpha^q}\right)^{\frac{1}{q}}$$

$$\le \delta^{1-\frac{p}{q}} (\delta + \alpha\|\hat{w}\|)^{\frac{1}{q}}$$

$$\le \delta^{1-\frac{p}{q}} (\delta + c\alpha^{\frac{p}{q+1}})^{\frac{1}{q}}$$

for some constant $c > 0$ and since $p < q+1$, we have

$$\frac{\delta}{\alpha} = O(\delta^m) \tag{3.11}$$

where $m = \frac{q+1-p}{q+1}$. To prove the convergence we first note that

$$\|w_\alpha^\delta - \hat{w}\| \le \|w_\alpha^\delta - w_\alpha\| + \|w_\alpha - \hat{w}\|.$$

Now, since

$$\|w_\alpha^\delta - w_\alpha\| \le \frac{\delta}{\alpha}$$

and

$$\|w_\alpha - \hat{w}\| = \|\alpha(A + \alpha I)^{-1}\hat{w}\|,$$

the result follows as in the proof of Theorem 2.1.1 (a) with $R_\alpha = \alpha(A + \alpha I)^{-1}$. ■

Theorem 3.1.3. Let $\hat{w} \in R(A^\nu)$, $0 < \nu \le 1$, $q > 0$, $p < q+1$ and $\alpha = \alpha(\delta)$ be chosen according to (3.6) with g^δ satisfying (3.7). Then

(i) $$\|\hat{w} - w_\alpha^\delta\| = O(\delta^s),$$

where $s = \min\{\frac{p\nu}{q+1}, 1 - \frac{p}{q+1}\}$.

In particular if $\frac{p}{q+1} = \frac{1}{\nu+1}$, then

(ii) $$\|\hat{w} - w_\alpha^\delta\| = O(\delta^{\nu/(\nu+1)}).$$

Proof: From (3.3) and (3.10) we have

$$\|\hat{w} - w_\alpha^\delta\| \le \|\hat{w} - w_\alpha\| + \|w_\alpha - w_\alpha^\delta\|$$

$$= O(\alpha^\nu) + O(\delta/\alpha),$$

so that the result in (i) follows from Theorem 3.1.2. If $\frac{p}{q+1} = \frac{1}{\nu+1}$ then $\frac{p\nu}{q+1} = \frac{q+1-p}{q+1}$ so that

$$O(\alpha^\nu) = O(\delta/\alpha) = O(\delta^{\nu/(\nu+1)})$$

proving (ii). ∎

Corollary 3.1.4. If $\hat{w} \in R(A^\nu)$, $0 < \nu \le 1$ and $\alpha = \alpha(\delta)$ is chosen according to (3.5) with g^δ satisfying (3.7), then

$$\|\hat{w} - w_\alpha^\delta\| = O(\delta^k)$$

where $k = \min\{2\nu/3, 1/3\}$. ∎

Remark 3.1.5. If the smoothness of the solution $\hat{w}$ is known, namely, $\hat{w} \in R(A^\nu)$, $0 < \nu \le 1$, then by taking $\frac{p}{q+1} = \frac{1}{\nu+1}$ our result provides the optimal rate $\mathcal{O}(\delta^{\nu/(\nu+1)})$. As a particular case, the discrepancy principle (3.5) gives the rate $\mathcal{O}(\delta^{1/3})$ for $\nu = \frac{1}{2}$, and the best rate $\mathcal{O}(\delta^{1/2})$ is achieved when $\nu = 1$ by taking $\frac{p}{q+1} = \frac{1}{2}$. The result for the case $\nu = 1$ has also been obtained by Guacaneme [19]. In fact, the proof of the main result of Guacaneme ([19], Theorem 2.3) is not complete as he used the estimate $\mathcal{O}(\delta^{1-p/(q+1)})$ for $\frac{\delta}{\alpha}$ which is not immediate from the estimate $\alpha = \mathcal{O}(\delta^{p/(q+1)})$ ([19], Lemma 2.1).

In the case of the general ill-posed problem (2.1) if $A = TT^*$, $g = y$, $g^\delta = y^\delta$ and $x_\alpha^\delta = T^*w_\alpha^\delta$, then x_α^δ is the Tikhonov regularized solution of (2.1) and the discrepancy principle (3.6) is the same as the one considered in Chapter 2, namely,

$$\|Tx_\alpha^\delta - y^\delta\| = \frac{\delta^p}{\alpha^q}, \quad p > 0, \; q \geq 0.$$

But the estimate in Theorem 3.1.3 does not help directly to deduce the estimate in Theorem 2.2.4 (ii). If we use a different definition of the noise level, namely, $\|y-y^\delta\| \leq \delta/c$ with $\|T^*\| \leq c$, then the discrepancy principle

$$\|T^*Tx_\alpha^\delta - T^*y^\delta\| = \frac{\delta^p}{\alpha^q},$$

considered by Engl [4] and Engl and Neubauer [5] is of the form (3.6) with $A = T^*T$, $g = T^*y$, $g^\delta = T^*y^\delta$ and $w_\alpha^\delta = x_\alpha^\delta$. The

estimate $\alpha = O(\delta^{\frac{p}{q+1}})$ of Theorem 3.1.2 can be used to obtain the optimal estimate of [4] and [7] as follows:
we observed that

$$w_\alpha - w_\alpha^\delta = (A + \alpha I)^{-1}(g - g^\delta)$$

$$= (T^*T + \alpha I)^{-1}T^*(y - y^\delta)$$

so that

$$\|w_\alpha - w_\alpha^\delta\| = O\left(\frac{\delta}{\sqrt{\alpha}}\right),$$

and hence from (3.9),

$$\frac{\delta^p}{\alpha^q} = O(\delta/\alpha + \alpha)$$

$$= O(\delta^{\frac{p}{q+1}}).$$

Therefore if $p < 2(q+1)$ and $0 < \nu \le 1$, then we have

$$\|\hat{w} - w_\alpha^\delta\| = O(\delta^h)$$

where $h = \min\{\frac{p\nu}{q+1}, 1 - \frac{p}{2(q+1)}\}$.

So that the optimal estimate $O(\delta^{2\nu/(2\nu+1)})$ is achieved for $\frac{p}{(q+1)} = \frac{2}{2\nu+1}$.

In general the simplified regularization is recommended when the operator T under consideration is positive self-adjoint operator, because in this case the method in Chapter 2 involves more computation, as for such operators we have $TT^* = T^2 = T^*T$.

3.2 A MODIFIED FORM OF GUACANEME'S METHOD

In this section we consider a parameter choice strategy, which is a modification of the one considered by Guacaneme [21], for simplified regularization of the operator equation $Aw = g$. The result of this section includes a result of Guacaneme [21], which he proved for compact positive self-adjoint operator A under the assumption that the minimal-norm solution $\hat{w}$ belongs to $R(A)$.

Let A, g, and g^δ are as in Section 3.1 and let Q be the orthogonal projection onto $\overline{R(A)}$, the closure of the range of A. For a fixed positive real number $\rho > 0$, consider the function φ defined by

$$\varphi(\alpha) = \alpha^{2(\rho+1)}\langle (A+\alpha I)^{-2(\rho+1)}Qg^\delta, Qg^\delta\rangle, \quad \alpha > 0. \tag{3.12}$$

We choose the regularization parameter $\alpha := \alpha(\delta)$ in (3.2), according to the discrepancy principle

$$\varphi(\alpha) = c\delta^2, \tag{3.13}$$

for a constant $c > 1$. In fact, for compact A Guacaneme [21], considered the discrepancy principle

$$\alpha^4\langle (A+\alpha I)^{-4}Qg^\delta, Qg^\delta\rangle = c\delta^2, \quad c > 1$$

and obtained the error estimate

$$\|\hat{w} - w_\alpha^\delta\| = O(\delta^{1/2}), \tag{3.14}$$

under the assumption $\hat{w} \in R(A)$.

Lemma 3.2.1. The function φ in (3.12) is continuous, strictly increasing,

$$\lim_{\alpha\to 0}\varphi(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha\to\infty}\varphi(\alpha) = \|Qg^\delta\|^2$$

Proof: Let $\{E_\lambda\}$ be the spectral family of the operator A. Then we have

$$\varphi(\alpha) = \int\left(\frac{\alpha}{\lambda+\alpha}\right)^{2(\rho+1)}d\langle E_\lambda Qg^\delta, Qg^\delta\rangle.$$

Now the map

$$\alpha \longmapsto f_\rho(\alpha,\lambda) := \left(\frac{\alpha}{\lambda+\alpha}\right)^{2(\rho+1)}$$

is strictly increasing for each $\lambda > 0$, and satisfies

$$f_\rho(\alpha,\lambda) \to 0 \quad \text{as} \quad \alpha \to 0$$

and

$$f_\rho(\alpha,\lambda) \to 1 \quad \text{as} \quad \alpha \to \infty.$$

Therefore the result follows using the Dominated Convergence Theorem. ■

Lemma 3.2.2. If g^δ satisfies

$$\|g-g^\delta\| \le \delta < \frac{\|Qg^\delta\|}{c^{1/2}}, \tag{3.15}$$

then the equation (3.13) has a unique solution $\alpha := \alpha(\delta)$ such that $\alpha(\delta) \to 0$ as $\delta \to 0$.

Proof: Using Lemma 3.2.1 and the Intermediate Value Theorem, the equation (3.13) has a unique solution $\alpha := \alpha(\delta)$. Now using the arguments as in Proposition 2.1.4, it follows that $\alpha(\delta) \to 0$ as $\delta \to 0$. ■

Lemma 3.2.3. Suppose that $g \neq 0$, g^δ satisfies (3.15), $c_1 = (c^{1/2}-1)^2$, $c_2 = (c^{1/2}+1)^2$ and $\alpha := \alpha(\delta)$ is chosen according to (3.13). Then

$$c_1\delta^2 \le \alpha^{2(\rho+1)}((A+\alpha I)^{-2(\rho+1)}g,g) \le c_2\delta^2.$$

Proof: For $\alpha > 0$, $\rho > 0$, let $B_\alpha = \alpha^{\rho+1}(A+\alpha I)^{-(\rho+1)}$. Then $\|B_\alpha\| \le 1$ and for each nonzero $g \in X$, we have

$$\|B_\alpha Q g\|^2 = \alpha^{2(\rho+1)}\langle (A+\alpha I)^{-2(\rho+1)}g, g\rangle .$$

Therefore,

$$\|B_\alpha Q g\| \ge \|B_\alpha Q g^\delta\| - \|B_\alpha Q(g-g^\delta)\|$$

$$\ge c^{1/2}\delta - \delta$$

and

$$\|B_\alpha Q g\| \le \|B_\alpha Q g^\delta\| + \|B_\alpha Q(g-g^\delta)\|$$

$$\le c^{1/2}\delta + \delta.$$

'his completes the proof. ∎

Theorem 3.2.4. Let $g \ne 0$, g^δ satisfies (3.15) and $\alpha := \alpha(\delta)$ is chosen according to (3.13). Then

$$w_\alpha^\delta \to \hat{w} \quad \text{as} \quad \delta \to 0.$$

Proof: From (3.2) and the fact that $g = A\hat{w}$, it follows that

$$\|\hat{w} - w_\alpha\| = \|\alpha(A + \alpha I)^{-1}\hat{w}\|$$

$$= \|R_\alpha \hat{w}\|$$

and

$$\|w_\alpha - w_\alpha^\delta\| \le \frac{\delta}{\alpha}$$

where $R_\alpha = \alpha(A + \alpha I)^{-1}$, $\alpha > 0$. Therefore it is enough to prove

(i) $R_{\alpha(\delta)}\hat{w} \to 0$ as $\delta \to 0$

and

(ii) $\frac{\delta}{\alpha} \to 0$ as $\delta \to 0$.

Now using Lemma 3.2.2 and arguments as in the proof of Theorem 2.1.1 (a) it follows that $R_{\alpha(\delta)}\hat{w} \to 0$ as $\delta \to 0$. To prove (ii) let

$$C_\alpha = \alpha^\rho(A + \alpha I)^{-(\rho+1)}A, \quad \alpha > 0.$$

Then for all $u \in R(A^\rho)$, with $u = A^\rho v$ for some $v \in X$,

$$\|C_\alpha u\| = \|C_\alpha A^\rho v\|$$

$$= \|\alpha^\rho(A + \alpha I)^{-(\rho+1)}A^{\rho+1}v\|$$

$$\le \alpha^\rho\|v\|,$$

for some $v \in X$. Since $\|C_\alpha\| \le 1$ for all $\alpha > 0$ and $R(A^\rho)$ is dense in $N(A)^\perp$, it follows that $C_{\alpha(\delta)}\hat{w} \to 0$ as $\delta \to 0$. Now by Lemma 3.2.3,

$$c_1\delta^2 \le \alpha^{2(\rho+1)}\langle(A + \alpha I)^{-2(\rho+1)}g, g\rangle$$

$$= \alpha^{2(\rho+1)} \langle (A + \alpha I)^{-2(\rho+1)} A^2 \hat{w}, \hat{w} \rangle$$

$$= \alpha^2 \| C_\alpha \hat{w} \|^2.$$

So that

$$\frac{\delta^2}{\alpha^2} \le \frac{1}{c_1} \| C_\alpha \hat{w} \|^2 \to 0 \quad \text{as} \quad \delta \to 0. \quad \blacksquare$$

Lemma 3.2.5. Let $g \neq 0$, g^δ satisfies (3.15) and $\alpha := \alpha(\delta)$ be chosen according to (3.13). Then we have the following

(i) $\alpha = O(\delta^{\frac{1}{\rho+1}})$

(ii) $\frac{\delta}{\alpha} = O(\delta^{\frac{\nu}{\rho+1}})$ if $\hat{w} \in R(A^\nu)$, $0 < \nu \le 1$ and $\nu \le \rho$

(iii) $\frac{\delta}{\alpha} = o(\delta^{\frac{\nu}{\rho+1}})$ if $\hat{w} \in R(A^\nu)$, $0 < \nu \le 1$ and $\nu < \rho$.

Proof: By Lemma 3.2.2 and 3.2.3 for all sufficiently small $\delta > 0$, we have

$$c_2 \delta^2 \ge \alpha^{2(\rho+1)} \| (A + \alpha I)^{-2(\rho+1)} g \|^2$$

$$\ge \frac{\alpha^{2(\rho+1)} \| g \|^2}{\| (A + \alpha I)^{\rho+1} \|^2}$$

$$\ge c\alpha^{2(\rho+1)}$$

for some constant $c > 0$. Thus $\alpha = O(\delta^{\frac{1}{\rho+1}})$ proving (i). If $\hat{w} \in R(A^{\nu})$, $0 < \nu \leq 1$, then, $\hat{w} = A^{\nu}x$ for some $x \in X$, so that $g = A\hat{w} = A^{\nu+1}x$. Therefore by using Lemma 3.2.3 we have,

$$c_1\delta^2 \leq \alpha^{2(\rho+1)}\langle (A + \alpha I)^{-2(\rho+1)}A^{2\nu+2}x,\ x\rangle$$

$$= \alpha^{2(\rho+1)}\|(A + \alpha I)^{-(\rho+1)}A^{\nu+1}x\|^2$$

$$\leq \|x\|^2 \sup_{\lambda > 0} \frac{\alpha^{2(\rho+1)}\lambda^{2\nu+2}}{(\lambda+\alpha)^{2(\rho+1)}}$$

$$\leq \|x\|^2\ \alpha^{2\nu+2} \sup_{\lambda > 0} \frac{(\lambda/\alpha)^{2\nu+2}}{(1+\lambda/\alpha)^{2(\rho+1)}}$$

$$\leq \alpha^{2\nu+2}\|x\|^2,$$

for $\nu \leq \rho$. The last inequality is a consequence of the relation $(\frac{\lambda}{\alpha})^{2\nu+2} \leq (1+\frac{\lambda}{\alpha})^{2\rho+1}$, for $\lambda > 0$, $\alpha > 0$ and $\nu \leq \rho$. Thus $\delta = O(\alpha^{\nu+1})$ and hence

$$\frac{\delta}{\alpha} = \delta^{\frac{\nu}{\nu+1}} \left(\frac{\delta}{\alpha^{\nu+1}}\right)^{\frac{1}{\nu+1}}$$

$$= O(\delta^{\frac{\nu}{\nu+1}}),$$

proving (ii). Now by (ii), for $\nu < \rho$, we have

$$\frac{\delta}{\alpha}\ \delta^{-\frac{\nu}{\rho+1}} = O(\delta^{\frac{(\rho-\nu)\nu}{(\nu+1)(\rho+1)}}) \to 0 \quad \text{as} \quad \delta \to 0.$$

