A STUDY OF GENERALIZATIONS OF FIBONACCI SEQUENCE

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BY

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DECLARATION

I, the undersigned, hereby declare that the thesis entitled " A Study of Generalizations of Fibonacci sequence" has been completed by me and has not previously formed the basis for the award of any diploma, degree or any other similar thesis.

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CERTIFICATE

This is to certify that Ms. Suchita Prabhakar Arolkar has successfully completed the thesis entitled " A Study of Generalizations of Fibonacci sequence" for the degree of Doctor of Philosophy in Mathematics under my guidance during the period 2012-2017 and to the best of my knowledge it has not previously formed the basis of award of any degree or diploma in Goa University or elsewhere.

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Chapter 1

Introduction

Leonardo Pisa popularly known as Fibonacci was famous for his book on *Liber Abaci* published around 1202. In this book, he states a recurrence relation which starts with 0 and 1 and the subsequent terms are obtained by adding the preceding two terms. Thus, we have

$$0, 1, 1, 2, 3, 5, \cdots \tag{1.1}$$

Equation (1.1) was later named as the Fibonacci sequence. This name was given by the French mathematician Edouard Lucas in 1876. These numbers can be mathematically expressed in terms of the recurrence relation ([7], [31] and [29]) by

$$F_{n+1} = F_n + F_{n-1}, \forall n \geq 1, \text{ with } F_0 = 0 \text{ and } F_1 = 1, \tag{1.2}$$

where F_n is the n^{th} Fibonacci number.

Edouard Lucas used the same recurrence relation of Fibonacci sequence with different seed values to generate a new sequence which is now called Lucas sequence. Lucas sequence is given by the recurrence relation

$$L_{n+1} = L_n + L_{n-1}, \forall n \geq 1, \text{ with } L_0 = 2 \text{ and } L_1 = 1, \tag{1.3}$$

where L_n is the n^{th} Lucas number. Identities similar to the identities of Fibonacci sequence can be also obtained for the equation (1.3) (see [31], [29]).

Fibonacci is the family surname in Italian and it means "son of the simpleton (Bonaccio)". He was born around the year 1170. Fibonacci studied Indo-Arabic numeration system and computation techniques from his school teacher. Although the Fibonacci sequence was described earlier in Indian mathematics, Fibonacci was the first person to introduce it to the world through his book on *Liber Abaci.* He also included arithmetic, elementary Algebra, Indo-numeration system, elementary algorithms and some examples of business problems. However today he is known to the world mostly for the Fibonacci sequence. People appreciated his work in Indo-Arabic system. Leonardo Fibonacci used this sequence to win a competition sponsored by Emperor Frederick II in 1225. The contest question was: Start with a pair of rabbits. Every month, every pair of rabbits who are over a month old gives birth to a new pair of rabbits. After 'n' months, how many pairs of rabbits are there? He found that solution for this problem was the Fibonacci sequence.

Kepler studied the Fibonacci sequence independently and also its properties [31]. One of the recurrence property he discovered is about the ratios of the consecutive terms of the Fibonacci sequence, that is
$\frac{1}{1} = 1, \frac{2}{1} = 2, \frac{3}{2} = 1.5, \frac{5}{3} = 1.666..., \frac{8}{5} = 1.6, \frac{13}{8} = 1.625, \frac{21}{13} = 1.61538\cdots$ He showed that this ratio approaches a number 1.618 (approx.) which is denoted by ϕ and is commonly known as the Golden ratio, named after the Greek sculptor Phidias, who used it in his artwork. This ratio has many applications. The rectangle in which the sides are in the ratio $\phi : 1$, is considered to be most pleasing to the human eye. There are many more applications in which this golden ratio appears. Like Fibonacci numbers, Tribonacci numbers also play an important role in problems of combinatorics ([16]) and also in the evaluation of determinants of circulant matrices ([6]).

It is this sequence which created interest within us to explore its extensions and learn various extended identities as (1.2) has wide varieties of interesting mathematical properties and various applications ranging from Nature to Technology.

The thesis is designed as follows:

Chapter 1 is Introduction and **Chapter 2** deals with an overview of literature work.

In **Chapter 3** we have introduced B-Tribonacci and B-Tri Lucas sequences, incomplete B-Tribonacci and B-Tri Lucas sequences. We also study various identities related to these sequences.

Chapter 4 deals with the q^{th} order linear recurrence relation as an extension of the ideas introduced in Chapter 3. Here $q \geq 2$ and $q \in \mathbb{N}$. In this chapter B-q bonacci, B-q Lucas, incomplete B-q bonacci and incomplete B-q Lucas sequences are introduced.

In **Chapter 5**, the generalized bivariate B-Tribonacci, B-Tri Lucas, B-q bonacci, B-q Lucas, incomplete B-Tribonacci, incomplete B-Tri Lucas, incomplete B-q bonacci and incomplete B-q Lucas polynomials are introduced. The results discussed in Chapter 3 and Chapter 4 are extended to these polynomials. In this chapter, the identities involving partial derivatives of these polynomials are included.

In **Chapter 6**, the Fibonacci functional equation is extended to the generalized linear Tribonacci functional equation and proven that its solution is associated with generalized Tribonacci sequence. Its stability in the class of functions $f : \mathbb{R} \to X$, where X is a real (or complex) Banach space is obtained. These results are further extended to the generalized linear q-bonacci functional equation.

At the end a brief summary of the work done is included. Few Python programming codes which are used to verify the identities are given in Appendix. This is followed by a list of publications. The thesis ends with a bibliography.

Chapter 2

Literature review

Let (a_n) be a real-valued sequence and $c_{i,}i = 1, 2, \cdots, n$ denote any real constants. The k^{th} order linear homogeneous recurrence relation with constant coefficients given by

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k},\ n, k \in \mathbb{N} \text{ and } k \leq n, \tag{2.1}$$

occur in various branches of Science and Social Science. There are numerous techniques of solving this equation. One technique which is stated in [32] and which we shall use in this thesis is listed below:
Consider the characteristic equation corresponding to (2.1),
$\lambda^n - c_1\lambda^{n-1} - c_2\lambda^{n-2} - c_3\lambda^{n-3} - \cdots - c_k\lambda^{n-k} = 0$ and let $\alpha_i, i = 1, 2, \cdots, n$ be the distinct roots of this characteristic equation. Then the solution of (2.1) is given by

$$a_n = \sum_{i=1}^{n} C_i \alpha_i^n, \text{where } C_i,\ i = 1, 2, \cdots, n, \text{ are any constants.} \tag{2.2}$$

For example, if $k = 2, c_1 = 1, c_2 = 1, a_0 = 0$ and $a_1 = 1$, then (2.1) reduces to the Fibonacci sequence defined by (1.2).

Another concept which we shall use in this thesis is of Generating function.

Generating functions are powerful tools used for solving linear recursion relations and identities relating to them. The function $g(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$ generates the terms of the recurrence relation defined by (2.1) and hence it is called the generating function of the sequence a_n. Thus we have the following definition.

Definition 2.0.1. *Let (a_n) be a sequence of real (or complex) numbers. If there exists a function $g : X \to \mathbb{R}$ such that*

$$g(x) = \sum_{i=0}^{\infty} a_i\, x^i \tag{2.3}$$

then g(x) is called the generating function of the sequence (a_n).

In 1718, the French mathematician Abraham De Moivre (1667-1754) used the generating function to generate the terms of the Fibonacci sequence (1.2) [31]. He proved that the function $f(x) = \frac{1}{1-x(1+x)}$ generates the terms of the Fibonacci sequence. The generating function of Lucas sequence is given by $g(x) = \frac{2-x}{1-x(1+x)}$.

Since (1.2) is a linear homogeneous recurrence relation of second degree, it can be solved using the characteristic equation

$$\lambda^2 - \lambda - 1 = 0 \tag{2.4}$$

If the distinct roots of (2.4) are ϕ_1 and ϕ_2, then the nth term of (1.2) is given by

$$F_n = \frac{\phi_1^n}{\phi_1 - \phi_2} + \frac{\phi_2^n}{\phi_2 - \phi_1} \tag{2.5}$$

Note that $\phi_1 = \frac{1+\sqrt{5}}{2}$ and $\phi_2 = \frac{1-\sqrt{5}}{2}$. Hence (2.5) reduces to

$$F_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n + \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n \tag{2.6}$$

Similarly, we have

$$L_n = \frac{(2\phi_1 - a)\phi_1^n}{\phi_1 - \phi_2} + \frac{(2\phi_2 - a)\phi_2^n}{\phi_2 - \phi_1}$$
$$= \phi_1^n + \phi_2^n. \tag{2.7}$$

In 1843, the French mathematician Jacques-Phillipe-Marie Binet [31] discovered this formula which is one of the techniques of finding the n^{th} term of (1.2). It is called the Binet's formula.

Some of the identities obtained in the thesis are in terms of falling factorial power $n^{\underline{k}}$ (read as n to the k falling) [3]. We define it below. For $n \in \mathbb{N} \cup \{0\}$,

$$n^{\underline{k}} = \begin{cases} n(n-1)\cdots(n-(k-1)), & \text{if } k \in \mathbb{N},\ k \leq n; \\ 0, & \text{if } k > n; \\ 1, & \text{if } k = 0; \\ \frac{1}{(n+1)(n+2)\cdots(n-k)}, & \text{if } k \text{ is a negative integer.} \end{cases} \tag{2.8}$$

The factorial of negative integers k is defined by [3]:

$$(-k)! = (-k)(-k+1)(-k+2)\cdots(-1). \tag{2.9}$$

For negative integer n and integer k ([19], [32]),

$$n^{\underline{k}} = \begin{cases} k!(-1)^{n-k}\,\frac{(-k-1)^{\underline{n-k}}}{(n-k)!}, & \text{if } k \leq n; \\ (-1)^k(-n+k-1)^{\underline{k}}, & \text{if } k \geq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{2.10}$$

We now state the identity related to nth term of (1.2) and (1.3) respectively.

$$F_n = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(n-1-r)^{\underline{r}}}{r!},\ \forall n \geq 1, \tag{2.11}$$

$$L_n = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-r} \frac{(n-r)^{\underline{r}}}{r!}, \ \forall n \geq 1. \tag{2.12}$$

2.1 Identities of Fibonacci sequence

In this section, we state some of the interesting properties of Fibonacci sequence.

(1) If 1 is added to the sum of n+1 terms of Fibonacci sequence with initial term F_0, the resultant sum is $(n+2)^{th}$ term. i.e.

$$1 + \textstyle\sum_{i=0}^{n} F_i = F_{n+2}.$$

(2) The sum of the first n terms with odd suffices with initial value F_1, is the $(2n)^{th}$ term which is the term with even suffix. i.e.

$$\textstyle\sum_{i=1}^{n} F_{2i-1} = F_{2n}.$$

On the other hand, if 1 is added to the sum of the first n+1 terms with even suffices with initial term F_0, the sum is $(2n+1)^{th}$ term. i.e.

$$1 + \textstyle\sum_{i=0}^{n} F_{2i} = F_{2n+1}.$$

(4) The sum of the squares of the first (n+1) terms with initial term F_0 of (1.2), is the product of n^{th} term and $(n+1)^{th}$ term of (1.2), i.e.

$$\textstyle\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}.$$

(5) Sum of the squares of n^{th} term and $(n+1)^{th}$ term is $(2n+1)^{th}$ term. i.e.

$$F_n^2 + F_{n+1}^2 = F_{2n+1}.$$

(6) The difference of the product of $(n+1)^{th}$ term and $(n-1)^{th}$ term, and the square of n^{th} term of the Fibonacci sequence is $(-1)^n$. i.e.

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

Remark: The above identities can be proved using the generating function, Binet's formula or by Mathematical induction on n.

Rewriting (1.2) as $F_{n-1} = F_{n+1} - F_n$ and using $F_0 = 0$ and $F_1 = 1$, we can obtain the following sequence.

$$F_0 = 0,\ F_1 = 1,\ F_{-1} = 1,\ F_{-2} = -1,\ F_{-3} = 2,\ F_{-4} = -3, \cdots.$$

Note that $F_{-n} = (-1)^{n+1}F_n$.

In [31], some of the identities of Fibonacci sequence stated above are proven for Lucas sequence.

Following identities show the relation between Fibonacci and Lucas sequences.

(1) The sum of $(n+1)^{th}$ and $(n-1)^{th}$ Fibonacci numbers is the n^{th} Lucas number. i.e

$$L_n = F_{n+1} + F_{n-1}.$$

(2) If $(n-2)^{th}$ Fibonacci number is subtracted from $(n+2)^{th}$ Fibonacci number then the resultant value is the n^{th} Lucas number. i.e.

$$L_n = F_{n+2} - F_{n-2}.$$

(3) The product of F_{n+1} and L_n is $F_{2n+1} - 1$, if n is odd and $F_{2n+1} + 1$, if n is even, i.e.

$$F_{n+1}L_n = \begin{cases} F_{2n+1} - 1, & \text{n is odd;} \\ F_{2n+1} + 1, & \text{n is even.} \end{cases}$$

2.2 Generalized Fibonacci sequence

In [31], the generalized Fibonacci sequence G_n with initial conditions $G_1 = a$ and $G_2 = b$ is defined by

$$G_{n+1} = G_n + G_{n-1} \tag{2.13}$$

The terms generated by this sequence are

$G_1 = a, G_2 = b, G_3 = a + b, G_4 = a + 2b, G_5 = 2a + 3b$ etc. It is interesting to see that the coefficients of the terms are the terms of classical Fibonacci sequence (1.2).

Thus, we have

$$G_{n+1} = F_{n-1}\, a + F_n\, b, \forall\, n \geq 1. \tag{2.14}$$

(2.14) can be proved using induction on n, see [31]. With a=1 and b=1, the sequence (2.14) reduces to $G_{n+1} = F_{n-1} + F_n$ which is Fibonacci sequence (1.2) and if a=2 and b=1, it reduces to Lucas sequence (1.3).

We state below some of the properties of (2.14).

(1) Sum of the first n terms:

$$\sum_{r=1}^{n} G_r = aF_n + bF_{n+1} - b, \forall n \geq 1. \tag{2.15}$$

(2) Binet's Formula:

$$G_n = c\,\frac{\phi_1^{n-2}}{\phi_1 - \phi_2} + d\frac{\phi_2^{n-2}}{\phi_2 - \phi_1}, \tag{2.16}$$

where $c = a + b\,\phi_1$ and $d = a + b\,\phi_2$.

In [8], the authors consider the set of all sequences (A_n) satisfying the following equation

$$A_{n+2} = aA_{n+1} + bA_n \tag{2.17}$$

with initial terms, A_0 and A_1 and later list various cases of this sequence by giving the choices for a, b, A_0 and A_1 including the generalized Fibonacci and Lucas sequences which are defined below respectively by:

$$F_{n+2} = aF_{n+1} + bF_n, \text{with } F_0 = 0 \text{ and } F_1 = 1, \tag{2.18}$$

$$L_{n+2} = aL_{n+1} + bL_n, \text{with } L_0 = 2 \text{ and } L_1 = a, \tag{2.19}$$

where a and b are fixed real constants. The authors in this paper have studied various properties of generalized Fibonacci sequence and Lucas sequence using the Difference operator.

First few terms of the sequence (2.18) are $F_0 = 0,\ F_1 = 1,\ F_2 = a, F_3 = a^2 + b,$ $F_4 = a^3 + 2ab, F_5 = a^4 + 3a^2b + b^2,\ F_6 = a^5 + 4a^3b + 3ab^2.$

For $0 \le n \le 4$, terms of the sequence (2.19) are $L_0 = 2,\ L_1 = a,\ L_2 = a^2 + 2b,$ $L_3 = a^3 + 3ab,\ L_4 = a^4 + 4a^2b + 2b^2.$

The terms of (2.18) can also be obtained by adding the anti-diagonal terms of the following Pascal type triangle.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & a & & b & & \\ & a^2 & & 2ab & & b^2 & \\ a^3 & & 3a^2b & & 3ab^2 & & b^3 \\ & & & \cdots & & & \end{array}$$

Rewriting equation (2.18), we get

$$F_{n-1} = \frac{1}{b}(F_{n+1} - a\ F_n),\ \text{with}\ \ F_0 = 0 \text{ and } F_1 = 1. \tag{2.20}$$

For $-2 \le n \le 0$, we obtain the terms $F_{-1} = \frac{1}{b}, F_{-2} = \frac{-a}{b^2}, F_{-3} = \frac{a^2+b}{b^3}.$

We list below some of the properties of (2.18) that fascinated us ([8] and [25]).

(1) The n^{th} number F_n is given by

$$F_n = \begin{cases} \frac{\phi_1^n}{\phi_1 - \phi_2} + \frac{\phi_2^n}{\phi_2 - \phi_1}, & a^2 + 4b \neq 0; \\ n\phi^{n-1} \quad , & a^2 + 4b = 0, \quad \phi_1 = \phi_2 = \phi, \end{cases} \tag{2.21}$$

where $\phi_1 = \frac{a+\sqrt{a^2+4b}}{2}$ and $\phi_2 = \frac{a-\sqrt{a^2+4b}}{2}$, for all $a, b \in \mathbb{R} \setminus \{0\}$, are roots of the equation $\lambda^2 - a\,\lambda - b = 0$.

(2) The generating function for Fibonacci sequence (2.18) is given by

$$G(x) = \frac{1}{1 - x(a + bx)}. \tag{2.22}$$

(3) The n^{th} number F_n is also given by

$$F_n = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(n-1-r)^{\underline{r}}}{r!} \; a^{n-1-2r}\, b^r, \; \forall n \geq 1. \tag{2.23}$$

With $a = k$ and $b = 1$, the result can be seen in [25].

(4) For all $n \geq 0$,

$$\sum_{r=0}^{n} F_r = \frac{bF_n + F_{n+1} - 1}{a + b - 1}, \tag{2.24}$$

provided $a + b \neq 1$.

Another form of the extended Fibonacci sequence defined in [25] and [21], the k-Fibonacci sequence can be obtained from (2.18) by substituting $a = k$ and $b = 1$. Various identities related to this sequence are included in this paper.

In Matrix form, the Fibonacci sequence (see [8]) is represented by

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix}.$$

Let A$= \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix} = \begin{bmatrix} F_0 & F_1 \\ bF_1 & F_2 \end{bmatrix}$, then $A^n = \begin{bmatrix} b\,F_{n-1} & F_n \\ b\,F_n & F_{n+1} \end{bmatrix}$.

Following identities can be proved by using the above matrix representation.

(5) (**Honsberger identity**)

For any $m, n \in \mathbb{Z}$,

$$F_{n+m-1} = bF_{n-1}F_{m-1} + \ F_nF_m. \tag{2.25}$$

With $m = n$ identity (2.25) reduces to

(a) $F_{2n-1} = b\ F_{n-1}^2 + F_n^2$.

With $m = n + 1$ identity (2.25) reduces to

(b) $F_{2n} = bF_{n-1}F_n + F_nF_{n+1}$.

Also using (2.18) and (5b), we can obtain, $F_{2n} = aF_n^2 + 2b\ F_{n-1}F_n$.

(6) (**General bilinear identity**)

For all $m_1, m_2, n_1, n_2 \in \mathbb{Z}$ with $m_1 + n_2 = m_2 + n_1$,

$$\begin{vmatrix} F_{m_1} & F_{n_1} \\ F_{m_2} & F_{n_2} \end{vmatrix} = (-b)^s \begin{vmatrix} F_{m_1-s} & F_{n_1-s} \\ F_{m_2-s} & F_{n_2-s} \end{vmatrix}. \tag{2.26}$$

(7) (**d'Ocagne identity**)

For all $m, n \in \mathbb{Z}$,

$$\begin{vmatrix} F_m & F_n \\ F_{m+1} & F_{n+1} \end{vmatrix} = (-b)^n F_{m-n}. \tag{2.27}$$

(8) (**Catalan identity**)

For all $n, r \in \mathbb{Z}$,

$$\begin{vmatrix} F_n & F_{n+r} \\ F_{n-r} & F_n \end{vmatrix} = -(-b)^n F_r\, F_{-r} = (-b)^{n-r} F_r^2. \tag{2.28}$$

putting $r = 1$, in (2.28) we get the following identity.

(9) (**Cassini identity**)

$$\begin{vmatrix} F_n & F_{n-1} \\ F_{n+1} & F_n \end{vmatrix} = (-b)^{n-1}, \forall n \in \mathbb{Z}. \tag{2.29}$$

2.3 Incomplete Fibonacci and Lucas sequences

Filipponi introduced the incomplete Fibonacci numbers F_n^l, incomplete Lucas numbers L_n^l as well as studied their various identities in [22]. Various identities related to the incomplete k-Fibonacci and k-Lucas numbers are studied in [12]. The author defines the incomplete k-Fibonacci and k-Lucas numbers respectively by

$$F_{k,n}^l = \sum_{i=0}^{l} \frac{(n-1-i)^{\underline{i}}}{i!}\ k^{n-1-2i},\ 0 \leq l \leq \lfloor \frac{n-1}{2} \rfloor,\ \ \forall n \geq 1, \tag{2.30}$$

$$L_{k,n}^l = \sum_{i=0}^{l} \frac{n}{n-i} \frac{(n-i)^{\underline{i}}}{i!}\ k^{n-2i},\ \ 0 \leq l \leq \lfloor \frac{n}{2} \rfloor,\ \forall n \geq 1, \tag{2.31}$$

where k is a positive real number.

He also studied various identities of these sequences.

We list below the properties of (2.30) and (2.31), (see [12]).

(1) For all $n \geq 2$,

$$F_{k,n+2}^{l+1} = k\ F_{k,n+1}^{l+1} + F_{k,n}^l,\ \ 0 \leq l \leq \lfloor \frac{n-2}{2} \rfloor. \tag{2.32}$$

Using (2.30), equation (2.32) can be rewritten as

$$F^{l}_{k,n+2} = k\ F^{l}_{k,n+1} + F^{l}_{k,n} - \frac{(n-1-l)^{\underline{l}}}{l!}\ \ k^{n-1-2l}. \tag{2.33}$$

(2) For all $s \geq 0$,

$$\sum_{i=0}^{s} \frac{s^{\underline{i}}}{i!}\ F^{l+i}_{k,n+i}\ k^{i} = F^{l+s}_{k,n+s},\ \ 0 \leq l \leq \lfloor \frac{n-s-1}{2} \rfloor. \tag{2.34}$$

(3) For all $s \geq 1$,

$$\sum_{i=0}^{s-1} F^{l}_{k,n+i} k^{s-1-i} = F^{l+1}_{k,n+s+1} - k^{s} F^{l+1}_{k,n+1}. \tag{2.35}$$

(4) For all $n \geq 2$,

$$L^{l+1}_{k,n+2} = k\ L^{l+1}_{k,n+1} + L^{l}_{k,n},\ \ 0 \leq l \leq \lfloor \frac{n-2}{2} \rfloor. \tag{2.36}$$

Using (2.31), equation (2.36) can be rewritten as

$$L^{l}_{k,n+2} = k\ L^{l}_{k,n+1} + L^{l}_{k,n} - \frac{(n-1-l)^{\underline{l}}}{l!}\ \ k^{n-1-2l}. \tag{2.37}$$

(5) For all $n \geq 1,\ s \geq 0$,

$$\sum_{i=0}^{s} \frac{s^{\underline{i}}}{i!}\ L^{l+i}_{k,n+i}\ k^{i} = L^{l+s}_{k,n+s},\ \ 0 \leq l \leq \lfloor \frac{n-s-1}{2} \rfloor. \tag{2.38}$$

(6) For all $s \geq 1$,

$$\sum_{i=0}^{s-1} L^{l}_{k,n+i}\ k^{s-1-i} = L^{l+1}_{k,n+s+1} - k^{s} L^{l+1}_{k,n+1}. \tag{2.39}$$

(7) For all $n \geq 2$,

$$L^{l}_{k,n} = F^{l-1}_{k,n-1} + F^{l}_{k,n+1},\ \ 0 \leq l \leq \lfloor \frac{n}{2} \rfloor. \tag{2.40}$$

2.4 Fibonacci polynomials

Fibonacci polynomials are the natural extensions of Fibonacci sequence. In [25] and [31], these polynomials are studied in one variable, where as Hongquan Yu and Chuan-guang Liang derives identities involving partial derivatives of bivariate Fibonacci and Lucas polynomials in [11]. In [25], Fibonacci polynomials are defined by

$$F_{n+1}(x) = \begin{cases} 1, & \text{when } n = 0, \\ x, & \text{when } n = 1, \\ xF_n(x) + F_{n-1}(x), & \text{when } n \geq 2, \end{cases} \tag{2.41}$$

with $F_0(x) = 0$, where $F_n(x)$ is the n^{th} Fibonacci polynomial.

In [15], Lucas polynomials in x are defined by

$$L_{n+1}(x) = xL_n(x) + L_{n-1}(x),\ \forall n \geq 1, \tag{2.42}$$

with $L_0(x) = 2$ and $L_1(x) = x$, where $L_n(x)$ is the n^{th} Lucas polynomial.

In [11], the bivariate Fibonacci and Lucas polynomials are respectively defined by

$$F_{n+1}(x, y) = xF_n(x, y) + yF_{n-1}(x, y),\ \forall n \geq 1, \tag{2.43}$$

with $F_0(x, y) = 0$ and $F_1(x, y) = 1$, where $F_n(x, y)$ is the n^{th} Fibonacci polynomial.

$$L_{n+1}(x, y) = xL_n(x, y) + yL_{n-1}(x, y),\ \forall n \geq 1, \tag{2.44}$$

with $L_0(x, y) = 2$ and $L_1(x, y) = x$, where $L_n(x, y)$ is the n^{th} Lucas polynomial.

Various properties related to the polynomials (2.43) and (2.44) are obtained in [31].

For simplicity, let F_n denote $F_n(x, y)$ and L_n denote $L_n(x, y)$.

The n^{th} term of (2.43) and (2.44) respectively as defined in [11] are given below

$$F_n = \sum_{i=0}^{[\frac{n-1}{2}]} \frac{(n-1-i)^{\underline{i}}}{i!} \; x^{n-2i-1}\, y^r, \;\; \forall n \geq 1. \tag{2.45}$$

$$L_n = \sum_{i=0}^{\left[\frac{n}{2}\right]} \left(\frac{n}{(n-i)} \frac{(n-i)^{\underline{i}}}{i!} \right) \; x^{n-2i}\, y^i, \;\; \forall n \geq 1. \tag{2.46}$$

Identities relating Fibonacci and Lucas polynomials (2.43) and (2.44) are

(1) $L_n = F_{n+1} + yF_{n-1}$.

(2) $L_n = 2F_{n+1} - xF_n$.

Following identities involving partial derivatives of F_n and L_n discussed in [11].

Let $F_n^{(k,j)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j}\big(F_n\big)$ and $L_n^{(k,j)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j}\big(L_n\big)$, $k, j \geq 0$.

we list the identities below:

(1) $L_n^{(k,j)} = yF_{n-1}^{(k,j)} + jF_{n-1}^{(k,j-1)} + F_{n+1}^{(k,j)}$.

(2) $F_n^{(k,j)} = xF_{n-1}^{(k,j)} + yF_{n-2}^{(k,j)} + kF_{n-1}^{(k-1,j)} + jF_{n-2}^{(k,j-1)}$.

(3) $L_n^{(k,j)} = xL_{n-1}^{(k,j)} + yL_{n-2}^{(k,j)} + kL_{n-1}^{(k-1,j)} + jL_{n-2}^{(k,j-1)}$.

(4) $nF_n^{(k,j)} = L_n^{(k+1,j)}$.

(5) $nF_{n-1}^{(k,j)} = L_n^{(k,j+1)}$.

2.5 Incomplete Fibonacci and Lucas polynomials

The incomplete $h(x)$-Fibonacci and $h(x)$-Lucas polynomials and their identities are introduced in [14], whereas in [13], the incomplete Tribonacci polynomials and their identities are studied.

Definition 2.5.1. *The incomplete $h(x)$-Fibonacci polynomials is defined by*

$$F_{h,n}^{l}(x) = \sum_{i=0}^{l} \frac{(n-1-i)^{\underline{i}}}{i!}\ h^{n-1-2i}(x), \ \ 0 \leq l \leq \lfloor \frac{n-1}{2} \rfloor. \tag{2.47}$$

Definition 2.5.2. *The incomplete $h(x)$-Lucas polynomials is defined by*

$$L_{h,n}^{l}(x) = \sum_{i=0}^{l} \frac{n}{n-i} \frac{(n-i)^{\underline{i}}}{i!}\ h^{n-2i}(x), \ \ 0 \leq l \leq \lfloor \frac{n}{2} \rfloor. \tag{2.48}$$

Identities similar to incomplete k-Fibonacci and k-Lucas sequences are obtained for the polynomials $F_{h,n}^{l}(x)$ and $L_{h,n}^{l}(x)$ in [14].

In [13], the Tribonacci numbers are defined by

$$t_{n+2} = t_{n+1} + t_n + t_{n-1}, \ \forall n \geq 1, \tag{2.49}$$

with $t_0 = 0$, $t_1 = 1$ and $t_2 = 1$.

In [13], Jose L.R. introduces the incomplete Tribonacci numbers and incomplete Tribonacci polynomials. These are respectively defined by

$$t_n^{l} = \sum_{i=0}^{l} \sum_{j=0}^{i} \frac{i^{\underline{j}}}{j!} \frac{(n-i-j-1)^{\underline{i}}}{i!}, \ \ 0 \leq l \leq \lfloor \frac{n-1}{2} \rfloor. \tag{2.50}$$

$$T_n^{l}(x) = \sum_{i=0}^{l} \sum_{j=0}^{i} \frac{i^{\underline{j}}}{j!} \frac{(n-i-j-1)^{\underline{i}}}{i!} x^{2n-2-3(i+j)}, \ \ 0 \leq l \leq \lfloor \frac{n-1}{2} \rfloor. \tag{2.51}$$

Various identities relating to (2.50) and (2.51) are discussed in [13].