From this we obtain (iii). ∎

Theorem 3.2.6. Let g^{δ} satisfies (3.15), $\alpha := \alpha(\delta)$ be chosen according to (3.13) $\hat{w} \in R(A^{\nu})$, $0 < \nu \leq 1$. Then

$$\text{(i)} \quad \|\hat{w} - w_{\alpha}^{\delta}\| = \begin{cases} O(\delta^{\frac{\nu}{\rho+1}}), & \nu \leq \rho \\ O(\delta^{\frac{\rho}{\rho+1}}), & \nu \geq \rho \end{cases}$$

If $0 < \nu < 1$ and $\nu < \rho$, then

$$\text{(ii)} \quad \|\hat{w} - w_{\alpha}^{\delta}\| = o(\delta^{\frac{\nu}{\nu+1}}).$$

In particular taking $\rho = 1$ in (3.12) we have

$$\text{(iii)} \quad \|\hat{w} - w_{\alpha}^{\delta}\| = \begin{cases} o(\delta^{\nu/2}), & 0 < \nu < 1 \\ O(\delta^{1/2}), & \nu = 1. \end{cases}$$

Proof: Let $\hat{w} \in R(A^{\nu})$, $0 < \nu \leq 1$. Then it is easy to see ([39]) that

$$\|\hat{w} - w_{\alpha}\| = \begin{cases} o(\alpha^{\nu}), & 0 < \nu < 1 \\ O(\alpha), & \nu = 1 \end{cases}$$

Also we observe that

$$\|w_{\alpha} - w_{\alpha}^{\delta}\| = \|(A + \alpha I)^{-1}(g-g^{\delta})\|$$

$$\leq \frac{\delta}{\alpha}.$$

Therefore

$$\|\hat{w} - w_\alpha^\delta\| = \begin{cases} o(\alpha^\nu) + \mathcal{O}(\frac{\delta}{\alpha}), & 0 < \nu < 1 \\ \mathcal{O}(\alpha) + \mathcal{O}(\frac{\delta}{\alpha}), & \nu = 1 \end{cases}$$

If $\nu \leq \rho$, then the result follows from Lemma 3.2.5 ((i), (ii)). If $\nu \geq \rho$, then $\hat{w} \in R(A^\rho)$, so that the result in this case is obtained by replacing ν by ρ in Lemma 3.2.5 ((i), (ii)). ■

Remark 3.2.7. (1) The result in Theorem 3.2.6 includes the result (3.14) of Guacaneme ([21], Theorem 3), which is proved when A is a compact positive self-adjoint operator and $\hat{w} \in R(A)$. Our proof does not require the compactness of A.

(2) If the smoothness of $\hat{w}$ is known å priorily, namely, $\hat{w} \in R(A^\nu)$, then Theorem 3.2.6 (i) gives the optimal rate $\mathcal{O}(\delta^{\frac{\nu}{\nu+1}})$ by taking $\rho = \nu$ in the discrepancy principle (3.13).

(3) By comparing the discrepancy principles (3.6) and (3.13), specifically the condition on δ, namely, (3.7) and (3.15), one can see that in (3.7) the upper bound of δ depends on the (unknown) exact data g, whereas in (3.15) the upper bound is in terms of (known) inexact data g^δ. Thus (3.13) is advantageous, over (3.6) in view of their applications.

3.3. ITERATED SIMPLIFIED REGULARIZATION

In 'iterated simplified regularization', one considers $w_\alpha^{\delta,j}$ obtained by solving the equations

(3.16) $$(A + \alpha I)w_\alpha^{\delta,i} = \alpha w_\alpha^{\delta,i-1} + g^\delta, \qquad w_\alpha^{\delta,0} = 0, \quad i = 1, \ldots, j,$$

iteratevely as approximations for the minimal norm solution $\hat{w}$ of the equation (3.1).

We consider the discrepancy principles (3.6) and (3.13) for choosing the parameter $\alpha := \alpha(\delta)$ in (3.16).

Theorem 3.3.1. If $\alpha := \alpha(\delta)$ is chosen according to (3.6), then $w_\alpha^{\delta,j} \to \hat{w}$ as $\delta \to 0$, for each $j = 1, 2, \ldots$

Proof: We observe that

$$w_\alpha^{\delta,j} = \sum_{i=1}^{j} \alpha^{i-1}(A + \alpha I)^{-i} g^\delta.$$

Let $w_\alpha^j = w_\alpha^{0,j}$, then

(3.17) $$\|w_\alpha^{\delta,j} - w_\alpha^j\| = \|\sum_{i=1}^{j} \alpha^{i-1}(A + \alpha I)^{-i}(g^\delta - g)\|$$

$$\leq j\,\frac{\delta}{\alpha}.$$

and

$$\hat{w} - w_\alpha^j = \hat{w} - \sum_{i=1}^{j} \alpha^{i-1}(A + \alpha I)^{-i} g$$

We note that

$$\hat{w} - (A + \alpha I)^{-1} g = \alpha (A + \alpha I)^{-1}\hat{w}.$$

Therefore, by induction, it follows that

$$\hat{w} - w_\alpha^j = \alpha^j (A + \alpha I)^{-j}\hat{w}. \tag{3.18}$$

where $g = A\hat{w}$. Now the result follows from the inequality

$$\|\hat{w} - w_\alpha^{\delta,j}\| \le \|\hat{w} - w_\alpha^j\| + \|w_\alpha^j - w_\alpha^{\delta,j}\|, \tag{3.19}$$

by using (3.17), (3.18) and arguments in the proof of Theorem 2.1.1 (a) with $R_\alpha = \alpha^j (A + \alpha I)^{-j}$. ■

Theorem 3.3.2. Let $\hat{w} \in R(A^\nu)$, $0 < \nu \le j$ for some fixed j, $q > 0$, $p < q + 1$ and $\alpha := \alpha(\delta)$ be chosen according to (3.6). Then

(i) $$\|\hat{w} - w_\alpha^{\delta,j}\| = O(\delta^s)$$

where $s = \min\left\{ \frac{p\nu}{q+1}, 1 - \frac{p}{q+1} \right\}$

In particular if $\frac{p}{q+1} = \frac{1}{\nu+1}$, then

(ii) $$\|\hat{w} - w_\alpha^{\delta,j}\| = O(\delta^{\frac{\nu}{\nu+1}}).$$

Proof: By (3.18), we have

$$\|\hat{w} - w_\alpha^j\| = \|\alpha^j(A + \alpha I)^{-j}\hat{w}\|$$

$$= \|\alpha^j(A + \alpha I)^{-j}A^\nu z\|$$

where $\hat{w} = A^\nu z$. Since

$$\|\alpha^j(A + \alpha I)^{-j}A^\nu z\| \le \|\alpha^j(A + \alpha I)^{-j}A^\nu\| \; \|z\|,$$

by (1.12) we have

$$(3.20) \qquad \|\alpha^j(A + \alpha I)^{-j}A^\nu z\| \le \sup_{\lambda > 0} \frac{\alpha^j \lambda^\nu}{(\lambda+\alpha)^j} \; \|z\|$$

$$= \alpha^\nu \sup_{\lambda > 0} \frac{(\lambda/\alpha)^\nu}{(1+\lambda/\alpha)^j} \|z\|$$

$$= O(\alpha^\nu).$$

The last step follows from the fact that $(\lambda/\alpha)^\nu \le (1 + \lambda/\alpha)^j$ for $0 < \nu \le j$. Thus from (3.19), (3.17) and (3.20) we have

$$\|\hat{w} - w_\alpha^{\delta,j}\| \le j\,\frac{\delta}{\alpha} + c\alpha^\nu$$

for some constant $c > 0$, independent of j. Now the result follows from (3.11) and the arguments used in the proof of Theorem 3.1.3. ∎

Theorem 3.3.3. Let $g \neq 0$, g^{δ} satisfies (3.15) and $\alpha := \alpha(\delta)$ is chosen according to (3.13). Then for each $j = 1, 2, \ldots,$

$$w_{\alpha}^{\delta,j} \to \hat{w} \quad \text{as} \quad \delta \to 0.$$

Proof: In view of (3.19), (3.17) and (3.18), the proof follows as in Theorem 3.2.4 with $R_{\alpha} = \alpha^{j}(A + \alpha I)^{-j}$. ∎

Proof of the following Theorem is analogous to the proof of Theorem 3.2.6.

Theorem 3.3.4. Let g^{δ} satisfies (3.15), $\alpha := \alpha(\delta)$ be chosen according to (3.13) and $\hat{w} \in R(A^{\nu})$, $0 < \nu \leq j$. Then

(i) $$\|\hat{w} - w_{\alpha}^{\delta,j}\| = \begin{cases} O(\delta^{\frac{\nu}{\nu+1}}), & \nu \leq \rho \\ O(\delta^{\frac{\rho}{\rho+1}}), & \nu \geq \rho \end{cases}$$

and if $0 < \nu < j$ and $\nu < \rho$, then

(ii) $$\|\hat{w} - w_{\alpha}^{\delta,j}\| = o(\delta^{\frac{\nu}{\nu+1}}).$$

In particular taking $\rho = j$ in (3.13) we have

(iii) $$\|\hat{w} - w_{\alpha}^{\delta,j}\| = \begin{cases} O(\delta^{\frac{\nu}{j+1}}), & 0 < \nu < j \\ O(\delta^{\frac{j}{j+1}}), & \nu = j \end{cases}$$ ∎

Remarks 3.3.5. (1) If we consider the discrepancy principle

$$\|Aw_\alpha^{\delta,j} - g^\delta\| = \frac{\delta^p}{\alpha^q}, \quad p > 0, \quad q > 0, \tag{3.21}$$

and $p < q+j$ for choosing the parameter α in (3.16), then by following the arguments in Section 3.1, we obtain that

$$\alpha = O(\delta^{\frac{p}{q+j}}) \quad \text{and} \quad \frac{\delta}{\alpha} = O(\delta^{1-\frac{p}{q(q+1)}(q+j-r)}),$$

where $\hat{w} \in R(A^\nu)$, $0 < \nu \le j$ and $r = \min\{\nu+1, j\}$. Thus if $j-1 \le \nu \le j$ and $\frac{p}{q+j} = \frac{1}{\nu+1}$, then

$$\|\hat{w} - w_\alpha^{\delta,j}\| = O(\delta^{\frac{\nu}{\nu+1}}).$$

Comparison of assumptions in this result with that of Theorem 3.3.2 (ii), shows the advantage of the discrepancy principle (3.6) over (3.21). More over, in order to obtain $\alpha(\delta,j)$ by (3.21), one has to solve the linear equation (3.16) and the nonlinear equation of the form (3.21), j times. But if one considers the discrepancy principle (3.6), then, one need to solve the linear equation (3.2) and the nonlinear equation (3.6), only once.

(2) Guacaneme [21] considered the iterated Simplified regularization with the regularization parameter α determined by

the discrepancy principle

$$\alpha^{2(1+j)}\langle (A+\alpha I)^{-2(1+j)} Qg^{\delta}, Qg^{\delta}\rangle = c j^2 \delta^2, \quad c > 1. \tag{3.22}$$

A generalization of the above procedure is

$$\alpha^{2(\rho+j)}\langle (A+\alpha I)^{-2(\rho+j)} Qg^{\delta}, Qg^{\delta}\rangle = c \delta^2, \quad c > 1, \tag{3.23}$$

for a fixed ρ such that $\rho+j > 0$. Following the arguments as Theorem 3.2.4, it can be seen that the condition required for ρ in this case is $\tau := \rho+j-1 > 0$. But, then (3.23) is reduced to the form (3.13). This in particular shows that the discrepancy principle (3.22) is included in the form (3.13) with $\rho = j$.

CHAPTER 4

REGULARIZATION WITH APPROXIMATELY SPECIFIED OPERATORS

In this chapter we consider the problem of solving the operator equation $Tx = y$ approximately when the data y, T are known only approximately. More precisely, we consider the regularization of $Tx = y$ with the help of the approximate data y^δ, T_h where $\|y-y^\delta\| \leq \delta$ and $\|T-T_h\| \leq \varepsilon_h$, $\varepsilon_h \to 0$ as $h \to 0$. The regularized equations and modified forms of the discrepancy principles (2.7), (3.6) and (3.13) are introduced in Section 4.1. The results corresponding to these discrepancy principles have been discussed in Sections 4.2, 4.3 and 4.4.

4.1. INTRODUCTION

We are concerned with the problem of solving the operator equation

$$Tx = y \tag{4.1}$$

approximately when the data y, T are known only approximately. In reality there are two occasions, where one has to consider an approximately specified operator T_h instead of T (e.g., [6], [33], [36], [37], [44]). One such occasion arises from the modelling error and the other when one considers numerical approximation of T.

If y and T are not known exactly, but instead some approximations y^δ and T_h are known, then a natural way to look for approximations to $\hat{x}$, the minimal norm solution of (4.1), is to solve

$$(4.2) \qquad (T_h^* T_h + \alpha I) x_{\alpha,h}^\delta = T_h^* y^\delta$$

instead of (2.2). Here $\{T_h\}_{h>0}$ is a family of bounded linear operators between Hilbert spaces X and Y. If $\|y-y^\delta\| \le \delta$ and $\|T-T_h\| \le \varepsilon_h$ with $\varepsilon_h \ge 0$ such that $\varepsilon_h \to 0$ as $h \to 0$, then one requires

$$(4.3) \qquad \|\hat{x} - x_{\alpha,h}^\delta\| \to 0 \quad \text{as} \quad \alpha \to 0, \quad \delta \to 0 \quad \text{and} \quad h \to 0.$$

But it can be shown that if $R(T)$ is not closed and $\varepsilon_h \to 0$ as $h \to 0$, then for every $h_0 > 0$, $\delta_0 > 0$, the set

$$\{x_{\alpha,h}^\delta : \|y-y^\delta\| \le \delta, \ \|T-T_h\| \le \varepsilon_h; \ 0 < \delta \le \delta_0, \ 0 < h \le h_0\}$$

is not bounded. Therefore it is important to choose the regularization parameter α in dependence of the error level δ and ε_h properly so as to satisfy (4.3). For this purpose we consider a class of discrepancy principles

$$(4.4) \qquad \|T_h x_{\alpha,h}^\delta - y^\delta\| = \frac{(\delta+\varepsilon_h)^p}{\alpha^q}, \quad p > 0, \quad q > 0$$

to compute $\alpha := \alpha(\delta,h)$. If $T_h = T$ and $\varepsilon_h = 0$, then the above discrepancy principle is reduced to (2.7) considered in Section 2.2.

Discrepancy principles with approximately specified operators have been considered in the literature (See [36], [45]). For example Neubauer [36] considered the discrepancy principle

$$(4.5) \qquad \alpha^3\langle(T_{h_m}T^*_{h_m} + \alpha I)^{-3}Q_m y^\delta, Q_m y^\delta\rangle = (d_1\delta + d_2\varepsilon_h)^2,$$

where $T_{h_m} = Q_m T_h$ and Q_m is the orthogonal projection onto a finite dimensional subspace W_m of Y such that Q_m converges to I pointwise and T is a compact operator. Our procedure can also be put in this setting with some modifications in the proof. It can be seen that the square of the left hand side of the equation (4.4) is $\alpha^2\langle(T_hT^*_h + \alpha I)^{-2}y^\delta, y^\delta\rangle$, so that the method (4.4) is simpler than the procedure of (4.5) of Neubauer [36]. Moreover the method (4.4) generalizes the procedure investigated in Section 2.2.

If $X = Y$ and the operator T is a positive self-adjoint operator on X, then as in Chapter 3, we use different notations for the operator and the data and consider the solution $w^\delta_{\alpha,h}$ of the equation

$$(4.6) \qquad (A_h + \alpha I)w^\delta_{\alpha,h} = g^\delta,$$

for obtaining approximations for $\hat{w}$, the minimal norm solution of the equation $Aw = g$. Here $(A_h)_{h>0}$ is a family of self-adjoint operators on X with $\|A-A_h\| \le \varepsilon_h$, $\varepsilon_h \to 0$ as $h \to 0$. In this case we consider the discrepancy principles

$$\|A_h w^{\delta}_{\alpha,h} - g^{\delta}\| = \frac{(\delta+\varepsilon_h)^p}{\alpha^q}, \quad p > 0, \qquad q > 0$$

and

$$\alpha^{2(\rho+1)}\langle (A_h + \alpha I)^{-2(\rho+1)} Q_h g^{\delta}, Q_h g^{\delta}\rangle = (c\delta + d\varepsilon_h)^2, \quad \rho > 0,$$

where c and d are properly chosen positive constants and Q_h is the orthogonal projection on to $\overline{R(A_h)}$, for obtaining convergence and error estimates.