2.6 Functional equations

A functional equation is an equation whose solutions are the functions [24]. Stability problems of functional equations have been extensively studied. (see [9], ([23]), ([26]) and references therein). The importance of the topic lies in the fact that stability of functional equation is associated with notions of Controlled Chaos [30] and Shadowing [33]. In [26], the author discusses the stability problem in Banach space for Fibonacci functional equation defined by $f(x) = f(x-1)+f(x-2)$, whereas in [27], he discusses the stability of the generalized functional equation defined by

$$f(x) = pf(x-1) - qf(x-2), \forall p, q \in \mathbb{R}, \tag{2.52}$$

in Banach space. In [20], the problem is discussed in Modular Functional space. In [4], k-Fibonacci functional equation is discussed whereas in [18] and [10] solution and stability of Tribonacci functional equation $f(x) = f(x-1) + f(x-2) + f(x-3)$ in non-Archimedean Banach spaces and 2-normed spaces have been discussed respectively. Stability of Tribonacci and k-Tribonacci functional equations in Modular spaces are discussed in [17]. In [28], authors investigate the solution of generalized linear Tribonacci functional equation in terms of Fibonacci numbers.

Chapter 3

B-Tribonacci and B-Tri Lucas sequences

This Chapter include the content of published paper P1.

Chapter 3

B-Tribonacci and B-Tri Lucas Sequences

3.1 Introduction

In this Chapter, we introduce a new extension of generalized Fibonacci sequence defined by the recurrence relation (2.18), namely, $F_{n+1} = aF_n + bF_{n-1}$, with $F_0 = 0$ and $F_1 = 1$. We consider the coefficient on the right hand side, namely a and b to be the terms of the binomial expansion of $(a + b)^1$. We rename this sequence as B-Fibonacci sequence.

Through out this Chapter, we denote a and b to be non-zero real numbers. We define B-Fibonacci sequence in terms of new notation as follows.

Definition 3.1.1. *Let $n \in \mathbb{N} \cup \{0\}$. The B-Fibonacci sequence is defined by*

$$({}^fB)_{n+1} = a\ ({}^fB)_n + b\ ({}^fB)_{n-1},\ \forall\ n \geq 1, \tag{3.1}$$

$$\textit{with } ({}^fB)_0 = 0 \textit{ and } ({}^fB)_1 = 1,$$

where $({}^fB)_n$ is the n^{th} term.

Rewriting equation (3.1),we get

$$({}^fB)_{n-1} = \frac{1}{b}\big(({}^fB)_{n+1} - a\ ({}^fB)_n\big), \tag{3.2}$$

$$with\ ({}^fB)_0 = 0\ and\ ({}^fB)_1 = 1.$$

Using this representation, we obtain the terms of $({}^fB)_n$ with negative integer suffix. Thus, (3.1) is true for all $n \in \mathbb{Z}$ and we have,

$$({}^fB)_{n+1} = a\ ({}^fB)_n + b\ ({}^fB)_{n-1}, \forall n \in \mathbb{Z}, \tag{3.3}$$

$$with\ ({}^fB)_0 = 0\ and\ ({}^fB)_1 = 1,$$

where $({}^fB)_n$ is the n^{th} term of the sequence defined by (3.3).

Note that equation (3.3) is equation (2.18) of Chapter 2 with the change in notation and hence all the identities stated there holds. The change in notation from F to $({}^fB)$ is made with expected further extensions.

The above idea is extended to Tribonacci sequence such that the n^{th} term is obtained by adding the preceding $(n-1)^{th}$, $(n-2)^{th}$, $(n-3)^{th}$ terms having coefficient a^2, $2ab$ and b^2 respectively. These coefficients are the terms of the binomial expansion of $(a+b)^2$. It is well known that the binomial coefficients carry a lot of combinatorial information in them. As Binomial expansion is an important tool in Combinatorics related fields, it is natural to expect some applications of such sequences.

In Section 2 of this Chapter, we study B-Tribonacci sequence and its various identities. In Section 3, we introduce B-Tri Lucas sequence and extend the identities of the B-Tribonacci sequence to B-Tri Lucas sequence. The last section deals with the incomplete B-Tribonacci sequence and incomplete B-Tri Lucas sequence. We also study their identities.

3.2 B-Tribonacci sequence

The Tribonacci sequence is an extension of the Fibonacci sequence where each term is the sum of the three preceding terms. This sequence has been extended in many ways. Here we extend the idea of introducing the sequence (3.3) to define a new sequence. We call it B-Tribonacci sequence and denote it by $({}^tB)_n$.

Definition 3.2.1. *Let $n \in \mathbb{N} \cup \{0\}$. The B-Tribonacci sequence is defined by*

$$({}^tB)_{n+2} = a^2({}^tB)_{n+1} + 2ab({}^tB)_n + b^2\ ({}^tB)_{n-1},\ n \geq 1, \tag{3.4}$$
$$\textit{with } ({}^tB)_0 = 0,\ ({}^tB)_1 = 0 \textit{ and } ({}^tB)_2 = 1,$$

where the coefficients on the right hand side are the terms of the binomial expansion of $(a+b)^2$ and $({}^tB)_n$ is the $\mathrm{n^{th}}$ *term.*

The first six terms of (3.4) are $({}^tB)_0 = 0$, $({}^tB)_1 = 0$, $({}^tB)_2 = 1$, $({}^tB)_3 = a^2$, $({}^tB)_4 = a^4 + 2ab$ and $({}^tB)_5 = a^6 + 4a^3b + b^2$.

Rewriting equation (3.4), we get

$$({}^tB)_{n-1} = \frac{1}{b^2}\Big[({}^tB)_{n+2} - a^2\ ({}^tB)_{n+1} - 2ab\ ({}^tB)_n\Big], \tag{3.5}$$
$$\textit{with } ({}^tB)_0 = 0,\ \ ({}^tB)_1 = 0 \textit{ and } ({}^tB)_2 = 1.$$

For $-3 \leq n \leq 0$, we have the terms of (3.5) as follows:

$({}^tB)_{-1} = \frac{1}{b^2}, ({}^tB)_{-2} = \frac{-2a}{b^3}, ({}^tB)_{-3} = \frac{3a^2}{b^4}, ({}^tB)_{-4} = \frac{-4a^3}{b^5} + \frac{1}{b^4} = \frac{1}{b^6}(-4a^3b + b^2)$.

Thus, Definition 3.2.1 can be extended as follows:

Definition 3.2.2. *The B-Tribonacci sequence is defined by*

$$({}^tB)_{n+2} = a^2({}^tB)_{n+1} + 2ab({}^tB)_n + b^2\ ({}^tB)_{n-1},\ \forall\ n \in \mathbb{Z}, \tag{3.6}$$

$$\textit{with } ({}^tB)_0 = 0,\ ({}^tB)_1 = 0 \textit{ and } ({}^tB)_2 = 1,$$

where $({}^tB)_n$ is the n^{th} term of (3.6).

We have following identities for the B-Tribonacci sequence.

The n^{th} term of Fibonacci type sequences can be obtained directly using Binet formula. We have similar type formula for B-Tribonacci sequence.

Theorem 3.2.3. *If ϕ_1, ϕ_2 and ϕ_3 are roots of the characteristic equation*

$$\lambda^3 - a^2\lambda^2 - 2ab\lambda - b^2 = 0 \tag{3.7}$$

corresponding to (3.6), *then the n^{th} term of B-Tribonacci sequence (3.6) is given by*

$$({}^tB)_n = \begin{cases} \frac{\phi_1^n}{(\phi_1-\phi_2)(\phi_1-\phi_3)} + \frac{\phi_2^n}{(\phi_2-\phi_1)(\phi_2-\phi_3)} + \frac{\phi_3^n}{(\phi_3-\phi_1)(\phi_3-\phi_2)}\ , & \phi_i\text{'s are all distinct,} \\ \\ \frac{\phi_1^n}{(\phi_2-\phi_1)^2} - \frac{\phi_2^n}{(\phi_2-\phi_1)^2} + \frac{n\phi_2^{n-1}}{(\phi_2-\phi_1)}, & \phi_i\text{'s are such that } \phi_1 \neq \phi_2 = \phi_3. \end{cases} \tag{3.8}$$

Proof. If ϕ_1, ϕ_2 and ϕ_3 are distinct roots of the characteristic equation (3.7), then the solution of (3.6) is given by

$$({}^tB)_n = C_1\phi_1^n + C_2\phi_2^n + C_3\phi_3^n, \text{ where } \mathrm{C_i},\ \mathrm{i} = 1,2,3 \text{ are real constants.} \tag{3.9}$$

If any two roots of the characteristic equation (3.7) are equal, say, $\phi_2 = \phi_3$, then its solution is given by

$$({}^tB)_n = C_1\phi_1^n + (C_2 + n\,C_3)\phi_2^n, \text{ where } \mathrm{C_i},\ \mathrm{i} = 1,2,3 \text{ are real constants.} \tag{3.10}$$

Equations (3.9) and (3.10), satisfying the conditions $({}^tB)_0 = 0, ({}^tB)_1 = 0$ and $({}^tB)_2 = 1$, leads to (3.8). □

Equation (3.8) is a Binet type formula for the B-Tribonacci sequence (3.6).

Remark: The case of all three roots of the characteristic equation being equal, is ruled out due to the choice of coefficients.

Theorem 3.2.4. *The n^{th} term of B-Tribonacci sequence* (3.4) *is given by*

$$({}^tB)_n = \sum_{r=0}^{\left\lfloor \frac{2n-4}{3} \right\rfloor} \frac{(2n-4-2r)^{\underline{r}}}{r!} \; a^{2n-4-3r} \; b^r, \; \forall n \geq 2. \tag{3.11}$$

Proof. By induction on n.

For $n = 2$, R.H.S. of (3.11) $= \sum_{r=0}^{0} \frac{(-2r)^{\underline{r}}}{r!} \; a^{-3r} \; b^r = 1 = ({}^tB)_2 =$ L.H.S., hence the result is true for $n = 2$.

Now let the result be true for $n \leqslant m$ and consider $n = m + 1$.

Let $k \geq 1$. We divide the proof into three cases, $m = 3k$, $3k+1$, $3k+2$.

Case (i) $m = 3k$,

Consider, $a^2({}^tB)_{3k} + 2ab({}^tB)_{3k-1} + b^2({}^tB)_{3k-2}$

$$= \sum_{r=0}^{2k-2} \frac{(6k-4-2r)^{\underline{r}}}{r!} \; a^{6k-2-3r} \; b^r + 2\sum_{r=0}^{2k-2} \frac{(6k-6-2r)^{\underline{r}}}{r!} \; a^{6k-5-3r} \; b^{r+1}$$

$$+ \sum_{r=0}^{2k-3} \frac{(6k-8-2r)^{\underline{r}}}{r!} \; a^{6k-8-3r} \; b^{r+2}$$

$$= \frac{(6k-4)^{\underline{0}}}{0!} \; a^{6k-2} b^0 + \left(\frac{(6k-6)^{\underline{1}}}{1!} + 2\right) a^{6k-5} \; b^1$$

$$+ \sum_{r=2}^{2k-1} \left(\frac{(6k-4-2r)^{\underline{r}}}{r!} + 2\frac{(6k-4-2r)^{\underline{r-1}}}{(r-1)!} + \frac{(6k-4-2r)^{\underline{r-2}}}{(r-2)!}\right) \; a^{6k-2-3r} \; b^r$$

$$= \frac{(6k-4)^{\underline{0}}}{0!} \; a^{6k-2} b^0 + \frac{(6k-4)^{\underline{1}}}{1!} \; a^{6k-5} \; b^1$$

$$+ \sum_{r=2}^{2k-1} \left(\frac{(6k-3-2r)^{\underline{r}}}{r!} + \frac{(6k-3-2r)^{\underline{r-1}}}{(r-1)!}\right) \; a^{6k-2-3r} \; b^r$$

$$= \frac{(6k-4)^{\underline{0}}}{0!} \; a^{6k-2} b^0 + \frac{(6k-4)^{\underline{1}}}{1!} \; a^{6k-5} \; b^1 + \sum_{r=2}^{2k-1} \frac{(6k-2-2r)^{\underline{r}}}{r!} \; a^{6k-2-3r} \; b^r$$

$$= \sum_{r=0}^{2k-1} \frac{(6k-2-2r)^{\underline{r}}}{r!} \; a^{6k-2-3r} \; b^r$$

$= ({}^tB)_{3k+1}.$

Similarly, we can prove the result for $m = 3k+1$ and $m = 3k+2$.
This completes the proof. $\square$

Using similar procedure we can prove the following Corollary.

Corollary 3.2.5. *The n^{th} term of B-Tribonacci sequence (3.5) is given by*

$$({}^tB)_n = \sum_{r=n-1}^{\left\lfloor \frac{2n-4}{3} \right\rfloor} \frac{(2n-4-2r)^{\underline{r}}}{r!} \; a^{2n-4-3r}\, b^r, \;\; \forall n \leq -1. \tag{3.12}$$

Theorem 3.2.6. *Sum of the first $n+1$ terms of B-Tribonacci sequence (3.4) is*

$$\sum_{r=0}^{n} ({}^tB)_r = \frac{({}^tB)_{n+1} + (b^2+2ab)({}^tB)_n + b^2({}^tB)_{n-1} - 1}{(a+b)^2 - 1}, \;\; \forall n \geq 0, \tag{3.13}$$

provided $a + b \neq 1, -1$.

Proof. Note that for $n = 0$, $\sum_{r=0}^{0} ({}^tB)_r = 0$.
Also, since $({}^tB)_{-1} = \frac{1}{b^2}$, R.H.S. of (3.13) $= 0$. Hence the result holds for $n = 0$.
For $n \geq 1$, we prove the result by induction on n.
Let $n = 1$, R.H.S.$= \frac{({}^tB)_2 + (b^2+2ab)({}^tB)_1 + b^2({}^tB)_0 - 1}{(a+b)^2-1} = 0 =$ L.H.S., as $({}^tB)_2 = 1$.
Therefore, the result holds for $n = 1$. Assume that the result is true for $n \leq m$.
Let $n = m+1$.

$$\sum_{r=0}^{m+1} ({}^tB)_r = \sum_{r=0}^{m} ({}^tB)_r + ({}^tB)_{m+1}$$

$$= \frac{({}^tB)_{m+1} + (b^2+2ab)({}^tB)_m + b^2({}^tB)_{m-1} - 1}{(a+b)^2-1} + ({}^tB)_{m+1}$$

$$= \frac{(b^2+2ab)({}^tB)_m + b^2({}^tB)_{m-1} - 1 + (a^2+2ab+b^2)({}^tB)_{m+1}}{(a+b)^2-1}$$

$$= \frac{({}^tB)_{m+2} + (b^2+2ab)({}^tB)_{m+1} + b^2({}^tB)_m - 1}{(a+b)^2 - 1}$$

Hence the result is true for $n = m+1$.

By mathematical induction, result follows. □

Similarly, we can prove the following Corollary.

Corollary 3.2.7. *Sum of the n terms of $({}^tB)_{-r}$, for $1 \leq r \leq n$ is*

$$\sum_{r=1}^{n} ({}^tB)_{-r} = -\frac{({}^tB)_{-n} + (b^2+2ab)({}^tB)_{-(n+1)} + b^2({}^tB)_{-(n+2)} - 1}{(a+b)^2 - 1}, \tag{3.14}$$

$\forall n \geq 1$, provided $a+b \neq 1, -1$.

Rewriting (3.14), we have

$$\sum_{r=-1}^{-n} ({}^tB)_{r} = -\frac{({}^tB)_{-n} + (b^2+2ab)({}^tB)_{-(n+1)} + b^2({}^tB)_{-(n+2)} - 1}{(a+b)^2 - 1}, \tag{3.15}$$

provided $a+b \neq 1, -1$.

Combining (3.13) and (3.15), we have

$$\sum_{r=-n}^{n} ({}^tB)_r = \frac{1}{(a+b)^2-1}\Big[\big(({}^tB)_{n+1} - ({}^tB)_{-n}\big)$$

$$+(b^2+2ab)\big(({}^tB)_n - ({}^tB)_{-(n+1)}\big) + b^2\big(({}^tB)_{n-1} - ({}^tB)_{-(n+2)}\big)\Big] \tag{3.16}$$

provided $a+b \neq 1, -1$.

Equation (3.16) gives the sum of the terms of (3.6) from $r = -n$ to $r = n$.

We now state the excluded cases in the above result.

(a) Let $a, b \in \mathbb{R}$. If $a+b = 1$, substituting $a = 1-b$ in (3.4) and then simplifying, we obtain the r^{th} term of B-Tribonacci sequence (3.4), given by

$$({}^tB)_r = \sum_{p=0}^{r-2} (-2b)^p + \sum_{p=0}^{\left\lfloor \frac{r-3}{2} \right\rfloor} \sum_{s=r-1-p}^{2r-4-3p} (-1)^s \frac{(2r-4-2p)^{\underline{s+p}}}{p!s!} b^{s+p},$$

where $\sum_{p=0}^{l}() = 0$, if $l < 0$.

Hence we have, $\sum_{r=2}^{n+1}({}^tB)_r$

$$= \sum_{r=2}^{n+1}\sum_{p=0}^{r-2}(-2b)^p + \sum_{r=2}^{n+1}\sum_{p=0}^{\left\lfloor\frac{r-3}{2}\right\rfloor}\sum_{s=r-1-p}^{2r-4-3p}(-1)^s\frac{(2r-4-2p)^{\underline{s+p}}}{p!s!}\,b^{s+p}$$

$$= \frac{1}{1+2b}\left(n + 2b\,\frac{1-(-2b)^n}{1+2b}\right) + \sum_{r=2}^{n+1}\sum_{p=0}^{\left\lfloor\frac{r-3}{2}\right\rfloor}\sum_{s=r-1-p}^{2r-4-3p}(-1)^s\frac{(2r-4-2p)^{\underline{s+p}}}{p!s!}\,b^{s+p},$$

provided $b \neq -\frac{1}{2}$.

If $b = -\frac{1}{2}$, then $a = \frac{3}{2}$ and equation (3.4) reduces to

$$({}^tB)_{n+2} = \frac{9}{4}({}^tB)_{n+1} - \frac{3}{2}({}^tB)_n + \frac{1}{4}({}^tB)_{n-1}.$$

The roots of the characteristic equation $\lambda^3 - \frac{9}{4}\lambda^2 + \frac{3}{2}\lambda - \frac{1}{4} = 0$ corresponding to (3.17) are $\phi_1 = \frac{1}{4}$, $\phi_2 = 1 = \phi_3$ and Binet type formula gives the r^{th} term,

$$({}^tB)_r = \frac{\phi_1^r}{(\phi_2-\phi_1)^2} - \frac{\phi_2^r}{(\phi_2-\phi_1)^2} + \frac{r\phi_2^{r-1}}{\phi_2-\phi_1}.$$

Hence,

$$\sum_{r=0}^{n+1}({}^tB)_r = \sum_{r=0}^{n+1}\frac{\phi_1^r}{(\phi_2-\phi_1)^2} - \frac{\phi_2^r}{(\phi_2-\phi_1)^2} + \frac{r\phi_2^{r-1}}{\phi_2-\phi_1}$$

$$= \left(\tfrac{4}{3}\right)^2\left[\tfrac{4}{3}\left(1-\left(\tfrac{1}{4}\right)^{n+2}\right) - (n+2)\right] + \tfrac{2}{3}(n+1)(n+2).$$

(b) If $a + b = -1$, then the n^{th} term of B-Tribonacci sequence (3.4) is given by

$$({}^tB)_n = \sum_{r=0}^{n-2}(2b)^r + \sum_{r=0}^{\left\lfloor\frac{n-3}{2}\right\rfloor}(-1)^r\sum_{s=n-1-r}^{2n-4-3r}\frac{(2n-4-2r)^{\underline{s+r}}}{r!s!}\,b^{s+r},$$

where $\sum_{r=0}^{l}() = 0$, if $l < 0$.

This case can also be discussed as above.

A similar type of cases can be studied for equation (3.5).

For $a = \frac{3}{4}$ and $b = \frac{1}{4}$ such that $a + b = 1$, we have the following graph for the sequence (3.4) defined by

$$({}^tB)_{n+2} = (\frac{3}{4})^2({}^tB)_{n+1} + 2(\frac{3}{4})(\frac{1}{4})({}^tB)_n + (\frac{1}{4})^2\ ({}^tB)_{n-1}. \tag{3.17}$$

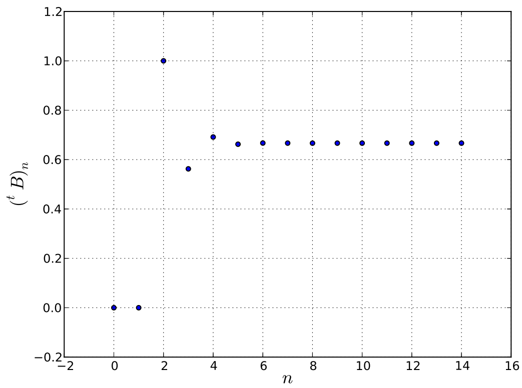

Figure 3-1: Graph showing terms of (3.17)

An interesting property of B-Tribonacci sequence is that like B-Fibonacci sequence, the ratio of successive B-Tribonacci sequence converges to one of the roots, say, ϕ_1 of the characteristics equation corresponding to the recurrence relation (3.5). Similarly, a ratio of preceding terms converges to $\frac{1}{\phi_1}$ which is also the root of characteristic equation corresponding to the recurrence relation (3.5). We have the following theorem.

Theorem 3.2.8. *Let the roots ϕ_1, ϕ_2 and ϕ_3 of (3.6) be distinct $\phi_1 \neq 0$ and $|\phi_1| > |\phi_2| > |\phi_3|$, then*

(i)

$$\lim_{n\to\infty} \frac{({}^tB)_n}{({}^tB)_{n-1}} = \phi_1. \tag{3.18}$$

(ii)

$$\lim_{n\to\infty} \frac{({}^tB)_{n-1}}{({}^tB)_n} = \frac{1}{\phi_1}. \tag{3.19}$$

Proof. By using Binet type formula (3.8).

$$\lim_{n\to\infty}\frac{({}^tB)_n}{({}^tB)_{n-1}} = \lim_{n\to\infty}\frac{\phi_1^n(\phi_2-\phi_3)-\phi_2^n(\phi_1-\phi_3)+\phi_3^n(\phi_1-\phi_2)}{\phi_1^{n-1}(\phi_2-\phi_3)-\phi_2^{n-1}(\phi_1-\phi_3)+\phi_3^{n-1}(\phi_1-\phi_2)}.$$

Since $|\phi_1| > |\phi_2| > |\phi_3|$, $\frac{|\phi_2|}{|\phi_1|} < 1$ and $\frac{|\phi_3|}{|\phi_1|} < 1$.
Hence $\lim_{n\to\infty}(\frac{|\phi_2|}{|\phi_1|})^n = 0$ and $\lim_{n\to\infty}(\frac{|\phi_3|}{|\phi_1|})^n = 0$.

Therefore we have,

$$\begin{aligned}\lim_{n\to\infty}\frac{({}^tB)_n}{({}^tB)_{n-1}} &= \lim_{n\to\infty}\frac{(\phi_2-\phi_3)-\left(\frac{\phi_2}{\phi_1}\right)^n(\phi_1-\phi_3)+\left(\frac{\phi_3}{\phi_1}\right)^n(\phi_1-\phi_2)}{\phi_1^{-1}(\phi_2-\phi_3)-(\frac{\phi_2}{\phi_1})^n\left(\frac{\phi_1-\phi_3}{\phi_2}\right)+(\frac{\phi_3}{\phi_1})^n\left(\frac{\phi_1-\phi_2}{\phi_3}\right)}\\ &= \frac{(\phi_2-\phi_3)}{\phi_1^{-1}(\phi_2-\phi_3)}\\ &= \phi_1.\end{aligned}$$

Again using Binet type formula, we can prove the equation (3.19). □

Theorem 3.2.9. *The terms of the equation (3.6) can be generated from the series*

$$\sum_{n=-\infty}^{\infty} z^n(a+bz)^{2n}.$$

Proof. $\sum_{n=-\infty}^{\infty} z^n(a+bz)^{2n} = \sum_{n=-\infty}^{-1} z^n(a+bz)^{2n} + \sum_{n=0}^{\infty} z^n(a+bz)^{2n}$

$= \sum_{n=-\infty}^{-1} z^n \ \sum_{k=0}^{\infty} \frac{(2n)\underline{k}}{k!}\ a^k b^{2n-k} z^{2n-k} + \sum_{n=0}^{\infty} z^n \ \sum_{k=0}^{2n} \frac{(2n)\underline{k}}{k!}\ a^{2n-k} b^k z^k$

$= \cdots + z^{-1}\frac{(-2)\underline{1}}{1!}\ a^1b^{-3}z^{-3} + z^{-1}\frac{(-3)\underline{0}}{0!}\ a^0b^{-2}z^{-2} + z^0(a+bz)^0 + z^1(a+bz)^2 + z^2(a+bz)^4 + \cdots$

$= \cdots - 2ab^{-3}z^{-4} + b^{-2}z^{-3} + 0\ z^{-2} + 0\ z^{-1} + 1\ z^0 + a^2z + (a^4+2ab)z^2 + \cdots$

$= \cdots + ({}^tB)_{-2}z^{-4} + ({}^tB)_{-1}z^{-3} + ({}^tB)_0\ z^{-2} + ({}^tB)_1\ z^{-1} + ({}^tB)_2\ z^0 + ({}^tB)_3\ z + ({}^tB)_4\ z^2 + \cdots.$

$= \sum_{n=-\infty}^{\infty}({}^tB)_{n+2}\ z^n.$

Hence the theorem is proved. □

Corollary 3.2.10. *(1) The generating function for B-Tribonacci sequence (3.4) is given by*

$$({}^tG)_1(z) = \frac{1}{1 - z(a+bz)^2)}, \tag{3.20}$$

provided $|z(a+bz)^2| < 1$.

(2) The generating function for B-Tribonacci sequence (3.5) is given by

$$({}^tG)_2(z) = \frac{1}{b^2}\Big(\frac{1}{1 - \frac{1}{b^2}(z^3 + b^2 - (az+b)^2)}\Big), \tag{3.21}$$

provided $|\frac{1}{b^2}(z^3 + b^2 - (az+b)^2)| < 1$.

The B-Tribonacci sequence (3.6) can be represented in Matrix form as follows:

$$\begin{bmatrix} ({}^tB)_n \\ ({}^tB)_{n+1} \\ ({}^tB)_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ b^2 & 2ab & a^2 \end{bmatrix} \begin{bmatrix} ({}^tB)_{n-1} \\ ({}^tB)_n \\ ({}^tB)_{n+1} \end{bmatrix}.$$

Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ b^2 & 2ab & a^2 \end{bmatrix}$, then

$$A^n = \begin{bmatrix} b^2({}^tB)_{n-1} & b^2({}^tB)_{n-2} + 2ab({}^tB)_{n-1} & ({}^tB)_n \\ b^2({}^tB)_n & b^2({}^tB)_{n-1} + 2ab({}^tB)_n & ({}^tB)_{n+1} \\ b^2({}^tB)_{n+1} & b^2({}^tB)_n + 2ab({}^tB)_{n+1} & ({}^tB)_{n+2} \end{bmatrix}. \tag{3.22}$$

Using (3.6), equation (3.22) can also be written as

$$A^n = \begin{bmatrix} b^2({}^tB)_{n-1} & ({}^tB)_{n+1} - a^2({}^tB)_n & ({}^tB)_n \\ b^2({}^tB)_n & ({}^tB)_{n+2} - a^2({}^tB)_{n+1} & ({}^tB)_{n+1} \\ b^2({}^tB)_{n+1} & ({}^tB)_{n+3} - a^2({}^tB)_{n+2} & ({}^tB)_{n+2} \end{bmatrix}. \tag{3.23}$$

Note that $A^0 = I$, is an identity matrix of order 3x3. Following result can be deduced from (3.22).

Theorem 3.2.11. (***Honsberger type identity***)
For any $m, n \in \mathbb{Z}$,

$$({}^tB)_{n+m-1} = b^2({}^tB)_{n-1}({}^tB)_{m-1} + ({}^tB)_m(b^2({}^tB)_{n-2} + 2ab({}^tB)_{n-1}) + ({}^tB)_n({}^tB)_{m+1}. \tag{3.24}$$

Proof. Equation (3.22) implies

$$A^{n+m} = \begin{bmatrix} b^2({}^tB)_{n+m-1} & b^2({}^tB)_{n+m-2} + 2ab({}^tB)_{n+m-1} & ({}^tB)_{n+m} \\ b^2({}^tB)_{n+m} & b^2({}^tB)_{n+m-1} + 2ab({}^tB)_{n+m} & ({}^tB)_{n+m+1} \\ b^2({}^tB)_{n+m+1} & b^2({}^tB)_{n+m} + 2ab({}^tB)_{n+m+1} & ({}^tB)_{n+m+2}. \end{bmatrix} \tag{3.25}$$

Let M_{11} denote the element of first row and first column of the matrix. Then, equating the element M_{11} of the matrix obtained by multiplying the matrices A^n and A^m with the element M_{11} of the matrix defined by (3.25), we get the required result. □

Following Corollary follows immediately.

Corollary 3.2.12. *For any $n \in \mathbb{Z}$,*

(1) $({}^tB)_{2n-1} = b^2({}^tB)^2_{n-1} + 2\ ({}^tB)_n({}^tB)_{n+1} - a^2({}^tB)^2_n$.

(2) $({}^tB)_{2n} = ({}^tB)^2_{n+1} + 2ab({}^tB)^2_n + 2b^2({}^tB)_n({}^tB)_{n-1}$.

Proof.

(1) Substituting $m = n$ in (3.24) and using (3.6), we get (1).

(2) Substituting $m = n + 1$ in (3.24) and using (3.6), we get (2). □

Theorem 3.2.13. (***General Trilinear identity***)

For all $m_i, n_i, r_i, s \in \mathbb{Z},\ 1 \leq i \leq 3,$

$$\begin{vmatrix} ({}^tB)_{m_1} & ({}^tB)_{n_1} & ({}^tB)_{r_1} \\ ({}^tB)_{m_2} & ({}^tB)_{n_2} & ({}^tB)_{r_2} \\ ({}^tB)_{m_3} & ({}^tB)_{n_3} & ({}^tB)_{r_3} \end{vmatrix} = [(-b)^2]^s \begin{vmatrix} ({}^tB)_{m_1-s} & ({}^tB)_{n_1-s} & ({}^tB)_{r_1-s} \\ ({}^tB)_{m_2-s} & ({}^tB)_{n_2-s} & ({}^tB)_{r_2-s} \\ ({}^tB)_{m_3-s} & ({}^tB)_{n_3-s} & ({}^tB)_{r_3-s} \end{vmatrix}, \tag{3.26}$$

provided $n_i + r_k = n_k + r_i, m_i + n_j = m_j + n_i, m_j + r_k = m_k + r_j$, *for distinct* i, j, k *such that* $i, j, k = 1, 2, 3$.

Proof. Let $\alpha_1 = \phi_2 - \phi_3,\ \ \alpha_2 = \phi_1 - \phi_3,\ \ \alpha_3 = \phi_1 - \phi_2$, where $\phi_i, i = 1, 2, 3$ are distinct roots of $\lambda^3 - a^2\lambda^2 - 2ab\lambda - b^2 = 0$. Therefore (3.8) implies

$$({}^tB)_n = \frac{\alpha_1\phi_1^n - \alpha_2\phi_2^n + \alpha_3\phi_3^n}{\alpha_1\alpha_2\alpha_3} = \frac{\sum_{i=1}^3(-1)^{i+1}\alpha_i\phi_i^n}{\prod_{i=1}^3 \alpha_i}.$$

Consider,

$$R.H.S. = [(-b)^2]^s \begin{vmatrix} \frac{\sum_{i=1}^3(-1)^{i+1}\alpha_i\phi_i^{m_1-S}}{\prod_{i=1}^3\alpha_i} & \frac{\sum_{i=1}^3(-1)^{i+1}\alpha_i\phi_i^{n_1-s}}{\prod_{i=1}^3\alpha_i} & \frac{\sum_{i=1}^3(-1)^{i+1}\alpha_i\phi_i^{r_1-S}}{\prod_{i=1}^3\alpha_i} \\ \frac{\sum_{i=1}^3(-1)^{i+1}\alpha_i\phi_i^{m_2-S}}{\prod_{i=1}^3\alpha_i} & \frac{\sum_{i=1}^3(-1)^{i+1}\alpha_i\phi_i^{n_2-S}}{\prod_{i=1}^3\alpha_i} & \frac{\sum_{i=1}^3(-1)^{i+1}\alpha_i\phi_i^{r_2-S}}{\prod_{i=1}^3\alpha_i} \\ \frac{\sum_{i=1}^3(-1)^{i+1}\alpha_i\phi_i^{m_3-S}}{\prod_{i=1}^3\alpha_i} & \frac{\sum_{i=1}^3(-1)^{i+1}\alpha_i\phi_i^{n_3-S}}{\prod_{i=1}^3\alpha_i} & \frac{\sum_{i=1}^3(-1)^{i+1}\alpha_i\phi_i^{r_3-S}}{\prod_{i=1}^3\alpha_i} \end{vmatrix}$$

$$= \frac{[(-b)^2]^s}{(\prod_{i=1}^3\alpha_i)^3}\Big[(\alpha_1\phi_1^{m_1-s} - \alpha_2\phi_2^{m_1-s} + \alpha_3\phi_3^{m_1-s})$$

$$\Big((\alpha_1\phi_1^{n_2-s} - \alpha_2\phi_2^{n_2-s} + \alpha_3\phi_3^{n_2-s})(\alpha_1\phi_1^{r_3-s} - \alpha_2\phi_2^{r_3-s} + \alpha_3\phi_3^{r_3-s})$$

$$-(\alpha_1\phi_1^{r_2-s} - \alpha_2\phi_2^{r_2-s} + \alpha_3\phi_3^{r_2-s})(\alpha_1\phi_1^{n_3-s} - \alpha_2\phi_2^{n_3-s} + \alpha_3\phi_3^{n_3-s})\Big)$$

$$-(\alpha_1\phi_1^{n_1-s} - \alpha_2\phi_2^{n_1-s} + \alpha_3\phi_3^{n_1-s})$$

$$\Big((\alpha_1\phi_1^{m_2-s}-\alpha_2\phi_2^{m_2-s}+\alpha_3\phi_3^{m_2-s})(\alpha_1\phi_1^{r_3-s}-\alpha_2\phi_2^{r_3-s}+\alpha_3\phi_3^{r_3-s})$$

$$-(\alpha_1\phi_1^{r_2-s}-\alpha_2\phi_2^{r_2-s}+\alpha_3\phi_3^{r_2-s})(\alpha_1\phi_1^{m_3-s}-\alpha_2\phi_2^{m_3-s}+\alpha_3\phi_3^{m_3-s})\Big)$$

$$+(\alpha_1\phi_1^{r_1-s}-\alpha_2\phi_2^{r_1-s}+\alpha_3\phi_3^{r_1-s})$$

$$\Big((\alpha_1\phi_1^{m_2-s}-\alpha_2\phi_2^{m_2-s}+\alpha_3\phi_3^{m_2-s})(\alpha_1\phi_1^{n_3-s}-\alpha_2\phi_2^{n_3-s}+\alpha_3\phi_3^{n_3-s})$$

$$-(\alpha_1\phi_1^{n_2-s}-\alpha_2\phi_2^{n_2-s}+\alpha_3\phi_3^{r_2-s})(\alpha_1\phi_1^{m_3-s}-\alpha_2\phi_2^{m_3-s}+\alpha_3\phi_3^{m_3-s})\Big)\Big]$$

$$=\frac{[(-b)^2]^s}{(\prod_{i=1}^3\alpha_i)^3}\Big[(\alpha_1\phi_1^{m_1-s}-\alpha_2\phi_2^{m_1-s}+\alpha_3\phi_3^{m_1-s})$$

$$\Big(-\alpha_1\alpha_2\big(\phi_1\phi_2\big)^{-s}(\phi_1^{n_2}\phi_2^{r_3}+\phi_1^{r_3}\phi_2^{n_2}-\phi_1^{r_2}\phi_2^{n_3}-\phi_1^{n_3}\phi_2^{r_2})$$

$$+\alpha_1\alpha_3\big(\phi_1\phi_3\big)^{-s}(\phi_1^{n_2}\phi_3^{r_3}+\phi_1^{r_3}\phi_3^{n_2}-\phi_1^{r_2}\phi_3^{n_3}-\phi_1^{n_3}\phi_3^{r_2})$$

$$-\alpha_2\alpha_3\big(\phi_2\phi_3\big)^{-s}(\phi_2^{n_2}\phi_3^{r_3}+\phi_2^{r_3}\phi_3^{n_2}-\phi_2^{r_2}\phi_3^{n_3}-\phi_2^{n_3}\phi_3^{r_2})\Big)$$

$$-(\alpha_1\phi_1^{n_1-s}-\alpha_2\phi_2^{n_1-s}+\alpha_3\phi_3^{n_1-s})$$

$$\Big(-\alpha_1\alpha_2\big(\phi_1\phi_2\big)^{-s}(\phi_1^{m_2}\phi_2^{r_3}+\phi_1^{r_3}\phi_2^{m_2}-\phi_1^{r_2}\phi_2^{m_3}-\phi_1^{m_3}\phi_2^{r_2})$$

$$+\alpha_1\alpha_3\big(\phi_1\phi_3\big)^{-s}(\phi_1^{m_2}\phi_3^{r_3}+\phi_1^{r_3}\phi_3^{m_2}-\phi_1^{r_2}\phi_3^{m_3}-\phi_1^{m_3}\phi_3^{r_2})$$

$$-\alpha_2\alpha_3\big(\phi_2\phi_3\big)^{-s}(\phi_2^{m_2}\phi_3^{r_3}+\phi_2^{r_3}\phi_3^{m_2}-\phi_2^{r_2}\phi_3^{m_3}-\phi_2^{m_3}\phi_3^{r_2})\Big)$$

$$+\left(\alpha_1\phi_1^{r_1-s}-\alpha_2\phi_2^{r_1-s}+\alpha_3\phi_3^{r_1-s}\right)$$

$$\Big(-\alpha_1\alpha_2\big(\phi_1\phi_2\big)^{-s}(\phi_1^{m_2}\phi_2^{n_3}+\phi_1^{n_3}\phi_2^{m_2}-\phi_1^{n_2}\phi_2^{m_3}-\phi_1^{m_3}\phi_2^{n_2})$$

$$+\alpha_1\alpha_3\big(\phi_1\phi_3\big)^{-s}(\phi_1^{m_2}\phi_3^{n_3}+\phi_1^{n_3}\phi_3^{m_2}-\phi_1^{n_2}\phi_3^{m_3}-\phi_1^{m_3}\phi_3^{n_2})$$

$$-\alpha_2\alpha_3\big(\phi_2\phi_3\big)^{-s}(\phi_2^{m_2}\phi_3^{n_3}+\phi_2^{n_3}\phi_3^{m_2}-\phi_2^{n_2}\phi_3^{m_3}-\phi_2^{m_3}\phi_3^{n_2})\Big)\Bigg],$$

since $n_i+r_k=n_k+r_i, m_i+n_j=m_j+n_i, m_j+r_k=m_k+r_j$, for distinct i,j,k such that $i,j,k=1,2,3,$

$$=\frac{[(-b)^2]^s}{(\prod_{i=1}^3\alpha_i)^3}\Bigg[\Big(-\alpha_1\alpha_2\alpha_3\big(\phi_1\phi_2\phi_3\big)^{-s}\phi_3^{m_1}(\phi_1^{n_2}\phi_2^{r_3}+\phi_1^{r_3}\phi_2^{n_2}-\phi_1^{r_2}\phi_2^{n_3}-\phi_1^{n_3}\phi_2^{r_2})$$

$$-\alpha_1\alpha_3\alpha_2\phi_2^{m_1}\big(\phi_1\phi_3\phi_2\big)^{-s}(\phi_1^{n_2}\phi_3^{r_3}+\phi_1^{r_3}\phi_3^{n_2}-\phi_1^{r_2}\phi_3^{n_3}-\phi_1^{n_3}\phi_3^{r_2})$$

$$-\alpha_2\alpha_3\alpha_1\phi_1^{m_1}\big(\phi_2\phi_3\phi_1\big)^{-s}(\phi_2^{n_2}\phi_3^{r_3}+\phi_2^{r_3}\phi_3^{n_2}-\phi_2^{r_2}\phi_3^{n_3}-\phi_2^{n_3}\phi_3^{r_2})\Big)$$

$$-\Big(-\alpha_1\alpha_2\alpha_3\phi_3^{n_1}\big(\phi_1\phi_2\phi_3\big)^{-s}(\phi_1^{m_2}\phi_2^{r_3}+\phi_1^{r_3}\phi_2^{m_2}-\phi_1^{r_2}\phi_2^{m_3}-\phi_1^{m_3}\phi_2^{r_2})$$

$$-\alpha_1\alpha_3\alpha_2\phi_2^{n_1}\big(\phi_1\phi_3\phi_2\big)^{-s}(\phi_1^{m_2}\phi_3^{r_3}+\phi_1^{r_3}\phi_3^{m_2}-\phi_1^{r_2}\phi_3^{m_3}-\phi_1^{m_3}\phi_3^{r_2})$$

$$-\alpha_2\alpha_3\alpha_1\phi_1^{n_1}\big(\phi_2\phi_3\phi_1\big)^{-s}(\phi_2^{m_2}\phi_3^{r_3}+\phi_2^{r_3}\phi_3^{m_2}-\phi_2^{r_2}\phi_3^{m_3}-\phi_2^{m_3}\phi_3^{r_2})\Big)$$

$$\Big(-\alpha_1\alpha_2\alpha_3\phi_3^{r_1}\big(\phi_1\phi_2\phi_3\big)^{-s}(\phi_1^{m_2}\phi_2^{n_3}+\phi_1^{n_3}\phi_2^{m_2}-\phi_1^{n_2}\phi_2^{m_3}-\phi_1^{m_3}\phi_2^{n_2})$$

$$-\alpha_1\alpha_3\alpha_2\phi_2^{r_1}\big(\phi_1\phi_3\phi_2\big)^{-s}(\phi_1^{m_2}\phi_3^{n_3}+\phi_1^{n_3}\phi_3^{m_2}-\phi_1^{n_2}\phi_3^{m_3}-\phi_1^{m_3}\phi_3^{n_2})$$

$$-\alpha_2\alpha_3\alpha_1\phi_1^{r_1}\big(\phi_2\phi_3\phi_1\big)^{-s}(\phi_2^{m_2}\phi_3^{n_3}+\phi_2^{n_3}\phi_3^{m_2}-\phi_2^{n_2}\phi_3^{m_3}-\phi_2^{m_3}\phi_3^{n_2})\Big)\Bigg],$$

since $n_i+r_k=n_k+r_i, m_i+n_j=m_j+n_i, m_j+r_k=m_k+r_j$, for distinct i,j,k such that $i,j,k=1,2,3$ and $\phi_1\phi_2\phi_3=b^2$,

$$=\frac{[(-b)^2]^s}{(\prod_{i=1}^3\alpha_i)^3}\Bigg[-\alpha_1\alpha_2\alpha_3\big(b^2\big)^{-s}\Big(\phi_3^{m_1}(\phi_1^{n_2}\phi_2^{r_3}+\phi_1^{r_3}\phi_2^{n_2}-\phi_1^{r_2}\phi_2^{n_3}-\phi_1^{n_3}\phi_2^{r_2})$$

$$+\phi_2^{m_1}(\phi_1^{n_2}\phi_3^{r_3}+\phi_1^{r_3}\phi_3^{n_2}-\phi_1^{r_2}\phi_3^{n_3}-\phi_1^{n_3}\phi_3^{r_2})$$

$$+\phi_1^{m_1}(\phi_2^{n_2}\phi_3^{r_3}+\phi_2^{r_3}\phi_3^{n_2}-\phi_2^{r_2}\phi_3^{n_3}-\phi_2^{n_3}\phi_3^{r_2})\Big)$$

$$-\alpha_1\alpha_2\alpha_3\big(b^2\big)^{-s}\Big(\phi_3^{n_1}(\phi_1^{m_2}\phi_2^{r_3}+\phi_1^{r_3}\phi_2^{m_2}-\phi_1^{r_2}\phi_2^{m_3}-\phi_1^{m_3}\phi_2^{r_2})$$

$$+\phi_2^{n_1}(\phi_1^{m_2}\phi_3^{r_3}+\phi_1^{r_3}\phi_3^{m_2}-\phi_1^{r_2}\phi_3^{m_3}-\phi_1^{m_3}\phi_3^{r_2})$$

$$+\phi_1^{n_1}(\phi_2^{m_2}\phi_3^{r_3}+\phi_2^{r_3}\phi_3^{m_2}-\phi_2^{r_2}\phi_3^{m_3}-\phi_2^{m_3}\phi_3^{r_2})\Big)$$

$$-\alpha_1\alpha_2\alpha_3\big(b^2\big)^{-s}\Big(\phi_3^{r_1}(\phi_1^{m_2}\phi_2^{n_3}+\phi_1^{n_3}\phi_2^{m_2}-\phi_1^{n_2}\phi_2^{m_3}-\phi_1^{m_3}\phi_2^{n_2})$$

$$+\phi_2^{r_1}(\phi_1^{m_2}\phi_3^{n_3}+\phi_1^{n_3}\phi_3^{m_2}-\phi_1^{n_2}\phi_3^{m_3}-\phi_1^{m_3}\phi_3^{n_2})$$

$$+\phi_1^{r_1}(\phi_2^{m_2}\phi_3^{n_3}+\phi_2^{n_3}\phi_3^{m_2}-\phi_2^{n_2}\phi_3^{m_3}-\phi_2^{m_3}\phi_3^{n_2})\Big)\Bigg]$$

$$=\frac{1}{(\prod_{i=1}^3\alpha_i)^3}(-\alpha_1\alpha_2\alpha_3)\Bigg[\phi_3^{m_1}(\phi_1^{n_2}\phi_2^{r_3}+\phi_1^{r_3}\phi_2^{n_2}-\phi_1^{r_2}\phi_2^{n_3}-\phi_1^{n_3}\phi_2^{r_2})$$

$$+\phi_2^{m_1}(\phi_1^{n_2}\phi_3^{r_3}+\phi_1^{r_3}\phi_3^{n_2}-\phi_1^{r_2}\phi_3^{n_3}-\phi_1^{n_3}\phi_3^{r_2})$$

$$+\phi_1^{m_1}(\phi_2^{n_2}\phi_3^{r_3}+\phi_2^{r_3}\phi_3^{n_2}-\phi_2^{r_2}\phi_3^{n_3}-\phi_2^{n_3}\phi_3^{r_2})$$

$$+\phi_3^{n_1}(\phi_1^{m_2}\phi_2^{r_3}+\phi_1^{r_3}\phi_2^{m_2}-\phi_1^{r_2}\phi_2^{m_3}-\phi_1^{m_3}\phi_2^{r_2})$$

$$+\phi_2^{n_1}(\phi_1^{m_2}\phi_3^{r_3}+\phi_1^{r_3}\phi_3^{m_2}-\phi_1^{r_2}\phi_3^{m_3}-\phi_1^{m_3}\phi_3^{r_2})$$

$$+\phi_1^{n_1}(\phi_2^{m_2}\phi_3^{r_3}+\phi_2^{r_3}\phi_3^{m_2}-\phi_2^{r_2}\phi_3^{m_3}-\phi_2^{m_3}\phi_3^{r_2})$$

$$+\phi_3^{r_1}(\phi_1^{m_2}\phi_2^{n_3}+\phi_1^{n_3}\phi_2^{m_2}-\phi_1^{n_2}\phi_2^{m_3}-\phi_1^{m_3}\phi_2^{n_2})$$

$$+\phi_2^{r_1}(\phi_1^{m_2}\phi_3^{n_3}+\phi_1^{n_3}\phi_3^{m_2}-\phi_1^{n_2}\phi_3^{m_3}-\phi_1^{m_3}\phi_3^{n_2})$$

$$+\phi_1^{r_1}(\phi_2^{m_2}\phi_3^{n_3}+\phi_2^{n_3}\phi_3^{m_2}-\phi_2^{n_2}\phi_3^{m_3}-\phi_2^{m_3}\phi_3^{n_2})\Big]$$

$$L.H.S.=\begin{vmatrix} \frac{\sum_{i=1}^{3}(-1)^{i+1}\alpha_i\phi_i^{m_1}}{\prod_{i=1}^{3}\alpha_i} & \frac{\sum_{i=1}^{3}(-1)^{i+1}\alpha_i\phi_i^{n_1}}{\prod_{i=1}^{3}\alpha_i} & \frac{\sum_{i=1}^{3}(-1)^{i+1}\alpha_i\phi_i^{r_1}}{\prod_{i=1}^{3}\alpha_i} \\ \frac{\sum_{i=1}^{3}(-1)^{i+1}\alpha_i\phi_i^{m_2}}{\prod_{i=1}^{3}\alpha_i} & \frac{\sum_{i=1}^{3}(-1)^{i+1}\alpha_i\phi_i^{n_2}}{\prod_{i=1}^{3}\alpha_i} & \frac{\sum_{i=1}^{3}(-1)^{i+1}\alpha_i\phi_i^{r_2}}{\prod_{i=1}^{3}\alpha_i} \\ \frac{\sum_{i=1}^{3}(-1)^{i+1}\alpha_i\phi_i^{m_3}}{\prod_{i=1}^{3}\alpha_i} & \frac{\sum_{i=1}^{3}(-1)^{i+1}\alpha_i\phi_i^{n_3}}{\prod_{i=1}^{3}\alpha_i} & \frac{\sum_{i=1}^{3}(-1)^{i+1}\alpha_i\phi_i^{r_3}}{\prod_{i=1}^{3}\alpha_i} \end{vmatrix}$$

$$=(\textstyle\prod_{i=1}^{3}\alpha_i)^{-3}\Big[(\alpha_1\phi_1^{m_1}-\alpha_2\phi_2^{m_1}+\alpha_3\phi_3^{m_1})$$

$$\Big((\alpha_1\phi_1^{n_2}-\alpha_2\phi_2^{n_2}+\alpha_3\phi_3^{n_2})(\alpha_1\phi_1^{r_3}-\alpha_2\phi_2^{r_3}+\alpha_3\phi_3^{r_3})$$

$$-(\alpha_1\phi_1^{r_2}-\alpha_2\phi_2^{r_2}+\alpha_3\phi_3^{r_2})(\alpha_1\phi_1^{n_3}-\alpha_2\phi_2^{n_3}+\alpha_3\phi_3^{n_3})\Big)$$

$$-(\alpha_1\phi_1^{n_1}-\alpha_2\phi_2^{n_1}+\alpha_3\phi_3^{n_1})$$

$$\Big((\alpha_1\phi_1^{m_2} - \alpha_2\phi_2^{m_2} + \alpha_3\phi_3^{m_2})(\alpha_1\phi_1^{r_3} - \alpha_2\phi_2^{r_3} + \alpha_3\phi_3^{r_3})$$

$$-(\alpha_1\phi_1^{r_2} - \alpha_2\phi_2^{r_2} + \alpha_3\phi_3^{r_2})(\alpha_1\phi_1^{m_3} - \alpha_2\phi_2^{m_3} + \alpha_3\phi_3^{m_3})\Big)$$

$$+(\alpha_1\phi_1^{r_1} - \alpha_2\phi_2^{r_1} + \alpha_3\phi_3^{r_1})$$

$$\Big((\alpha_1\phi_1^{m_2} - \alpha_2\phi_2^{m_2} + \alpha_3\phi_3^{m_2})(\alpha_1\phi_1^{n_3} - \alpha_2\phi_2^{n_3} + \alpha_3\phi_3^{n_3})$$

$$-(\alpha_1\phi_1^{n_2} - \alpha_2\phi_2^{n_2} + \alpha_3\phi_3^{n_2})(\alpha_1\phi_1^{m_3} - \alpha_2\phi_2^{m_3} + \alpha_3\phi_3^{m_3})\Big)\Big]$$

$$= (\textstyle\prod_{i=1}^3 \alpha_i)^{-3}\Big[(\alpha_1\phi_1^{m_1} - \alpha_2\phi_2^{m_1} + \alpha_3\phi_3^{m_1})$$

$$\Big(- \alpha_1\alpha_2(\phi_1^{n_2}\phi_2^{r_3} + \phi_1^{r_3}\phi_2^{n_2} - \phi_1^{r_2}\phi_2^{n_3} - \phi_1^{n_3}\phi_2^{r_2})$$

$$+\alpha_1\alpha_3(\phi_1^{n_2}\phi_3^{r_3} + \phi_1^{r_3}\phi_3^{n_2} - \phi_1^{r_2}\phi_3^{n_3} - \phi_1^{n_3}\phi_3^{r_2})$$

$$-\alpha_2\alpha_3(\phi_2^{n_2}\phi_3^{r_3} + \phi_2^{r_3}\phi_3^{n_2} - \phi_2^{r_2}\phi_3^{n_3} - \phi_2^{n_3}\phi_3^{r_2})\Big)$$

$$-(\alpha_1\phi_1^{n_1} - \alpha_2\phi_2^{n_1} + \alpha_3\phi_3^{n_1})\Big(- \alpha_1\alpha_2(\phi_1^{m_2}\phi_2^{r_3} + \phi_1^{r_3}\phi_2^{m_2} - \phi_1^{r_2}\phi_2^{m_3} - \phi_1^{m_3}\phi_2^{r_2})$$

$$+\alpha_1\alpha_3(\phi_1^{m_2}\phi_3^{r_3} + \phi_1^{r_3}\phi_3^{m_2} - \phi_1^{r_2}\phi_3^{m_3} - \phi_1^{m_3}\phi_3^{r_2})$$

$$-\alpha_2\alpha_3(\phi_2^{m_2}\phi_3^{r_3} + \phi_2^{r_3}\phi_3^{m_2} - \phi_2^{r_2}\phi_3^{m_3} - \phi_2^{m_3}\phi_3^{r_2})\Big)$$

$$+(\alpha_1\phi_1^{r_1} - \alpha_2\phi_2^{r_1} + \alpha_3\phi_3^{r_1})\Big(- \alpha_1\alpha_2(\phi_1^{m_2}\phi_2^{n_3} + \phi_1^{n_3}\phi_2^{m_2} - \phi_1^{n_2}\phi_2^{m_3} - \phi_1^{m_3}\phi_2^{n_2})$$

$$+\alpha_1\alpha_3(\phi_1^{m_2}\phi_3^{n_3} + \phi_1^{n_3}\phi_3^{m_2} - \phi_1^{n_2}\phi_3^{m_3} - \phi_1^{m_3}\phi_3^{n_2})$$

$$-\alpha_2\alpha_3\big(\phi_2^{m_2}\phi_3^{n_3}+\phi_2^{n_3}\phi_3^{m_2}-\phi_2^{n_2}\phi_3^{m_3}-\phi_2^{m_3}\phi_3^{n_2})\Big)\Bigg]$$

$$=(\textstyle\prod_{i=1}^3\alpha_i)^{-3}\Bigg[\Big(-\alpha_1\alpha_2\alpha_3\phi_3^{m_1}(\phi_1^{n_2}\phi_2^{r_3}+\phi_1^{r_3}\phi_2^{n_2}-\phi_1^{r_2}\phi_2^{n_3}-\phi_1^{n_3}\phi_2^{r_2})$$

$$-\alpha_1\alpha_3\alpha_2\phi_2^{m_1}(\phi_1^{n_2}\phi_3^{r_3}+\phi_1^{r_3}\phi_3^{n_2}-\phi_1^{r_2}\phi_3^{n_3}-\phi_1^{n_3}\phi_3^{r_2})$$

$$-\alpha_2\alpha_3\alpha_1\phi_1^{m_1}(\phi_2^{n_2}\phi_3^{r_3}+\phi_2^{r_3}\phi_3^{n_2}-\phi_2^{r_2}\phi_3^{n_3}-\phi_2^{n_3}\phi_3^{r_2})\Big)$$

$$-\Big(-\alpha_1\alpha_2\alpha_3\phi_3^{n_1}(\phi_1^{m_2}\phi_2^{r_3}+\phi_1^{r_3}\phi_2^{m_2}-\phi_1^{r_2}\phi_2^{m_3}-\phi_1^{m_3}\phi_2^{r_2})$$

$$-\alpha_1\alpha_3\alpha_2\phi_2^{n_1}(\phi_1^{m_2}\phi_3^{r_3}+\phi_1^{r_3}\phi_3^{m_2}-\phi_1^{r_2}\phi_3^{m_3}-\phi_1^{m_3}\phi_3^{r_2})$$

$$-\alpha_2\alpha_3\alpha_1\phi_1^{n_1}(\phi_2^{m_2}\phi_3^{r_3}+\phi_2^{r_3}\phi_3^{m_2}-\phi_2^{r_2}\phi_3^{m_3}-\phi_2^{m_3}\phi_3^{r_2})\Big)$$

$$\Big(-\alpha_1\alpha_2\alpha_3\phi_3^{r_1}(\phi_1^{m_2}\phi_2^{n_3}+\phi_1^{n_3}\phi_2^{m_2}-\phi_1^{n_2}\phi_2^{m_3}-\phi_1^{m_3}\phi_2^{n_2})$$

$$-\alpha_1\alpha_3\alpha_2\phi_2^{r_1}(\phi_1^{m_2}\phi_3^{n_3}+\phi_1^{n_3}\phi_3^{m_2}-\phi_1^{n_2}\phi_3^{m_3}-\phi_1^{m_3}\phi_3^{n_2})$$

$$-\alpha_2\alpha_3\alpha_1\phi_1^{r_1}(\phi_2^{m_2}\phi_3^{n_3}+\phi_2^{n_3}\phi_3^{m_2}-\phi_2^{n_2}\phi_3^{m_3}-\phi_2^{m_3}\phi_3^{n_2})\Big)\Bigg]$$

$$=(\textstyle\prod_{i=1}^3\alpha_i)^{-3}\Bigg[-\alpha_1\alpha_2\alpha_3\Big(\phi_3^{m_1}(\phi_1^{n_2}\phi_2^{r_3}+\phi_1^{r_3}\phi_2^{n_2}-\phi_1^{r_2}\phi_2^{n_3}-\phi_1^{n_3}\phi_2^{r_2})$$

$$+\phi_2^{m_1}(\phi_1^{n_2}\phi_3^{r_3}+\phi_1^{r_3}\phi_3^{n_2}-\phi_1^{r_2}\phi_3^{n_3}-\phi_1^{n_3}\phi_3^{r_2})$$

$$+\phi_1^{m_1}(\phi_2^{n_2}\phi_3^{r_3}+\phi_2^{r_3}\phi_3^{n_2}-\phi_2^{r_2}\phi_3^{n_3}-\phi_2^{n_3}\phi_3^{r_2})\Big)$$

$$-\Big(\phi_3^{n_1}(\phi_1^{m_2}\phi_2^{r_3}+\phi_1^{r_3}\phi_2^{m_2}-\phi_1^{r_2}\phi_2^{m_3}-\phi_1^{m_3}\phi_2^{r_2})$$

$$+\phi_2^{n_1}(\phi_1^{m_2}\phi_3^{r_3}+\phi_1^{r_3}\phi_3^{m_2}-\phi_1^{r_2}\phi_3^{m_3}-\phi_1^{m_3}\phi_3^{r_2})$$

$$+\phi_1^{n_1}(\phi_2^{m_2}\phi_3^{r_3}+\phi_2^{r_3}\phi_3^{m_2}-\phi_2^{r_2}\phi_3^{m_3}-\phi_2^{m_3}\phi_3^{r_2})\Big)$$

$$\Big(\phi_3^{r_1}(\phi_1^{m_2}\phi_2^{n_3}+\phi_1^{n_3}\phi_2^{m_2}-\phi_1^{n_2}\phi_2^{m_3}-\phi_1^{m_3}\phi_2^{n_2})$$

$$+\phi_2^{r_1}(\phi_1^{m_2}\phi_3^{n_3}+\phi_1^{n_3}\phi_3^{m_2}-\phi_1^{n_2}\phi_3^{m_3}-\phi_1^{m_3}\phi_3^{n_2})$$

$$+\phi_1^{r_1}(\phi_2^{m_2}\phi_3^{n_3}+\phi_2^{n_3}\phi_3^{m_2}-\phi_2^{n_2}\phi_3^{m_3}-\phi_2^{m_3}\phi_3^{n_2})\Big)\Bigg]$$

$$=(\textstyle\prod_{i=1}^3\alpha_i)^{-3}(-\alpha_1\alpha_2\alpha_3)\Bigg[\phi_3^{m_1}(\phi_1^{n_2}\phi_2^{r_3}+\phi_1^{r_3}\phi_2^{n_2}-\phi_1^{r_2}\phi_2^{n_3}-\phi_1^{n_3}\phi_2^{r_2})$$

$$+\phi_2^{m_1}(\phi_1^{n_2}\phi_3^{r_3}+\phi_1^{r_3}\phi_3^{n_2}-\phi_1^{r_2}\phi_3^{n_3}-\phi_1^{n_3}\phi_3^{r_2})$$

$$+\phi_1^{m_1}(\phi_2^{n_2}\phi_3^{r_3}+\phi_2^{r_3}\phi_3^{n_2}-\phi_2^{r_2}\phi_3^{n_3}-\phi_2^{n_3}\phi_3^{r_2})$$

$$+\phi_3^{n_1}(\phi_1^{m_2}\phi_2^{r_3}+\phi_1^{r_3}\phi_2^{m_2}-\phi_1^{r_2}\phi_2^{m_3}-\phi_1^{m_3}\phi_2^{r_2})$$

$$+\phi_2^{n_1}(\phi_1^{m_2}\phi_3^{r_3}+\phi_1^{r_3}\phi_3^{m_2}-\phi_1^{r_2}\phi_3^{m_3}-\phi_1^{m_3}\phi_3^{r_2})$$

$$+\phi_1^{n_1}(\phi_2^{m_2}\phi_3^{r_3}+\phi_2^{r_3}\phi_3^{m_2}-\phi_2^{r_2}\phi_3^{m_3}-\phi_2^{m_3}\phi_3^{r_2})$$

$$+\phi_3^{r_1}(\phi_1^{m_2}\phi_2^{n_3}+\phi_1^{n_3}\phi_2^{m_2}-\phi_1^{n_2}\phi_2^{m_3}-\phi_1^{m_3}\phi_2^{n_2})$$

$$+\phi_2^{r_1}(\phi_1^{m_2}\phi_3^{n_3}+\phi_1^{n_3}\phi_3^{m_2}-\phi_1^{n_2}\phi_3^{m_3}-\phi_1^{m_3}\phi_3^{n_2})$$

$$+\phi_1^{r_1}(\phi_2^{m_2}\phi_3^{n_3}+\phi_2^{n_3}\phi_3^{m_2}-\phi_2^{n_2}\phi_3^{m_3}-\phi_2^{m_3}\phi_3^{n_2})\Bigg].$$

$\therefore L.H.S = R.H.S.$

Hence the theorem is proved. □

Following identities can be deduced from Theorem 3.2.13.