4.2. ON THE APPLICATION OF GENERALIZED ARCANGELI'S METHOD FOR TIKHONOV REGULARIZATION

Let $T \in BL(X,Y)$, $y \in R(T)$ and let $\hat{x}$ be the minimal norm solution of the equation (4.1). Let H be a bounded subset of positive reals such that zero is a limit point of H. Let $\{T_h\}_{h\in H}$ be a family of bounded linear operators between X and Y, such that $\|T-T_h\| \le \varepsilon_h$, $h \in H$, where $\{\varepsilon_h\}_{h\in H}$ is a set of non-negative real numbers satisfying $\varepsilon_h \to 0$ as $h \to 0$. For $\delta > 0$, let D^{δ} be as in Section 2.1, i.e., $D^{\delta} = \{u \in Y : \|u-y\| \le \delta\}$.

In the following, x_{α} is the solution of (4.2) with exact data (y,T) in place of (y^{δ},T_h) and $x^{\delta}_{\alpha,h}$ is the solution of (4.2) for

$y^{\delta} \in D^{\delta}$.

Hereafter we assume that there exists $\varepsilon_0 > 0$ such that $\varepsilon_h \le \varepsilon_0$ for all $h \in H$. This is the case when $\{T_h\}_{h \in H}$ is a uniformly bounded family. Let δ_0 be such that $0 < \delta_0 \le \frac{\|y\|}{2}$.

Theorem 4.2.1. For a fixed pair p, q of positive reals, and for each $\delta \in (0, \delta_0]$, $h \in H$, and $y^{\delta} \in D^{\delta}$, there exists a unique $\alpha := \alpha(\delta, h) > 0$ satisfying (4.4). Moreover,

(i) $\{\alpha(\delta, h) : 0 < \delta \le \delta_0,\ h \in H\}$ is a bounded set of reals,

(ii) $\alpha(\delta, h) \le c(\delta + \varepsilon_h)^{\frac{p}{q+1}}$ for some constant $c > 0$.

(iii) $\frac{p}{q+1} < \frac{4q}{2q+1}$ and $\varepsilon_h \to 0$ as $h \to 0$, imply

$$\|\hat{x} - x^{\delta}_{\alpha,h}\| \to 0 \quad \text{as} \quad \delta \to 0, \quad h \to 0.$$

Proof: The existence and uniqueness of $\alpha := \alpha(\delta, h)$ satisfying (4.4) follows as in Proposition 2.1.4.

If the set $\{\alpha(\delta, h) : 0 < \delta \le \delta_0,\ h \in H\}$ is not bounded then there exist sequences (δ_n) and (h_n) with $0 < \delta_n \le \delta_0$, $h_n \in H$ such that

$$\alpha_n := \alpha(\delta_n, h_n) \to \infty \quad \text{as} \quad n \to \infty.$$

Now since

$$(4.7) \qquad \frac{\alpha_n^q \|y\|}{2(1+M/\alpha_n)} \le \alpha_n^q \|T_{h_n} x^{\delta_n}_{\alpha_n, h_n} - y^{\delta_n}\| = (\delta_n + \varepsilon_{h_n})^p,$$

where $M \ge (\varepsilon_0 + \|T\|)^2$, we have

$$\alpha_n^q \le \frac{2(\delta_n+\varepsilon_n)^p(1+M/\alpha_n)}{\|y\|}.$$

This leads to a contradiction. Thus (i) is proved.

Again from (4.7), by using (i) we have

$$\alpha_n^{q+1} \le \frac{2(\delta+\varepsilon_h)^p(\alpha_n+M)}{\|y\|} \le c(\delta+\varepsilon_h)^p,$$

proving (ii).

If $\varepsilon_h \to 0$ as $h \to 0$, then it follows from (ii) that $\alpha(\delta,h) \to 0$ as $\delta \to 0$, $h \to 0$. It can be seen as in Neubauer [36] that

$$(4.8) \qquad \|\hat{x} - x^{\delta}_{\alpha,h}\| \le c(\|\hat{x} - x_\alpha\| + \frac{\delta+\varepsilon_h}{\sqrt{\alpha}}).$$

Therefore to prove (iii) it is enough to show that

$$(4.9) \qquad \frac{\delta+\varepsilon_h}{\sqrt{\alpha(\delta,h)}} \to 0 \quad \text{as} \quad \delta \to 0, \quad h \to 0$$

and

$$(4.10) \qquad \|\hat{x} - x_\alpha\| \to 0 \quad \text{as} \quad \delta \to 0, \quad h \to 0.$$

We note that

$$\frac{(\delta+\varepsilon_h)^p}{\alpha(\delta,h)^q} = \|T_h x_{\alpha,h}^{\delta} - y^{\delta}\| = \|\alpha(T_h T_h^* + \alpha I)^{-1} y^{\delta}\|$$

$$\le \|\alpha(T_h T_h^* + \alpha I)^{-1}(y^{\delta} - y)\| + \|(T_h T_h^* + \alpha I)^{-1} y\|$$

(4.11) $$\le \delta + \|\alpha(T_h T_h^* + \alpha I)^{-1} y\|$$

where

(4.12) $$\alpha(T_h T_h^* + \alpha I)^{-1} y = \alpha(T_h T_h^* + \alpha I)^{-1}(TT^* - T_h T_h^*)(TT^* + \alpha I)^{-1} y$$

$$+ \alpha(TT^* + \alpha I)^{-1} y$$

$$= \alpha(T_h T_h^* + \alpha I)^{-1}(T - T_h)(T^*T + \alpha I)^{-1} T^* y$$

$$+ \alpha(T_h T_h^* + \alpha I)^{-1} T_h (T^* - T_h^*)(TT^* + \alpha I)^{-1} y$$

$$+ \alpha(TT^* + \alpha I)^{-1} T\hat{x}.$$

Now using the relations

$$\|T_h T_h^* + \alpha I)^{-1} T_h\| \le \frac{1}{2\sqrt{\alpha}},$$

$$\|(T^*T + \alpha I)^{-1} T^* y\| \le \|\hat{x}\|,$$

$$\|\alpha(T_h T_h^* + \alpha I)^{-1}\| \le 1 \quad \text{and} \quad \|(TT^* + \alpha I)^{-1} y\| \le \frac{\|\hat{x}\|}{2\sqrt{\alpha}},$$

it follows that

(4.13) $$\|\alpha(T_h T_h^* + \alpha I)^{-1} y\| \le 2\|\hat{x}\|\varepsilon_h + \|\alpha(TT^* + \alpha I)^{-1} y\|.$$

Now by Lemma 2.2.1, we have

$$\|\alpha(TT^* + \alpha I)^{-1}T\hat{x}\| \le c\alpha^{\omega} \tag{4.14}$$

where $\omega = \min\{1, \nu+1/2\}$ for $\hat{x} \in R((T^*T)^{\nu})$, $0 < \nu \le 1$ so that (4.13) implies

$$\frac{(\delta+\varepsilon_h)^p}{\alpha^q} \le \max\{1, 2\|\hat{x}\|\}(\delta+\varepsilon_h) + \|\alpha(TT^* + \alpha I)^{-1}T\hat{x}\|, \tag{4.15}$$

$$\le c\left((\delta+\varepsilon_h) + (\delta+\varepsilon_h)^{\frac{p}{2(q+1)}}\right)$$

Therefore

$$\left(\frac{\delta+\varepsilon_h}{\sqrt{\alpha}}\right)^{2q} = (\delta+\varepsilon_h)^{2q-p}\left(\frac{(\delta+\varepsilon_h)^p}{\alpha^q}\right) \tag{4.16}$$

$$\le c\left((\delta+\varepsilon_h)^{2q-p+1} + (\delta+\varepsilon_h)^{2q-p+\frac{p}{2(q+1)}}\right)$$

Now the assumption $\frac{p}{q+1} < \frac{4q}{2q+1}$ implies $2q-p+1 \ge 2q-p + \frac{p}{2(q+1)} > 0$, so that $\frac{\delta+\varepsilon_h}{\sqrt{\alpha}} \to 0$ as $\delta \to 0$, proving (4.9) and (4.10) follows from (ii) and arguments used in the proof of Theorem 2.1.1 (a). ■

Theorem 4.2.2. Let $\hat{x} \in R((T^*T)^{\nu})$, $0 < \nu \le 1$, $\omega = \min\{1, \nu+1/2\}$. If $\frac{p}{q+1} \le \min\{\frac{1}{\omega}, \frac{2}{1+(1-\omega)/q}\}$ and $\alpha := \alpha(\delta,h)$ is chosen according to (4.4) for $0 < \delta \le \delta_0$, $h \in H$. Then

(i) $\frac{\delta+\varepsilon_h}{\sqrt{\alpha}} \le c_1(\delta+\varepsilon_h)^{\mu}$

(ii) $\quad \|\hat{x} - x^{\delta}_{\alpha,h}\| \le c_2(\delta+\varepsilon_h)^{r}$,

where $\mu = 1 - \frac{p}{2(q+1)}(1+\frac{1-\omega}{q})$, $\quad r = \min\{\mu, \frac{p\nu}{q+1}\}$.

In particular, if $\frac{p}{q+1} = \frac{2}{2\nu+1+(1-\omega)/q}$, then

(iii) $\quad \|\hat{x} - x^{\delta}_{\alpha,h}\| \le c_3(\delta+\varepsilon_h)^{\frac{p\nu}{q+1}}$.

Proof: In view of (4.8), Theorem 4.2.1 (ii) and Theorem 2.1.1(i), the result in (ii) and (iii) will follow, once (i) is proved. The proof of (i) is a consequence of the relations in (4.14), (4.15), (4.16) and Theorem 4.2.1 (ii). ∎

Proof of the following Corollary is along the same lines as the proof of the Corollary 2.2.5.

Corollary 4.2.3 Let p, q be positive reals satisfying $\frac{p}{q+1} \le 1$ and let $\hat{x} \in R((T^*T)^{\nu})$, $0 < \nu \le 1$, $\omega = \min\{1, \nu+1/2\}$, $l = \min\{\frac{1}{\omega}, \frac{2}{2\nu+1+(1-\omega)/q}\}$. If $\alpha := \alpha(\delta,h)$ is chosen according to (4.4), for $0 < \delta \le \delta_0$; $h \in H$, then

$$\|\hat{x} - x^{\delta}_{\alpha,h}\| \le c(\delta+\varepsilon_h)^{\tau}$$

where

$$\tau = \begin{cases} \frac{p}{q+1}, & \frac{p}{q+1} \le l \\ 1-\frac{p}{2(q+1)}, & \frac{p}{q+1} \ge l. \end{cases}$$ ∎

Remark 4.2.4. We note that if $\hat{x} \in R((T^*T)^\nu)$, $1/2 \le \nu \le 1$, then Theorem 4.2.2, provides the order $O((\delta+\varepsilon_h)^r)$ with $r = \min\{1-\frac{p}{2(q+1)}, \frac{p\nu}{q+1}\}$ for $\frac{p}{q+1} \le 1$. Also if $\hat{x} \in R((T^*T)^\nu)$, $0 < \nu \le 1$ and ν_1 is any estimate for ν such that $\nu \le \nu_1$ and $\nu_1 \ge 1/2$ then by taking $\frac{p}{q+1} = \frac{2}{2\nu_1+1}$, we obtain the rate $O((\delta+\varepsilon_h)^{\frac{2\nu}{2\nu_1+1}})$. In particular for $\frac{p}{q+1} = \frac{2}{3}$, i.e., with $\nu_1 = 1$, the rate $O((\delta+\varepsilon_h)^{\frac{2\nu}{3}})$ is guaranteed. This case includes the Arcangeli's type discrepancy principle, i.e.,

$$\|T_h x_{\alpha,h}^{\hat{\delta}} - y^{\hat{\delta}}\| = \frac{\delta+\varepsilon_h}{\sqrt{\alpha}},$$

for $p = 1$, $q = 1/2$.

In Chapter 5, we consider a special case of the operator T_h, namely, $T_h = TP_h$, where $\{P_h\}_{h\in H}$ is a sequence of orthogonal projections on X. This case, under certain conditions, leads to improved accuracy.

4.3. ON THE APPLICATION OF GENERALIZED ARCANGELI'S METHOD FOR SIMPLIFIED REGULARIZATION

Let $A \in BL(X)$, $g \in R(A)$ and let $\hat{w}$ be the minimal norm solution of the operator equation

(4.17) $$Aw = g.$$

Let H be as in Section 4.2 and $\{A_h\}_{h\in H}$ be a family of self-adjoint operators on X satisfying

(4.18) $$\|A-A_h\| \leq \varepsilon_h, \quad h \in H,$$

where $\{\varepsilon_h\}_{h\in H}$ is a set of non-negative real numbers satisfying $\varepsilon_h \to 0$ as $h \to 0$.

In case A_h is not positive, then one may consider the operator $B_h = A_h + \varepsilon_h I$ (See [45]) and $2\varepsilon_h$ in place of A_h and ε_h respectively. Then B_h is a positive self adjoint operator satisfying $\|B_h-A\| \leq 2\varepsilon_h$. This is seen as follows. From (4.18) it is clear that $\|B_h-A\| \leq 2\varepsilon_h$. Now using the fact that A is positive self adjoint operator, we have

$$\langle A_h x, x\rangle \geq \langle Ax, x\rangle - \langle \varepsilon_h x, x\rangle$$

$$\geq -\langle \varepsilon_h x, x\rangle,$$

so that

$$\langle B_h x, x\rangle = \langle A_h x, x\rangle + \langle \varepsilon_h x, x\rangle$$

$$\geq -\langle \varepsilon_h x, x\rangle + \langle \varepsilon_h x, x\rangle$$

$$\geq 0.$$

Thus without loss of generality we may assume that $\{A_h\}_{h\in H}$ is a family of positive self-adjoint operators on X.

For $\delta > 0$, let $F^\delta := \{u \in X : \|u-g\| \le \delta\}$. Let w_α be the solution of

$$(4.19) \qquad (A + \alpha I)w_\alpha = g,$$

and $w^\delta_{\alpha,h}$ be the solution of the equation

$$(4.20) \qquad (A_h + \alpha I)w^\delta_{\alpha,h} = g^\delta, \quad g^\delta \in F^\delta.$$

We assume that there exists $\varepsilon_0 > 0$ such that $\varepsilon_h \le \varepsilon_0$ for all $h \in H$. Let δ_0 be such that $0 < \delta_0 \le \frac{\|g\|}{2}$.

Theorem 4.3.1 Let $\hat{w} \in R(A^\nu)$, $0 < \nu \le 1$ and $w^\delta_{\alpha,h}$ be defined as in (4.20). Then

$$\|\hat{w} - w^\delta_{\alpha,h}\| \le c_1\frac{\delta+\varepsilon_h}{\alpha} + c_2\alpha^\nu$$

where c_1 and c_2 are positive constants.

Proof: Using triangle inequality, we have

$$\|\hat{w} - w^\delta_{\alpha,h}\| \le \|\hat{w} - w_\alpha\| + \|w_\alpha - w_{\alpha,h}\| + \|w_{\alpha,h} - w^\delta_{\alpha,h}\|$$

where $w_{\alpha,h} := w^0_{\alpha,h}$. Thus by the definition of w_α, $w_{\alpha,h}$ and the fact that $g = A\hat{w}$, it follows that

(4.21) $$\|w_\alpha - w_{\alpha,h}\| \le \frac{\varepsilon_h}{\alpha}\|\hat{w}\|$$

(4,22) $$\|w_{\alpha,h} - w^\delta_{\alpha,h}\| \le \frac{\delta}{\alpha}.$$

The result follows from the above relations together with estimate (3.3), i.e., $\|\hat{w} - w_\alpha\| = O(\alpha^\nu)$. ■

From the above Theorem it is clear that if $\alpha := c(\delta+\varepsilon_h)^{\frac{1}{\nu+1}}$ for some constant $c > 0$, then

(4.23) $$\|\hat{w} - w^\delta_{\alpha,h}\| = O((\delta+\varepsilon_h)^{\frac{\nu}{\nu+1}})$$

and this order is 'optimal' (See. (3.4)) in the sense that in general it can not be improved.

In order to obtain the convergence of $w^\delta_{\alpha,h}$ to $\hat{w}$ and to obtain the order in (4.23) we suggest the discrepancy principle

(4.24) $$\|A_h w^\delta_{\alpha,h} - g^\delta\| = \frac{(\delta+\varepsilon_h)^p}{\alpha^q}, \quad p > 0, \quad q > 0$$

and $\delta \in (0,\delta_0]$.