Theorem 3.2.14. (***d'Ocagne type identity***)

For any $m, n, r \in \mathbb{Z}$,

$$\begin{vmatrix} ({}^tB)_m & ({}^tB)_n & ({}^tB)_r \\ ({}^tB)_{m+1} & ({}^tB)_{n+1} & ({}^tB)_{r+1} \\ ({}^tB)_{m+2} & ({}^tB)_{n+2} & ({}^tB)_{r+2} \end{vmatrix} \tag{3.27}$$

$$= [(-b)^2]^r\Big(({}^tB)_{m-r}({}^tB)_{n-r+1} - ({}^tB)_{m-r+1}({}^tB)_{n-r}\Big).$$

Proof. Substitute $m_i = m+i-1$, $n_i = n+i-1$, $r_i = r+i-1$, for $i = 1,2,3$ and taking $s = r$ in (3.26), we get

$$\begin{vmatrix} ({}^tB)_m & ({}^tB)_n & ({}^tB)_r \\ ({}^tB)_{m+1} & ({}^tB)_{n+1} & ({}^tB)_{r+1} \\ ({}^tB)_{m+2} & ({}^tB)_{n+2} & ({}^tB)_{r+2} \end{vmatrix} = [(-b)^2]^r \begin{vmatrix} ({}^tB)_{m-r} & ({}^tB)_{n-r} & ({}^tB)_0 \\ ({}^tB)_{m+1-r} & ({}^tB)_{n+1-r} & ({}^tB)_1 \\ ({}^tB)_{m+2-r} & ({}^tB)_{n+2-r} & ({}^tB)_2 \end{vmatrix}$$

$$= [(-b)^2]^r\Big(({}^tB)_{m-r}({}^tB)_{n-r+1} - ({}^tB)_{m-r+1}({}^tB)_{n-r}\Big).$$ □

Theorem 3.2.15. (***Catalan type identity***)

For all $n, r \in \mathbb{Z}$,

$$\begin{vmatrix} ({}^tB)_n & ({}^tB)_{n+r} & ({}^tB)_{n+2r} \\ ({}^tB)_{n-r} & ({}^tB)_n & ({}^tB)_{n+r} \\ ({}^tB)_{n-2r} & ({}^tB)_{n-r} & ({}^tB)_n \end{vmatrix} \tag{3.28}$$

$$= [(-b)^2]^n \Big(({}^tB)_r^2 ({}^tB)_{-2r} + ({}^tB)_{-r}^2 ({}^tB)_{2r} \Big).$$

Proof. Substitute $m_i = n + (1-i)r, n_i = n + (2-i)r, r_i = n + (3-i)r$, for $i = 1, 2, 3$ and $s = n$ in (3.26), we get

$$\begin{vmatrix} ({}^tB)_n & ({}^tB)_{n+r} & ({}^tB)_{n+2r} \\ ({}^tB)_{n-r} & ({}^tB)_n & ({}^tB)_{n+r} \\ ({}^tB)_{n-2r} & ({}^tB)_{n-r} & ({}^tB)_n \end{vmatrix} = [(-b)^2]^n \begin{vmatrix} ({}^tB)_0 & ({}^tB)_r & ({}^tB)_{2r} \\ ({}^tB)_{-r} & ({}^tB)_0 & ({}^tB)_r \\ ({}^tB)_{-2r} & ({}^tB)_{-r} & ({}^tB)_0 \end{vmatrix}$$

$$= [(-b)^2]^n \Big(({}^tB)_r^2 ({}^tB)_{-2r} + ({}^tB)_{-r}^2 ({}^tB)_{2r} \Big), \text{ since } ({}^tB)_0 = 0.$$

□

Theorem 3.2.16. (***Cassini type identity***)

For all $n \in \mathbb{Z}$,

$$\begin{vmatrix} ({}^tB)_n & ({}^tB)_{n+1} & ({}^tB)_{n+2} \\ ({}^tB)_{n-1} & ({}^tB)_n & ({}^tB)_{n+1} \\ ({}^tB)_{n-2} & ({}^tB)_{n-1} & ({}^tB)_n \end{vmatrix} = [(-b)^2]^{n-2}. \tag{3.29}$$

Proof. Substitute $r = 1$ in (3.28). Using the fact that $({}^tB)_{-1} = \frac{1}{b^2}, ({}^tB)_1 = 0$ and $({}^tB)_2 = 1$, we get the required result. □

Theorem 3.2.17. (***Extended form of Cassini type identity***)

For any $n, r \in \mathbb{Z}$,

$$\begin{vmatrix} ({}^tB)_n & ({}^tB)_{n-1} & ({}^tB)_{n-2} \\ ({}^tB)_{n+1} & ({}^tB)_n & ({}^tB)_{n-1} \\ ({}^tB)_{n+r} & ({}^tB)_{n+r-1} & ({}^tB)_{n+r-2} \end{vmatrix} = [(-b)^2]^{n-2} ({}^tB)_r. \tag{3.30}$$

Proof. Substitute $m_i = n + i - 1, n_i = n + i - 2, r_i = n + i - 3, 1 \leq i \leq 2$, $m_3 = n + r, n_3 = n + r - 1, r_3 = n + r - 2$ and $s = n$ in (3.26). Using the condition that $({}^tB)_0 = 0 = ({}^tB)_1$ and $({}^tB)_{-1} = \frac{1}{b^2}$, we get the required result. □

We now state two results based on the relation between the nth terms $(^tB)_n$ and $(^tB)_{-n}$ of (3.4) and (3.5) respectively.

Theorem 3.2.18. *The n^{th} terms $(^tB)_n$ and $(^tB)_{-n}$ satisfy the following relations.*

(1)

$$(^tB)_n^2 - (^tB)_{n-1}(^tB)_{n+1} = b^{2n-2}(^tB)_{-(n-1)}. \tag{3.31}$$

(2)

$$(^tB)_n(^tB)_{n+1} - (^tB)_{n+2}(^tB)_{n-1} = b^{2n-2}\big((^tB)_{-(n-2)} - a^2(^tB)_{-(n-1)}\big). \tag{3.32}$$

Proof. (1) Let $\alpha_1 = \phi_2 - \phi_3$, $\alpha_2 = \phi_1 - \phi_3$ and $\alpha_3 = \phi_1 - \phi_2$, where $\phi_i,\ i = 1, 2, 3$, are all distinct. Then using Binet type formula (3.8), we get

$(^tB)_n^2 - (^tB)_{n-1}(^tB)_{n+1} = \frac{1}{(\alpha_1\alpha_2\alpha_3)^2}\big(\alpha_1\phi_1^n - \alpha_2\phi_2^n + \alpha_3\phi_3^n\big)^2$

$-\frac{1}{(\alpha_1\alpha_2\alpha_3)^2}\big(\alpha_1\phi_1^{n-1} - \alpha_2\phi_2^{n-1} + \alpha_3\phi_3^{n-1}\big)\big(\alpha_1\phi_1^{n+1} - \alpha_2\phi_2^{n+1} + \alpha_3\phi_3^{n+1}\big)$

$= \frac{1}{(\alpha_1\alpha_2\alpha_3)^2}$

$\Big(\alpha_1\alpha_2(\phi_1\phi_2)^{n-1}(\phi_1 - \phi_2)^2 - \alpha_1\alpha_3(\phi_1\phi_3)^{n-1}(\phi_1 - \phi_3)^2 + \alpha_2\alpha_3(\phi_2\phi_3)^{n-1}(\phi_2 - \phi_3)^2\Big)$

$= \frac{1}{(\alpha_1\alpha_2\alpha_3)^2}\Big(\alpha_1\alpha_2(\phi_1\phi_2)^{n-1}(\alpha_3)^2 - \alpha_1\alpha_3(\phi_1\phi_3)^{n-1}(\alpha_2)^2 + \alpha_2\alpha_3(\phi_2\phi_3)^{n-1}(\alpha_1)^2\Big)$

$= \frac{1}{(\alpha_1\alpha_2\alpha_3)}\Big(\alpha_3(b^2\phi_3^{-1})^{n-1} - \alpha_2(b^2\phi_2^{-1})^{n-1} + \alpha_1(b^2\phi_1^{-1})^{n-1}\Big)$

$= b^{2n-2}(^tB)_{-(n-1)}$.

Hence the proof.

(2) Replace n in (3.31) by $n-1$ and then multiply through out by b^2 to get,

$$b^2({}^tB)_{n-1}^2 - b^2({}^tB)_{n-2}({}^tB)_n = b^{2n-2}({}^tB)_{-(n-2)}$$

Also, multiplying (3.31) by $-a^2$, we get

$$-a^2({}^tB)_n^2 + a^2({}^tB)_{n-1}({}^tB)_{n+1} = -b^{2n-2}\ a^2({}^tB)_{-(n-1)}$$

Adding the above two equations and using (3.4), we get the required result. □

Analogous to Pythagorean triples of B-Fibonacci sequence [1], we have them for B-Tribonacci sequence.

Theorem 3.2.19. *If* $({}^tB)_n$ *is the* n^{th} *term of* B*-Tribonacci sequence* (3.6)*, then*

$$\Big[b^2({}^tB)_{n-1}\Big(2\ ({}^tB)_{n+2} - b^2({}^tB)_{n-1}\Big)\Big]^2 + \Big[2({}^tB)_{n+2}\Big(({}^tB)_{n+2} - b^2({}^tB)_{n-1}\Big)\Big]^2$$

$$= \Big[b^4({}^tB)_{n-1}^2 + 2\ ({}^tB)_{n+2}(({}^tB)_{n+2} - b^2({}^tB)_{n-1})\Big]^2. \qquad (3.33)$$

Proof. Since $({}^tB)_{n+2} = a^2\ ({}^tB)_{n+1} + 2ab\ ({}^tB)_n + b^2\ ({}^tB)_{n-1}$,

$$b^4\ ({}^tB)_{n-1}^2 = \Big[({}^tB)_{n+2} - \Big(a^2\ ({}^tB)_{n+1} + 2ab\ ({}^tB)_n\Big)\Big]^2$$

Hence, squaring both sides, we get

$$b^4\ ({}^tB)_{n-1}^2 + 2\ ({}^tB)_{n+2}\Big(a^2\ ({}^tB)_{n+1} + 2ab\ ({}^tB)_n\Big)$$

$$= ({}^tB)_{n+2}^2 + \Big(a^2\ ({}^tB)_{n+1} + 2ab\ ({}^tB)_n\Big)^2$$

Again squaring both sides,

$$\left[b^4\ ({}^tB)_{n-1}^2 + 2\ ({}^tB)_{n+2}\Big(a^2\ ({}^tB)_{n+1} + 2ab\ ({}^tB)_n\Big)\right]^2$$

$$= ({}^tB)_{n+2}^4 + \Big(a^2\ ({}^tB)_{n+1} + 2ab\ ({}^tB)_n\Big)^4 + 2\ ({}^tB)_{n+2}^2\Big(a^2\ ({}^tB)_{n+1} + 2ab\ ({}^tB)_n\Big)^2$$

$$= \left[({}^tB)_{n+2}^2 - \Big(a^2\ ({}^tB)_{n+1} + 2ab\ ({}^tB)_n\Big)^2\right]^2 + 4\ ({}^tB)_{n+2}^2\Big(a^2\ ({}^tB)_{n+1} + 2ab\ ({}^tB)_n\Big)^2$$

$$= \left[\Big(({}^tB)_{n+2} - \Big(a^2\ ({}^tB)_{n+1} + 2ab\ ({}^tB)_n\Big)\Big)\Big(({}^tB)_{n+2} + \Big(a^2\ ({}^tB)_{n+1} + 2ab\ ({}^tB)_n\Big)\Big)\right]^2$$

$$+\left[2\ ({}^tB)_{n+2}\Big(a^2\ ({}^tB)_{n+1} + 2ab\ ({}^tB)_n\Big)\right]^2$$

Thus, $\left[b^4\ ({}^tB)_{n-1}^2 + 2\ ({}^tB)_{n+2}\Big(({}^tB)_{n+2} - b^2\ ({}^tB)_{n-1}\Big)\right]^2$

$$= \left[b^2\ ({}^tB)_{n-1}\Big(2\ ({}^tB)_{n+2} - b^2\ ({}^tB)_{n-1}\Big)\right]^2 + \left[2\ ({}^tB)_{n+2}\Big(({}^tB)_{n+2} - b^2\ ({}^tB)_{n-1}\Big)\right]^2.$$

□

3.3 B-Tri Lucas sequence

In this section, we discuss B-Tri Lucas sequence and obtain the various identities related to it. We also prove the relation between the n^{th} term of B-Tribonacci sequence and B-Tri Lucas sequence.

We first define the new sequence.

Definition 3.3.1. *Let $n \in \mathbb{N} \cup \{0\}$. The B-Tri Lucas sequence is defined by*

$$({}^tL)_{n+2} = a^2\ ({}^tL)_{n+1} + 2ab\ ({}^tL)_n + b^2({}^tL)_{n-1},\ \forall n \geq 1, \tag{3.34}$$

with $({}^tL)_0 = 0, ({}^tL)_1 = 2$ and $({}^tL)_2 = a^2$, *where* $({}^tL)_n$ *is the* n^{th} *term.*

For $0 \leq n \leq 5$, the terms of (3.34) are $({}^tL)_0 = 0$, $({}^tL)_1 = 2, ({}^tL)_2 = a^2$, $({}^tL)_3 = a^4 + 4ab$, $({}^tL)_4 = a^6 + 6a^3b + 2b^2$ and $({}^tL)_5 = a^8 + 8a^3b + 11a^2b^2$.

Equation (3.34) can be expressed as

$$({}^tL)_{n-1} = \frac{1}{b^2}\Big[({}^tL)_{n+2} - a^2\ ({}^tL)_{n+1} - 2ab\ ({}^tL)_n\Big], \tag{3.35}$$

$$\text{with } ({}^tL)_0 = 0,\ ({}^tL)_1 = 2 \text{ and } ({}^tL)_2 = a^2.$$

For $-4 \leq n \leq 0$, we have terms of (3.35) as follows:
$({}^tL)_{-1} = \frac{-a^2}{b^2}$, $({}^tL)_{-2} = \frac{2}{b^3}(a^3 + b), ({}^tL)_{-3} = \frac{-a}{b^4}(3a^3 + 4b)$, $({}^tL)_{-4} = \frac{a^2}{b^5}(4a^3 + 5b)$, $({}^tL)_{-5} = \frac{1}{b^6}(-5a^6 - 4a^3b + 2b^2)$.

Thus, we define B-Tri Lucas sequence for all integers n.

Definition 3.3.2. *The B-Tri Lucas sequence is defined by*

$$({}^tL)_{n+2} = a^2\ ({}^tL)_{n+1} + 2ab\ ({}^tL)_n + b^2({}^tL)_{n-1},\ \forall n \in \mathbb{Z}, \tag{3.36}$$

$$\textit{with } ({}^tL)_0 = 0,\ ({}^tL)_1 = 2 \textit{ and } ({}^tL)_2 = a^2,$$

where $({}^tL)_n$ *is the* n^{th} *term.*

We have the following identities of B-Tri Lucas sequence.

Theorem 3.3.3.

$$({}^tL)_n = \frac{(2\phi_1 - a^2)}{(\phi_1 - \phi_2)(\phi_1 - \phi_3)}\ \phi_1^n + \frac{(2\phi_2 - a^2)}{(\phi_2 - \phi_1)(\phi_2 - \phi_3)}\ \phi_2^n + \frac{(2\phi_3 - a^2)}{(\phi_3 - \phi_1)(\phi_3 - \phi_2)}\ \phi_3^n, \tag{3.37}$$

where $\phi_i, i = 1, 2, 3$ *are the distinct roots of the characteristic equation corresponding to (3.36) given by*

$$\lambda^3 - a^2\lambda^2 - 2ab\lambda - b^2 = 0. \tag{3.38}$$

Equation (3.37) is a Binet type Formula for (3.36).

Proof. If ϕ_1, ϕ_2 and ϕ_3 are distinct roots of the characteristic equation (3.38), then solution of (3.36) is given by

$$({}^tL)_n = C_1\phi_1^n + C_2\phi_2^n + C_3\phi_3^n \tag{3.39}$$

Using the conditions $({}^tL)_0 = 0$, $({}^tL)_1 = 2$ and $({}^tL)_2 = a^2$, we get (3.37). □

Remark 3.3.4. *The case of repeated roots is excluded here.*

The next two theorems give the relationship between (3.6) and (3.36).

Theorem 3.3.5. *The n^{th} term $({}^tL)_n$ of (3.36) is given by*

$$({}^tL)_n = 2({}^tB)_{n+1} - a^2({}^tB)_n, \forall n \in \mathbb{Z}. \tag{3.40}$$

Proof. Equation (3.37) implies

$$({}^tL)_n = \frac{(2\phi_1 - a^2)}{(\phi_1-\phi_2)(\phi_1-\phi_3)}\,\phi_1^n + \frac{(2\phi_2 - a^2)}{(\phi_2-\phi_1)(\phi_2-\phi_3)}\,\phi_2^n + \frac{(2\phi_3 - a^2)}{(\phi_3-\phi_1)(\phi_3-\phi_2)}\,\phi_3^n,$$

where $\phi_i, i = 1, 2, 3$ are the distinct roots of $\lambda^3 - a^2\lambda^2 - 2ab\lambda - b^2 = 0$.

Therefore,

$$\begin{aligned} ({}^tL)_n &= 2\left(\frac{\phi_1^{n+1}}{(\phi_1-\phi_2)(\phi_1-\phi_3)} + \frac{\phi_2^{n+1}}{(\phi_2-\phi_1)(\phi_2-\phi_3)} + \frac{\phi_3^{n+1}}{(\phi_3-\phi_1)(\phi_3-\phi_2)}\right) \\ &\quad -a^2\left(\frac{\phi_1^{n}}{(\phi_1-\phi_2)(\phi_1-\phi_3)} + \frac{\phi_2^{n}}{(\phi_2-\phi_1)(\phi_2-\phi_3)} + \frac{\phi_3^{n}}{(\phi_3-\phi_1)(\phi_3-\phi_2)}\right) \\ &= 2({}^tB)_{n+1} - a^2({}^tB)_n, \text{ from (3.8).} \end{aligned}$$

Hence the theorem is proved. □

Corollary 3.3.6. *The n^{th} term of (3.36) is given by*

$$({}^tL)_n = ({}^tB)_{n+1} + 2ab({}^tB)_{n-1} + b^2({}^tB)_{n-2}, \forall n \in \mathbb{Z}. \tag{3.41}$$

Proof. From (3.40), we have

$$\begin{aligned}({}^tL)_n &= 2({}^tB)_{n+1} - a^2({}^tB)_n \\ &= ({}^tB)_{n+1} + a^2({}^tB)_n + 2ab\ ({}^tB)_{n-1} + b^2({}^tB)_{n-2} - a^2({}^tB)_n, \text{ using (3.6).} \\ &= ({}^tB)_{n+1} + 2ab({}^tB)_{n-1} + b^2({}^tB)_{n-2}.\end{aligned}$$

Hence the Corollary is proved. □

Theorem 3.3.7. *The n^{th} term of* (3.34) *is given by*

$$({}^tL)_n = \sum_{r=0}^{\left\lfloor \frac{2n-2}{3} \right\rfloor} \left(\frac{(2n-2)}{(2n-2-2r)} \frac{(2n-2-2r)^{\underline{r}}}{r!} - r(r-1)\frac{(2n-4-2r)^{\underline{r-2}}}{r!} \right) a^{2n-2-3r}\ b^r, \tag{3.42}$$

$\forall n \geq 2$.

Proof. Equations (3.11) and (3.40) implies,

$$({}^tL)_n = 2({}^tB)_{n+1} - a^2({}^tB)_n$$

$$= 2\ \sum_{r=0}^{\left\lfloor \frac{2n-2}{3} \right\rfloor} \frac{(2n-2-2r)^{\underline{r}}}{r!}\ a^{2n-2-3r}\ b^r - a^2 \sum_{r=0}^{\left\lfloor \frac{2n-4}{3} \right\rfloor} \frac{(2n-4-2r)^{\underline{r}}}{r!}\ a^{2n-4-3r}\ b^r$$

$$= \sum_{r=0}^{\left\lfloor \frac{2n-2}{3} \right\rfloor} \left(2\ \frac{(2n-2-2r)^{\underline{r}}}{r!} - \frac{(2n-4-2r)^{\underline{r}}}{r!}\right)\ a^{2n-2-3r}\ b^r$$

$$= \sum_{r=0}^{\left\lfloor \frac{2n-2}{3} \right\rfloor} \frac{(2n-4-2r)^{\underline{r-2}}}{r!}$$

$$\Big(2\ (2n-2-2r)(2n-3-2r) - (2n-2-3r)(2n-3-3r)\Big)\ a^{2n-2-3r}\ b^r$$

$$= \sum_{r=0}^{\left\lfloor \frac{2n-2}{3} \right\rfloor} \frac{(2n-4-2r)^{\underline{r-2}}}{r!}$$

$$\Big(2\ (2n-2-2r)(2n-3-2r) - (2n-2-2r-r)(2n-3-2r-r)\Big)\ a^{2n-2-3r}\ b^r$$

$$= \sum_{r=0}^{\left\lfloor \frac{2n-2}{3} \right\rfloor} \frac{(2n-4-2r)^{\underline{r-2}}}{r!}$$

$$\Big((2n-2-2r)(2n-3-2r) + 2r(2n-3-2r) - r + r^2\Big) \; a^{2n-2-3r} \; b^r$$

$$= \sum_{r=0}^{\left\lfloor \frac{2n-2}{3} \right\rfloor} \frac{(2n-4-2r)^{\underline{r-2}}}{r!} \Big((2n-3-2r)(2n-2-2r+2r) - r(r-1)\Big) \; a^{2n-2-3r} \; b^r$$

$$= \sum_{r=0}^{\left\lfloor \frac{2n-2}{3} \right\rfloor} \Big(\frac{(2n-2)(2n-3-2r)^{\underline{r-1}}}{r!} - r(r-1)\frac{(2n-4-2r)^{\underline{r-2}}}{r!} \Big) \; a^{2n-2-3r} \; b^r$$

$$= \sum_{r=0}^{\left\lfloor \frac{2n-2}{3} \right\rfloor} \Big(\frac{(2n-2)(2n-3-2r)^{\underline{r-1}}}{r!} - r(r-1)\frac{(2n-4-2r)^{\underline{r-2}}}{r!} \Big) \; a^{2n-2-3r} \; b^r$$

$$= \sum_{r=0}^{\left\lfloor \frac{2n-2}{3} \right\rfloor} \Big(\frac{(2n-2)}{(2n-2-2r)} \frac{(2n-2-2r)^{\underline{r}}}{r!} - r(r-1)\frac{(2n-4-2r)^{\underline{r-2}}}{r!} \Big) \; a^{2n-2-3r} \; b^r. \qquad \square$$

Similarly, we can prove the following Corollary.

Corollary 3.3.8. *Let $n \in \mathbb{Z}^-$, the set of negative integers. The n^{th} term of (3.35) is given by*

$$({}^tL)_n = \sum_{r=n-1}^{\left\lfloor \frac{2n-2}{3} \right\rfloor} \Big(\frac{(2n-2)}{(2n-2-2r)} \frac{(2n-2-2r)^{\underline{r}}}{r!} - r(r-1)\frac{(2n-4-2r)^{\underline{r-2}}}{r!} \Big) \; a^{2n-2-3r} \; b^r, \tag{3.43}$$

$\forall n \leq -1$.

Equation (3.43) can be rewritten as

$$({}^tL)_n = \sum_{r=n-1}^{\left\lfloor \frac{2n-2}{3} \right\rfloor} \Big(\frac{(2n-2)}{r} \frac{(2n-3-2r)^{\underline{r-1}}}{(r-1)!} - \frac{(2n-4-2r)^{\underline{r-2}}}{(r-2)!} \Big) \; a^{2n-2-3r} \; b^r.$$

For $n = -1$,

$$({}^tL)_{-1} = \sum_{r=-2}^{\left\lfloor \frac{-4}{3} \right\rfloor} \Big(\frac{-4}{r} \frac{(-5-2r)^{\underline{r-1}}}{(r-1)!} - \frac{(-6-2r)^{\underline{r-2}}}{(r-2)!} \Big) \; a^{-4-3r} \; b^r$$

$$= \Big(\frac{-4}{-2} \frac{(-1)^{\underline{-3}}}{(-3)!} - \frac{(-2)^{\underline{-4}}}{(-4)!} \Big) \; a^2 \; b^{-2}$$

$$= \left(\frac{-4}{-2} \frac{(-3)!(-1)^{-1+3} \frac{(3-1)^{-1+3}}{(-1+3)!}}{(-3)!} - \frac{(-4)!(-1)^{-2+4} \frac{(4-1)^{-2+4}}{(-2+4)!}}{(-4)!} \right) a^2\, b^{-2}, \text{form (2.10).}$$

$$= \left(2\, \frac{2^2}{2!} - \frac{3^2}{2!} \right) a^2\, b^{-2}$$

$$= (2-3)a^2\, b^{-2}$$

$$= -a^2\, b^{-2}.$$

Similarly, other terms of $({}^tL)_n$, $n \leq -1$ can be calculated.

Following theorem give the sum of terms of B-Tri Lucas sequence. This theorem can be proved by a similar procedure to the one used in Section 2 of this Chapter. We give here alternative proof using Binet type formula (3.37).

Theorem 3.3.9. *The sum of the first $n+1$ terms of (3.34) is*

$$\sum_{r=0}^{n} ({}^tL)_r = \frac{({}^tL)_{n+1} + (b^2+2ab)({}^tL)_n + b^2({}^tL)_{n-1} + ({}^tL)_2 - ({}^tL)_1}{(a+b)^2-1}, \tag{3.44}$$

provided $a+b \neq 1, -1$ *and* $n \geq 0$.

Proof. Consider, $\sum_{r=0}^{n} ({}^tL)_r$

$$= 2\sum_{r=0}^{n}({}^tB)_{r+1} - a^2 \sum_{r=0}^{n}({}^tB)_r, \text{ from (3.37)}$$

$$= 2\sum_{r=0}^{n+1}({}^tB)_r - a^2 \sum_{r=0}^{n}({}^tB)_r$$

$$= 2\frac{({}^tB)_{n+2} + (b^2+2ab)({}^tB)_{n+1} + b^2({}^tB)_n - 1}{(a+b)^2-1}$$

$$-a^2 \frac{({}^tB)_{n+1} + (b^2+2ab)({}^tB)_n + b^2({}^tB)_{n-1} - 1}{(a+b)^2-1}$$

$$= \frac{1}{(a+b)^2-1} \Big[\big(2({}^tB)_{n+2} - a^2({}^tB)_{n+1} \big)$$

$$+(b^2+2ab)\big(2(^tB)_{n+1}-a^2(^tB)_n\big)+b^2\big((^tB)_n-a^2(^tB)_{n-1}\big)-2+a^2\Big]$$

$$=\frac{(^tL)_{n+1}+(b^2+2ab)(^tL)_n+b^2(^tL)_{n-1}+(^tL)_2-(^tL)_1}{(a+b)^2-1}.$$

□

Similarly, we can prove the following Corollary.

Corollary 3.3.10. *The sum of the first $n+1$ terms of (3.35) is*

$$\sum_{r=-1}^{-n}(^tL)_r=-\frac{(^tL)_{-n}+(b^2+2ab)(^tL)_{-(n+1)}+b^2(^tL)_{-(n+2)}+(^tL)_2-(^tL)_1}{(a+b)^2-1}, \quad (3.45)$$

provided $a+b\neq 1,-1$ *and* $n\geq 1$.

Combining (3.44) and (3.45), we have

$$\sum_{r=-n}^{n}(^tL)_r=\frac{1}{(a+b)^2-1}\Big[((^tL)_{n+1}-(^tL)_{-n})$$

$$+(b^2+2ab)((^tL)_n-(^tL)_{-(n+1)})+b^2((^tL)_{n-1}-(^tL)_{-(n+2)})\Big], \quad (3.46)$$

provided $a+b\neq 1,-1$.

The next theorem is based on the ratio of successive and preceding terms of B-Tri Lucas sequence.

Theorem 3.3.11. *Let the roots ϕ_1,ϕ_2 and ϕ_3 of (3.38) be distinct, $\phi_1\neq 0$ and $|\phi_1|>|\phi_2|>|\phi_3|$, then*

(i)

$$\lim_{n\to\infty}\frac{(^tL)_n}{(^tL)_{n-1}}=\phi_1\big(2-a^2\big), \quad (3.47)$$

(ii)

$$\lim_{n\to\infty}\frac{(^tL)_{n-1}}{(^tL)_n}=\frac{1}{\phi_1}\big(2-a^2\big). \quad (3.48)$$

Proof.

$$\lim_{n\to\infty} \frac{({}^tL)_n}{({}^tL)_{n-1}}$$

$$= 2 \lim_{n\to\infty} \frac{({}^tB)_{n+1}}{({}^tB)_n} - a^2 \lim_{n\to\infty} \frac{({}^tB)_n}{({}^tB)_{n-1}}, \text{ from (3.40)}$$

$$= \phi_1\big(2 - a^2\big), \text{ using equation (3.18).}$$

Similarly, using (3.19) we can prove (3.48). □

Following theorem gives the generating function of B-Tri Lucas sequence (3.6).

Theorem 3.3.12. *The terms of the B-Tri Lucas sequence (3.36) can be generated from the series*

$$(2 - a^2 z) \sum_{r=-\infty}^{\infty} z^r (a + bz)^{2r}. \tag{3.49}$$

Proof of Theorem 3.3.12 is similar to that of Theorem 3.2.9.

We now establish some of the identities of B-Tri Lucas sequence as presented in Section 2 for the B-Tribonacci sequence.

Theorem 3.3.13. (***Honsberger type identity***)

For any $m, n \in \mathbb{Z}$,

$$({}^tL)_{n+m-1} = b^2({}^tB)_{n-1}({}^tL)_{m-1} + \big(b^2({}^tB)_{n-2} + 2ab({}^tB)_{n-1}\big)({}^tL)_m + ({}^tB)_n({}^tL)_{m+1}. \tag{3.50}$$

Proof. Equation (3.24) implies

$$({}^tB)_{n+m-1} = b^2({}^tB)_{n-1}({}^tB)_{m-1} + \big(b^2({}^tB)_{n-2} + 2ab({}^tB)_{n-1}\big)({}^tB)_m + ({}^tB)_n({}^tB)_{m+1}$$

Therefore,

$$({}^tB)_{n+m} = b^2({}^tB)_{n-1}({}^tB)_m + (b^2({}^tB)_{n-2} + 2ab({}^tB)_{n-1})({}^tB)_{m+1} + ({}^tB)_n({}^tB)_{m+2}$$

From (3.40), we have

$$({}^tL)_{n+m-1} = 2\ ({}^tB)_{n+m} - a^2({}^tB)_{n+m-1}$$

$$= b^2({}^tB)_{n-1}\Big(2({}^tB)_m - a^2({}^tB)_{m-1}\Big)$$

$$+(b^2({}^tB)_{n-2}+2ab({}^tB)_{n-1})\Big(2({}^tB)_{m+1}-a^2({}^tB)_m\Big)+({}^tB)_n\Big(2({}^tB)_{m+2}-a^2({}^tB)_{m+1}\Big)$$

Therefore, using again (3.40) we have

$$({}^tL)_{n+m-1} = b^2({}^tB)_{n-1}({}^tL)_{m-1} + \big(b^2({}^tB)_{n-2} + 2ab({}^tB)_{n-1}\big)({}^tL)_m + ({}^tB)_n({}^tL)_{m+1}.$$

Hence the theorem is proved. □

Theorem 3.3.14. (***General Trilinear identity***)

For all $m_i, n_i, r_i, s \in \mathbb{Z},\ 1 \leq i \leq 3$,

$$\begin{vmatrix} ({}^tL)_{m_1} & ({}^tL)_{n_1} & ({}^tL)_{r_1} \\ ({}^tL)_{m_2} & ({}^tL)_{n_2} & ({}^tL)_{r_2} \\ ({}^tL)_{m_3} & ({}^tL)_{n_3} & ({}^tL)_{r_3} \end{vmatrix} = [(-b)^2]^s \begin{vmatrix} ({}^tL)_{m_1-s} & ({}^tL)_{n_1-s} & ({}^tL)_{r_1-s} \\ ({}^tL)_{m_2-s} & ({}^tL)_{n_2-s} & ({}^tL)_{r_2-s} \\ ({}^tL)_{m_3-s} & ({}^tL)_{n_3-s} & ({}^tL)_{r_3-s} \end{vmatrix}. \quad (3.51)$$

Proof. Let $\alpha_1 = \phi_2 - \phi_3,\ \ \alpha_2 = \phi_1 - \phi_3,\ \ \alpha_3 = \phi_1 - \phi_2$ and $\beta_1 = \alpha_1(2\phi_1 - a^2)$, $\beta_2 = \alpha_2(2\phi_2 - a^2)$ and $\beta_3 = \alpha_3(2\phi_3 - a^2)$, where $\phi_i, i = 1, 2, 3$ are distinct roots of $\lambda^3 - a^2\lambda^2 - 2ab\lambda - b^2 = 0$.

Therefore (3.37) implies,

$$({}^tL)_n = \frac{\beta_1\phi_1^n - \beta_2\phi_2^n + \beta_3\phi_3^n}{\alpha_1\alpha_2\alpha_3} = \frac{\sum_{i=1}^{3}(-1)^{i+1}\beta_i\phi_i^n}{\prod_{i=1}^{3}\alpha_i}.$$

Using the procedure similar to the one used to prove Theorem 3.2.13, we get the required result. □

Following identities can be deduced from general Trilinear identity for B-Tri Lucas sequence.

Theorem 3.3.15. (***d'Ocagne type identity***)

For any $m, n, r \in \mathbb{Z}$, *then*

$$\begin{vmatrix} ({}^tL)_m & ({}^tL)_n & ({}^tL)_r \\ ({}^tL)_{m+1} & ({}^tL)_{n+1} & ({}^tL)_{r+1} \\ ({}^tL)_{m+2} & ({}^tL)_{n+2} & ({}^tL)_{r+2} \end{vmatrix}$$

$$= [(-b)^2]^r \Big(a^2 \big(({}^tL)_{m-r} ({}^tL)_{n-r+1} - ({}^tL)_{n-r} ({}^tL)_{m-r+1} \big)$$

$$-2 \big(({}^tL)_{m-r} ({}^tL)_{n-r+2} - ({}^tL)_{n-r} ({}^tL)_{m-r+2} \big) \Big). \qquad (3.52)$$

Proof. Substitute $m_i = m + i - 1, n_i = n + i - 1, r_i = r + i - 1$, for $i = 1, 2, 3$ and taking $s = r$ in (3.51). Using the procedure similar to the one used for proving Theorem 3.2.14 and the condition that $({}^tL)_0 = 0, ({}^tL)_1 = 2, ({}^tL)_2 = a^2$, we get the required result. □

Theorem 3.3.16. (***Catalan type identity***)

For any $n, r \in \mathbb{Z}$,

$$\begin{vmatrix} ({}^tL)_n & ({}^tL)_{n+r} & ({}^tL)_{n+2r} \\ ({}^tL)_{n-r} & ({}^tL)_n & ({}^tL)_{n+r} \\ ({}^tL)_{n-2r} & ({}^tL)_{n-r} & ({}^tL)_n \end{vmatrix} = [(-b)^2]^n \Big(({}^tL)_r^2 ({}^tL)_{-2r} + ({}^tL)_{-r}^2 ({}^tL)_{2r} \Big). \qquad (3.53)$$

Proof. Take $m_i = n + (1 - i)r, n_i = n + (2 - i)r, r_i = n + (3 - i)r$, for $i = 1, 2, 3$ and $s = n$ in identity (3.51). This implies

$$\begin{vmatrix} ({}^tL)_n & ({}^tL)_{n+r} & ({}^tL)_{n+2r} \\ ({}^tL)_{n-r} & ({}^tL)_n & ({}^tL)_{n+r} \\ ({}^tL)_{n-2r} & ({}^tL)_{n-r} & ({}^tL)_n \end{vmatrix} = [(-b^2)]^n \begin{vmatrix} ({}^tL)_{n-n} & ({}^tL)_{n+r-n} & ({}^tL)_{n+2r-n} \\ ({}^tL)_{n-r-n} & ({}^tL)_{n-n} & ({}^tL)_{n+r-n} \\ ({}^tL)_{n-2r-n} & ({}^tL)_{n-r-n} & ({}^tL)_{n-n} \end{vmatrix}$$

$$= [(-b)^2]^n \begin{vmatrix} ({}^tL)_0 & ({}^tL)_r & ({}^tL)_{2r} \\ ({}^tL)_{-r} & ({}^tL)_0 & ({}^tL)_r \\ ({}^tL)_{-2r} & ({}^tL)_{-r} & ({}^tL)_0 \end{vmatrix}$$

$$= [(-b)^2]^n \Big(({}^tL)_{-r}^2 ({}^tL)_{2r} + ({}^tL)_r^2 ({}^tL)_{-2r} \Big), \text{ since } ({}^tL)_0 = 0. \qquad \square$$

Theorem 3.3.17. (***Cassini type identity***)

For all $n \in \mathbb{Z}$,

$$\begin{vmatrix} ({}^tL)_n & ({}^tL)_{n+1} & ({}^tL)_{n+2} \\ ({}^tL)_{n-1} & ({}^tL)_n & ({}^tL)_{n+1} \\ ({}^tL)_{n-2} & ({}^tL)_{n-1} & ({}^tL)_n \end{vmatrix} = [(-b)^2]^{n-2} \Big(({}^tL)_4 + 2ab({}^tL)_2 + 3b^2({}^tL)_1 \Big). \tag{3.54}$$

Proof. Substitute $r = 1$ in (3.53) and using the condition that $({}^tL)_{-2} = \frac{2}{b^3}(a^3 + b)$, $({}^tL)_{-1} = \frac{-a^2}{b^2}, ({}^tL)_1 = 2, ({}^tL)_2 = a^2$, we get the required result. $\square$

Theorem 3.3.18. (***Extended form of Cassini type identity***)

For any $n, r \in \mathbb{Z}$,

$$\begin{vmatrix} ({}^tL)_n & ({}^tL)_{n-1} & ({}^tL)_{n-2} \\ ({}^tL)_{n+1} & ({}^tL)_n & ({}^tL)_{n-1} \\ ({}^tL)_{n+r} & ({}^tL)_{n+r-1} & ({}^tL)_{n+r-2} \end{vmatrix} \tag{3.55}$$

$$= [(-b)^2]^{n-2} \Big(a^4({}^tL)_r + 4b(a^3 + b)({}^tL)_{r-1} + 2a^2b^2({}^tL)_{r-2}) \Big).$$

Proof. The result follows by Substituting $m_i = n+i-1, n_i = n+i-2, r_i = n+i-3$, $1 \leq i \leq 2$, $m_3 = n + r, n_3 = n + r - 1, r_i = n + r - 2$ and $s = n$ in (3.51). $\square$

3.4 Incomplete B-Tribonacci and B-Tri Lucas sequences

In this section, we study incomplete B-Tribonacci, incomplete B-Tri Lucas sequences and various properties related to them. We first define the extension of incomplete

Fibonacci sequence and incomplete Lucas sequence defined in [12] and call them incomplete B-Tribonacci sequence and incomplete B-Tri Lucas sequence respectively.

Definition 3.4.1. *Let $n \in \mathbb{N}$. The incomplete B-Tribonacci sequence is defined by*

$$({}^{t}B)_n^l = \sum_{r=0}^{l} \frac{(2n-4-2r)_{\underline{r}}}{r!} \; a^{2n-4-3r}\, b^r, \;\forall\; 0 \le l \le \lfloor \frac{2n-4}{3} \rfloor \text{ and } n \ge 2. \tag{3.56}$$

For $l = 0,\ 1,\ 2$ and $\lfloor \frac{2n-4}{3} \rfloor$, $({}^{t}B)_n^l$ are listed below:

$({}^{t}B)_n^0 = a^{2n-4},\ \forall\, n \ge 2.$

$({}^{t}B)_n^1 = a^{2n-4} + (2n-6)a^{2n-7}\, b,\ \forall\, n \ge 4.$

$({}^{t}B)_n^2 = a^{2n-4} + (2n-6)a^{2n-7}\, b + \frac{(2n-8)(2n-9)}{2}\, a^{2n-10}\, b^2,\ \forall\, n \ge 5.$

$({}^{t}B)_n^{\lfloor \frac{2n-4}{3} \rfloor} = ({}^{t}B)_n.$

Following table give the terms of incomplete B-Tribonacci sequence.

l / n	0	1	2	3
2	1			
3	a^2			
4	a^4	$a^4 + 2ab$		
5	a^6	$a^6 + 4a^3b$	$a^6 + 4a^3b + b^2$	
6	a^8	$a^8 + 6a^5b$	$a^8 + 6a^5b + 6a^2b^2$	
7	a^{10}	$a^{10} + 8a^7b$	$a^{10} + 8a^7b + 15a^4b^2$	$a^{10} + 8a^5b + 15a^4b^2 + 4ab^3$
8	a^{12}	$a^{12} + 10a^9b$	$a^{12} + 10a^9b + 28a^6b^2$	$a^{12} + 10a^9b + 28a^6b^2 + 20a^3b^3$

Table 3.1: Terms of incomplete B-Tribonacci sequence

The next three theorems give the results on the recurrence properties of incomplete B-Tribonacci sequence (3.56).

Theorem 3.4.2. *The recurrence relation of the incomplete B-Tribonacci sequence $({}^{t}B)_n^l$ is given by*

$$({}^{t}B)_{n+3}^{l+2} = a^2({}^{t}B)_{n+2}^{l+2} + 2ab\; ({}^{t}B)_{n+1}^{l+1} + b^2({}^{t}B)_n^l,\; 0 \le l \le \lfloor \frac{2n-6}{3} \rfloor \text{ and } n \ge 3. \tag{3.57}$$

Proof. R.H.S of (3.57)$= a^2({}^tB)^{l+2}_{n+2} + 2ab\ ({}^tB)^{l+1}_{n+1} + b^2({}^tB)^l_n$

$$= \sum_{r=0}^{l+2} \frac{(2n-2r)^{\underline{r}}}{r!}\ a^{2n+2-3r}b^r + 2\ \sum_{r=0}^{l+1} \frac{(2n-2-2r)^{\underline{r}}}{r!}\ \ a^{2n-1-3r}b^{r+1}$$

$$+ \sum_{r=0}^{l} \frac{(2n-4-2r)^{\underline{r}}}{r!}\ a^{2n-4-3r}b^{r+2}$$

$$= a^{2n+2} + (2n-2+2)a^{2n-1}b + \sum_{r=2}^{l+2} \frac{(2n-2r)^{\underline{r}}}{r!}\ a^{2n+2-3r}b^r$$

$$+2\sum_{r=2}^{l+2} \frac{(2n-2r)^{\underline{r-1}}}{(r-1)!}\ a^{2n+2-3r}b^r + \sum_{r=2}^{l+2} \frac{(2n-2r)^{\underline{r-2}}}{(r-2)!}\ a^{2n+2-3r}b^r$$

$$= a^{2n+2} + (2n)a^{2n-1}b$$

$$+ \sum_{r=2}^{l+2} \left[\frac{(2n-2r)^{\underline{r}}}{r!} + 2\ \frac{(2n-2r)^{\underline{r-1}}}{(r-1)!} + \frac{(2n-2r)^{\underline{r-2}}}{(r-2)!}\right] a^{2n+2-3r}b^r$$

$$= \sum_{r=0}^{l+2} \frac{(2n+2-2r)^{\underline{r}}}{r!}\ \ a^{2n+2-3r}b^r$$
$$= ({}^tB)^{l+2}_{n+3} = \text{L.H.S.}$$

Hence the theorem is proved. □

Theorem 3.4.3. *For all* $s \geq 1$,

$$\sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!}\ ({}^tB)^{l+i}_{n+i}\ a^i\ b^{2s-i} = ({}^tB)^{l+2s}_{n+3s},\ \ 0 \leq l \leq \left\lfloor \frac{2n-4-2s}{3} \right\rfloor. \tag{3.58}$$

Proof. We prove it by mathematical induction on s.

Let $s = 1$. Then L.H.S. of (3.58) $= \sum_{i=0}^{2} \frac{2^{\underline{i}}}{i!}\ ({}^tB)^{l+i}_{n+i}\ a^i b^{2-i} = ({}^tB)^{l+2}_{n+3} = R.H.S.$

Thus, the theorem is true for $s = 1$.

Assume that the result is true for all $s \leq m$.

L.H.S. of (3.58) $= \sum_{i=0}^{2m+2} \frac{(2m+2)^{\underline{i}}}{i!}\ ({}^tB)^{l+i}_{n+i}\ a^i\ b^{2m+2-i}$

$$= \sum_{i=0}^{2m+2} \Big(\tfrac{(2m-2)^{\underline{i-2}}}{(i-2)!} \ ({}^tB)_{n+i}^{l+i} \ a^i \ b^{2m+2-i} + 2 \ \tfrac{(2m-1)^{\underline{i-1}}}{(i-1)!} \ ({}^tB)_{n+i}^{l+i} \ a^i \ b^{2m+2-i}$$

$$+ \tfrac{(2m)^{\underline{i}}}{i!} \ ({}^tB)_{n+i}^{l+i} \ a^i \ b^{2m+2-i} \Big)$$

$$= \sum_{i=0}^{2m} \Big(\tfrac{(2m)^{\underline{i}}}{i!} \ ({}^tB)_{n+i+2}^{l+i+2} \ a^{i+2} \ b^{2m-i} + 2 \ \tfrac{(2m)^{\underline{i}}}{i!} ({}^tB)_{n+i+1}^{l+i+1} \ a^{i+1} \ b^{2m-i+1}$$

$$+ \tfrac{(2m)^{\underline{i}}}{i!} \ ({}^tB)_{n+i}^{l+i} \ a^i \ b^{2m-i+2} \Big)$$

$$= a^2({}^tB)_{n+3m+2}^{l+2m+2} + 2ab({}^tB)_{n+3m+1}^{l+2m+1} + b^2({}^tB)_{n+3m}^{l+2m}$$

$$= ({}^tB)_{n+3m+3}^{l+2m+2} = \text{R.H.S.}$$

Hence the result is true for $s = m + 1$. Thus by mathematical induction, the theorem is proved. □

Theorem 3.4.4. *For* $0 \le l \le \left\lfloor \frac{2n-2}{3} \right\rfloor$ *and* $s \ge 1$,

$$\sum_{i=0}^{s-1} \Big(2 \ a^{2s-1-2i} \ b \ ({}^tB)_{n+1+i}^{l+1} + a^{2s-2-2i} \ b^2 ({}^tB)_{n+i}^{l} \Big) = ({}^tB)_{n+2+s}^{l+2} - a^{2s} ({}^tB)_{n+2}^{l+2}. \quad (3.59)$$

Proof. By induction on s.

Note that (3.59) clearly holds for $s = 1$.

Now let the result be true for $s \le m$. we prove it for $s = m + 1$.

Consider, $\sum_{i=0}^{m} \Big(2a^{2m+1-2i} \ b \ ({}^tB)_{n+1+i}^{l+1} + a^{2m-2i} \ b^2 ({}^tB)_{n+i}^{l} \Big)$

$$= \sum_{i=0}^{m-1} \Big(2a^{2m+1-2i} b \ ({}^tB)_{n+1+i}^{l+1} + a^{2m-2i} b^2 ({}^tB)_{n+i}^{l} \Big)$$

$$+ \Big(2ab \ ({}^tB)_{n+1+m}^{l+1} + b^2 ({}^tB)_{n+m}^{l} \Big)$$

$$= a^2 \Big(\sum_{i=0}^{m-1} \big(2 \ a^{2m-1-2i} b \ ({}^tB)_{n+1+i}^{l+1} + a^{2m-2-2i} b^2 ({}^tB)_{n+i}^{l} \big) \Big)$$

$$+\Big(2ab\ ({}^tB)^{l+1}_{n+1+m} + b^2({}^tB)^{l}_{n+m}\Big)$$

$$= a^2\Big(({}^tB)^{l+2}_{n+2+m} - a^{2m}({}^tB)^{l+2}_{n+2}\Big) + 2ab\ ({}^tB)^{l+1}_{n+1+m} + b^2\ ({}^tB)^{l}_{n+m}$$

$$= a^2({}^tB)^{l+2}_{n+2+m} - a^{2m+2}({}^tB)^{l+2}_{n+2} + 2ab\ ({}^tB)^{l+1}_{n+1+m} + b^2({}^tB)^{l}_{n+m}$$

$$= ({}^tB)^{l+2}_{n+3+m} - a^{2m+2}({}^tB)^{l+2}_{n+2}, \text{ from (3.57).}$$

Hence by mathematical induction, the theorem is proved. □

Definition 3.4.5. *The incomplete B-Tri Lucas sequence is defined by*

$({}^tL)^l_n$

$$= \sum_{r=0}^{l} \Big(\frac{(2n-2)}{(2n-2-2r)} \frac{(2n-2-2r)^{\underline{r}}}{r!} - r(r-1)\frac{(2n-4-2r)^{\underline{r-2}}}{r!} \Big)\ a^{2n-2-3r}\ b^r,$$

$$\forall\ 0 \le l \le \lfloor \frac{2n-2}{3} \rfloor \text{ and } n \ge 2. \tag{3.60}$$

Some special cases of (3.60) are listed below:

$({}^tL)^0_n = a^{2n-2},\ \forall\ n \ge 2.$

$({}^tL)^1_n = a^{2n-2} + (2n-2)a^{2n-5}\ b,\ \forall\ n \ge 3.$

$({}^tL)^2_n = a^{2n-2} + (2n-2)a^{2n-5}\ b + \frac{[(2n-2)(2n-7)-2]}{2}a^{2n-8}\ b^2,\ \forall\ n \ge 4.$

$({}^tL)^{\lfloor \frac{2n-2}{3} \rfloor}_n = ({}^tL)_n.$

The following table gives the terms of B-Tri Lucas sequence.

We now establish the relation between incomplete B-Tribonacci sequence and incomplete B-Tri Lucas sequence. These results are used to prove the recurrence relation for B-Tri Lucas sequence.

l / n	0	1	2	3
2	a^2			
3	a^4	a^4+4ab		
4	a^6	a^6+6a^3b	$a^6+6a^3b+2b^2$	
5	a^8	a^8+8a^5b	$a^8+8a^3b+11a^2b^2$	
6	a^{10}	$a^{10}+10a^7b$	$a^{10}+10a^7b+24a^4b^2$	$a^{10}+10a^7b+24a^4b^2+8ab^3$
7	a^{12}	$a^{12}+12a^9b$	$a^{12}+12a^9b+41a^6b^2$	$a^{12}+12a^9b+41a^6b^2+36a^3b^3$

Table 3.2: Terms of incomplete B-Tri Lucas sequence

Theorem 3.4.6. *For* $2 \leq l \leq \left\lfloor \frac{2n-2}{3} \right\rfloor$,

$$({}^tL)_n^l = ({}^tB)_{n+1}^l + 2ab\ ({}^tB)_{n-1}^{l-1} + b^2\ ({}^tB)_{n-2}^{l-2}. \tag{3.61}$$

Proof. From (3.56), the R.H.S. of (3.61)

$$= \sum_{r=0}^{l} \frac{(2n-2-2r)^{\underline{r}}}{r!}\ a^{2n-2-3r}b^r + 2ab\ \sum_{r=0}^{l-1} \frac{(2n-6-2r)^{\underline{r}}}{r!}\ a^{2n-6-3r}b^r$$

$$+b^2\ \sum_{r=0}^{l-2} \frac{(2n-8-2r)^{\underline{r}}}{r!}\ \ a^{2n-8-3r}b^r$$

$$=\sum_{r=0}^{l} \frac{(2n-2-2r)^{\underline{r}}}{r!}\ a^{2n-2-3r}b^r + 2\sum_{r=1}^{l} \frac{(2n-4-2r)^{\underline{r-1}}}{(r-1)!}\ a^{2n-2-3r}b^r$$

$$+\sum_{r=2}^{l} \frac{(2n-4-2r)^{\underline{r-2}}}{(r-2)!}\ \ a^{2n-2-3r}b^r$$

$$=\sum_{r=0}^{l} \left[\frac{(2n-2-2r)^{\underline{r}}}{r!} + 2\Big(\frac{(2n-4-2r)^{\underline{r-1}}}{(r-1)!} + \frac{(2n-4-2r)^{\underline{r-2}}}{(r-2)!}\Big) - \frac{(2n-4-2r)^{\underline{r-2}}}{(r-2)!}\right] a^{2n-2-3r}b^r$$

$$=\sum_{r=0}^{l} \left[\frac{(2n-2-2r)^{\underline{r}}}{r!} + 2\Big(\frac{(2n-3-2r)^{\underline{r-1}}}{(r-1)!}\Big) - \frac{(2n-4-2r)^{\underline{r-2}}}{(r-2)!}\right] a^{2n-2-3r}b^r$$

$$= \sum_{r=0}^{l} \left[\frac{(2n-2-2r)^{\underline{r}}}{r!}\Big(1 + \frac{2\,r}{2n-2-2r}\Big) - \frac{(2n-4-2r)^{\underline{r-2}}}{(r-2)!}\right] a^{2n-2-3r}b^r$$

$$= \sum_{r=0}^{l} \left[\frac{2n-2}{2n-2-2r}\Big(\frac{(2n-2-2r)^{\underline{r}}}{r!}\Big) - \frac{(2n-4-2r)^{\underline{r-2}}}{(r-2)!}\right] a^{2n-2-3r}b^r$$

$= \sum_{r=0}^{l}\left[\frac{2n-2}{2n-2-2r}(\frac{(2n-2-2r)^{\underline{r}}}{r!}) - r(r-1)\frac{(2n-4-2r)^{\underline{r-2}}}{r!}\right]a^{2n-2-3r}b^r$

$= ({}^tL)_n^l.$ □

We have the following Corollary.

Corollary 3.4.7. *For* $0 \leq l \leq \lfloor \frac{2n-2}{3} \rfloor$,

$$({}^tL)_n^l = 2\ ({}^tB)_{n+1}^l - a^2\ ({}^tB)_n^l. \tag{3.62}$$

Proof. For $l = 0$, $2\ ({}^tB)_{n+1}^0 - a^2\ ({}^tB)_n^0 = 2\ a^{2n-2} - a^{2n-2} = ({}^tL)_n^0$.

Also, if $l = 1$, then

$$\begin{aligned} &2\ ({}^tB)_{n+1}^1 - a^2\ ({}^tB)_n^1 \\ &\quad = 2\ a^{2n-2} + (2n-4)a^{2n-5}b - a^2(a^{2n-4} + (2n-6)a^{2n-7}b) \\ &\quad = ({}^tL)_n^1. \end{aligned}$$

Hence (3.62) is true for $l = 0, 1$.

For $l \geq 2$, the result follows from equations (3.57) and (3.61). □

Theorem 3.4.8. *The recurrence relation of the incomplete B-Tri Lucas sequence* $({}^tL)_n^l$ *is given by*

$$({}^tL)_{n+3}^{l+2} = a^2({}^tL)_{n+2}^{l+2} + 2ab({}^tL)_{n+1}^{l+1} + b^2({}^tL)_n^l,\ 0 \leq l \leq \lfloor \frac{2n-4}{3} \rfloor. \tag{3.63}$$

Proof. From (3.61) we have,

$$\begin{aligned} ({}^tL)_{n+3}^{l+2} &= ({}^tB)_{n+4}^{l+2} + 2ab\ ({}^tB)_{n+2}^{l+1} + b^2\ ({}^tB)_{n+1}^l \\ &= a^2({}^tB)_{n+3}^{l+2} + 2ab\ ({}^tB)_{n+2}^{l+1} + b^2({}^tB)_{n+1}^l \\ &\quad +2ab\Big(a^2\ ({}^tB)_{n+1}^{l+1} + 2ab\ ({}^tB)_n^l + b^2\ ({}^tB)_{n-1}^{l-1}\Big) \\ &\quad +a^2({}^tB)_n^l + 2ab({}^tB)_{n-1}^{l-1} + b^2({}^tB)_{n-2}^{l-2} \end{aligned}$$

$$= a^2\ ({}^tL)^{l+2}_{n+2} + 2ab\ ({}^tL)^{l+1}_{n+1} + b^2\ ({}^tL)^{l}_{n}, \text{ from (3.61).} \qquad \square$$

Theorem 3.4.9. *For $s \geq 1$,*

$$\sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!}\ ({}^tL)^{l+i}_{n+i}\ a^i b^{2s-i} = ({}^tL)^{l+2s}_{n+3s},\quad 0 \leq l \leq \lfloor \frac{2n-2-2s}{3} \rfloor. \tag{3.64}$$

Proof. Consider, L.H.S. of (3.64),

$$\sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!}\ ({}^tL)^{l+i}_{n+i}\ a^i b^{2s-i}$$

$$= \sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!} \Big(2\ ({}^tB)^{l+i}_{n+1+i} - a^2({}^tB)^{l+i}_{n+i}\Big) a^i b^{2s-i}$$

$$= 2\ \sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!}\ ({}^tB)^{l+i}_{n+1+i}\ a^i b^{2s-i} - a^2 \sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!}\ ({}^tB)^{l+i}_{n+i} a^i b^{2s-i}$$

$$= 2\ ({}^tB)^{l+2s}_{n+1+3s} - a^2({}^tB)^{l+3s}_{n+2s}, \text{ from (3.58).}$$

$$= ({}^tL)^{l+2s}_{n+3s}. \qquad \square$$

Theorem 3.4.10. *For $n \geq \lfloor \frac{3l+6}{2} \rfloor$,*

$$\sum_{i=0}^{s-1} \Big(2\ a^{2s-1-2i}({}^tL)^{l+1}_{n+1+i} + a^{2s-2-2i}({}^tL)^{l}_{n+i}\Big) = ({}^tL)^{l+2}_{n+2+s} - a^{2s}({}^tL)^{l+2}_{n+2}. \tag{3.65}$$

Proof. Consider, L.H.S. of (3.65)

$$\sum_{i=0}^{s-1} \Big(2\ a^{2s-1-2i}({}^tL)^{l+1}_{n+1+i} + a^{2s-2-2i}({}^tL)^{l}_{n+i}\Big)$$

$$= \sum_{i=0}^{s-1} 2\ a^{2s-1-2i}\Big(2({}^tB)^{l+1}_{n+2+i} - a^2({}^tB)^{l+1}_{n+1+i}\Big)$$

$$+ \sum_{i=0}^{s-1} a^{2s-2-2i}\Big(2({}^tB)^{l}_{n+1+i} - a^2({}^tB)^{l}_{n+i}\Big), \text{ from (3.62).}$$

$$= 2\ \sum_{i=0}^{s-1}\left(2\ a^{2s-1-2i}({}^tB)_{n+2+i}^{l+1} + a^{2s-2-2i}({}^tB)_{n+1+i}^{l}\right)$$

$$-a^2\sum_{i=0}^{s-1}\left(2\ a^{2s-1-2i}({}^tB)_{n+1+i}^{l+1} + a^{2s-2-2i}({}^tB)_{n+i}^{l}\right)$$

$$= 2\ \left(({}^tB)_{n+3+s}^{l+2} - a^{2s}({}^tB)_{n+3}^{l+2}\right) - a^2\left(({}^tB)_{n+2+s}^{l+2} - a^{2s}({}^tB)_{n+2}^{l+2}\right), \text{ from (3.59).}$$

$$= 2\ ({}^tB)_{n+3+s}^{l+2} - a^2({}^tB)_{n+2+s}^{l+2} - a^{2s}\left(2({}^tB)_{n+3}^{l+2} - a^2({}^tB)_{n+2}^{l+2}\right)$$

$$= ({}^tL)_{n+2+s}^{l+2} - a^{2s}({}^tL)_{n+2}^{l+2}.$$

Hence the theorem is proved. □

Chapter 4

B-q bonacci and B-q Lucas Sequences

This Chapter include the content of published paper (P4).