Theorem 4.3.2. For a fixed pair p, q of positive reals and for each $\delta \in (0,\delta_0]$, $h \in H$ and $g^\delta \in F^\delta$, there exist a unique

$\alpha := \alpha(\delta,h)$ satisfying (4.24). Moreover

(i) $\{\alpha(\delta,h) : 0 < \delta \le \delta_0, \; h \in H\}$ is a bounded set of reals,

(ii) $\alpha(\delta,h) \le c_1(\delta+\varepsilon_h)^{\frac{p}{q+1}}$ and $\dfrac{\delta+\varepsilon_h}{\alpha(\delta,\varepsilon_h)} = c_2(\delta+\varepsilon_h)^{1-\frac{p}{q+1}}$

for some constants $c_1 > 0$ and $c_2 > 0$,

(iii) $p \le q+1$ and $\varepsilon_h \to 0$ as $h \to 0$ imply

$$\|\hat{w} - w^\delta_{\alpha,h}\| \to 0 \quad \text{as} \quad \delta \to 0, \quad h \to 0.$$

Proof: The existence and uniqueness of $\alpha := \alpha(\delta,h)$ satisfying (4.24) follows as in Lemma 3.1.2. The boundedness of the set $\{\alpha(\delta,h) : 0 < \delta \le \delta_0, \; h \in H\}$ and the estimate for $\alpha(\delta,\varepsilon_h)$ follow by using similar arguments as in Theorem 4.2.1 ((i) and (ii)). To obtain estimate for $\dfrac{\delta+\varepsilon_h}{\alpha(\delta,h)}$ and the convergence of $w^\delta_{\alpha,h}$ to $\hat{w}$, we first note that

$$\frac{(\delta+\varepsilon_h)^p}{\alpha^q} = \|A_h w^\delta_{\alpha,h} - g^\delta\|$$

$$= \|\alpha(A_h + \alpha I)^{-1} g^\delta\|$$

$$\le \|\alpha(A_h + \alpha I)^{-1}(g^\delta - g)\| + \|\alpha[(A_h + \alpha I)^{-1} - (A + \alpha I)^{-1}]g\|$$

$$+ \|\alpha(A + \alpha I)^{-1}g\|$$

where

$$\|\alpha(A_h + \alpha I)^{-1}(g^{\delta}-g)\| \le \delta,$$

$$\|\alpha[A_h + \alpha I)^{-1} - (A + \alpha I)^{-1}]g\| = \|\alpha(A_h + \alpha I)^{-1}(A-A_h)(A + \alpha I)^{-1}g\|$$

$$\le \varepsilon_h\|\hat{w}\|$$

and

$$\|\alpha(A + \alpha I)^{-1}g\| \le \alpha\|\hat{w}\|.$$

Thus

$$\frac{(\delta+\varepsilon_h)^p}{\alpha^q} \le c_1(\delta+\varepsilon_h) + c_2\alpha$$

where c_1 and c_2 are positive constants. Therefore by (ii), we have

$$\frac{\delta+\varepsilon_h}{\alpha} = (\delta+\varepsilon_h)^{1-\frac{p}{q}}\left(\frac{(\delta+\varepsilon_h)^p}{\alpha^q}\right)^{\frac{1}{q}}$$

$$\le c_1(\delta+\varepsilon_h)^{1-\frac{p}{q}+\frac{1}{q}} + c_2(\delta+\varepsilon_h)^{1-\frac{p}{q}+\frac{p}{q(q+1)}}.$$

Now since $1-\frac{p}{q}+\frac{p}{q(q+1)} = 1-\frac{p}{q+1}$ and $p < q+1$, we have

$$\frac{\delta+\varepsilon_h}{\alpha} = O((\delta+\varepsilon_h)^{1-\frac{p}{q+1}}).$$

Now the convergence of $w^{\delta}_{\alpha,h}$ to $\hat{w}$ follows as in Theorem 3.1.3 (iii), once we prove that

$$\|\hat{w} - w^{\delta}_{\alpha,h}\| \le c_1\frac{\delta+\varepsilon_h}{\alpha} + \|\alpha(A + \alpha I)^{-1}\hat{w}\|.$$

But this is clear from (4.21), (4,22) and the inequality

$$(4.25) \qquad \|\hat{w} - w^{\delta}_{\alpha,h}\| \le \|\hat{w} - w_{\alpha}\| + \|w_{\alpha} - w_{\alpha,h}\| + \|w_{\alpha,h} - w^{\delta}_{\alpha,h}\|.$$

This completes the proof. ■

Theorem 4.3.3. Let $\hat{w} \in R(A^{\nu})$, $0 < \nu \le 1$, $q > 0$, $p < q+1$ and $\alpha := \alpha(\delta,h)$ be chosen according to (4.24) for $0 < \delta \le \delta_0$, $h \in H$. Then for some constant $c > 0$, we have

$$(i) \qquad \|\hat{w} - w^{\delta}_{\alpha,h}\| \le c(\delta+\varepsilon_h)^{r}$$

$$r = \min\{\frac{p\nu}{q+1}, 1- \frac{p}{q+1}\}.$$

In particular if $\frac{p}{q+1} = \frac{1}{\nu+1}$, then,

$$(ii) \qquad \|\hat{w} - w^{\delta}_{\alpha,h}\| \le c(\delta+\varepsilon_h)^{\frac{\nu}{\nu+1}}.$$

Proof: Proof of (i) follows from Theorems 4.3.1 and 4.3.2 (ii), and proof of (ii) is a consequence of (i) and the fact that $\frac{p\nu}{q+1} = 1- \frac{p}{q+1}$ if and only if $\frac{p}{q+1} = \frac{1}{\nu+1}$, and in this case $\frac{p\nu}{q+1} = \frac{\nu}{\nu+1}$. ■

4.4. ON THE APPLICATION OF MODIFIED GUACANEME'S METHOD FOR SIMPLIFIED REGULARIZATION

In this section we study the analogoue of the parameter choice strategy considered in Section 3.2 for simplified regularization with approximately specified operator. Specifically, for a fixed real number $\rho > 0$, we consider the discrepancy principle

$$(4.26) \qquad \alpha^{2(\rho+1)}\langle (A_h + \alpha I)^{-2(\rho+1)} Q_h g^\delta, Q_h g^\delta\rangle = (c\delta + d\varepsilon_h)^2 , \quad \alpha > 0,$$

where c and d are constants and Q_h is the orthogonal projection onto $\overline{R(A_h)}$ for choosing the parameter 'α' in (4.20). Here also $g^\delta \in F^\delta$ and A_h, $h \in H$ are as in Section 4.3. If in addition g^δ satisfies $\|Q_h g^\delta\| \geq c\delta + d\varepsilon_h$, then as in Lemma 3.2.2 one can prove that there exists a unique $\alpha := \alpha(\delta,h)$ satisfying (4.26). The following result is used to prove our main result of this section.

Lemma 4.4.1. Let $\alpha := \alpha(\delta,h)$ be the solution of (4.26) with $c > 1$ and $d > e = (2+\rho)\|\hat{w}\|$, where $\hat{w}$ is the minimal norm solution of (4.17). Then

$$[(c-1)\delta + (d-e)\varepsilon_h]^2 \leq \alpha^{2(\rho+1)}\langle (A + \alpha I)^{-2(\rho+1)} g, g\rangle$$

$$\leq [(c+1)\delta + (d+e)\varepsilon_h]^2.$$

Proof: We note that

$$\alpha^{2(\rho+1)}\langle (A + \alpha I)^{-2(\rho+1)} g,\ g\rangle = \|\alpha^{\rho+1}(A + \alpha I)^{-(\rho+1)} g\|^2.$$

Also

$$(A + \alpha I)^{-(\rho+1)} g = (A_h + \alpha I)^{-(\rho+1)} Q_h g^{\delta} + (A_h + \alpha I)^{-(\rho+1)} Q_h (g - g^{\delta})$$

$$+ (A_h + \alpha I)^{-(\rho+1)} (I - Q_h) g$$

$$+ [(A + \alpha I)^{-(\rho+1)} - (A_h + \alpha I)^{-(\rho+1)}] g.$$

Therefore

$$\|\alpha^{\rho+1}(A + \alpha I)^{-(\rho+1)} g\| \le \|\alpha^{\rho+1}(A_h + \alpha I)^{-(\rho+1)} Q_h g^{\delta}\|$$

$$+ \|\alpha^{\rho+1}(A_h + \alpha I)^{-(\rho+1)} Q_h (g - g^{\delta})\|$$

$$+ \|\alpha^{\rho+1}(A_h + \alpha I)^{-(\rho+1)} (I - Q_h) g\|$$

$$+ \|\alpha^{\rho+1}[(A + \alpha I)^{-(\rho+1)} - (A_h + \alpha I)^{-(\rho+1)}] g\|,$$

and

$$\|\alpha^{\rho+1}(A_h + \alpha I)^{-(\rho+1)} g\| \ge \|\alpha^{\rho+1}(A_h + \alpha I)^{-(\rho+1)} Q_h g^{\delta}\|$$

$$- \|\alpha^{\rho+1}(A_h + \alpha I)^{-(\rho+1)} Q_h (g - g^{\delta})\|$$

$$- \|\alpha^{\rho+1}(A_h + \alpha I)^{-(\rho+1)} (I - Q_h) g\|$$

$$- \|\alpha^{\rho+1}[(A + \alpha I)^{-(\rho+1)} - (A_h + \alpha I)^{-(\rho+1)}] g\|,$$

Now,

$$\|\alpha^{\rho+1}(A_h + \alpha I)^{-(\rho+1)} Q_h (g - g^{\delta})\| = [\alpha^{2(\rho+1)}\langle (A_h$$

$$+ \alpha I)^{-2(\rho+1)}Q_h g^\delta, Q_h g^\delta\rangle]^{1/2},$$

$$= c\delta + d\varepsilon_h,$$

$$\|\alpha^{\rho+1}(A_h + \alpha I)^{-(\rho+1)}Q_h(g-g^\delta)\| \le \delta$$

and

$$\|\alpha^{\rho+1}(A_h + \alpha I)^{-(\rho+1)}(I-Q_h)g\| \le \|\alpha^{\rho+1}(A_h + \alpha I)^{-(\rho+1)}(I-Q_h)A\hat{w}\|$$

$$\le \|\alpha^{\rho+1}(A_h + \alpha I)^{-(\rho+1)}(I-Q_h)(A-A_h)\hat{w}\|$$

$$\le \|(I-Q_h)(A-A_h)\hat{w}\|$$

$$\le \|I-Q_h\|\|(A-A_h)\hat{w}\|$$

$$\le \|(A-A_h)\hat{w}\|$$

$$\le \varepsilon_h\|\hat{w}\|.$$

Therefore the Lemma will follow once we prove

$$\|\alpha^{\rho+1}f(\alpha,h)\| \le (\rho+1)\varepsilon_h\,\|\hat{w}\|,$$

where

$$f(\alpha,h) = [(A + \alpha I)^{-(\rho+1)} - (A_h + \alpha I)^{-(\rho+1)}]g.$$

To prove this first we note that

$$f(\alpha,h) = (A_h + \alpha I)^{-(\rho+1)}[(A_h + \alpha I)^{(\rho+1)}$$

$$- (A + \alpha I)^{(\rho+1)}](A + \alpha I)^{-(\rho+1)}g$$

$$= (A_h + \alpha I)^{-(\rho+1)}[(A_h + \alpha I)(A_h + \alpha I)^{\rho}$$

$$- (A + \alpha I)(A + \alpha I)^{\rho}](A + \alpha I)^{-(\rho+1)}g$$

$$= (A_h + \alpha I)^{-(\rho+1)}[A_h(A_h + \alpha I)^{\rho} \quad -A(A + \alpha I)^{\rho}](A + \alpha I)^{-(\rho+1)}g$$

$$+ \alpha(A_h + \alpha I)^{-(\rho+1)}[(A_h + \alpha I)^{\rho} - (A + \alpha I)^{\rho}](A + \alpha I)^{-(\rho+1)}g$$

$$= (A_h + \alpha I)^{-(\rho+1)}A_h[(A_h + \alpha I)^{\rho} \quad - (A + \alpha I)^{\rho}](A + \alpha I)^{-(\rho+1)}g$$

$$+ (A_h + \alpha I)^{-(\rho+1)}(A_h - A)(A + \alpha I)^{-1}g$$

$$+ \alpha(A_h + \quad \alpha I)^{-(\rho+1)}[(A_h + \alpha I)^{\rho} - (A + \alpha I)^{\rho}](A + \alpha I)^{-(\rho+1)}g$$

$$= (A_h + \alpha I)^{-\rho}[(A_h + \alpha I)^{\rho} \quad - (A + \alpha I)^{\rho}](A + \alpha I)^{-(\rho+1)}g$$

$$+ (A_h + \alpha I)^{-(\rho+1)}(A_h - A)(A + \alpha I)^{-1}g.$$

Thus

$$\|\alpha^{\rho+1}f(\alpha,h)\| \quad \le \quad \|U(\alpha,h)\|$$

$$+ \|\alpha^{\rho+1}(A_h + \alpha I)^{-(\rho+1)}(A_h - A)(A + \alpha I)^{-1}g\|$$

where

$$U(\alpha,h) = \alpha^{\rho+1}(A_h + \alpha I)^{-\rho}[(A_h + \alpha I)^{\rho} - (A + \alpha I)^{\rho}](A + \alpha I)^{-(\rho+1)}g.$$

Now since

$$\|\alpha^{\rho+1}(A_h + \alpha I)^{-(\rho+1)}(A_h - A)(A + \alpha I)^{-1}g\| \le \varepsilon_h\|\hat{w}\|,$$

we have

$$\|\alpha^{\rho+1}f(\alpha,h)\| \le \|U(\alpha,h)\| + \varepsilon_h\|\hat{w}\|. \tag{4.27}$$

We note that if $\rho = 1$, then

$$\|U(\alpha,h)\| \le \|\alpha^2(A_h + \alpha I)^{-1}(A_h - A)(A + \alpha I)^{-2}g\|$$

$$\le \varepsilon_h\|w\|,$$

so that in this case

$$\|\alpha^{\rho+1}f(\alpha,h)\| \le 2\varepsilon_h\|\hat{w}\|.$$

Now consider the case when $0 < \rho < 1$. In this case,

$$U(\alpha,h) = \alpha^{\rho}(A_h + \alpha I)^{-\rho}(\alpha^{1-\rho}[(A_h + \alpha I)^{\rho} - (A + \alpha I)^{\rho}])(\alpha^{\rho}(A + \alpha I)^{-(\rho+1)}g), \tag{4.28}$$

so that

$$\|U(\alpha,h)\| \le \|\alpha^{\rho}(A_h + \alpha I)^{-\rho}\|\|\alpha^{1-\rho}[(A_h + \alpha I)^{\rho} - (A + \alpha I)^{\rho}](\alpha^{\rho}(A + \alpha I)^{-(\rho+1)}g)\|$$

$$(4.29) \qquad \leq \|\alpha^{1-\rho}[(A_h + \alpha I)^{\rho} - (A + \alpha I)^{\rho}](\alpha^{\rho}(A + \alpha I)^{-(\rho+1)}g)\|.$$

Now recall the formula ([25] page 287),

$$B^z x = \frac{\sin\pi z}{\pi} \int_0^\infty \lambda^z[(\lambda I + B)^{-1}x - \frac{\theta(\lambda)}{\lambda}x + \ldots + (-1)^n \frac{\theta(\lambda)B^{n-1}x}{\lambda^n}]\,d\lambda$$

$$+ \frac{\sin\pi z}{\pi}[\frac{x}{z} - \frac{Bx}{z-1} + \ldots + (-1)^{n-1}\frac{B^{n-1}x}{z-n+1}], \quad x \in X$$

where

$$\theta(\lambda) = \begin{cases} 0 & \text{if } 0 \leq \lambda \leq 1 \\ 1 & \text{if } 1 < \lambda < \omega, \end{cases}$$

for any positive self adjoint operator B and for complex number z such that $0 < \operatorname{Re} z < n$. Taking $z = \rho$, $0 < \rho < 1$, we have

$$B^{\rho}x = \frac{\sin\pi z}{\pi}[\frac{x}{\rho} + \int_0^\infty \lambda^{\rho}(\lambda I + B)^{-1}x\,d\lambda - \int_1^\infty \frac{x}{\lambda^{1-\rho}}d\lambda]$$

Using the above formula, taking

$$\xi = \alpha^{\rho}(A + \alpha I)^{-(\rho+1)}g,$$

and

$$Q_{\alpha,h} = (A_h + \alpha I)^{\rho} - (A + \alpha I)^{\rho},$$

we obtain

$$Q_{\alpha,h}\xi = \frac{\sin\pi\rho}{\pi}\int_0^\infty t^{\rho}[(A_h + (t+\alpha)I)^{-1} - (A + (t+\alpha)I)^{-1}]\,\xi\,dt.$$

$$= \frac{\sin \pi\rho}{\pi} \int_0^\infty t^\rho (A_h + (t+\alpha)I)^{-1}(A-A_h)(A + (t+\alpha)I)^{-1}\xi dt.$$

Thus

$$\|Q_{\alpha,h}\xi\| \le \frac{\sin \pi\rho}{\pi} \int_0^\infty \|t^\rho (A_h + (t+\alpha)I)^{-1}(A-A_h)(A + (t+\alpha)I)^{-1}\xi\| dt.$$

Now since

$$\|(A_h + (t+\alpha)I)^{-1}\| \le \frac{1}{\alpha+t} \quad \text{and} \quad \|(A + (t+\alpha)I)^{-1}\| \le \frac{1}{\alpha+t},$$

we have

$$\|\alpha^{1-\rho}Q_{\alpha,h}\xi\| \le \alpha^{1-\rho} \frac{\sin \pi\rho}{\pi} \int_0^\infty \frac{t^\rho}{(\alpha+t)^2} dt \; \|A-A_h\| \; \|\xi\|$$

(4.30) $$\le \frac{\sin \pi\rho}{\pi} \int_0^\infty \frac{\beta^\rho}{(1+\beta)^2} d\beta \; \varepsilon_h \|\xi\|,$$

where $\beta = \frac{t}{\alpha}$.

It can be seen that

$$\int_0^\infty \frac{\beta^\rho}{(1+\beta)^2} d\beta = \rho \int_0^\infty \frac{ds}{s^{1-\rho}(1+s)}$$

and

$$\int_0^\infty \frac{ds}{s^{1-\rho}(1+s)} = \frac{\pi}{\sin \pi\rho},$$

so that

$$\int_0^\infty \frac{\beta^\rho}{(1+\beta)^2} d\beta = \frac{\pi\rho}{\sin \pi\rho}.$$

Therefore, from (4.30), we have

(4.31) $$\|\alpha^{1-\rho}[(A_h + \alpha I)^{\rho} - (A + \alpha I)^{\rho}]\xi\| \le \rho\varepsilon_h \|\xi\|.$$

where

$$\|\xi\| = \|\alpha^{\rho}(A + \alpha I)^{-(\rho+1)}g\| = \|\alpha^{\rho}(A + \alpha I)^{-(\rho+1)}A\hat{w}\| \le \|\hat{w}\|.$$

Thus from (4.27), (4.29) and (4.30), we have

$$\|\alpha^{\rho+1}[(A + \alpha I)^{-(\rho+1)} - (A_h + \alpha I)^{-(\rho+1)}]g\| \le (\rho+1)\varepsilon_h \|\hat{w}\|.$$

This completes the proof. ∎

Theorem 4.4.2. Let $g^{\delta} \in F^{\delta}$, $h \in H$ and let $\alpha = \alpha(\delta,h)$ is chosen according to (4.26) with c and d are as in Lemma 4.4.1. Then

$$w^{\delta}_{\alpha,h} \to \hat{w} \quad \text{as} \quad \delta \to 0, \quad h \to 0.$$

Proof: Note that

$$\|\hat{w} - w^{\delta}_{\alpha,h}\| \le \|\hat{w} - w_{\alpha}\| + \|w_{\alpha} - w_{\alpha,h}\| + \|w_{\alpha,h} - w^{\delta}_{\alpha,h}\|.$$

Thus from (4.21) and (4.22), we have

$$\|\hat{w} - w^{\delta}_{\alpha,h}\| \le \max\{1, \|\hat{w}\|\}\frac{\delta+\varepsilon_h}{\alpha} + \|\hat{w} - w_{\alpha}\|.$$

From this we obtain the result by using the arguments used in the proof of Theorem 3.2.4. ∎

Lemma 4.4.3 Let $g^\delta \in F^\delta$, $h \in H$ and let $\alpha := \alpha(\delta,h)$ is the unique solution of (4.26) with c and d are as in Lemma 4.4.1. Let $\hat{w} \in R(A^\nu)$, $0 < \nu \le 1$. Then we have the following

(i) $\alpha(\delta,h) = O((\delta+\varepsilon_h)^{\frac{1}{\nu+1}})$

(ii) $\frac{\delta+\varepsilon_h}{\alpha(\delta,h)} = O((\delta+\varepsilon_h)^{\frac{\nu}{\nu+1}})$, if $0 < \nu \le 1$ and $\nu \le \rho$.

(iii) $\frac{\delta+\varepsilon_h}{\alpha(\delta,h)} = o((\delta+\varepsilon_h)^{\frac{\nu}{\nu+1}})$, if $0 < \nu < 1$ and $\nu < \rho$. ∎

Proof: Proof of the Lemma follows in the line of the proof of Lemma 3.2.5 with $((c+1)\delta + (d+e)\varepsilon_h)^2$ in place of $c_2\delta^2$ and $((c-1)\delta + (d-e)\varepsilon_h)^2$ in place of $c_1\delta^2$.

Theorem 4.4.4. Let $g^\delta \in F^\delta$, $h \in H$ and let $\alpha := \alpha(\delta,h)$ be the unique solution of (4.26) with c and d are as in Lemma 4.4.1. Let $\hat{w} \in R(A^\nu)$, $0 < \nu \le 1$. Then

(i) $$\|\hat{w} - w^\delta_{\alpha,h}\| = \begin{cases} O((\delta+\varepsilon_h)^{\frac{\nu}{\nu+1}}), & \nu \le \rho \\ O((\delta+\varepsilon_h)^{\frac{\rho}{\rho+1}}), & \nu \ge \rho \end{cases}$$

If $0 < \nu < 1$ and $\nu < \rho$, then

(ii) $\|\hat{w} - w^\delta_{\alpha,h}\| = o((\delta+\varepsilon_h)^{\frac{\nu}{\nu+1}})$.

In particular if $\rho = 1$ in (4.26) then

(iii) $$\|\hat{w} - w^{\delta}_{\alpha,h}\| = \begin{cases} O((\delta+\varepsilon_h)^{\frac{\nu}{2}}), & 0 < \nu < 1 \\ O((\delta+\varepsilon_h)^{\frac{1}{2}}), & \nu = 1 \end{cases}$$

Proof: The proof of the Theorem follows from Theorem 4.3.1 and Lemma 4.4.3 by using the arguments used in the proof of Theorem 3.2.6. ■

CHAPTER 5

REGULARIZED PROJECTION METHOD AND NUMERICAL APPROXIMATION

In practical implimentation of regularization methods for obtaining approximations for the minimal norm least-square solution of the equation $Tx = y$, one uses finite dimensional subspaces rather than the space X itself. This amounts to, for example, the projection methods for solving the regularized equations

$$(T^*T + \alpha I)x_\alpha^\delta = T^*y^\delta \quad \text{and} \quad (A + \alpha I)w_\alpha^\delta = g^\delta.$$

In Section 5.1, first we consider the projection method for Tikhonov regularization of the equation $Tx = y$ with a modified form of the discrepancy principle (4.4) for choosing the regularization parameter. We show that this procedure leads to a generalization and modification of the Marti's method ([28], [29], [30]). In this case the results include, and in certain cases improve the conclusions of Engl and Neubauer [6] under weaker conditions. Then projection method is applied to Simplified regularization with corresponding modified form of the discrepancy principles (4.24) and (4.26) considered in Chapter 4. In Section 5.2 we present the Algorithms to impliment the methods of Section 5.1. Finally in Section 5.3 we present results of some numerical experiments which confirm the Theoretical results presented in Section 5.1.

5.1. REGULARIZED PROJECTION METHOD

Let $\{P_h\}_{h>0}$ be a family of orthogonal projections on X. Our aim in this Section is to obtain an approximate solution for the equation

$$(5.1) \qquad Tx = y, \quad y \in R(T)$$

in the finite dimensional space $R(P_h)$. For the results that follow, we impose the conditions

$$\eta_h := \|(I-P_h)\hat{x}\| \to 0 \quad \text{and} \quad \gamma_h := \|T(I-P_h)\| \to 0 \text{ as } h \to \infty.$$

on P_h and $\hat{x}$, where $\hat{x}$ is the minimal norm solution of (5.1). The above conditions are satisfied if, for example, $P_h \to I$ pointwise and if T is a compact operator.

Projection Method for Tikhonov Regularization:-

The projection method for the regularized equation

$$(T^*T + \alpha I)x_\alpha^\delta = T^*y^\delta,$$

consists of solving the equation

$$(5.2) \qquad (P_hT^*TP_h + \alpha I)x_{\alpha,h}^\delta = P_hT^*y^\delta,$$

where $y^\delta \in D^\delta = \{u \in Y: \|u-y\| \le \delta\}$. The unique solution $x^\delta_{\alpha,h}$, of the equation (5.2) can be interpreted as the unique element satisfying

$$\langle (T^*T + \alpha I)x^\delta_{\alpha,h} , u\rangle = \langle T^*y^\delta , u\rangle \quad \text{for all} \quad u \in R(P_h).$$

In fact equation (5.2) is a particular case of (4.2) obtained by taking $T_h = TP_h$. Here after we use the notation T_h instead of TP_h. It is proved in Groetsch [12] (Lemma 4.2.3) that

$$\|x_{\alpha,h}-x_\alpha\| \le \sqrt{(1+(\gamma_h^2/\alpha)}\, \|(I-P_h)x_\alpha\|, \tag{5.3}$$

where $x_\alpha = (T^*T + \alpha I)^{-1}T^*y$ and $x_{\alpha,h} = x^0_{\alpha,h}$.

Note that

$$\hat{x} - x^\delta_{\alpha,h} = \hat{x} - x_\alpha + x_\alpha - x_{\alpha,h} + x_{\alpha,h} - x^\delta_{\alpha,h},$$

so that

$$\|\hat{x}-x^\delta_{\alpha,h}\| \le \|\hat{x}-x_\alpha\| + \|x_\alpha-x_{\alpha,h}\| + \|x_{\alpha,h}-x^\delta_{\alpha,h}\|.$$

Now by (5.3) and the fact that

$$\|x_{\alpha,h}-x^\delta_{\alpha,h}\| = \|(T_h^*T_h + \alpha I)^{-1}T_h^*(y-y^\delta)\| \le \frac{\delta}{\sqrt{\alpha}},$$

we have

$$\|\hat{x}-x^\delta_{\alpha,h}\| \le \|\hat{x}-x_\alpha\| + \sqrt{(1+(\gamma_h^2/\alpha)}\|(I-P_h)x_\alpha\| + \frac{\delta}{\sqrt{\alpha}}$$

$$\le \|\hat{x}-x_\alpha\| + \sqrt{(1+(\gamma_h^2/\alpha)}(\|(I-P_h)(x_\alpha-\hat{x})\| + \|(I-P_h)\hat{x}\|) + \frac{\delta}{\sqrt{\alpha}}.$$

Therefore

$$\|\hat{x}-x_{\alpha,h}^{\delta}\| \le (2+\frac{\gamma}{\sqrt{\alpha}})\|\hat{x}-x_\alpha\| + (1+\frac{\gamma}{\sqrt{\alpha}})\eta_h + \frac{\delta}{\sqrt{\alpha}}. \tag{5.4}$$

From (4.11) and (4.12), we have

$$\|T_h x_{\alpha,h}^{\delta}-y^{\delta}\| \le \delta + \|\alpha(T_hT_h^* + \alpha I)^{-1}(T-T_h)(T^*T + \alpha I)^{-1}T^*y\|$$

$$+ \|\alpha(T_hT_h^* + \alpha I)^{-1}T_h(T^*-T^*_h)(TT^* + \alpha I)^{-1}y\|$$

$$+ \|\alpha(TT^* + \alpha I)^{-1}T\hat{x}\|.$$

Note that

$$T_h(T^*-T^*_h) = TP_h(T^*-P_hT^*)$$

$$= 0$$

so that

$$\|T_h x_{\alpha,h}^{\delta}-y^{\delta}\| \le \delta + \|\alpha(T_hT_h^* + \alpha I)^{-1}(T-T_h)(T^*T + \alpha I)^{-1}T^*y\|$$

$$+ \|\alpha(TT^* + \alpha I)^{-1}T\hat{x}\|,$$

where

$$\|\alpha(T_hT_h^* + \alpha I)^{-1}(T-T_h)(T^*T + \alpha I)^{-1}T^*y\| \le \|(T-T_h)(T^*T + \alpha I)^{-1}T^*y\|$$

$$\le \|T(I-P_h)x_\alpha\|$$

$$\leq \|T(I-P_h)(x_\alpha-\hat{x})\| + \|T(I-P_h)\hat{x}\|$$

$$\leq \|T(I-P_h)(x_\alpha-\hat{x})\| + \|T(I-P_h)(I-P_h)\hat{x}\|$$

$$\leq \gamma_h(\|\hat{x}-x_\alpha\| + \eta_h).$$

Therefore,

$$\|T_h x^\delta_{\alpha,h}-y^\delta\| \leq \delta + \gamma_h(\|\hat{x}-x_\alpha\| + \eta_h) + \|R_\alpha \hat{x}\|$$

where $R_\alpha := \alpha(TT^*+\alpha I)^{-1}T$ satisfies $\|R_\alpha\hat{x}\| \leq c\alpha^\omega$, with $\omega = \min\{1, \nu+1/2\}$, whenever $\hat{x} \in R((T^*T)^\nu)$, $\nu > 0$. For choosing the regularization parameter α, we consider a modified form of the discrepancy principle (4.4), namely,

$$\|T_h x^\delta_{\alpha,h}-y^\delta\| = \frac{(\delta+b_h)^p}{\alpha^q},$$

where $\{b_h\}_{h\in H}$ is a set of positive reals such that $b_h \to 0$ as $h \to 0$. Note that, since $T_h = TP_h$ and $x^\delta_{\alpha,h} \in R(P_h)$, the above equation can be written as

$$\|Tx^\delta_{\alpha,h}-y^\delta\| = \frac{(\delta+b_h)^p}{\alpha^q}, \tag{5.5}$$

Here and below, as in Section 4.2, H is a bounded subset of non-negative reals such that zero is a limit point of H. Imitating the proof of Theorem 4.2.1 (i) and (ii), it can be seen that there exists a unique $\alpha := \alpha(\delta,h)$ such that (5.5) is satisfied and that

$$\alpha(\delta,h) \le c(\delta+b_h)^{\frac{p}{q+1}}, \quad 0 < \delta \le \delta_0, \; h \in H.$$

Theorem 5.1.1. Let $\eta_h = O(b_h^k)$ and $\gamma_h = O(b_h^\lambda)$ for some positive reals k and λ. If $\frac{p}{q+1} \le \min\{2k, \frac{4q\lambda}{2q+1}\}$ and α is chosen according to (5.5) then we have the following

(i) If $\frac{p}{q+1} < \frac{4q}{2q+1}$, then $\|\hat{x}-x_{\alpha,h}^\delta\| \to 0$ as $h \to 0$, $\delta \to 0$.

(ii) If $\hat{x} \in R((T^*T)^\nu)$, $0 < \nu \le 1$ and $\frac{p}{q+1} \le \frac{1}{\omega}$, then

$$\|\hat{x}-x_{\alpha,h}^\delta\| \le c(\|\hat{x}-x_\alpha\| + \eta_h + (\delta+b_h)^{\mathit{l}}),$$

$$= O((\eta_h + (\delta+b_h)^\tau))$$

where $\omega = \min\{1, \nu+1/2\}$, $\mathit{l} = 1 - \frac{p}{2(q+1)}(1+(1-\omega)/q)$ and $\tau = \min\{\frac{p\nu}{q+1}, \mathit{l}\}$.