Chapter 4

B-q bonacci and B-q Lucas Sequences

4.1 Introduction

The extension of Fibonacci recurrence relation to m^{th} order linear recurrence relation is studied in [5]. The author has obtained two properties. In this Chapter we intend to extend the work done in Chapter 3 to q^{th} order linear recurrence relation, where $q \geq 2$ and $q \in \mathbb{N}$. In this recurrence relation, the n^{th} term is the sum of the preceding q terms with coefficients $\frac{(q-1)_r}{r!}\ a^{q-1-r}b^r,\ r = 1, 2, \cdots q$. These coefficients are the terms of binomial expansion of $(a + b)^{q-1}$, where a and b are fixed real numbers and $q \geq 2$ and $q \in \mathbb{N}$. We call these class of sequences, the B-q bonacci sequences.

In Section 2, we define B-q bonacci sequence and study its various identities which are the extension of the identities of B- Tribonacci sequence discussed in Chapter 3. B-q Lucas sequence and its identities similar to the identities of B-q bonacci sequence are discussed in Section 3. In Section 4, the incomplete B-q bonacci sequence and incomplete B-q Lucas sequence, and their identities are discussed.

4.2 B-q bonacci sequence

We define an extension of (3.4), (3.5) and (3.6). Let $q \geq 2$ and $q \in \mathbb{N}$.

Definition 4.2.1. *Let $n \in \mathbb{N}\bigcup\{0\}$. The B-q bonacci sequence is defined by,*

$$({}^{q}B)_{n+q-1} = \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ a^{q-1-r} \ b^{r} \ ({}^{q}B)_{n+q-2-r}, \ \forall \ n \geq 1, \tag{4.1}$$
$$\text{with } ({}^{q}B)_{i} = 0, \ i = 0, 1, 2, 3, \cdots, q-2 \text{ and } ({}^{q}B)_{q-1} = 1,$$

where $({}^{q}B)_{n}$ is n^{th} term.

For $n = 1, 2, 3, 4$, we list below the terms of (4.1).
$({}^{q}B)_{q} = a^{q-1}$, $({}^{q}B)_{q+1} = a^{2(q-1)} + (q-1)a^{q-2} \ b$,
$({}^{q}B)_{q+2} = a^{3(q-1)} + \frac{(2(q-1))^{\underline{1}}}{1!} \ a^{2q-3} \ b + \frac{(q-1)^{\underline{2}}}{2!} \ a^{q-3}b^{2}$,
$({}^{q}B)_{q+3} = a^{4(q-1)} + \frac{(3(q-1))^{\underline{1}}}{1!} \ a^{3q-4} \ b + \frac{(2(q-1))^{\underline{2}}}{2!} \ a^{2q-4} \ b^{2} + \frac{(q-1)^{\underline{3}}}{3!} \ a^{q-4} \ b^{3}$,
$({}^{q}B)_{q+4} = a^{5(q-1)} + \frac{(4(q-1))^{\underline{1}}}{1!} \ a^{4q-5} \ b + \frac{(3(q-1))^{\underline{2}}}{2!} \ a^{3q-5} \ b^{2} + \frac{(2(q-1))^{\underline{3}}}{3!} \ a^{2q-5} \ b^{3} + \frac{(q-1)^{\underline{4}}}{4!} a^{q-5} \ b^{4}$.

We rearrange terms of (4.1) as follows to obtain terms for the negative integer values of n.

$$({}^{q}B)_{n-1} = \frac{1}{b^{q-1}} \left[({}^{q}B)_{n+q-1} - \sum_{r=0}^{q-2} \frac{(q-1)^{\underline{r}}}{r!} \ a^{q-1-r} \ b^{r} \ ({}^{q}B)_{n+q-2-r} \right], \tag{4.2}$$
$$\text{with } ({}^{q}B)_{i} = 0, \ i = 0, 1, 2, 3, \cdots q-2 \text{ and } ({}^{q}B)_{q-1} = 1.$$

For $n = -2, -1, 0$, the terms of (4.2) are, $({}^{q}B)_{-3} = (\frac{q(q-1)}{2})\frac{a^{2}}{b^{q+1}}$,
$({}^{q}B)_{-2} = -(q-1)\frac{a}{b^{q}}$ and $({}^{q}B)_{-1} = \frac{1}{b^{q-1}}$.

We now define the B-q bonacci sequence for all $n \in \mathbb{Z}$.

Definition 4.2.2. *Let $n \in \mathbb{Z}$. The B-q bonacci sequence is defined by*

$$({}^{q}B)_{n+q-1} = \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ a^{q-1-r} \ b^{r} \ ({}^{q}B)_{n+q-2-r}, \tag{4.3}$$
$$\text{with } ({}^{q}B)_{i} = 0, \ i = 0, 1, 2, 3, \cdots, q-2 \text{ and } ({}^{q}B)_{q-1} = 1,$$

where $({}^{q}B)_{n}$ is n^{th} term.

We have following theorems related to the B-q bonacci sequence.

Similar to the Binet type formula, a sum of the terms and generating a formula for the B-Tribonacci sequence, we have them also for B-q bonacci sequence.

Theorem 4.2.3. (***Binet type formula***) *The n^{th} term of (4.3) is given by*

$$({}^qB)_n = \frac{\sum_{k=1}^{q}(-1)^{k+1}\prod_{1\leq i<j\leq q, i,j\neq k}(\phi_i - \phi_j)\phi_k^n}{\prod_{1\leq i<j\leq q}(\phi_i - \phi_j)}, \tag{4.4}$$

where $\phi_p, p = 1, 2, \cdots, q$ are q distinct roots of the characteristic equation $\lambda^q - \sum_{r=0}^{q-1}\frac{(q-1)^r}{r!}\ a^{(q-1)-r}\ b^r\lambda^r = 0$ corresponding to (4.3).

Proof. Since the roots $\phi_p, p = 1, 2, \cdots, q$ are q distinct roots of the characteristic equation $\lambda^q - \sum_{r=0}^{q-1}\frac{(q-1)^r}{r!}\ a^{(q-1)-r}\ b^r\lambda^r = 0$ corresponding to (4.3), solution of (4.3) is given by

$$({}^qB)_n = \sum_{r=0}^{q} C_r\phi_r^n. \tag{4.5}$$

Using the conditions $({}^qB)_i = 0,\ i = 0, 1, 2, 3, \cdots q-2$ and $({}^qB)_{q-1} = 1$, we get equation (4.4). □

Remark 4.2.4. *The case of repeated roots is excluded.*

Theorem 4.2.5. *The n^{th} term of (4.1) is given by*

$$({}^qB)_n = \sum_{r=0}^{\left\lfloor\frac{(q-1)(n-(q-1))}{q}\right\rfloor} \frac{\Big((q-1)(n-(q-1)-r)\Big)^{r}}{r!}\ a^{(q-1)(n-(q-1)-r)-r}\ b^r, \tag{4.6}$$

$\forall\, n\ \geq\ q-1$ *and* $q\ \geq\ 2$.

Proof. We prove the theorem by induction on n.

For $n = q-1$, $({}^qB)_{q-1} = \sum_{r=0}^{0}\frac{(-(q-1)r)^r}{r!}\ \ a^{-qr}\ b^r = 1$.

Therefore, the theorem is true for $n = q-1$.

Now let us assume that the theorem is true for $n \leq m$.

We shall prove (4.6) for $n = m+1$, by dividing the proof into q cases depending upon the form of m, where $m = qk,\ qk+1, \cdots,\ qk+q-1$ and $k \geq 1$.

Case (i) $m = qk$

To prove, $({}^qB)_{qk+1}$

$$= \sum_{r=0}^{\left\lfloor \frac{(q-1)\left((qk+1)-(q-1)\right)}{q} \right\rfloor} \frac{\left((q-1)(qk+1-(q-1)-r)\right)^{\underline{r}}}{r!} \; a^{\left((q-1)(qk+1-(q-1)-r)\right)-r} \; b^r.$$

That is to prove,

$$({}^qB)_{qk+1} = \sum_{r=0}^{(q-1)k-(q-2)} \frac{\left((q-1)(qk+1-(q-1)-r)\right)^{\underline{r}}}{r!} \; a^{\left((q-1)(qk+1-(q-1))\right)-qr} \; b^r.$$

Since the result is true for $n \leq m = qk$, we have

$({}^qB)_{qk+1} = \sum_{s=0}^{q-1} \frac{(q-1)^{\underline{s}}}{s!} \; a^{(q-1)-s} \; b^s \; ({}^qB)_{qk-s}$

$$= \sum_{s=0}^{q-1} \frac{(q-1)^{\underline{s}}}{s!}$$

$$\sum_{r=0}^{\left\lfloor \frac{(q-1)\left((qk-s)-(q-1)\right)}{q} \right\rfloor} \frac{\left((q-1)\left((qk-s)-(q-1)-r\right)\right)^{\underline{r}}}{r!} \; a^{(q-1)\left((qk+1-s)-(q-1)-r\right)-r-s} \; b^{r+s}$$

$$= \sum_{s=0}^{q-1} \frac{(q-1)^{\underline{s}}}{s!}$$

$$\sum_{r=0}^{\left\lfloor (q-1)k-(q-2)-\frac{(q-1)s}{q} \right\rfloor} \frac{\left((q-1)\left(qk-(q-1)-(r+s)\right)\right)^{\underline{r}}}{r!} \; a^{(q-1)\left(qk+1-(q-1)\right)-q(r+s)} \; b^{r+s}$$

$$= \sum_{s=0}^{q-1} \frac{(q-1)^{\underline{s}}}{s!} \sum_{m=s}^{\left\lfloor (q-1)k-(q-2)+\frac{s}{q} \right\rfloor} \frac{\left((q-1)\left(qk-(q-1)-m\right)\right)^{\underline{m-s}}}{(m-s)!} \; a^{\left((q-1)(qk+1-(q-1))\right)-qm} \; b^m.$$

Since $0 \leq s \leq (q-1)$, we have

$$({}^qB)_{qk+1} = \sum_{m=0}^{(q-1)k-(q-2)} \frac{\left((q-1)(qk+1-(q-1)-m)\right)^{\underline{m}}}{m!} \; a^{\left((q-1)(qk+1-(q-1))\right)-qm} \; b^m.$$

The proof for the other cases can be given using similar procedure.

Hence the theorem is proved. □

Similarly, we can prove the following Corollary which gives the nth term of (4.2).

Corollary 4.2.6. *The n^{th} term of* (4.2) *is given by*

$$({}^{q}B)_n = \sum_{r=n-(q-2)}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} \frac{\Big((q-1)(n-(q-1)-r)\Big)^{\underline{r}}}{r!} \, a^{(q-1)(n-(q-1)-r)-r} \, b^{r}, \qquad (4.7)$$

$\forall \, n \leq \; -1$ *and* $q \; \geq \; 2$.

Theorem 4.2.7. (*i*) *For* $n \geq q-1$, *we have*

$$\sum_{r=0}^{n} ({}^{q}B)_r = \frac{({}^{q}B)_{n+1} + \sum_{i=0}^{q-2} \left(\sum_{r=1+i}^{q-1} \frac{(q-1)\underline{r}}{r!} \, a^{q-1-r} \, b^{r} \right) ({}^{q}B)_{n-i} - 1}{(a+b)^{q-1} - 1}, \qquad (4.8)$$

$$\textit{provided} \begin{cases} a+b \neq 1, \text{if q is even;} \\ \\ a+b \neq \pm 1, \text{if q is odd.} \end{cases}$$

(*ii*) *For* $n \geq 1$, *we have*

$$\sum_{r=-1}^{-n} ({}^{q}B)_r = -\frac{({}^{q}B)_{-n} + \sum_{i=0}^{q-2} \left(\sum_{r=1+i}^{q-1} \frac{(q-1)\underline{r}}{r!} \, a^{q-1-r} \, b^{r} \right) ({}^{q}B)_{-(n+1+i)} - 1}{(a+b)^{q-1} - 1}, \qquad (4.9)$$

$$\textit{provided} \begin{cases} a+b \neq 1, \text{if q is even;} \\ \\ a+b \neq \pm 1, \text{if q is odd.} \end{cases}$$

Proof. (i) For $n = q-1$, R.H.S. of (4.8)

$$= \frac{({}^{q}B)_q + \sum_{i=0}^{q-2} \sum_{r=1+i}^{q-1} \frac{(q-1)\underline{r}}{r!} \, a^{q-1-r} \, b^{r} \, ({}^{q}B)_{q-1-i} - 1}{(a+b)^{q-1} - 1}$$

$$= \frac{a^{q-1} + \sum_{r=1}^{q-1} \frac{(q-1)\underline{r}}{r!} \, a^{q-1-r} \, b^{r} - 1}{(a+b)^{q-1} - 1}$$

$$= \frac{\sum_{r=0}^{q-1} \frac{(q-1)\underline{r}}{r!} \, a^{q-1-r} \, b^{r} - 1}{(a+b)^{q-1} - 1}$$

$$= 1$$

$$= \sum_{r=0}^{q-1} ({}^{q}B)_r = \text{L.H.S.}$$

Therefore, the theorem holds for $n = q-1$.

Assume that the result is true for $n \leq m$.

Consider, $\sum_{r=0}^{m+1}({}^qB)_r = \sum_{r=0}^{m}({}^qB)_r + ({}^qB)_{m+1}$

$$= \frac{({}^qB)_{m+1}+\sum_{i=0}^{q-2}\left(\sum_{r=1+i}^{q-1}\frac{(q-1)_{\underline{r}}}{r!}\;a^{q-1-r}\;b^r\right)\;({}^qB)_{m-i}-1}{(a+b)^{q-1}-1} + ({}^qB)_{m+1}$$

$$= \frac{({}^qB)_{m+1}+\sum_{i=0}^{q-2}\left(\sum_{r=1+i}^{q-1}\frac{(q-1)_{\underline{r}}}{r!}\;a^{q-1-r}\;b^r\right)\;({}^qB)_{m-i}-1+\left((a+b)^{q-1}-1\right)({}^qB)_{m+1}}{(a+b)^{q-1}-1}$$

$$= \frac{\sum_{i=0}^{q-2}\left(\sum_{r=1+i}^{q-1}\frac{(q-1)_{\underline{r}}}{r!}\;a^{q-1-r}\;b^r\right)\;({}^qB)_{m-i}-1+(a+b)^{q-1}({}^qB)_{m+1}}{(a+b)^{q-1}-1}$$

$$= \frac{\sum_{i=0}^{q-2}\left(\sum_{r=1+i}^{q-1}\frac{(q-1)_{\underline{r}}}{r!}\;a^{q-1-r}\;b^r\right)\;({}^qB)_{m-i}-1+\sum_{i=0}^{q-1}\frac{(q-1)_{\underline{i}}}{i!}\;a^{q-1-i}\;b^i({}^qB)_{m+1}}{(a+b)^{q-1}-1}$$

$$= \frac{({}^qB)_{m+2}+\sum_{i=0}^{q-2}\sum_{r=1+i}^{q-1}\left(\frac{(q-1)_{\underline{r}}}{r!}\;a^{q-1-r}\;b^r\right)\;({}^qB)_{m+1-i}-1}{(a+b)^{q-1}-1}.$$

Hence the result is true.

Using similar procedure we can prove (4.9). □

Combining (4.8) and (4.9) we have

$$\sum_{r=-n}^{n}({}^qB)_r$$

$$= \frac{\left(({}^qB)_{n+1} - ({}^qB)_{-n}\right) + \sum_{i=0}^{q-2}\sum_{r=1+i}^{q-1}\left(\frac{(q-1)_{\underline{r}}}{r!}\;a^{q-1-r}\;b^r\right)\;\left(({}^qB)_{n-i} - ({}^qB)_{-(n+1+i)}\right)}{(a+b)^{q-1}-1},$$

provided $\begin{cases} a+b \neq 1, \text{if q is even;} \\ a+b \neq \pm 1, \text{if q is odd,} \end{cases}$

For $q = 4$, $a = \frac{3}{4}$ and $b = \frac{1}{4}$, we have the following graph for the sequence

$$({}^qB)_{n+3} = (\frac{3}{4})^3({}^qB)_{n+2} + 3(\frac{3}{4})^2(\frac{1}{4})({}^qB)_{n+1} + 3(\frac{3}{4})(\frac{1}{4})^2({}^qB)_n + (\frac{1}{4})^3\;({}^qB)_{n-1}. \quad (4.10)$$

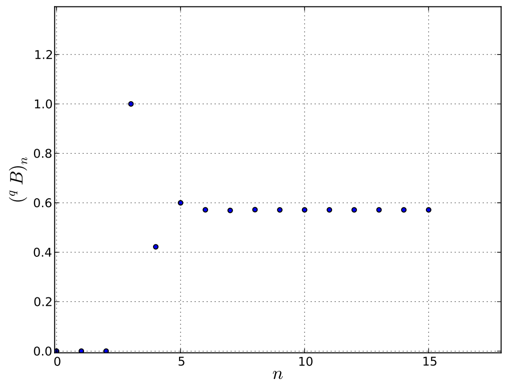

Figure 4-1: Graph showing terms of (4.10).

The next theorem is based on the ratio of successive B-q bonacci sequence and the ratio of preceding terms.

Theorem 4.2.8. *Let $\phi_i, i = 1, 2, \cdots, q$ be the distinct roots of (4.4) such that $\phi_1 \neq 0$ and $|\phi_1| > |\phi_2| > \cdots > |\phi_q|$, then*

(i)

$$\lim_{n\to\infty} \frac{({}^qB)_n}{({}^qB)_{n-1}} = \phi_1. \tag{4.11}$$

(ii)

$$\lim_{n\to\infty} \frac{({}^qB)_{n-1}}{({}^qB)_n} = \frac{1}{\phi_1}. \tag{4.12}$$

Proof. Equation (4.9) implies

$({}^qB)_n = \frac{\sum_{k=1}^{q}(-1)^{k+1}\prod_{1\le i<j\le q,i,j\neq k}(\phi_i-\phi_j)\phi_k^n}{\prod_{1\le i<j\le q}(\phi_i-\phi_j)}$

$= \frac{\prod_{2\le i<j\le q}(\phi_i-\phi_j)\,\phi_1^n+\sum_{k=2}^{q}(-1)^{k+1}\prod_{1\le i<j\le q,i,j\neq k}(\phi_i-\phi_j)\phi_k^n}{\prod_{1\le i<j\le q}(\phi_i-\phi_j)}$

Therefore, $\lim_{n\to\infty}\frac{({}^qB)_n}{({}^qB)_{n-1}}$

$= \lim_{n\to\infty}\frac{\prod_{2\le i<j\le q}(\phi_i-\phi_j)\,\phi_1^n+\sum_{k=2}^{q}(-1)^{k+1}\prod_{1\le i<j\le q,i,j\neq k}(\phi_i-\phi_j)\,\phi_k^n}{\prod_{2\le i<j\le q}(\phi_i-\phi_j)\,\phi_1^{n-1}+\sum_{k=2}^{q}(-1)^{k+1}\prod_{1\le i<j\le q,i,j\neq k}(\phi_i-\phi_j)\,\phi_k^{n-1}}$

Since $\phi_1 \neq 0$, dividing numerator and denominator by ϕ_1^n, we get

$\lim_{n\to\infty} \frac{({}^qB)_n}{({}^qB)_{n-1}}$

$$= \lim_{n\to\infty} \frac{\prod_{1\le i<j\le q, i,j\neq k} (\phi_i-\phi_j) + \sum_{k=2}^{q} (-1)^{k+1} \prod_{1\le i<j\le q, i,j\neq k} (\phi_i-\phi_j) \left(\frac{\phi_k}{\phi_1}\right)^n}{\prod_{1\le i<j\le q, i,j\neq k} (\phi_i-\phi_j)\, \phi_1^{-1} + \sum_{k=2}^{q} (-1)^{k+1} \prod_{1\le i<j\le q, i,j\neq k} (\phi_i-\phi_j) \phi_k^{-1} \left(\frac{\phi_k}{\phi_1}\right)^n}$$

$$= \lim_{n\to\infty} \frac{\prod_{1\le i<j\le q, i,j\neq k} (\phi_i-\phi_j)}{\prod_{1\le i<j\le q, i,j\neq k} (\phi_i-\phi_j)\, \phi_1^{-1}}, \text{ since } |\phi_1| > |\phi_i|, i = 2, 3, \cdots, q$$

$$= \phi_1.$$

Similarly, we can prove the equation (4.12). □

Theorem 4.2.9. *The terms of the equation (4.3) can be generated from the series*

$$\sum_{n=-\infty}^{\infty} z^n (a+bz)^{(q-1)n}$$

Proof. $\sum_{n=-\infty}^{\infty} z^n (a+bz)^{(q-1)n}$

$$= \sum_{n=-\infty}^{-1} z^n \sum_{k=0}^{\infty} \frac{\big((q-1)n\big)^{\underline{k}}}{k!} a^k\, b^{(q-1)n-k} z^{(q-1)n-k}$$

$$+ \sum_{n=0}^{\infty} z^n \sum_{k=0}^{(q-1)n} \frac{\big((q-1)n\big)^{\underline{k}}}{k!} a^k\, b^{(q-1)n-k} z^{(q-1)n-k}$$

$$= \sum_{n=-\infty}^{-1} \sum_{k=0}^{\infty} \frac{\big((q-1)n\big)^{\underline{k}}}{k!} a^k\, b^{(q-1)n-k} z^{nq-k}$$

$$+ \sum_{n=0}^{\infty} \sum_{k=0}^{(q-1)n} \frac{\big((q-1)n\big)^{\underline{k}}}{k!} a^k\, b^{(q-1)n-k} z^{nq-k}$$

$$= \cdots + \sum_{k=0}^{\infty} \frac{\big(-2(q-1)\big)^{\underline{k}}}{k!} a^k\, b^{-2(q-1)-k} z^{-2q-k} + \sum_{k=0}^{\infty} \frac{\big(-(q-1)\big)^{\underline{k}}}{k!} a^k\, b^{-(q-1)-k} z^{-q-k}$$

$$+1 + \sum_{k=0}^{(q-1)} \frac{\big((q-1)\big)^{\underline{k}}}{k!} a^k\, b^{(q-1)-k} z^{q-k} + \sum_{k=0}^{2(q-1)} \frac{\big(2(q-1)\big)^{\underline{k}}}{k!} a^k\, b^{2(q-1)-k} z^{2q-k} + \cdots$$

$$= \cdots + \frac{\big((q-1)\big)^{\underline{1}}}{1!} a^1\, b^{q-2} z^{-q-1} + b^{-(q-1)} z^{-q} + \cdots$$

$$+1 + a^{q-1} z + \big(a^{2(q-1)} + (q-1)\, a^{q-2} b\big) z^2 + \big(a^{3(q-1)} + 2(q-1)\, a^{2q-3} b + b^2\big) z^3 + \cdots$$

$$= \cdots + ({}^{q}B)_{-2}\, a^{1}\, b^{q-2} z^{-q-1} + \; ({}^{q}B)_{-1} z^{-q} + \cdots$$

$$+({}^{q}B)_{q-1} z^{0} + ({}^{q}B)_{q}\; z + ({}^{q}B)_{q+1} z^{2} + ({}^{q}B)_{q+2} z^{3} + \cdots .$$

$$= \textstyle\sum_{n=-\infty}^{\infty} ({}^{q}B)_{n}\; z^{n-(q-1)}. \qquad \square$$

In Matrix Form (4.1) is represented as

$$\begin{bmatrix} ({}^{q}B)_{n} \\ ({}^{q}B)_{n+1} \\ ({}^{q}B)_{n+2} \\ \cdots \\ ({}^{q}B)_{n+q-2} \\ ({}^{q}B)_{n+q-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ b^{q-1} & \frac{(q-1)^{\underline{1}}}{1!}\, ab^{q-2} & \frac{(q-1)^{\underline{2}}}{2!}\, a^{2}b^{q-3} & \frac{(q-1)^{\underline{3}}}{3!}\, a^{3}b^{q-4} & \cdots & a^{q-1} \end{bmatrix} \begin{bmatrix} ({}^{q}B)_{n-1} \\ ({}^{q}B)_{n} \\ ({}^{q}B)_{n+1} \\ \cdots \\ ({}^{q}B)_{n+q-3} \\ ({}^{q}B)_{n+q-2} \end{bmatrix}.$$

$$\text{Let } A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ b^{q-1} & \frac{(q-1)^{\underline{1}}}{1!} ab^{q-2} & \frac{(q-1)^{\underline{2}}}{2!} a^{2}b^{q-3} & \frac{(q-1)^{\underline{3}}}{3!} a^{3}b^{q-4} & \cdots & a^{q-1} \end{bmatrix}$$

$$= \begin{bmatrix} b^{q-1}({}^{q}B)_{0} & \cdots & \sum_{r=q-j}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\, a^{(q-1)-r}\, b^{r}({}^{q}B)_{q-r-j} & \cdots & ({}^{q}B)_{1} \\ \cdots & & & & \\ b^{q-1}({}^{q}B)_{i-1} & \cdots & \sum_{r=q-j}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\, a^{(q-1)-r}\, b^{r}({}^{q}B)_{q-1-r-j+i} & \cdots & ({}^{q}B)_{i} \\ \cdots & & & & \\ b^{q-1}({}^{q}B)_{q-1} & \cdots & \sum_{r=q-j}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\, a^{(q-1)-r}\, b^{r}({}^{q}B)_{q-1-r-j} & \cdots & ({}^{q}B)_{q} \end{bmatrix},$$

$1 \le i,\ j \le q$. Then

$$A^n = \begin{bmatrix} b^{q-1}({}^qB)_{n-1} \cdots & \sum_{r=q-j}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\, a^{(q-1)-r}\, b^r\, ({}^qB)_{n+q-1-r-j} & \cdots ({}^qB)_n \\ \cdots & & \\ b^{q-1}({}^qB)_{n+(i-2)} & \sum_{r=q-j}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\, a^{(q-1)-r}\, b^r\, ({}^qB)_{n+q-2-r-j+i} & \cdots ({}^qB)_{n+(i-1)} \\ \cdots & & \\ b^{q-1}({}^qB)_{n+q-2} \cdots & \sum_{r=q-j}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\, a^{(q-1)-r}\, b^r\, ({}^qB)_{n+2q-2-r-j} & \cdots ({}^qB)_{n+q-1} \end{bmatrix}. \tag{4.13}$$

We have following results, the particular case (i.e. for $q = 3$) of which is discussed in Chapter 3.

Theorem 4.2.10. (***Honsberger type identity***)
For any $m, n \in \mathbb{Z}$,

$$({}^qB)_{n+m-1} = \sum_{r=0}^{q-1} \Big(\sum_{s=0}^{r} \frac{(q-1)^{\underline{s}}}{s!}\, b^{q-1-s}\, a^s ({}^qB)_{n-1+s-r} \Big)\, ({}^qB)_{m-1+r}. \tag{4.14}$$

Proof. Let M_{11} be the element of the matrix in the first row and first column. Replacing n with $n+m$ in (4.13), we obtain the matrix A^{n+m}. Also multiplying A^n and A^m, we obtain the another form of the matrix A^{n+m}. The required result can be now obtained by equating the M_{11} of these two forms of the matrix A^{n+m}. □

Corollary 4.2.11. *For $n \in \mathbb{Z}$,*

(i) $({}^qB)_{2n-1} = \sum_{r=0}^{q-1} \big(\sum_{s=0}^{r} \frac{(q-1)^{\underline{s}}}{s!}\, b^{q-1-s}\, a^s ({}^qB)_{n-1+s-r} \big)\, ({}^qB)_{n-1+r}$.

(ii) $({}^qB)_{2n} = \sum_{r=0}^{q-1} \big(\sum_{s=0}^{r} \frac{(q-1)^{\underline{s}}}{s!}\, b^{q-1-s}\, a^s ({}^qB)_{n-1+s-r} \big)\, ({}^qB)_{n+r}$.