Proof: We recall from Theorem 2.1.1 (i) and Lemma 2.2.2 that for $\hat{x} \in R((T^*T)^\nu)$, $0 < \nu \le 1$,

$$\|\hat{x}-x_\alpha\| \le c\alpha^\nu \quad \text{and} \quad \|R_\alpha \hat{x}\| \le \alpha^\omega.$$

Therefore using the assumption $\frac{p}{q+1} \le \frac{1}{\omega}$ it follows from (4.15) that

$$\frac{(\delta+b_h)^p}{\alpha^q} \le c(\delta+b_h)^{\frac{p\omega}{q+1}},$$

so that for any $s > 0$, we have

$$\frac{(\delta+b_h)^s}{\sqrt{\alpha}} \le (\delta+b_h)^{s-\frac{p}{2q}}\left(\frac{(\delta+b_h)^p}{\alpha^q}\right)^{1/2q}$$

$$\le c(\delta+b_h)^{s-\frac{p}{2q}+\frac{p\omega}{2q(q+1)}}$$

From this we have

$$\frac{\gamma_h}{\sqrt{\alpha}} \le \frac{(\delta+b_h)^\lambda}{\alpha^q} = O(1) \quad \text{and} \quad \frac{\delta}{\sqrt{\alpha}} \le \frac{\delta+b_h}{\sqrt{\alpha}} = O((\delta+b_h)^l),$$

where $l = 1 - \frac{p}{2(q+1)}(1+(1-\omega)/q)$. Note that $l \ge 1 - \frac{(2q+1)p}{4q(q+1)}$, so that by the assumption $\frac{p}{q+1} < \frac{4q}{2q+1}$, we have $\frac{\delta}{\sqrt{\alpha}} = o(1)$. Now the result in (i) follows from (5.4) by using the arguements used in Theorem 2.2.1 and (ii) follows from (5.4).

Remarks 5.1.2. We note that if $\nu \ge 1/2$, then, $\hat{x} \in R(T^*)$, so that $\|(I-P_h)\hat{x}\| \le c\ \|T(I-P_h)\|$. Thus we may take $k \ge \lambda$. We consider two special cases.

Case (i) $\hat{x} \in R(T^*)$, i.e., $\nu = 1/2$:- Let $\frac{p}{q+1} = 1$. Taking $k \ge \lambda = \frac{2q+1}{4q}$ in Theorem 5.1.1 (ii), we obtain the rate

$$\|\hat{x}-x^\delta_{\alpha,h}\| = O((\delta+b_h)^{1/2}).$$

For obtaing the same result Engl and Neubauer [6] requires the condition $\lambda = \frac{q+1}{2q}$, which is stronger than ours. As in [6], from Theorem 5.1.1 (ii), we also obtain the rate arbitrarily close to

$\mathcal{O}(\gamma_h)$ for large values of q provided

$$b_h \sim \gamma_h^{\frac{4q}{2q+1}} \quad \text{and} \quad \delta = \delta_h = \mathcal{O}(b_h).$$

Case (ii) $\hat{x} \in R(T^*T)$, i.e., $\nu = 1$:- Let $\frac{p}{q+1} = \frac{2}{3}$. Taking $k \geq \lambda := \frac{2q+1}{6q}$, the rate in Theorem 5.1.1 (ii) becomes $\mathcal{O}((\delta+b_h)^s)$ with $s = \min\{2/3, k\}$. In this case, if $\delta = \delta_h \leq cb_h \sim \gamma_h^{\frac{6q}{2q+1}}$ then the rate is

$$\|\hat{x}-x^{\delta}_{\alpha,h}\| = \mathcal{O}(\gamma_h^t), \quad t = \frac{2q}{2q+1}\min\{2, 3k\}.$$

Note that if $q \geq 1/2$, then $t \geq 1$, and if $q \geq 1/2$ and $k > \lambda := \frac{2q+1}{6q}$ then $t > 1$. In particular if $k \geq 2/3$ then the rate

$$\|\hat{x}-x^{\delta}_{\alpha,h}\| = \mathcal{O}(\gamma_h^{\frac{4q}{2q+1}})$$

is arbitrarily close to $\mathcal{O}(\gamma_h^2)$ for large values of q, where as the result in [6] can give only up to $\mathcal{O}(\gamma_h)$. Since T^*T is self-adjoint, we have $\|(I-P_h)\hat{x}\| = \mathcal{O}(\|T^*T(I-P_h)\|)$, so that the condition $k \geq 2/3$ is satisfied if the operator T has the property $\|T^*T(I-P_h)\| = \mathcal{O}(\|T(I-P_h)\|^2)$ for then one can take $k = 2\lambda = \frac{2}{3} + \frac{1}{3q}$. Such cases do occure. For example, suppose that T is an injective compact operator with $R(T)$ dende in Y. Let $\{\sigma_k\}$ be the set of singular values of T satisfying $\sigma_1 \geq \sigma_2 \geq \ldots$, and $\{u_k\}$ and $\{v_k\}$ be orthogonal basis of X and Y respectively such that $Tu_k = \sigma_k u_k$, $T^*v_k = \sigma_k v_k$ for $k = 1,2,\ldots$. If $h = 1/n$, $n = 1,2,\ldots$, P_h

is the orthogonal projection of X onto $V_n := \text{span}\{u_1,\dots,u_n\}$ then it can be seen that $\|T^*T(I-P_h)\| = \|T(I-P_h)\|^2 = \sigma^2_{n+1}$.

Marti (See [28], [29], [30]), used an algorithm to compute approximate solution for the equation (5.1). In this method, a sequence of finite dimensional subspaces $V_1 \subset V_2 \subset \dots$ of X with

$$\overline{\bigcup_{n\in\mathbb{N}} V_n} = X$$

is used to obtain an approximate solution x_n of (5.1). More precisely, let for $n \in \mathbb{N}$

$$a_n = \inf\{\|Tx-y\| : x \in V_n\},$$

P_h, $h = 1/n$, be the orthogonal projection of X onto V_n, and $b_h > 0$ be chosen such that

$$\lim_{n\to\infty} \frac{\|P_h\hat{x} - \hat{x}\|}{b_h} = 0, \qquad \lim_{n\to\infty} b_h = 0.$$

Then x_n is defined by

$$x_n \in V_n,$$

(5.6) $$\|Tx_n - y\|^2 \le a_n^2 + b_h^2,$$

$$\|x_n\| = \inf\{\|x\| : x \in V_n \text{ and satisfies (5.6)}\}$$

In [12, Section 4.3], Groetsch has reformulated, Marti's method as solving

$$\alpha x_n + T_h^* T_h x_n = T_h^* y,$$

for x_n with the regularization parameter α is determined by

$$\|T x_n - y\|^2 = a_n^2 + b_h^2. \tag{5.7}$$

Thus the method (5.2), (5.5) is a generalization and modification of Marti's method.

Projection Method for Simplified Regularization:-

Here we consider the case when the operator under consideration is a positive self adjoint operator. More precisely we consider the operator equation

$$Aw = g, \qquad g \in R(A), \tag{5.8}$$

where A is a positive self adjoint operator on X. As earlier, let $\{P_h\}_{h>0}$ is a family of finite rank orthogonal projection on X. In this case the projection method for the equation

$$(A + \alpha I)w_\alpha^\delta = g^\delta,$$

will take the form

$$(5.9) \qquad (P_hAP_h + \alpha I)u^{\delta}_{\alpha,h} = P_h g^{\delta},$$

where $g^{\delta} \in F^{\delta} = \{u \in X: \|u-g\| \le \delta\}$.

Note that

$$u^{\delta}_{\alpha,h} = P_h w^{\delta}_{\alpha,h},$$

where $w^{\delta}_{\alpha,h}$ is the solution of the equation

$$(5.10) \qquad (P_hAP_h + \alpha I)w^{\delta}_{\alpha,h} = g^{\delta}.$$

In fact the equation (5.10) is a particular case of (4.21) obtained by taking $A_h = P_hAP_h$. Here after we use the notation A_h instead of P_hAP_h. Let $\|A-AP_h\| = O(b_h)$, where $\{b_h\}_{h>0}$ is a family of positive reals such that $b_h \to 0$ as $h \to 0$.

The following Theorem is a companion result of Theorem 4.3.1.

Theorem 5.1.3. Let $u^{\delta}_{\alpha,h}$ be defined as in (5.9) and $\hat{w}$ be the minimal norm solution of the equation (5.8). If $\hat{w} \in R(A^{\nu})$, $0 < \nu \le 1$, then

$$\|\hat{w}-u^{\delta}_{\alpha,h}\| \le c_1\frac{\delta+b_h}{\alpha} + c_2\alpha^{\nu}$$

where c_1 and c_2 are positive constants.

Proof: Note that

(5.11) $$\|\hat{w}-u^{\delta}_{\alpha,h}\| \le \|\hat{w}-w^{\delta}_{\alpha,h}\| + \|w^{\delta}_{\alpha,h}-u^{\delta}_{\alpha,h}\|.$$

Now from (5.9) and (5.10), we have

$$\alpha\|u^{\delta}_{\alpha,h}-w^{\delta}_{\alpha,h}\| = \|(P_h - I)g^{\delta}\|$$

$$\le \|(P_h-I)(g^{\delta}-g)\| + \|(P_h-I)g\|$$

$$\le \|g^{\delta}-g\| + \|(P_h-I)A\hat{w}\|$$

$$\le \delta + \|(P_h-I)(A-P_hA)\hat{w}\|$$

$$\le \delta + \|(A-P_hA)\hat{w}\|$$

Since $\|A-AP_h\| = \|A-P_hA\|$, we obtain

$$\alpha\|u^{\delta}_{\alpha,h}-w^{\delta}_{\alpha,h}\| \le \delta + b_h\|\hat{w}\|.$$

Thus from (5.11), we have

(5.12) $$\|\hat{w}-u^{\delta}_{\alpha,h}\| \le \|\hat{w}-w^{\delta}_{\alpha,h}\| + \max\{1,\|\hat{w}\|\}\frac{\delta+b_h}{\alpha}$$

Now the result follows from Theorem 4.3.1, with $A_h = P_hAP_n$ and $\varepsilon_h = O(b_h)$. ∎

For choosing the regularization parameter α in (5.9), we first consider the discrepancy principle

(5.13) $$\|P_h A u^{\delta}_{\alpha,h} - P_h g^{\delta}\| = \frac{(\delta+b_h)^p}{\alpha^q}, \quad p > 0, \quad q > 0,$$

which is a modified form of (4.24). Imitating the proof of Theorem 4.3.2 (i) and (ii), it can be seen that there exists a unique $\alpha := \alpha(\delta,h)$ such that (5.13) is satisfied and that

(5.14) $$\alpha(\delta,h) \le c(\delta+b_h)^{\frac{p}{q+1}}, \quad 0 < \delta \le \delta_0, \quad h \in H.$$

Theorem 5.1.4. Let $\alpha = \alpha(\delta,h)$ be chosen according to (5.13). Then

(i) If $p < q+1$ and $b_h \to 0$ as $h \to 0$, then

$$\|\hat{w} - u^{\delta}_{\alpha,h}\| \to 0 \quad \text{as} \quad \delta \to 0, \quad h \to 0.$$

(ii) $$\|\hat{w} - u^{\delta}_{\alpha,h}\| = O((\delta+b_h)^{r})$$

where $r = \min(\frac{p\nu}{q+1}, 1-\frac{p}{q+1})$.

In particular if $\frac{p}{q+1} = \frac{1}{\nu+1}$, then

(iii) $$\|\hat{w} - u^{\delta}_{\alpha,h}\| = O((\delta+b_h)^{\frac{\nu}{\nu+1}}).$$

Proof: Note that

(5.15) $$\frac{(\delta+b_h)^p}{\alpha^q} = \|P_h A u^{\delta}_{\alpha,h} - P_h g^{\delta}\|$$

$$= \|\alpha(P_hA+\alpha I)^{-1}P_hg^\delta\|$$

$$\le \|\alpha(P_hA+\alpha I)^{-1}P_h(g^\delta-g)\| + \|\alpha(P_hA+\alpha I)^{-1}P_hg\|$$

$$\le \delta + \|\alpha[(P_hA+\alpha I)^{-1}P_h - (A+\alpha I)^{-1}]g\|$$

$$+ \|\alpha(A+\alpha I)^{-1}g\|$$

$$\le \delta + \|\alpha(P_hA+\alpha I)^{-1}[P_h(A+\alpha I) -(P_hA+\alpha I)]$$

$$(A+\alpha I)^{-1}]g\| + \|\alpha(A+\alpha I)^{-1}g\|$$

$$\le \delta + \|\alpha^2(A_h+\alpha I)^{-1}(P_h-I)(A+\alpha I)^{-1}g\|$$

$$+ \|\alpha(A+\alpha I)^{-1}g\|$$

Note that

$$\|\alpha^2(A_h+\alpha I)^{-1}(P_h-I)(A+\alpha I)^{-1}g\| = \|\alpha^2(A_h+\alpha I)^{-1}(P_h-I)A(A+\alpha I)^{-1}\hat{w}\|$$

$$\le b_h\|\hat{w}\|$$

and

$$\|\alpha(A+\alpha I)^{-1}g\| \le \alpha\|\hat{w}\|.$$

Therefore from (5.15) it follows that

$$\frac{(\delta+b_h)^p}{\alpha^q} \le c_1(\delta+b_h) + c_2\alpha,$$

where c_1 and c_2 are positive constants. Now from (5.14) and the fact that $p < q+1$, we have

$$\frac{(\delta+b_h)^p}{\alpha^q} = O((\delta+b_h)^{\frac{p}{q+1}}).$$

Therefore

$$\frac{\delta+b_h}{\alpha} = (\delta+b_h)^{1-\frac{p}{q}}\left(\frac{(\delta+b_h)^p}{\alpha^q}\right)^{1/q}$$

$$= O((\delta+b_h)^{1-\frac{p}{q}+\frac{p}{q(q+1)}}),$$

i.e.,

(5.16) $$\frac{\delta+b_h}{\alpha} = O((\delta+b_h)^{1-\frac{p}{q+1}}).$$

Therefore $\frac{\delta+b_h}{\alpha} \to 0$ as $\delta \to 0$, $h \to 0$. Also by Theorem 4.3.2 (iii) we have $\|\hat{w}-w^{\delta}_{\alpha,h}\| \to 0$. Thus by (5.12), $\|\hat{u}-u^{\delta}_{\alpha,h}\| \to 0$ as $\delta \to 0$, $h \to 0$. Now (ii) follows by applying the estimates in (5.14), (5.16) to the estimate in Theorem 5.1.3. The proof of (iii) is a consequence of (ii) and the fact that $\frac{p\nu}{q+1} = 1-\frac{p}{q+1}$ if and only if $\frac{p}{q+1} = \frac{1}{\nu+1}$, and in that case $\frac{p}{q+1} = \frac{\nu}{\nu+1}$.

Remark 5.1.5. As in Remark 3.1.5, the above method for Simplified regularization can be used for Tikhono regularization also by taking $A = T^*T$, $g = T^*y$ and $g^{\delta} = T^*y^{\delta}$, $\|y-y^{\delta}\| \le \frac{\delta}{c}$ where $c \ge \|T^*\|$. In this case the estimate $\alpha = O((\delta+b_h)^{\frac{p}{q+1}})$ of (5.14) can be used to obtain the estimate

$$\frac{\delta+b_h}{\sqrt{\alpha}} = O((\delta+b_h)^{1-\frac{p}{2(q+1)}}).$$

Therefore if $p < 2(q+1)$ and $\hat{x} \in R((T^*T)^{\nu})$, $0 < \nu \le 1$, then we have

$$\|\hat{x}-x^{\delta}_{\alpha,h}\| = O((\delta+b_h)^{m})$$

where $m = \min\{\frac{p\nu}{q+1}, 1-\frac{p}{2(q+1)}\}$.

This, in particular, gives the optimal estimate $O((\delta+b_h)^{\frac{2\nu}{2\nu+1}})$ for $\frac{p}{q+1} = \frac{2}{2\nu+1}$.

Next we use a modified form of the discrepancy principle (4.26), namely,

$$\alpha^{2(p+1)}\langle (P_hAP_h+\alpha I)^{-2(p+1)}P_hg^{\delta}, P_hg^{\delta}\rangle = (c\delta+db_h)^2, \tag{5.17}$$

where c and d are positive constants, for choosing the regularization parameter α in (5.9). Before proving the existence and uniqueness of α satisfying (5.17) we prove the following result.

Proposition 5.1.6. Let $g \neq 0$ and $g^{\delta} \in F^{\delta}$. Then there exists $\delta_0 > 0$, $h_0 > 0$ such that

$$\|(I-Q_h)P_hg^{\delta}\| \le c\delta+db_h \le \|P_hg^{\delta}\|$$

for all $\delta \le \delta_0$ and $h \le h_0$, where $c > 1$, $d > \|\hat{w}\|$ are constants and Q_h is the orthogonal projection onto $\overline{R(P_hAP_h)}$.

Proof: Since $P_h \to I$ pointwise as $h \to 0$ and $g^\delta \to g$ as $\delta \to 0$ we have

$$\|P_hg^\delta\| \to \|g\| \neq 0 \quad \text{and} \quad c\delta + db_h \to 0 \quad \text{as} \quad \delta \to 0,\ h \to 0 .$$

Therefore there exists $\delta_0 > 0$ and $h_0 > 0$ such that

$$c\delta + db_h \leq \|P_hg^\delta\|, \quad \text{for all} \quad \delta < \delta_0 \quad \text{and} \quad h < h_0.$$

Also, since P_h and Q_h are orthogonal projections and $g = A\hat{w}$,

$$\|(I-Q_h)P_hg^\delta\| \leq \|(I-Q_h)P_h(g^\delta - g)\| + \|(I-Q_h)P_hg\|$$

$$\leq \delta + \|(I-Q_h)P_hA\hat{w}\|$$

But $\|(I-Q_h)P_hAP_h\| = 0$, so that

$$\|(I-Q_h)P_hg^\delta\| \leq \delta + \|(I-Q_h)P_h(A - AP_h)\hat{w}\|$$

$$\leq \delta + b_h\|\hat{w}\|$$

for all $\delta > 0$ and $h > 0$. This completes the proof of the Proposition. ■

Lemma 5.1.7. Let $\delta_0 > 0$ and $h_0 > 0$ be as in Proposition 5.1.6. Then for $\delta \leq \delta_0$, $h \leq h_0$, there exists a unique $\alpha := \alpha(\delta, h)$ satisfying (5.17).

Proof: For fixed $\delta \le \delta_0$, $h \le h_0$, let

$$\varphi(\alpha) = \alpha^{2(\rho+1)}\langle (P_hAP_h+\alpha I)^{-2(\rho+1)}P_hg^\delta, P_hg^\delta\rangle .$$

Then as in Lemma 3.2.1,

$$\varphi(\alpha) = \int \left(\frac{\alpha}{\alpha+\lambda}\right)^{2(\rho+1)} d\langle E_\lambda P_hg^\delta,\ P_hg^\delta\rangle .$$

where $\{E_\lambda\}$, is the spectral family of the operator P_hAP_h. Now the map

$$\alpha \longmapsto f_\rho(\alpha,\lambda) = \left(\frac{\alpha}{\alpha+\lambda}\right)^{2(\rho+1)}$$

is strictly increasing for each $\lambda > 0$, and satisfies

$$f_\rho(\alpha,\lambda) \to 0 \quad \text{as} \quad \alpha \to 0$$

and

$$f_\rho(\alpha,\lambda) \to 1 \quad \text{as} \quad \alpha \to \infty.$$

Therefore by Dominated Convergence Theorem we have

(5.18) $$\varphi(\alpha) \to \|E_0P_hg^\delta\|^2 \quad \text{as} \quad \alpha \to 0$$

where E_0 is the projection on to $\overline{R(P_hAP_h)}^\perp$ and

(5.19) $$\varphi(\alpha) \to \|P_hg^\delta\|^2.$$

Now since

$$E_0P_hg^\delta = E_0Q_hP_hg^\delta + E_0(I-Q_h)P_hg^\delta$$

$$= E_0(I-Q_h)P_hg^\delta.$$

Thus

$$\|E_0P_hg^\delta\| = \|E_0(I-Q_h)P_hg^\delta\|$$

$$\le \|(I-Q_h)P_hg^\delta\|.$$

This together with Proposition 5.1.6, gives

$$\|E_0P_hg^\delta\| \le c\delta+db_h \le \|P_hg^\delta\|$$

for all $\delta \le \delta_0$, $h \le h_0$. Now the Lemma follows by Intermidiate value Theorem by using (5.18) and (5.19). ∎

We note that

$$\|A-P_hAP_h\| \le \|A-AP_h\| + \|(A-P_hA)P_h\|$$

$$\le 2\|A-AP_h\|$$

$$\le 2b_h.$$

Therefore, if $\alpha := \alpha(\delta,h)$ satisfies (5.17), then Lemma 4.4.1 and Lemma 4.4.3 holds with $2b_h$ in place of ε_h. Thus in view of Theorem 5.1.3, we have the following result which is same as Theorem

4.4.4 with $u^{\delta}_{\alpha,h}$, in place of $w^{\delta}_{\alpha,h}$.

Theorem 5.1.8. Let $g^{\delta} \in F^{\delta}$, $h \in H$ and let $\alpha := \alpha(\delta,h)$ be the unique solution of (5.17) with $c > 1$ and $d > e = 2(2+\rho)\|\hat{w}\|$. Let $\hat{w} \in R(A^{\nu})$, $0 < \nu \leq 1$. Then

$$\text{(i)} \qquad \|\hat{w}-u^{\delta}_{\alpha,h}\| = \begin{cases} O((\delta+b_h)^{\frac{\nu}{\rho+1}}, & \nu \leq \rho \\ O((\delta+b_h)^{\frac{\rho}{\rho+1}}, & \nu \geq \rho \end{cases}$$

If $0 < \nu \leq 1$ and $\nu < \rho$, then

$$\text{(ii)} \qquad \|\hat{w}-u^{\delta}_{\alpha,h}\| = o((\delta+b_h)^{\frac{\nu}{\rho+1}}).$$

In particular if $\rho = 1$ in (5.17), then

$$\text{(iii)} \qquad \|\hat{w}-u^{\delta}_{\alpha,h}\| = \begin{cases} O((\delta+b_h)^{\frac{\nu}{2}}, & 0 < \nu < 1 \\ O((\delta+b_h)^{\frac{1}{2}}, & \nu = 1. \end{cases} \qquad \blacksquare$$

5.2. ALGORITHMS

In this Section we give algorithms for implimenting the methods considered in Sections 5.1. Let (V_n) be a sequence of finite dimensional subspaces of X and P_n denote the orthogonal projection on X with $R(P_n) = V_n$. We assume that $\dim V_n = n$, and

(5.20) $$\|P_n x - x\| \to 0 \quad \text{as} \quad n \to 0,$$

for all $x \in X$. Let $\{v_1, \ldots, v_n\}$ be a basis of V_n, $n = 1, 2, \ldots$.

Algorithm 5.2.1. Let $T \in BL(X,Y)$ be a compact operator and let $T_h = TP_n$ where $h = \frac{1}{n}$. Now by assumption (5.20) and the fact that T^* is compact, we have $\|T - T_h\| = \|T - TP_n\| = \|(I - P_n)T^*\| \to 0$ as $n \to \infty$.

With the above notation (5.2) takes the form

(5.21) $$(P_n T^* T P_n + \alpha I) x^{\delta}_{\alpha,h} = P_n T^* y^{\delta}.$$

From (5.21) it follows that

$$x^{\delta}_{\alpha,h} = \frac{1}{\alpha}(P_n T^* y^{\delta} - P_n T^* T P_n x^{\delta}_{\alpha,h})$$

$$= \frac{1}{\alpha} P_n (T^* y^{\delta} - T^* T P_n x^{\delta}_{\alpha,h}) \in V_n.$$

Thus $x^{\delta}_{\alpha,h}$ is of the form $\sum_{i=1}^{n} \lambda_i v_i$ for some scalars $\lambda_1, \ldots, \lambda_n$. It can be seen that $x^{\delta}_{\alpha,h} = \sum_{i=1}^{n} \lambda_i v_i$ is the solution of (5.21) if and only if $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n)^T$ is the unique solution of

(5.22) $$(M_n + \alpha B_n)\bar{\lambda} = W_n$$

where

$$M_n = (\langle Tv_i \, , \, Tv_j \rangle), \quad i, j = 1, \ldots, n,$$

$$B_n = (\langle v_i \, , \, v_j \rangle), \quad i, j = 1, \ldots, n,$$

and

$$W_n = (\langle y^\delta \, , \, Tv_1 \rangle, \ldots, \langle y^\delta \, , \, Tv_n \rangle)^T.$$

Here and below $(\beta_1, \ldots \beta_n)^T$ denotes the transpose of $(\beta_1, \ldots, \beta_n)$.

Note that (5.22) is uniquely solvable because M_n is a positive definite matrix (i.e., $xM_nx^\perp > 0$ for all non-zero vector x) and B_n is an invertible matrix. The parameter α in (5.22) is chosen according to (5.5), which is same as

(5.23) $$\|Tx^\delta_{\alpha,h} - y^\delta\| = \frac{(\delta+b_h)^p}{\alpha^q}.$$

This is equivalent to solving the non linear equation

(5.24) $$f(\alpha) := \alpha^{2q}[-\alpha\overline{\lambda(\alpha)}^T B_n \overline{\lambda(\alpha)} - W_n^T \overline{\lambda(\alpha)} + \langle y^\delta, y^\delta \rangle] - (\delta+b_h)^{2p} = 0$$

where $\overline{\lambda(\alpha)}$ is the solution of (5.22). The parameter $\alpha = \alpha(\delta, b_h)$ satisfying (5.24) can be found as follows.

Step 1 For some initial (good) approximation $\alpha_0 > 0$ find $\overline{\lambda(\alpha_0)}$ satisfying (5.22). Specifically we use cholesky decomposition to compute $\overline{\lambda(\alpha_0)}$.

Step 2 (Newtons method) Using $\overline{\lambda(\alpha_0)}$ of Step 1 compute

$$\alpha_1 = \alpha_0 - \frac{f(\alpha_0)}{f'(\alpha_0)}$$

where

$$f(\alpha) = \alpha^{2q}[-\alpha\overline{\lambda(\alpha)}^T B_n \overline{\lambda(\alpha)} - W_n^T \overline{\lambda(\alpha)} + \langle y^\delta, y^\delta \rangle] - (\delta + b_h)^{2p}$$

and

$$f'(\alpha) = 2q\alpha^{2q-1}[-\alpha\overline{\lambda(\alpha)}^T B_n \overline{\lambda(\alpha)} - W_n^T \overline{\lambda(\alpha)} + \langle y^\delta, y^\delta \rangle]$$

$$+ \alpha^{2q}[-\overline{\lambda(\alpha)}^T B_n \overline{\lambda(\alpha)} - 2\alpha\overline{\lambda(\alpha)}^T B_n \overline{\lambda(\alpha)}' - W_n^T \overline{\lambda(\alpha)}'] .$$

Repeat Step 1 with α_1 in place of α_0 and Step 2 with $\overline{\lambda(\alpha_1)}$ in place of $\overline{\lambda(\alpha_0)}$ and so on. In the $2m^{th}$ step, we obtain

$$\alpha_m = \alpha_{m-1} - \frac{f(\alpha_{m-1})}{f'(\alpha_{m-1})}. \tag{5.25}$$

For sufficiently good initial approximation, the iterates in (5.25) converges to $\alpha(\delta,h)$, the zero of the function $f(\alpha)$.

Algorithm 5.2.2. Let $A \in BL(X)$ be a compact positive self-adjoint operator and let $A_h = P_n A P_n$ where $h = \frac{1}{n}$. In this case we consider the equation (5.9), i.e,

$$P_n A P_n u^\delta_{\alpha,h} + \alpha u^\delta_{\alpha,h} = P_n g^\delta. \tag{5.26}$$

As in Algorithms 5.2.1, it can be seen that $u^\delta_{\alpha,h} \in V_n$. Thus

$u^{\delta}_{\alpha,h}$ is of the form $\sum_{i=1}^{n}\mu_i v_i$ for some scalars $\mu_1, \ldots, \mu_n$. We note that $u^{\delta}_{\alpha,h} = \sum_{i=1}^{n}\mu_i v_i$ is the solution (5.26) if and only if $\bar{\mu} = (\mu_1, \ldots, \mu_n)^T$ is the solution of

$$(5.27) \qquad (A_n + \alpha B_n)\bar{\mu} = Y_n$$

where

$$A_n = (\langle Av_i , v_j\rangle), \quad i, j = 1, \ldots, n,$$

$$B_n = (\langle v_i , v_j\rangle), \quad i, j = 1, \ldots, n,$$

and

$$Y_n = (\langle y^{\delta} , v_1\rangle, \ldots, \langle y^{\delta} , v_n\rangle)^T.$$

The parameter α in (5.27) is chosen according to (5.13), i.e.,

$$(5.28) \qquad \|P_n A u^{\delta}_{\alpha,h} - P_n g^{\delta}\| = \frac{(\delta+b_h)^p}{\alpha^q}.$$

This is equivalent to solving the non linear equation

$$(5.29) \qquad g(\alpha) := \alpha^{2q+2}\overline{\mu(\alpha)}^T B_n \overline{\mu(\alpha)} - (\delta+b_h)^{2p} = 0.$$

The parameter $\alpha = \alpha(\delta, b_h)$ satisfying (5.29) can be found as follows.

Step 1 For some initial approximation $\alpha_0 > 0$ find $\overline{\mu(\alpha_0)}$ satisfying (5.27).

Step 2 Using $\overline{\mu(\alpha_0)}$ of step 1 compute

$$\alpha_1 = \alpha_0 - \frac{g(\alpha_0)}{g'(\alpha_0)}$$

where

$$g(\alpha) = \alpha^{2q+2}\overline{\mu(\alpha)}^T B_n \overline{\mu(\alpha)} - (\delta + b_h)^{2p}$$

and

$$g'(\alpha) = 2(q+1)\alpha^{2q+1}\overline{\mu(\alpha)}^T B_n \overline{\mu(\alpha)} + 2\alpha^{2q+2}\overline{\mu(\alpha)}^T B_n \overline{\mu(\alpha)} ,$$

As in Algorithm 5.2.1, in the $2m^{th}$ step we have

$$\alpha_m = \alpha_{m-1} - \frac{g(\alpha_{m-1})}{g'(\alpha_{m-1})}. \tag{5.30}$$

For sufficiently good initial approximation, the iterates (5.30) converges to $\alpha(\delta,h)$ the zero of the function $g(\alpha)$. We note that the procedure in the above Algorithm is similar to the one given in Engl and Neubauer [6] with $A = T^*T$ and $g^\delta = T^*y^\delta$.

Algorithm 5.2.3. Let A be as in Algorithm 5.2.2. We choose the regularization parameter according to the discrepancy principle (5.17), i.e.,

$$\alpha^{2(\rho+1)}\langle (A + \alpha I)^{-2(\rho+1)} P_n g^\delta , P_n g^\delta \rangle = (c\delta + db_h)^2. \tag{5.31}$$

We consider only two values of ρ, namely, $\rho = 1/2$ and $\rho = 1$.

Case 1 Let $\rho = 1/2$. Then (5.31) takes the form

$$\alpha^3\langle (A + \alpha I)^{-3} P_n g^\delta \ , \ P_n g^\delta \rangle = (c\delta + db_h)^2. \tag{5.32}$$

In this case, choosing the parameter α satisfying (5.32) is equivalent to solving the non linear equation

$$h(\alpha) := \alpha^3 \overline{\zeta(\alpha)}^T B_n \overline{\mu(\alpha)} - (c\delta + db_h)^2 = 0 \tag{5.33}$$

where $\overline{\mu(\alpha)}$ is the solution of the equation (5.27) and $\overline{\zeta(\alpha)} := (\zeta_1(\alpha), \ldots, \zeta_n(\alpha))$ is the solution of

$$(A_n + \alpha B_n)\overline{\zeta(\alpha)} = \overline{\mu(\alpha)} B_n. \tag{5.34}$$

Now the parameter $\alpha := \alpha(\delta, b_n)$ satisfying (5.33) can be found as follows.

Step 1 For $\alpha_0 > 0$ find $\overline{\mu(\alpha_0)}$ satisfying (5. 27). We use cholosky decomposition for computing $\overline{\mu(\alpha_0)}$.

Step 2 Using $\overline{\mu(\alpha_0)}$ compute, $\overline{\zeta(\alpha_0)}$ satisfying (5.34). Here also we use cholosky decomposition.

Step 3 Using $\overline{\mu(\alpha_0)}$ and $\overline{\zeta(\alpha_0)}$ compute

$$\alpha_1 = \alpha_0 - \frac{h(\alpha_0)}{h'(\alpha_0)}$$

where

$$h(\alpha) = \alpha^3 \overline{\zeta(\alpha)}^T B_n \overline{\mu(\alpha)} - (c\delta + db_h)^2$$

and

$$h'(\alpha) = 3\alpha^2\overline{\zeta(\alpha)}^T B_n\overline{\mu(\alpha)} + \alpha^3(\overline{\zeta(\alpha)}^{T'} B_n\overline{\mu(\alpha)} + \overline{\zeta(\alpha)}^T B_n\overline{\mu(\alpha)}').$$

Repeat the process with α_1 to compute α_2 and so on. In the $3m^{th}$ step we have

$$\alpha_m = \alpha_{m-1} - \frac{h(\alpha_{m-1})}{h'(\alpha_{m-1})}. \tag{5.35}$$

For sufficiently good initial approximation, the iterate in (5.35) converges to the zero of $h(\alpha)$.