Proof. Substituting $m = n$ in (4.14), we obtain identity (i). Taking $m = n+1$ in (4.14), we obtain identity (ii). □

Theorem 4.2.12. (***General q-linear identity***)

For any $a_{i_pj} \in \mathbb{Z}$, $1 \le i_p, j, p \le q$ with distinct i_p and the following $({}^qC_2)^2$ equations $a_{i_11} + a_{i_22} = a_{i_21} + a_{i_12}, \cdots, a_{i_11} + a_{i_qq} = a_{i_q1} + a_{i_1q}$, $a_{i_22} + a_{i_33} = a_{i_32} + a_{i_23}, \cdots,$ $a_{i_22} + a_{i_qq} = a_{i_q2} + a_{i_2q}, \cdots, a_{i_{q-1}(q-1)} + a_{i_qq} = a_{i_q(q-1)} + a_{i_{q-1}q}$, we have

$$\begin{vmatrix} ({}^qB)_{a_{11}} & \cdots & ({}^qB)_{a_{1j}} & \cdots & ({}^qB)_{a_{1q}} \\ \cdots & & & & \\ ({}^qB)_{a_{i1}} & \cdots & ({}^qB)_{a_{ij}} & \cdots & ({}^qB)_{a_{iq}} \\ \cdots & & & & \\ ({}^qB)_{a_{q1}} & \cdots & ({}^qB)_{a_{qj}} & \cdots & ({}^qB)_{a_{qq}} \end{vmatrix}$$

$$= [(-b)^{q-1}]^m \begin{vmatrix} ({}^qB)_{a_{11}-m} & \cdots & ({}^qB)_{a_{1j}-m} & \cdots & ({}^qB)_{a_{1q}-m} \\ \cdots & & & & \\ ({}^qB)_{a_{i1}-m} & \cdots & ({}^qB)_{a_{ij}-m} & \cdots & ({}^qB)_{a_{iq}-m} \\ \cdots & & & & \\ ({}^qB)_{a_{q1}-m} & \cdots & ({}^qB)_{a_{qj}-m} & \cdots & ({}^qB)_{a_{qq}-m} \end{vmatrix}$$

$$= [(-b)^{q-1}]^m \sum_{\sigma \epsilon S_q} \prod_{i=1}^{q} sign(\sigma) ({}^qB)_{a_{i\ \sigma(i)-m}}, \forall m \in \mathbb{Z}, \tag{4.15}$$

ranging over the symmetric group S_q, where

$$sign(\sigma) = \begin{cases} +1, & if\ \sigma\ is\ an\ even\ permutation, \\ -1 & if\ \sigma\ is\ an\ odd\ permutation. \end{cases}$$

Proof. Use (4.4) and the procedure similar to the one used in Theorem 3.2.13. □

The following identities can be deduced from general q linear identity.

Theorem 4.2.13. (***d'Ocagne type identity***)

For any $m_k \in \mathbb{Z},\ k = 1, 2, ..., q$ *and* $1 \le i,\ j \le q$,

$$\begin{vmatrix} ({}^qB)_{m_1} & \cdots & ({}^qB)_{m_j} & \cdots & ({}^qB)_{m_q} \\ \cdots & & & & \\ ({}^qB)_{m_1+i-1} & \cdots & ({}^qB)_{m_j+i-1} & \cdots & ({}^qB)_{m_q+i-1} \\ \cdots & & & & \\ ({}^qB)_{m_1+(q-1)} & \cdots & ({}^qB)_{m_j+(q-1)} & \cdots & ({}^qB)_{m_q+(q-1)} \end{vmatrix}$$

$$= [(-b)^{q-1}]^{m_q} \sum_{\sigma \epsilon S_q} \prod_{j=1}^{q} \quad sign(\sigma)({}^qB)_{m_j - m_q + \sigma(j) - 1}. \tag{4.16}$$

Proof. Substitute $a_{ij} = m_j + i - 1, 1 \le i, j \le q$, $m = m_q$ in general q-linear identity and evaluating the resulting determinant, we get the result. □

Theorem 4.2.14. (***Catalan type identity***)

For any $n, r \in \mathbb{Z}$,

$$\begin{vmatrix} ({}^qB)_n & \cdots & ({}^qB)_{n+(j-1)r} & \cdots & ({}^qB)_{n+(q-1)r} \\ \cdots & & & & \\ ({}^qB)_{n+(1-i)r} & \cdots & ({}^qB)_{n+(j-i)r} & \cdots & ({}^qB)_{n+(q-i)r} \\ \cdots & & & & \\ ({}^qB)_{n+(1-q)r} & \cdots & ({}^qB)_{n+(j-q)r} & \cdots & ({}^qB)_n \end{vmatrix}$$

$$= [(-b)^{q-1}]^{n} \sum_{\sigma \epsilon S_q} \prod_{i=1}^{q} \quad sgn(\sigma)({}^qB)_{(i-\sigma(i))r}. \tag{4.17}$$

Proof. Substitute $a_{ij} = n + (j - i)r, 1 \le i, j \le q$, $m = n$ in general q-linear identity (4.15) and evaluating the resulting determinant, we get the result (4.17). □

Remark: When q is odd, it is seen that the contribution of anti-diagonal elements to the determinant value is zero. Hence the R.H.S. of the above identity takes the simpler form $[(-b)^{q-1}]^n \sum_{j=1}^{q-1} ({}^qB)_{jr}^{q-j} ({}^qB)_{-(q-j)r}^{j}$.

Theorem 4.2.15. (***Cassini type identity***)

For any $n \in \mathbb{Z},\ 1 \le i \le q,$

$$\begin{vmatrix} ({}^qB)_n & \cdots & ({}^qB)_{n+j-1} & \cdots & ({}^qB)_{n+(q-1)} \\ \cdots & & & & \\ ({}^qB)_{n+(1-i)} & \cdots & ({}^qB)_{n+(j-i)} & \cdots & ({}^qB)_{n+(q-i)} \\ \cdots & & & & \\ ({}^qB)_{n+(1-q)} & \cdots & ({}^qB)_{n+(j-q)} & \cdots & ({}^qB)_n \end{vmatrix} = [(-b)^{q-1}]^{n-(q-1)}. \qquad (4.18)$$

Proof. Substitute $r = 1$ in (4.17), the required result can be obtained. □

Theorem 4.2.16. (***Extended form of Cassini type identity***)

For all $n \in \mathbb{Z}$ *and* $0 \le j \le q-2,$

$$\begin{vmatrix} ({}^qB)_n & \cdots & ({}^qB)_{n+j} & \cdots & ({}^qB)_{n+r} \\ \cdots & & & & \\ ({}^qB)_{n-j} & \cdots & ({}^qB)_n & \cdots & ({}^qB)_{n+r-j} \\ \cdots & & & & \\ ({}^qB)_{n-(q-2)} & \cdots & ({}^qB)_{n+j-(q-2)} & \cdots & ({}^qB)_{n+r-(q-2)} \\ ({}^qB)_{n-(q-1)} & \cdots & ({}^qB)_{n+j-(q-1)} & \cdots & ({}^qB)_{n+r-(q-1)} \end{vmatrix} = [(-b)^{q-1}]^{n-(q-1)}({}^qB)_r. \qquad (4.19)$$

Proof. Substitute $a_{ij} = n + j - i$, $a_{iq} = n + r - i, \forall\ 1 \le i \le q$ and $1 \le j \le q-1$ in general q-linear identity (4.15) and evaluating the resulting determinant, we get the result. □

Similar to the Pythagorean result of B-Tribonacci sequence, we have it for B-q bonacci sequence.

Theorem 4.2.17. *For all $n \in \mathbb{Z}$,*

$$\Big[b^{q-1}({}^qB)_{n-1}\big(2\ ({}^qB)_{n+q-1} - b^{q-1}({}^qB)_{n-1}\big)\Big]^2 + \Big[2({}^qB)_{n+q-1}\big(({}^qB)_{n+q-1} - b^{q-1}({}^qB)_{n-1}\big)\Big]^2$$

$$= \Big[b^{2(q-1)}({}^qB)^2_{n-1} + 2\ ({}^qB)_{n+q-1}\big(({}^qB)_{n+q-1} - b^{q-1}({}^qB)_{n-1}\big)\Big]^2. \qquad (4.20)$$

Proof. Consider $({}^qB)_{n+q-1} = \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!}\ a^{q-1-r}\ b^r\ ({}^qB)_{n+q-2-r}$

Therefore, $({}^qB)_{n+q-1} - \sum_{r=0}^{q-2} \frac{(q-1)^r}{r!}\ a^{q-1-r}\ b^r\ ({}^qB)_{n+q-2-r} = b^{q-1}\ ({}^qB)_{n-1}$

This implies $({}^qB)_{n+q-1} - \Big(({}^qB)_{n+q-1} - b^{q-1}\ ({}^qB)_{n-1}\Big) = b^{q-1}\ ({}^qB)_{n-1}$

Squaring both sides,

$$b^{2(q-1)}\ ({}^qB)^2_{n-1} + 2\ ({}^qB)_{n+q-1}\ \Big(({}^qB)_{n+q-1} - b^{q-1}\ ({}^qB)_{n-1}\Big)$$

$$= ({}^qB)^2_{n+q-1} + \Big(({}^qB)_{n+q-1} - b^{q-1}\ ({}^qB)_{n-1}\Big)^2$$

Again squaring again both sides,

$$\Big[b^{2(q-1)}\ ({}^qB)^2_{n-1} + 2\ ({}^qB)_{n+q-1}\ \big(({}^qB)_{n+q-1} - b^{q-1}({}^qB)_{n-1}\big)\Big]^2$$

$$= ({}^qB)^4_{n+q-1} + \Big(({}^qB)_{n+q-1} - b^{q-1}\ ({}^qB)_{n-1}\Big)^4$$

$$+2({}^qB)^2_{n+q-1}\Big(({}^qB)_{n+q-1} - b^{q-1}\ ({}^qB)_{n-1}\Big)^2$$

Thus, $\Big[b^{2(q-1)}\ ({}^qB)^2_{n-1} + 2\ ({}^qB)_{n+q-1}\ \big(({}^qB)_{n+q-1} - b^{q-1}({}^qB)_{n-1}\big)\Big]^2$

$$= \Big[({}^qB)^2_{n+q-1} - \big(({}^qB)_{n+q-1} - b^{q-1}({}^qB)_{n-1}\big)^2\Big]^2$$

$$+4\Big({}^qB)^2_{n+q-1}\ \big(({}^qB)_{n+q-1} - b^{q-1}\ ({}^qB)_{n-1}\big)\Big)^2$$

Therefore, $\Big[b^{2(q-1)}\ ({}^qB)^2_{n-1} + 2\ ({}^qB)_{n+q-1}\ \big(({}^qB)_{n+q-1} - b^{q-1}({}^qB)_{n-1}\big)\Big]^2$

$$= \Big[b^{q-1}\ {}^{q}B)_{n-1}\Big(2({}^{q}B)_{n+q-1} - b^{q-1}\ ({}^{q}B)_{n-1}\Big)\Big]^2$$

$$+\Big[2\ ({}^{q}B)_{n+q-1}\ \Big(({}^{q}B)_{n+q-1} - b^{q-1}\ ({}^{q}B)_{n-1}\Big)\Big]^2.$$

Hence the theorem is proved. □

4.3 B-q Lucas sequence

In this section, we introduce B-q Lucas sequence and obtain some identities of this sequence.

Definition 4.3.1. *Let $n \in \mathbb{N} \cup \{0\}$. The B-q Lucas sequence is defined by*

$$({}^{q}L)_{n+q-1} = \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\ a^{q-1-r}\ b^r\ ({}^{q}L)_{n+q-2-r},\ \forall n \geq 1, \tag{4.21}$$

with $({}^{q}L)_i = 0,\ i = 0, 1, 2, 3, \cdots q-3\ (q \geq 3),\ ({}^{q}L)_{q-2} = 2$ *and* $({}^{q}L)_{q-1} = a^{q-1}$,

where $({}^{q}L)_n$ is n^{th} term of B-q Lucas sequence (4.21).

Terms of (4.21) for $q-2 \leq n \leq q+1$ are:

$({}^{q}L)_{q-2} = 2$, $({}^{q}L)_{q-1} = a^{q-1}$, $({}^{q}L)_q = a^{2(q-1)} + 2(q-1)a^{q-2}\ b$,
$({}^{q}L)_{q+1} = a^{3(q-1)} + 3(q-1)\ a^{2q-3}\ b + (q-1)(q-2)\ a^{q-3}b^2$.

Rearranging the terms of (4.21) as follows, we obtain the terms $({}^{q}L)_n$, where n is a negative integer.

$$({}^{q}L)_{n-1} = \frac{1}{b^{q-1}}\Big[({}^{q}L)_{n+q-1} - \sum_{r=0}^{q-2} \frac{(q-1)^{\underline{r}}}{r!}\ a^{q-1-r}\ b^r\ ({}^{q}L)_{n+q-2-r}\Big], \tag{4.22}$$

with $({}^{q}L)_i = 0,\ i = 0, 1, 2, 3, \cdots, q-3 (q \geq 3),\ ({}^{q}L)_{q-2} = 2$ *and* $({}^{q}L)_{q-1} = a^{q-1}$.

Few terms of (4.22) are given below.

$({}^{q}L)_{-1} = -\frac{a^{q-1}}{b^{q-1}}$, $({}^{q}L)_{-2} = \frac{2}{b^{q-1}} + (q-1)\ \frac{a^q}{b^q}$, $({}^{q}L)_{-3} = \frac{-2(q-1)a}{b^q} + \frac{q^{\underline{2}}}{2!}\ \frac{a^{q+1}}{b^{q+1}}$.

Now we define B-q Lucas sequence for all $n \in \mathbb{Z}$.

Definition 4.3.2. *The B-q Lucas sequence is defined by*

$$({}^qL)_{n+q-1} = \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ a^{q-1-r} \ b^r \ ({}^qL)_{n+q-2-r}, \forall n \in \mathbb{Z}, \tag{4.23}$$

with $({}^qL)_i = 0, \ i = 0, 1, 2, 3, \cdots q-3 (q \geq 3), \ ({}^qL)_{q-2} = 2$ and $({}^qL)_{q-1} = a^{q-1}$,

where $({}^qL)_n$ is n^{th} term.

We have Binet type formula for (4.23).

Theorem 4.3.3. *The n^{th} term of (4.23) is given by*

$$({}^qL)_n = \frac{\sum_{k=1}^{q} (-1)^{k+1} \prod_{1 \leq i < j \leq q, i,j \neq k} (2\phi_i - a^{q-1})(\phi_i - \phi_j)\phi_k^n}{\prod_{1 \leq i < j \leq q} (\phi_i - \phi_j)} \tag{4.24}$$

where $\phi_p, \ p = 1, 2, \cdots, q$ are q distinct roots of the characteristic equation $\lambda^q - \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} a^{(q-1)-r} b^r \lambda^r = 0$ corresponding to (4.23).

Proof. Proof is similar to that of Binet type formula (4.4). □

The following theorem gives the relationship between (4.3) and (4.23).

Theorem 4.3.4. *The n^{th} term $({}^qL)_n$ of (4.23) is given by*

$$({}^qL)_n = 2({}^qB)_{n+1} - a^{q-1}({}^qB)_n, \forall \text{ integer n.} \tag{4.25}$$

Proof. Equation (4.24) implies

$$\begin{aligned} ({}^qL)_n = 2 \ & \frac{\sum_{k=1}^{q} (-1)^{k+1} \prod_{1 \leq i < j \leq q, i,j \neq k} (\phi_i - \phi_j)\phi_k^{n+1}}{\prod_{1 \leq i < j \leq q} (\phi_i - \phi_j)} \\ & -a^{q-1} \frac{\sum_{k=1}^{q} (-1)^{k+1} \prod_{1 \leq i < j \leq q, i,j \neq k} (2\phi_i - a^{q-1})(\phi_i - \phi_j)\phi_k^n}{\prod_{1 \leq i < j \leq q} (\phi_i - \phi_j)} \\ = \ & 2({}^qB)_{n+1} - a^{q-1}({}^qB)_n, \text{ using Binet type formula (4.4).} \end{aligned}$$

□

Corollary 4.3.5. *The n^{th} term $({}^{q}L)_n$ of (4.23) is given by*

$$({}^{q}L)_n = ({}^{q}B)_{n+1} + \sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ a^{q-1-r} \ b^r \ ({}^{q}B)_{n+q-1-r}, \forall n \in \mathbb{Z}. \tag{4.26}$$

Proof. The proof of the theorem follows from equations (4.3) and (4.25). □

We have the following identities of B-q Lucas sequence similar to the identities of B-q bonacci sequence.

Theorem 4.3.6. *The n^{th} term of B-q Lucas sequence (4.21) is given by*

$$({}^{q}L)_n = \sum_{r=0}^{p} \Big[\frac{(q-1)\big(n-(q-2)\big)}{(q-1)\big(n-(q-2)-r\big)} \frac{\Big((q-1)\big(n-(q-2)-r\big)\Big)^{\underline{r}}}{r!}\Big] a^{(q-1)\big(n-(q-2)\big)-qr} b^r$$

$$-\sum_{r=2}^{p}\Big[\sum_{s=1}^{q-1}(s-1)\frac{\Big((q-1)\big(n-(q-1)-r\big)+s-2\Big)^{\underline{r-2}}}{(r-2)!}\Big] \ a^{(q-1)\big(n-(q-2)\big)-qr} b^r, \tag{4.27}$$

$\forall n > q-2$ *and* $p = \lfloor \frac{(q-1)(n-(q-2))}{q} \rfloor$.

Proof. Let $n > q-2$. We divide the proof in to q cases by taking $n = qk - r$, where $r = 0, 1, 2, \cdots q-1$, and use equations (4.6) and (4.25).

Let $n = qk$ and consider,

$$({}^{q}L)_n = 2({}^{q}B)_{n+1} - a^{q-1}({}^{q}B)_n.$$

$$= 2 \ \sum_{r=0}^{\lfloor \frac{(q-1)\left(qk+1-(q-1)\right)}{q} \rfloor} \ \frac{(q-1)(qk+1-(q-1)-r)^{\underline{r}}}{r!} \ a^{(q-1)(qk+1-(q-1)-r)-r} \ b^r$$

$$-a^{q-1} \ \sum_{r=0}^{\lfloor \frac{(q-1)\left(qk-(q-1)\right)}{q} \rfloor} \ \frac{(q-1)(qk-(q-1)-r)^{\underline{r}}}{r!} \ a^{(q-1)(qk-(q-1)-r)-r} \ b^r$$

$$= \sum_{r=0}^{(q-1)\left(k-(q-2)\right)} 2\frac{(q-1)(qk+1-(q-1)-r)^{\underline{r}}}{r!} \ a^{(q-1)(qk+1-(q-1)-r)-r} \ b^r$$

$$-a^{q-1} \ \sum_{r=0}^{(q-1)\left(k-(q-2)\right)} \frac{(q-1)(qk-(q-1)-r)^{\underline{r}}}{r!} \ a^{(q-1)(qk-(q-1)-r)-r} \ b^r$$

$$= \sum_{r=0}^{(q-1)\left(k-(q-2)\right)} \left(2\ \frac{(q-1)(qk+1-(q-1)-r)^{\underline{r}}}{r!} - \frac{(q-1)(qk-(q-1)-r)^{\underline{r}}}{r!}\right)\ a^{(q-1)(qk-(q-1)-r)-r}\ b^r$$

$$= \sum_{r=0}^{(q-1)\left(k-(q-2)\right)} \left(\frac{(q-1)(qk+1-(q-1)-r)^{\underline{r}}}{r!} + \frac{(q-1)(qk+1-(q-1)-r)^{\underline{r}}}{r!}\right.$$

$$\left. - \frac{(q-1)(qk-(q-1)-r)^{\underline{r}}}{r!}\right) a^{(q-1)(qk-(q-1)-r)-r} b^r$$

$$= \sum_{r=0}^{(q-1)\left(k-(q-2)\right)} \left[\frac{(q-1)\left(qk-(q-2)\right)}{(q-1)\left(qk-(q-2)-r\right)} \frac{\left((q-1)\left(qk-(q-2)-r\right)\right)^{\underline{r}}}{r!}\right] a^{(q-1)\left(qk-(q-2)\right)-qr} b^r$$

$$- \sum_{r=2}^{(q-1)\left(k-(q-2)\right)} \left[\sum_{s=1}^{q-1}(s-1)\frac{\left((q-1)\left(qk-(q-1)-r\right)+s-2\right)^{\underline{r-2}}}{(r-2)!}\right] a^{(q-1)\left(qk-(q-2)\right)-qr} b^r.$$

Hence the theorem is proved. □

Following theorem gives the nth term of (4.22).

Corollary 4.3.7. *The n^{th} term of (4.22) is given by*

$$({}^qL)_n = \sum_{r=n-(q-2)}^{p} \left[\left[\frac{(q-1)\left(n-(q-2)\right)}{(q-1)\left(n-(q-2)-r\right)} \frac{\left((q-1)\left(n-(q-2)-r\right)\right)^{\underline{r}}}{r!}\right]\right.$$

$$\left. -\left[\sum_{s=1}^{q-1}(s-1)\frac{\left((q-1)\left(n-(q-1)-r\right)+s-2\right)^{\underline{r-2}}}{(r-2)!}\right]\right]\ a^{(q-1)\left(n-(q-2)\right)-qr} b^r, \quad (4.28)$$

$\forall n \leq -1$ and $p = \left\lfloor \frac{(q-1)(n-(q-2))}{q} \right\rfloor$.

Theorem 4.3.8. *(i) For $n \geq 0$, we have*

$$\sum_{r=0}^{n}({}^qL)_r = \frac{({}^qL)_{n+1} + \left(\sum_{i=0}^{q-2}\sum_{r=1+i}^{q-1} \frac{(q-1)\underline{r}}{r!}\ a^{q-1-r}\ b^r\right)({}^qL)_{n-i} + ({}^qL)_{q-1} - ({}^qL)_{q-2}}{(a+b)^{q-1}-1}, \quad (4.29)$$

provided $\begin{cases} a+b \neq 1, & \text{if q is even;} \\ a+b \neq \pm 1, & \text{if q is odd.} \end{cases}$

(ii) For $n \geq 1$, we have

$$\sum_{r=-1}^{-n} ({}^qL)_r = -\frac{({}^qL)_{-n} + \big(\sum_{i=0}^{q-2}\sum_{r=1+i}^{q-1} \frac{(q-1)r}{r!}\ a^{q-1-r}\ b^r\big)\ ({}^qL)_{-(n+1+i)} + ({}^qL)_{q-1} - ({}^qL)_{q-2}}{(a+b)^{q-1}-1}, \tag{4.30}$$

$$\textit{provided} \begin{cases} a+b \neq 1, & \text{if q is even;} \\ a+b \neq \pm 1, & \text{if q is odd.} \end{cases}$$

Proof. (i). We use equations(4.25) and (4.8) to prove (4.29).

Consider, $\sum_{r=0}^{n}({}^qL)_r = 2\sum_{r=0}^{n}({}^qB)_{r+1} - a^{q-1}\sum_{r=0}^{n}({}^qB)_r$

$$= \frac{2}{(a+b)^{q-1}-1}\Big(({}^qB)_{n+2} + \big(\sum_{i=0}^{q-2}\sum_{r=1+i}^{q-1} \frac{(q-1)r}{r!}\ a^{q-1-r}\ b^r\big)\ ({}^qB)_{n+1-i} - 1\Big)$$

$$-\frac{a^{q-1}}{(a+b)^{q-1}-1}\ \Big(({}^qB)_{n+1} + \big(\sum_{i=0}^{q-2}\sum_{r=1+i}^{q-1} \frac{(q-1)r}{r!}\ a^{q-1-r}\ b^r\big)\ ({}^qB)_{n-i} - 1\Big)$$

$$= \frac{1}{(a+b)^{q-1}-1}\Big(\big(2({}^qB)_{n+2} - a^{q-1}({}^qB)_{n+1}\big)$$

$$+\big(\sum_{i=0}^{q-2}\sum_{r=1+i}^{q-1} \frac{(q-1)r}{r!}\ a^{q-1-r}\ b^r\big)\big(2({}^qB)_{n+1-i} - a^{q-1}({}^qB)_{n-i}\big) - 2 + a^{q-1}\Big)$$

$$= \frac{1}{(a+b)^{q-1}-1}\Big(({}^qL)_{n+1} + \big(\sum_{i=0}^{q-2}\sum_{r=1+i}^{q-1} \frac{(q-1)r}{r!}\ a^{q-1-r}\ b^r\big)({}^qL)_{n-i} + ({}^qL)_{q-1} - ({}^qL)_{q-2}\Big)$$

Similarly, using equations (4.25) and (4.9) we can prove (4.30). □

Combining (4.29) and (4.30) we have

$\sum_{r=-n}^{n}({}^qL)_r$

$$= \frac{\big(({}^qL)_{n+1} - ({}^qL)_{-n}\big) + \sum_{i=0}^{q-2}\sum_{r=1+i}^{q-1}\big(\frac{(q-1)r}{r!}\ a^{q-1-r}\ b^r\big)\ \Big(({}^qL)_{n-i} - ({}^qL)_{-(n+1+i)}\Big)}{(a+b)^{q-1}-1},$$

$$\text{provided} \begin{cases} a+b \neq 1, \text{if q is even;} \\ a+b \neq \pm 1, \text{if q is odd.} \end{cases}$$

The following theorems can be proved using the procedure similar to that used to prove the related results of B-Tribonacci sequence in Chapter 3.

Theorem 4.3.9. (***Honsberger type identity***)

For any $m, n \in \mathbb{Z}$,

$$({}^{q}L)_{n+m-1} = \sum_{r=0}^{q-1} \Big(\sum_{s=0}^{r} \frac{(q-1)^{\underline{s}}}{s!}\, b^{q-1-s}\; a^{s} ({}^{q}B)_{n-1+s-r} \Big)\, ({}^{q}L)_{m-1+r}. \tag{4.31}$$

Proof. The proof follows from Honsberger type identity (4.14) for $({}^{q}B)_n$ and equation (4.25). □

Theorem 4.3.10. ***(General q-linear formula)***

For any $a_{i_n j} \in \mathbb{Z}, 1 \leq i_n, j, n \leq q$ with distinct i_n and the following $({}^{q}C_2)^2$ equations $a_{i_1 1} + a_{i_2 2} = a_{i_2 1} + a_{i_1 2}, \cdots, a_{i_1 1} + a_{i_q q} = a_{i_q 1} + a_{i_1 q}$ $a_{i_2 2} + a_{i_3 3} = a_{i_3 2} + a_{i_2 3}, \cdots,$ $a_{i_2 2} + a_{i_q q} = a_{i_q 2} + a_{i_2 q}, \cdots, a_{i_{q-1}(q-1)} + a_{i_q q} = a_{i_q (q-1)} + a_{i_{q-1} q}$, we have,

$$\begin{vmatrix} ({}^{q}L)_{a_{11}} & \cdots & ({}^{q}L)_{a_{1j}} & \cdots & ({}^{q}L)_{a_{1q}} \\ \cdots & & & & \\ ({}^{q}L)_{a_{i1}} & \cdots & ({}^{q}L)_{a_{ij}} & \cdots & ({}^{q}L)_{a_{iq}} \\ \cdots & & & & \\ ({}^{q}L)_{a_{q1}} & \cdots & ({}^{q}L)_{a_{qj}} & \cdots & ({}^{q}L)_{a_{qq}} \end{vmatrix}$$

$$= [(-b)^{q-1}]^{m} \begin{vmatrix} ({}^{q}L)_{a_{11}-m} & \cdots & ({}^{q}L)_{a_{1j}-m} & \cdots & ({}^{q}L)_{a_{1q}-m} \\ \cdots & & & & \\ ({}^{q}L)_{a_{i1}-m} & \cdots & ({}^{q}L)_{a_{ij}-m} & \cdots & ({}^{q}L)_{a_{iq}-m} \\ \cdots & & & & \\ ({}^{q}L)_{a_{q1}-m} & \cdots & ({}^{q}L)_{a_{qj}-m} & \cdots & ({}^{q}L)_{a_{qq}-m} \end{vmatrix}$$

$$= [(-b)^{q-1}]^{m} \sum_{\sigma \epsilon S_q} \prod_{i=1}^{q} \; sign(\sigma) ({}^{q}L)_{a_{i\sigma(i)-m}}, \tag{4.32}$$

ranging over the symmetric group S_q, *where*

$$sign(\sigma) = \begin{cases} +1, & if\ \sigma\ is\ an\ even\ permutation; \\ -1 & if\ \sigma\ is\ an\ odd\ permutation. \end{cases}$$

Theorem 4.3.11. ***(d'Ocagne type identity)***

For any $m_k \in \mathbb{Z}, k = 1, 2, ..., q, 0 \leq i \leq q-1$ *and* $1 \leq j \leq q$,

$$\begin{vmatrix} ({}^qL)_{m_1} & \cdots & ({}^qL)_{m_j} & \cdots & ({}^qL)_{m_q} \\ ({}^qL)_{m_1+1} & \cdots & ({}^qL)_{m_j+1} & \cdots & ({}^qL)_{m_q+1} \\ \cdots & & & & \\ ({}^qL)_{m_1+i} & \cdots & ({}^qL)_{m_j+i} & \cdots & ({}^qL)_{m_q+i} \\ \cdots & & & & \\ ({}^qL)_{m_1+(q-1)} & \cdots & ({}^qL)_{m_j+(q-1)} & \cdots & ({}^qL)_{m_q+(q-1)} \end{vmatrix}$$

$$= [(-b)^{q-1}]^{m_q} \sum_{\sigma \epsilon S_q} \prod_{i=1}^{q} \ sign(\sigma)({}^qL)_{m_i - m_q + \sigma(i) - 1}. \tag{4.33}$$

Theorem 4.3.12. ***(Catalan type identity)***

For any $n, r \in \mathbb{Z}$,

$$\begin{vmatrix} ({}^qL)_{n} & \cdots & ({}^qL)_{n+(j-1)r} & \cdots & ({}^qL)_{n+(q-1)r} \\ \cdots & & & & \\ ({}^qL)_{n+(1-i)r} & \cdots & ({}^qL)_{n+(j-i)r} & \cdots & ({}^qL)_{n+(q-i)r} \\ \cdots & & & & \\ ({}^qL)_{n+(1-q)r} & \cdots & ({}^qL)_{n+(j-q)r} & \cdots & ({}^qL)_{n} \end{vmatrix}$$

$$= [(-b)^{q-1}]^{n} \sum_{\sigma \epsilon S_q} \prod_{i=1}^{q} \ sign(\sigma)({}^qL)_{(i-\sigma(i))r}. \tag{4.34}$$

Theorem 4.3.13. ***(Cassini type identity)***

For any $n \in \mathbb{Z}$,

$$\begin{vmatrix} ({}^qL)_n & \cdots & ({}^qL)_{n+(j-1)} & \cdots & ({}^qL)_{n+(q-1)} \\ \cdots & & & & \\ ({}^qL)_{n+(1-i)} & \cdots & ({}^qL)_{n+(j-i)} & \cdots & ({}^qL)_{n+(q-i)} \\ \cdots & & & & \\ ({}^qL)_{n+(1-q)} & \cdots & ({}^qL)_{n+(j-q)} & \cdots & ({}^qL)_n \end{vmatrix}$$

$$= [(-b)^{q-1}]^n \sum_{\sigma \epsilon S_q} \prod_{i=1}^{q} \quad sign(\sigma)({}^qL)_{(i-\sigma(i))}. \tag{4.35}$$

We have Pythagorean result for B-q Lucas sequence. This result can be proved using the procedure similar to that used to prove Theorem 4.2.17. Hence omitted.