Case 2 Let $\rho = 1$. In this case (5.31) becomes

$$\alpha^4\langle(A + \alpha I)^{-4}P_n g^\delta , P_n g^\delta\rangle = (c\delta + db_h)^2. \tag{5.36}$$

Now choosing the parameter α satisfying (5.36) is equivalent to solving the equation

$$k(\alpha) = \alpha^4\overline{\zeta(\alpha)}^T B_n\overline{\zeta(\alpha)} - (c\delta + db_h)^2 = 0 \tag{5.37}$$

where $\overline{\zeta(\alpha)}$ is the solution of the equation (5.34).

The parameter $\alpha := \alpha(\delta, b_n)$ satisfying (5.37) can be found as follows.

Step 1 For $\alpha_0 > 0$ find $\overline{\mu(\alpha_0)}$ satisfying (5.27).

Step 2 Using cholosky decomposition and $\overline{\mu(\alpha_0)}$ of step 1, compute $\overline{\zeta(\alpha_0)}$ satisfying (5.34).

Step 3 Using $\overline{\zeta(\alpha_0)}$ compute

$$\alpha_1 = \alpha_0 - \frac{k(\alpha)}{k'(\alpha_0)}$$

where

$$k(\alpha) = \alpha^4 \overline{\zeta(\alpha)}^T B_n \overline{\zeta(\alpha)} - (c\delta + db_h)^2$$

and

$$k'(\alpha) = 4\alpha^3 \overline{\zeta(\alpha)}^T B_n \overline{\zeta(\alpha)} + 2\alpha^4 \overline{\zeta(\alpha)}^T B_n \overline{\zeta(\alpha)}' .$$

Now as in Case 1, in the 3mth step we have

(5.38) $$\alpha_m = \alpha_{m-1} - \frac{k(\alpha_{m-1})}{k'(\alpha_{m-1})}.$$

Here also, for sufficiently good initial approximation, the iterates in (5.38) converges to the zero of $k(\alpha)$.

5.3. NUMERICAL EXAMPLES

In order to illustrate the methods considered in Section 5.1, we consider the space $X = Y = L^2[0,1]$ and consider the Fredholm integral equations of the first kind

(5.39) $$\int_0^1 k(s,t)x(t)dt = y(s)$$

with $k(s,t)$ defined by

$$k(s,t) = \begin{cases} s(1-t), & s \le t \\ t(1-s), & s > t \end{cases} \tag{5.40}$$

We apply the Algotithms in Section 5.2 by choosing V_n as the space of linear splines in a uniform grid of $n+1$ points in $[0,1]$. Specifically for fixed n we consider $t_i = \frac{i-1}{n}$, $i = 1, 2, \ldots, n+1$ as the grid points. We take the basis function v_i, $i = 1, \ldots, n+1$ of V_n as follows:

$$v_1(t) = \begin{cases} \frac{t_2-t}{t_2} & \text{if } 0 = t_1 \le t \le t_2 \\ 0 & \text{if } t_2 \le t \le t_{n+1} = 1 \end{cases}$$

for $j = 2, \ldots, n$,

$$v_j(t) = \begin{cases} 0 & \text{if } 0 \le t \le t_2 \\ \frac{t-t_{j-1}}{t_j-t_{j-1}} & \text{if } t_{j-1} \le t \le t_j \\ \frac{t_{j+1}-t}{t_{j+1}-t_j} & \text{if } t_j \le t \le t_{j+1} \\ 0 & \text{if } t_{j+1} \le t \le t_{n+1} = 1 \end{cases}$$

and

$$v_{n+1}(t) = \begin{cases} 0 & \text{if } 0 \le t \le t_n \\ \frac{t-t_n}{t_{n+1}-t_n} & \text{if } t_n \le t \le t_{n+1} = 1 \end{cases}$$

Let P_n be the orthogonal projection onto V_n. We note that for $x \in C[0,1]$

$$\|P_n x - x\|_2 = \text{dist}(x, R(P_n))$$

$$\leq \|\pi_n x - x\|_2$$

$$\leq \|\pi_n x - x\|_\infty$$

where π_n is the (piecewise linear) interpolatory projection onto V_n. It is known [27] that $\|\pi_n x - x\|_\infty \to 0$ as $n \to \infty$. Therefore using the fact that $C[0,1]$ is dense in $L^2[0,1]$, it follows that

$$\|P_n x - x\|_2 \to 0$$

for all $x \in L^2[0,1]$.

The elements Tv_i, $i = 1, \ldots, n+1$ and the entries of the matrix B_n needed in the Algorithms are computed explicitly. Finally the scalar product, $\langle Tv_i, Tv_j\rangle$ and $\langle y^\delta, Tv_j\rangle$, $i, j = 1, \ldots, n+1$ are computed by trapezoidal rule. For the operator T defined by (5.39) and (5.40), $\gamma_n = \|T - TP_n\| = O(n^{-2})$ (see [17]). We take $y^\delta(s) = y(s) + \delta$, $0 \leq s \leq 1$. The iterations in the algorithms have been stoped as soon as $|\alpha_m - \alpha_{m-1}| \leq 10^{-7}$.

In the tables in Examples 5.3.1 and 5.3.2, $e = \|\hat{x} - x_{\alpha,h}\|$, $\bar{e} = \|\hat{x} - x^\delta_{\alpha,h}\|$ and the last column shows that we obtain the expected convergence rates.

Example 5.3.1 Here we use Algorithm 5.2.1.

a) Let $y(s) = \frac{1}{24}(s - 2s^3 + s^4)$. Then the exact solution is $\hat{x} = T^{\dagger}y(t) = \frac{1}{2}(t - t^2) \in R(T^*)$, since $\hat{x} = T^*1$ (See [6]). In this example we take $p = 2$, $q = 1$ and $b_h = 10^{-1/2}n^{-2}$ where $h = 1/n$. According to Remark 5.1.2 (i) we should obtain the rate $O(n^{-1}10^{-1/4})$. The computational result are as follows.

n	α	e	$e.n.10^{1/4}$
4	5.080676E-02	7.597033E-02	5.403859E-01
8	6.692923E-03	3.616450E-02	5.144848E-01
16	1.386671E-03	1.131647E-02	3.219815E-01
32	3.300157E-04	3.820369E-03	2.173979E-01
64	8.122311E-05	1.702979E-03	1.938159E-01

b) We take y, p, q, b_h are as in (a) and $\delta = n^{-2}10^{-1/2}\|y\|$. According to Remark 5.1.2 we should obtain the rate $O(n^{-1}10^{-1/4})$.

n	α	δ	$\delta.n.10^{1/4}$
4	6.312341E-02	7.690701E-02	5.470486E-01
8	8.384762E-03	3.947685E-02	5.616070E-01
16	1.999828E-03	1.457269E-02	4.146292E-01
32	7.363359E-04	6.595434E-03	3.753128E-01
64	3.437066E-04	3.866229E-03	4.400151E-01

c) Let $y(s) = \frac{1}{30}(3s - 5s^3 + 3s^5 - s^6)$. Then $\hat{x} = T^{\dagger}y(t) = (t - 2t^3 + t^4) \in R(T^*T)$ (See [6]). Here we take $p = 1$, $q = 1/2$ (i.e., Arcangeli's method) and $b_h = 10^{-1}n^{-2}$. In this case we should get the rate $\mathcal{O}(n^{-4/3}10^{-2/3})$.

n	α	e	$e.n^{4/3}.10^{2/3}$
4	9.496383E-02	2.001868E-01	5.899955
8	1.428025E-02	1.290556E-01	9.584369
16	3.934675E-03	6.146955E-02	11.503240
32	1.367126E-03	2.608137E-02	12.298830
64	5.157226E-04	1.064097E-02	12.644100

d) Here y, p, q are as in (c) and $b_h = (n^{-2})^{\frac{6q}{2q+1}}$. Then by Remark 5.1.2 we should obtain the rate $\mathcal{O}(\gamma_h) = \mathcal{O}(n^{-2})$.

n	α	e	$e.n^2$
4	5.032420E-01	2.173905E-01	3.478248
8	1.835899E-02	1.422737E-01	9.105517
16	2.708180E-03	4.631227E-02	11.855940
32	6.017325E-04	1.230339E-02	12.598670
64	1.462539E-04	3.222171E-03	13.198010

e) We take y, p, q are as in (c), $b_h = 10^{-1}n^{-2}$ and $\delta = 10^{-1}n^{-2}\|y\|$. According to Remark 5.1.2 we should get the rate $\mathcal{O}(n^{-4/3}10^{-2/3})$.

n	α	$\bar{e}$	$\bar{e}.n^{4/3}.10^{2/3}$
4	9.764295E-02	2.006114E-01	5.912469
8	1.460831E-02	1.301518E-01	9.665779
16	4.005746E-03	6.221194E-02	11.642170
32	1.389137E-03	2.643386E-02	12.465050
64	5.236521E-04	1.079201E-02	12.823570

f) Let y, p, q and b_h be as in (d) and $\delta = n^{-3}\|y\|$. Again by Remark 5.1.2 we should obtain the rate $O(\gamma_h) = O(n^{-2})$.

n	α	$\bar{e}$	$\bar{e}.n^2$
4	5.114884E-01	2.174024E-01	3.478438
8	1.880284E-02	1.433523E-01	9.174546
16	2.754756E-03	4.690156E-02	12.006800
32	6.108578E-04	1.247299E-02	12.772350
64	1.484457E-04	3.263526E-03	13.36740

g) Let $y(s) = \frac{1}{6}(s - s^3)$. Then $\hat{x} = T^{\dagger}y(t) = t \in R((T^*T)^{\nu})$ for all $\nu < \frac{1}{8}$ (See [36]). Here we take $p = 1$, $q = \frac{1}{2}$, $b_h = 10^{-1/2}n^{-2}$ and $\nu = \frac{1}{8}$. According to Theorem 4.2.2 (ii), we should obtain the rate $O(n^{-1/6}10^{-1/12})$.

n	α	e	$e.n^{1/6}.10^{1/12}$
4	2.032875E-01	6.933492E-01	1.058348
8	2.345335E-02	6.264760E-01	1.073378
16	5.452919E-03	5.571122E-01	1.071427
32	1.624781E-03	5.220351E-01	1.126915
64	4.782124E-04	4.951615E-01	1.199804

h) Let y, p, q and b_h be as in (g) and $\delta = 10^{-1/2}n^{-2}\|y\|$.

n	α	e	$\bar{e}.n^{1/6}.10^{1/12}$
4	1.989054E-01	6.886654E-01	1.051199
8	2.911166E-02	6.317720E-01	1.082452
16	6.711536E-03	5.633281E-01	1.083382
32	1.996030E-03	5.266258E-01	1.136825
64	5.931330E-04	4.994133E-01	1.210106

Now we illustrate the use of Algorithms 5.2.2 and 5.2.3 by considering the operator equation

$$T^*Tx = T^*y$$

where $T: L^2[0,1] \longrightarrow L^2[0,1]$ is given by

$$(Tx)(s) = \int_0^1 k(s,t)x(t)dt.$$

Note that, the Simplified regularization of the above equation is the Tikhonov regularization of the equation $Tx = y$. With this observation we have the following Example.

Example 5.3.2. In the following cases, (a)-(c) and (d), we take y as the corresponding y in (a)-(c) and (e) of Example 5.3.1. We use Algorithm 5.2.2 to compute the regularization parameter α in (5.27).

a) We take $p = 2$, $q = 1$ and $b_h = 10^{-1}n^{-2}$ so that in view of Remark 5.1.5 we should obtain the rate $O(n^{-1}10^{-1/2})$. The following table gives the numerical results.

n	α	e	e.n.10^{1/2}
4	5.022955E-02	7.582424E-02	9.591092E-01
8	6.638567E-03	3.598772E-02	9.104252E-01
16	1.377367E-03	1.125443E-02	5.694289E-01
32	3.284572E-04	3.808466E-03	3.853897E-01
64	8.114147E-05	1.702159E-03	3.444927E-01

b) We take $p = 2$, $q = 1$, $b_h = 10^{-1}n^{-2}$ and $\delta = 10^{-1}n^{-2}\|y\|$. Again in view of Remark 5.1.5 we should obtain the rate $O(n^{-1}10^{-1/2})$.

n	α	$\bar{e}$	$\bar{e}.n.10^{1/2}$
4	5.379350E-02	7.481332E-02	9.463220E-01
8	7.374059E-03	3.651361E-02	9.237294E-01
16	1.525748E-03	1.1543763-02	5.840729E-01
32	3.632755E-04	3.827736E-03	3.873397E-01
64	8.970418E-05	1.688976E-03	3.418248E-01

c) In this case p, q are as in (a) and $b_h = 10^{-3/4}n^{-2}$ so that we should obtain the rate $O(n^{-2}10^{-3/4})$.

n	α	e	$e.n^2 10^{3/4}$
4	6.306100E-02	1.907738E-01	17.164800
8	7.833364E-03	9.601407E-02	34.555320
16	1.584625E-03	2.967627E-02	42.721780
32	3.754008E-04	7.853163E-03	45.221460
64	9.259419E-05	2.237586E-03	51.539440

d) We take $p = 2$, $q = 1$, $b_h = 10^{-3/4}n^{-2}$ and $\delta = 10^{-3/4}n^{-2}\|y\|$. Here we should get the rate $O(n^{-2}10^{-3/4})$.

n	α	$\bar{e}$	$\bar{e}.n^2 10^{3/4}$
4	6.377255E-02	1.907804E-01	17.165390
8	7.932854E-03	9.639260E-02	34.691550
16	1.603083E-03	2.985455E-02	42.978430
32	3.796421E-04	7.902579E-03	45.506020
64	9.363182E-05	2.242575E-03	51.654350

Example 5.3.3. The kernel $k(s,t)$ and y in (a)-(d) are as that of corresponding part of Example 5.3.2. We choose the regularization parameter α in (5.27) according to Algorithm 5.2.3. In the tables below $e = \|\hat{w} - u_{\alpha,h}\|$ and $\bar{e} = \|\hat{w} - u^{\delta}_{\alpha,h}\|$.

a) Here we take $\rho = 1/2$, $b_h = 10^{-2}n^{-2}$ and $d = 1.5$. According to Theorem 4.4.3 (i), we should get the rate $O((n10)^{-2/3})$.

n	α	e	$e.(n.10)^{2/3}$
4	7.923149E-02	8.083386E-02	9.454386E-01
8	5.457997E-03	3.184877E-02	5.913156E-01
16	1.638970E-03	1.296893E-02	3.822236E-01
32	5.931827E-04	5.758975E-03	2.694296E-01
64	2.271117E-04	3.032624E-03	2.252191E-01

b) We take y, ρ, b_h are as in (a) and $\delta = 10^{-2}n^{-2}$. Let $c = 1.5$ and $d = 0.5$. By Theorem 5.1.5 (i), we should obtain the rate $O((n.10)^{-2/3})$.

n	α	e	$\bar{e}.(n.10)^{2/3}$
4	1.550571E-01	8.530082E-02	9.976845E-01
8	6.247536E-03	3.382691E-02	6.280424E-01
16	1.827964E-03	1.386662E-02	4.086806E-01
32	6.562740E-04	6.124841E-03	2.865463E-01
64	2.505496E-04	3.186567E-03	2.366518E-01

c) In this case $\rho = 1$, b_h and c are as in (a). According to theory we should get the rate $O(n^{-1}10^{-1})$.

n	α	e	e.n.10
4	1.449626E-02	1.299059E-01	5.196238
8	4.181230E-03	6.421530E-02	5.137224
16	1.739352E-03	3.215186E-02	5.144298
32	8.020217E-04	1.609211E-02	5.149476
64	3.859191E-04	8.075576E-03	5.168369

d) Let $\rho = 1$, b_h, c, d be as in (b) and $\delta = 10^{-2}n^{-2}$. Here also we should get the rate $O(n^{-1}.10^{-1})$.

n	α	e	$\bar{e}$.n.10
4	1.694081E-02	1.361748E-01	5.446991
8	4.616672E-03	6.787931E-02	5.430345
16	1.891412E-03	3.422990E-02	5.476784
32	8.667071E-04	1.720542E-02	5.505734
64	4.158478E-04	8.650644E-03	5.536412

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