Theorem 4.3.14. *For all* $n \in \mathbb{Z}$,

$$[b^{q-1}({}^qL)_{n-1}(2\ ({}^qL)_{n+q-1} - b^{q-1}({}^qL)_{n-1})]^2 + [2({}^qL)_{n+q-1}(({}^qL)_{n+q-1} - b^{q-1}({}^qL)_{n-1})]^2$$

$$= [b^{2(q-1)}({}^qL)_{n-1}^2 + 2\ ({}^qL)_{n+q-1}(({}^qL)_{n+q-1} - b^{q-1}({}^qL)_{n-1})]^2. \tag{4.36}$$

4.4 Incomplete B-q bonacci and B-q Lucas sequences

In this section, we extend the incomplete B-Tribonacci sequence and incomplete B-Tri Lucas sequence to q^{th} order and call them the incomplete B-q bonacci sequence and incomplete B-q Lucas sequence respectively.

Definition 4.4.1. *The incomplete B-q bonacci sequence is defined by*

$$({}^qB)_n^l = \sum_{r=0}^{l} \frac{\Big((q-1)(n-(q-1)-r)\Big)^{\underline{r}}}{r!}\ a^{(q-1)(n-(q-1)-r)-r}\ b^r, \tag{4.37}$$

$\forall\ 0 \leq l \leq \lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor$ and $n \geq q-1$.

We list below $({}^qB)_n^l$, for $l = 0, 1, 2$ and $\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor$.

$({}^qB)_n^0 = a^{(q-1)\left(n-(q-1)\right)} = 1,\ \forall\ n \geq q-1.$

$({}^qB)_n^1 = a^{(q-1)\left(n-(q-1)\right)} + \big((q-1)(n-q)\big)\ a^{(q-1)(n-q)-1}\ b, \forall n \geq q.$

$({}^qB)_n^2 = a^{(q-1)\left(n-(q-1)\right)} + \big((q-1)(n-q)\big) a^{(q-1)(n-q)-1}\ b$

$$+ \frac{\big((q-1)(n-(q+1))\big)\big((q-1)(n-(q+2))\big)}{2}\ a^{(q-1)(n-(q+3))}\ b^2, \forall n \geq q+1.$$

$({}^qB)_n^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor} = ({}^qB)_n.$

We prove below some recurrence properties of the sequence, $({}^qB)_n^l$.

Theorem 4.4.2. *The recurrence relation of the incomplete B-q bonacci sequence $({}^qB)_n^l$ is given by*

$$({}^qB)_{n+q}^{l+q-1} = \sum_{k=0}^{q-1} \frac{(q-1)^{\underline{k}}}{k!} ({}^qB)_{n+q-1-k}^{l+q-1-k}\ a^{q-1-k}\ b^k, \tag{4.38}$$

$\forall\ 0 \leq l \leq \left\lfloor \frac{(q-1)(n-q)}{q} \right\rfloor$ and $n \geq q-1$.

Proof. Consider, $\sum_{k=0}^{q-1} \frac{(q-1)^{\underline{k}}}{k!}\ ({}^qB)_{n+q-1-k}^{l+q-1-k}\ a^{q-1-k}\ b^k$

$= \sum_{k=0}^{q-1} \frac{(q-1)^{\underline{k}}}{k!}$

$\sum_{r=0}^{l+q-1-k} \frac{\left((q-1)\left((n+q-1-k)-(q-1)-r\right)\right)^{\underline{r}}}{r!}\ a^{(q-1)\left((n+q-1-k)-(q-1)-r\right)-r+q-1-k}\ b^{k+r}$

$= \sum_{k=0}^{q-1} \frac{(q-1)^{\underline{k}}}{k!}$

$\sum_{r=0}^{l+q-1-k} \frac{\left((q-1)\left(n-(k+r)\right)\right)^{\underline{r}}}{r!}\ a^{(q-1)\left(n+1-(k+r)\right)-(k+r)}\ b^{k+r}$

$= \sum_{k=0}^{q-1} \frac{(q-1)^{\underline{k}}}{k!} \sum_{s=k}^{l+q-1} \frac{\Big((q-1)(n-s)\Big)^{\underline{s-k}}}{(s-k)!} \; a^{(q-1)(n+1-s)-s}\, b^s$, taking $k+r=s$,

$= \sum_{s=0}^{l+q-1} \frac{\Big((q-1)(n+1-s)\Big)^{\underline{s}}}{s!} \; a^{(q-1)(n+1-s)-s}\, b^s$, since $\frac{n^{\underline{s}}}{s!} + \frac{n^{\underline{s-1}}}{(s-1)!} = \frac{(n+1)^{\underline{s}}}{s!}$,

$= \sum_{s=0}^{l+q-1} \frac{\Big((q-1)(n+q-(q-1)-s)\Big)^{\underline{s}}}{s!} \; a^{(q-1)(n+q-(q-1)-s)-s}\, b^s$

$= ({}^qB)_{n+q}^{l+q-1}$. □

Theorem 4.4.3. *For all* $s \geq 1$,

$$({}^qB)_{n+qs}^{l+(q-1)s} = \sum_{i=0}^{(q-1)s} \frac{((q-1)s)^{\underline{i}}}{i!} \; ({}^qB)_{n+(q-1)s-i}^{l+(q-1)s-i} \; a^{(q-1)s-i} b^i. \tag{4.39}$$

Proof. $\sum_{i=0}^{(q-1)s} \frac{((q-1)s)^{\underline{i}}}{i!} \; ({}^qB)_{n+(q-1)s-i}^{l+(q-1)s-i} \; a^{(q-1)s-i} b^i$

$= \sum_{i=0}^{(q-1)s} \frac{((q-1)s)^{\underline{i}}}{i!}$

$\sum_{r=0}^{l+(q-1)s-i} \frac{\bigg((q-1)\Big(n+(q-1)s-i-(q-1)-r\Big)\bigg)^{\underline{r}}}{r!} \; a^{(q-1)\Big(n+(q-1)s-i-(q-1)-r\Big)-r+(q-1)s-i} \; b^{r+i}$

$= \sum_{i=0}^{(q-1)s} \frac{((q-1)s)^{\underline{i}}}{i!}$

$\sum_{r=0}^{l+(q-1)s-i} \frac{\bigg((q-1)\Big(n+(q-1)(s-1)-(i+r)\Big)\bigg)^{\underline{r}}}{r!} \; a^{(q-1)\Big(n+qs-(q-1)-(i+r)\Big)-(i+r)} \; b^{i+r}$

Taking $i+r=m$, we get

$\sum_{i=0}^{(q-1)s} \frac{((q-1)s)^{\underline{i}}}{i!} \; ({}^qB)_{n+(q-1)s-i}^{l+(q-1)s-i} \; a^{(q-1)s-i} b^i$

$= \sum_{i=0}^{(q-1)s} \frac{((q-1)s)^{\underline{i}}}{i!}$

$$\sum_{m=i}^{l+(q-1)s} \frac{\left((q-1)\left(n+(q-1)(s-1)-m\right)\right)^{\underline{m-i}}}{(m-i)!} a^{(q-1)\left(n+qs-(q-1)-m\right)-m} b^m$$

$$= \sum_{m=0}^{l+(q-1)s} \frac{\left((q-1)\left(n+(q-1)(s-1)+(q-1)s-m\right)\right)^{\underline{m}}}{m!} a^{(q-1)\left(n+qs-(q-1)-m\right)-m} b^m$$

$$= \sum_{m=0}^{l+(q-1)s} \frac{\left((q-1)\left(n+qs-(q-1)-m\right)\right)^{\underline{m}}}{m!} a^{(q-1)\left(n+qs-(q-1)-m\right)-m} b^m,$$

$$\text{since } \frac{n^{\underline{s}}}{s!} + \frac{n^{\underline{s-1}}}{(s-1)!} = \frac{(n+1)^{\underline{s}}}{s!}$$

$$= ({}^qB)_{n+qs}^{l+(q-1)s}. \qquad \square$$

Theorem 4.4.4. *For* $0 \le l \le \left\lfloor \frac{(q-1)(n-q))}{q} \right\rfloor$ *and* $s \ge 1$,

$$({}^qB)_{n+(q-1)+s}^{l+(q-1)} - a^{(q-1)s}({}^qB)_{n+(q-1)}^{l+(q-1)} = \sum_{i=0}^{s-1}\sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\left(a^{(q-1)(s-i)-r}\, b^r ({}^qB)_{n+(q-1)+i-r}^{l+(q-1)-r}\right). \tag{4.40}$$

Proof. By mathematical induction on s.

Equation (4.38) implies, (4.40) holds for $s = 1$. Now let the result be true for $s \le m$. Let $s = m + 1$ and consider,

$$\sum_{i=0}^{m}\sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \left(a^{(q-1)(m+1-i)-r}\, b^r\, ({}^qB)_{n+(q-1)+i-r}^{l+(q-1)-r}\right)$$

$$= \sum_{i=0}^{m-1}\sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \left(\left(a^{(q-1)(m+1-i)-r}\, b^r\, ({}^qB)_{n+(q-1)+i-r}^{l+(q-1)-r}\right)\right.$$

$$\left. + \sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \left(a^{(q-1)(m+1-m)-r}\, b^r\, ({}^qB)_{n+(q-1)+m+1-r}^{l+(q-1)-r}\right)\right)$$

$$= a^{q-1} \sum_{i=0}^{m-1} \sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\left(a^{(q-1)(m-i)-r}\, b^r\, ({}^qB)_{n+(q-1)+i-r}^{l+(q-1)-r}\right)$$

$$+ \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\left(a^{(q-1)-r}\, b^r\, ({}^qB)_{n+q+m-r}^{l+(q-1)-r}\right) - \left(a^{(q-1)}\, ({}^qB)_{n+q+m}^{l+(q-1)}\right)$$

$$= a^{q-1}\,({}^{q}B)^{l+(q-1)}_{n+(q-1)+m} - a^{(q-1)(m+1)}({}^{q}B)^{l+(q-1)}_{n+(q-1)}$$

$$+({}^{q}B)^{l+(q-1)-r}_{n+q+m+1} - a^{(q-1)}\,({}^{q}B)^{l+(q-1)}_{n+q+m}, \text{ by induction assumption.}$$

$$= ({}^{q}B)^{l+(q-1)}_{n+(q-1)+m+1} - a^{(q-1)(m+1)}({}^{q}B)^{l+q-1}_{n+q-1}. \qquad \square$$

We define below the incomplete B-q Lucas sequence and study the various results related to it.

Definition 4.4.5. *The incomplete B-q Lucas sequence is defined by*

$$({}^{q}L)^{l}_{n}$$

$$= \sum_{r=0}^{l}\Big[\frac{(q-1)\big(n-(q-2)\big)}{(q-1)\big(n-(q-2)-r\big)}\,\frac{\Big((q-1)\big(n-(q-2)-r\big)\Big)^{\underline{r}}}{r!}\Big]a^{(q-1)\big(n-(q-2)\big)-qr}b^{r}$$

$$-\sum_{r=2}^{l}\Big[\sum_{s=1}^{q-1}(s-1)\frac{\Big((q-1)\big(n-(q-1)-r\big)+s-2\Big)^{\underline{r-2}}}{(r-2)!}\Big]a^{(q-1)\big(n-(q-2)\big)-qr}b^{r}, \quad (4.41)$$

$0 \le l \le \left\lfloor \frac{(q-1)(n-(q-2))}{q} \right\rfloor$ *and* $\forall n \ge q-1$.

We state below the relation between n^{th} terms $({}^{q}B)^{l}_{n}$ and $({}^{q}L)^{l}_{n}$ of (4.37) and (4.41) respectively. The proof of the Theorem 4.4.6 can be obtained using the procedure similar to the procedure used in Theorem 3.4.6.

Theorem 4.4.6. *The relation between the n^{th} terms $({}^{q}L)^{l}_{n}$ and $({}^{q}B)^{l}_{n}$ is given by*

$$({}^{q}L)^{l}_{n} = ({}^{q}B)^{l}_{n+1} + \sum_{r=1}^{q-1}\frac{(q-1)^{\underline{r}}}{r!}\,a^{q-1-r}b^{r}({}^{q}B)^{l-r}_{n-r}, \qquad (4.42)$$

$q-1 \le l \le \left\lfloor \frac{(q-1)(n-(q-2))}{q} \right\rfloor, n \ge 2(q-1)$.

The following result can be obtained from (4.37) and (4.42).

Corollary 4.4.7. *For* $0 \leq l \leq \left\lfloor \frac{(q-1)(n-(q-2))}{q} \right\rfloor$,

$$ ({}^{q}L)_{n}^{l} = 2\ ({}^{q}B)_{n+1}^{l} - a^{q-1}\ ({}^{q}B)_{n}^{l}, \tag{4.43} $$

$n \geq q-1$.

The next three theorems give the results on the recurrence properties of incomplete B-q Lucas sequence (4.41).

Theorem 4.4.8. *The recurrence relation of the incomplete B-q Lucas sequence* $({}^{q}L)_{n}^{l}$ *is given by*

$$ ({}^{q}L)_{n+q}^{l+q-1} = \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} ({}^{q}L)_{n+q-1-r}^{l+q-1-r}\ a^{q-1-r} b^{r}, \tag{4.44} $$

$\forall\ 0 \leq l \leq \left\lfloor \frac{(q-1)(n-(q-2))}{q} \right\rfloor$ and $n \geq q-2$.

Proof. L.H.S of (4.44)$= \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} ({}^{q}L)_{n+q-1-r}^{l+q-1-r} a^{q-1-r} b^{r}$,

$= \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} 2 \left(({}^{q}B)_{n+q-r}^{l+q-1-r} - a^{q-1}\ ({}^{q}B)_{n+q-1-r}^{l+q-1-r} \right) a^{q-1-r} b^{r}$, from (4.43)

$= 2\ \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} a^{q-1-r} b^{r}\ ({}^{q}B)_{n+q-r}^{l+q-1-r} - a^{q-1}\ \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} a^{q-1-r} b^{r} ({}^{q}B)_{n+q-1-r}^{l+q-1-r}$

$= 2\ \sum_{r=0}^{q-1} ({}^{q}B)_{n+q+1}^{l+q-1} - a^{q-1}\ ({}^{q}B)_{n+q-1}^{l+q-1}$, from (4.38).

$= ({}^{q}L)_{n+q}^{l+q-1}$ □

Theorem 4.4.9. *For all* $0 \leq l \leq \left\lfloor \frac{(q-1)\left(n-(q-2)-s\right)}{q} \right\rfloor$,

$$ ({}^{q}L)_{n+qs}^{l+(q-1)s} = \sum_{i=0}^{(q-1)s} \frac{((q-1)s)^{\underline{i}}}{i!}\ ({}^{q}L)_{n+(q-1)s-i}^{l+(q-1)s-i}\ a^{(q-1)s-i} b^{i}. \tag{4.45} $$

Proof. R.H.S. of (4.45) $= \sum_{i=0}^{(q-1)s} \frac{((q-1)s)^{\underline{i}}}{i!}\ ({}^{q}L)_{n+(q-1)s-i}^{l+(q-1)s-i}\ a^{(q-1)s-i} b^{i}$

$= \sum_{i=0}^{(q-1)s} \frac{((q-1)s)^{\underline{i}}}{i!} \left(2\ ({}^{q}B)_{n+(q-1)s+1-i}^{l+(q-1)s-i} - a^{q-1}\ ({}^{q}B)_{n+(q-1)s-i}^{l+(q-1)s-i}\right) a^{(q-1)s-i}b^{i}$

$= 2\ \sum_{i=0}^{(q-1)s} \frac{((q-1)s)^{\underline{i}}}{i!}\ ({}^{q}B)_{n+(q-1)s+1-i}^{l+(q-1)s-i}\ a^{(q-1)s-i}b^{i}$

$-a^{q-1}\ \sum_{i=0}^{(q-1)s} \frac{((q-1)s)^{\underline{i}}}{i!}\ ({}^{q}B)_{n+(q-1)s-i}^{l+(q-1)s-i}\ a^{(q-1)s-i}b^{i}$

$= 2\ ({}^{q}B)_{n+qs+1}^{l+(q-1)s} - a^{q-1}\ ({}^{q}B)_{n+qs}^{l+(q-1)s}$, from (4.39).

$= ({}^{q}L)_{n+qs}^{l+(q-1)s}$.

□

Theorem 4.4.10. *For* $n \geq \lfloor \frac{ql}{q-1} + q - 2 \rfloor$,

$$({}^{q}L)_{n+(q-1)+s}^{l+(q-1)} - a^{(q-1)s}({}^{q}L)_{n+(q-1)}^{l+(q-1)} = \sum_{i=0}^{s-1}\sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} a^{(q-1)(s-i)-r} b^{r} ({}^{q}L)_{n+(q-1)+i-r}^{l+(q-1)-r}. \tag{4.46}$$

Proof. R.H.S. of (4.46) $= \sum_{i=0}^{s-1}\sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} a^{(q-1)(s-i)-r} b^{r} ({}^{q}L)_{n+(q-1)+i-r}^{l+(q-1)-r}$

$= \sum_{i=0}^{s-1}\sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} a^{(q-1)(s-i)-r} b^{r} \Big(2({}^{q}B)_{n+1+(q-1)+i-r}^{l+(q-1)-r}$

$-a^{q-1}({}^{q}B)_{n+(q-1)+i-r}^{l+(q-1)-r}\Big)$, from (4.43)

$= 2\ \left(({}^{q}B)_{n+1+(q-1)+s}^{l+(q-1)} - a^{(q-1)s}({}^{q}B)_{n+1+(q-1)}^{l+(q-1)}\right)$

$-a^{q-1}\left(({}^{q}B)_{n+(q-1)+s}^{l+(q-1)} - a^{(q-1)s}({}^{q}B)_{n+(q-1)}^{l+(q-1)}\right)$

$= \left(2({}^{q}B)_{n+1+(q-1)+s}^{l+(q-1)} - a^{q-1}\ ({}^{q}B)_{n+(q-1)+s}^{l+(q-1)}\right)$

$$-a^{(q-1)s}\Big(2\ ({}^{q}B)^{l+(q-1)}_{n+1+(q-1)} - a^{(q-1)}({}^{q}B)^{l+(q-1)}_{n+(q-1)}\Big)$$

$$= ({}^{q}L)^{l+(q-1)}_{n+(q-1)+s} - a^{(q-1)s}({}^{q}L)^{l+(q-1)}_{n+(q-1)}.$$

Hence the result is proved. □

Chapter 5

Generalized Bivariate B-q bonacci and B-q Lucas Polynomials

This Chapter includes the content of published papers (P2), (P3) and (E1).

Chapter 5

Generalized Bivariate B-q bonacci and B-q Lucas Polynomials

5.1 Introduction

It is known that one way of studying the extensions of Fibonacci sequence is the study of polynomials associated with it. In this Chapter, we generalize and extend bivariate Fibonacci polynomials defined by (2.43). The coefficients x and y of F_n and F_{n-1} in (2.43) is generalized to non-zero polynomials $h(x)$ and $g(y)$ with real coefficients respectively. Thus, we rewrite (2.43) and (2.44) respectively as

$$({}^fB)_{h,g,n+1}(x,y) = h(x)\ ({}^fB)_{h,g,n}(x,y) + g(y)\ ({}^fB)_{h,g,n-1}(x,y), \tag{5.1}$$

$$\text{with } ({}^fB)_{h,g,0}(x,y) = 0,\ ({}^fB)_{h,g,1}(x,y) = 1.$$

and

$$({}^fL)_{h,g,n+1}(x,y) = h(x)\ ({}^fL)_{h,g,n}(x,y) + g(y)\ ({}^fL)_{h,g,n-1}(x,y), \tag{5.2}$$

$$\text{with } ({}^fL)_{h,g,0}(x,y) = 2,\ ({}^fL)_{h,g,1}(x,y) = x.$$

We call (5.1) and (5.2), generalized bivariate B-Fibonacci polynomials and generalized bivariate B-Lucas polynomials respectively. With $g(y) = 1$, identities of (5.1) and (5.2) can be seen in [14] and [2]. In this Chapter, we extend and generalized (5.1) and (5.2). This extension is such that the n^{th} polynomial is constructed by adding the preceding three terms having the coefficients as the terms of the binomial expansion of $\big(h(x)+g(y)\big)^2$. We call them, generalized bivariate B-Tribonacci polynomials and generalized bivariate B-Tri Lucas polynomials respectively. We also extend and generalized incomplete Fibonacci and Lucas polynomials defined by (2.47) and (2.48) respectively. Further they are extended to q^{th} order polynomials.

In Section 2, we introduce and obtain various identities relating generalized bivariate B-Tribonacci polynomials. Section 3 deals with B-Tri Lucas polynomials and their identities. In Section 4 and Section 5, we introduce incomplete generalized bivariate B-Tribonacci polynomials and incomplete generalized bivariate B-Tri Lucas polynomials respectively. Section 6 deals with a generalized bivariate B-q bonacci polynomials. In Section 7, we study generalized bivariate B-q Lucas polynomials. In Section 8 and Section 9, we study incomplete generalized bivariate B-q bonacci polynomials and incomplete generalized bivariate B-q Lucas polynomials respectively. We also study their various identities. Throughout this Chapter we take $h(x)$ and $g(y)$ to be two non-zero polynomials in x and y with real coefficients respectively.

5.2 Generalized bivariate B-Tribonacci polynomials

We define now generalized bivariate B-Tribonacci polynomials.

Definition 5.2.1. *The generalized bivariate B-Tribonacci polynomials are defined by*

$$({}^tB)_{h,g,n+2}(x,y)$$

$$= h^2(x)({}^tB)_{h,g,n+1}(x,y) + 2h(x)g(y)({}^tB)_{h,g,n}(x,y) + g^2(y)({}^tB)_{h,g,n-1}(x,y), \forall n \in \mathbb{N}, \tag{5.3}$$

$$\text{with } ({}^tB)_{h,g,0}(x,y) = 0, ({}^tB)_{h,g,1}(x,y) = 0 \text{ and } ({}^tB)_{h,g,2}(x,y) = 1,$$

where the coefficients of the terms on right hand side of (5.3) are the terms of the binomial expansion of $(h(x)+g(y))^2$ *and* $({}^tB)_{h,g,n}(x,y)$ *is the* n^{th} *polynomial.*

For $0 \le n \le 6$, the terms of (5.3) are $({}^tB)_{h,g,0}(x,y) = 0, ({}^tB)_{h,g,1}(x,y) = 0$, $({}^tB)_{h,g,2}(x,y) = 1$, $({}^tB)_{h,g,3}(x,y) = h^2(x)$, $({}^tB)_{h,g,4}(x,y) = h^4(x) + 2h(x)g(y)$, $({}^tB)_{h,g,5}(x,y) = h^6(x) + 4h^3(x)g(y) + g^2(y)$ and $({}^tB)_{h,g,6}(x,y) = h^8(x) + 6h^5(x)g^2(y) + 6h^2(x)g^2(y)$.

In particular, if $g(y) = 1$, then (5.3) with $({}^tB)_{h,1,n}(x,y)$ written as $({}^tB)_{h,n}(x)$, reduces to (1.1) of (P3), namely

$$({}^tB)_{h,n+2}(x) = h^2(x)({}^tB)_{h,n+1}(x) + 2h(x)({}^tB)_{h,n}(x) + ({}^tB)_{h,n-1}(x),\ \forall n \in \mathbb{N}, \quad (5.4)$$

$$\text{with } ({}^tB)_{h,0}(x) = 0, ({}^tB)_{h,1}(x) = 0 \text{ and } ({}^tB)_{h,2}(x) = 1.$$

For $0 \le n \le 6$, the terms of (5.4) are $({}^tB)_{h,0}(x) = 0$, $({}^tB)_{h,1}(x) = 0$, $({}^tB)_{h,2}(x) = 1$, $({}^tB)_{h,3}(x) = h^2(x)$, $({}^tB)_{h,4}(x) = h^4(x) + 2h(x)$, $({}^tB)_{h,5}(x) = h^6(x) + 4h^3(x) + 1$ and $({}^tB)_{h,6}(x) = h^8(x) + 6h^5(x) + 6h^2(x)$.

If $h(x) = 1$, then (5.4) reduce to B-Tribonacci sequence (3.4) with $a = 1$ and $b = 1$, namely,

$$({}^tB)_{1,n+2} = ({}^tB)_{1,n+1} + 2({}^tB)_{1,n} + ({}^tB)_{1,n-1}, \forall n \ge 1, \quad (5.5)$$

$$\text{with } ({}^tB)_{1,0} = 0, ({}^tB)_{1,1} = 0 \text{ and } ({}^tB)_{1,2} = 1.$$

First few terms of (5.5) are $({}^tB)_{1,0} = 0$, $({}^tB)_{1,1} = 0$, $({}^tB)_{1,2} = 1$, $({}^tB)_{1,3} = 1$, $({}^tB)_{1,4} = 3$, $({}^tB)_{1,5} = 6$, $({}^tB)_{1,6} = 13$, $({}^tB)_{1,7} = 28$ and $({}^tB)_{1,8} = 60$.

For simplicity, we use $({}^tB)_{h,g,n}(x,y) = ({}^tB)_{h,g,n}$, $({}^tB)_{h,n}(x) = ({}^tB)_{h,n}$, $h(x) = h$ and $g(y) = g$.

The following table shows the coefficients of $({}^tB)_{h,n}$ defined by (5.4) arranged in ascending order and also the terms of the sequence $({}^tB)_{1,n}$ defined by (5.5).

n	h^0	h^1	h^2	h^3	h^4	h^5	h^6	h^7	h^8	h^9	h^{10}	h^{11}	h^{12}	$({}^tB)_{1,n}$
0	0													0
1	0													0
2	1													1
3	0	0	1											1
4	0	2	0	0	1									3
5	1	0	0	4	0	0	1							6
6	0	0	6	0	0	6	0	0	1					13
7	0	4	0	0	15	0	0	8	0	0	1			28
8	1	0	0	20	0	0	28	0	0	10	0	0	1	60

Table 5.1: Coefficients of $({}^tB)_{h,n}$ and terms of $({}^tB)_{1,n}$.

In Table 5.1, the sum of the n^{th} row is the n^{th} term of the sequence $({}^tB)_{1,n}$. Also, for n $\geq$ 2, sum of the elements in the anti-diagonal of corresponding (2n-3)x(2n-3) matrix is $2^{2(n-2)}$.

We state below theorems on the n^{th} term $({}^tB)_{h,g,n}$ defined by (5.3). These theorems can be proved using the procedure similar to that used to prove theorems in Section 2 of Chapter 3 and hence omitted.

Theorem 5.2.2. *The n^{th} term of* (5.3) *is given by*

$$({}^tB)_{h,g,n} = \frac{(\alpha-\beta)\gamma^n - (\alpha-\gamma)\beta^n + (\beta-\gamma)\alpha^n}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}, \tag{5.6}$$

where α, β and γ are the distinct roots of the characteristics equation $\lambda^3 - h^2\lambda^2 - 2hg\lambda - g^2 = 0$ corresponding to (5.3).

Equation (5.6) is called the Binet type identity for (5.3).

Theorem 5.2.3. *The n^{th} term $({}^tB)_{h,g,n}$ of (5.3) is given by*

$$({}^tB)_{h,g,n} = \sum_{r=0}^{\lfloor \frac{2n-4}{3} \rfloor} \frac{(2n-4-2r)^{\underline{r}}}{r!} \; h^{2n-4-3r} \, g^r, \; \forall n \geq 2. \tag{5.7}$$

Theorem 5.2.4. *The sum of the first $n+1$ terms of (5.3) is*

$$\sum_{r=0}^{n}({}^{t}B)_{h,g,r} = \frac{({}^{t}B)_{h,g,n+1} + \big(g^2+2hg\big)({}^{t}B)_{h,g,n} + g^2({}^{t}B)_{h,g,n-1} - 1}{(h+g)^2-1}, \tag{5.8}$$

provided $h+g \neq \pm 1$.

Theorem 5.2.5. *The generating function for (5.3) is given by*

$$({}^{t}G_{(B)})_{h,g}(z) = \frac{1}{1-z(h+gz)^2}. \tag{5.9}$$

The next two theorems are related to the recurrence properties of $({}^{t}B)_{h,g,n}$.

Theorem 5.2.6. *For all $s \geq 1$,*

$$\sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!}\, ({}^{t}B)_{h,g,n+i}\, h^i g^{2s-i} = ({}^{t}B)_{h,g,n+3s}. \tag{5.10}$$

Proof. We prove the theorem by mathematical induction on n. For $s=1$,

L.H.S. of (5.10) $= \sum_{i=0}^{2} \frac{(2)^{\underline{i}}}{i!}({}^{t}B)_{h,g,n+i}\, h^i g^{2-i}$

$$= g^2({}^{t}B)_{h,g,n} + 2hg\ ({}^{t}B)_{h,g,n+1} + h^2\ ({}^{t}B)_{h,g,n+2}$$

$$= ({}^{t}B)_{h,g,n+3} = \text{R.H.S.}$$

Therefore (5.10) is true for $s=1$. Assume that the result holds for all $s \leq m$.

Consider, $\sum_{i=0}^{2m+2} \frac{(2m+2)^{\underline{i}}}{i!}\ ({}^{t}B)_{h,g,n+i}\ h^i g^{2m+2-i}$

$$= \sum_{i=0}^{2m+2} \left(\frac{(2m)^{\underline{i-2}}}{(i-2)!} + 2\,\frac{(2m)^{\underline{i-1}}}{(i-1)!} + \frac{(2m)^{\underline{i}}}{i!}\right) ({}^{t}B)_{h,g,n+i}\ h^i g^{2m+2-i}$$

$$= \sum_{i=-2}^{2m} \frac{(2m)^{\underline{i}}}{i!}({}^{t}B)_{h,g,n+i+2}\ h^{i+2} g^{2m-i}$$

$$+2 \sum_{i=-1}^{2m} \frac{(2m)^{\underline{i}}}{i!} ({}^tB)_{h,g,n+i+1}\, h^{i+1} g^{2m-i+1} + \sum_{i=0}^{2m} \frac{(2m)^{\underline{i}}}{i!} ({}^tB)_{h,g,n+i}\, h^i g^{2m-i+2}$$

$$= \sum_{i=0}^{2m} \frac{(2m)^{\underline{i}}}{i!} h^i\, g^{2m-i} \Big(h^2\, ({}^tB)_{h,g,n+i+2} + 2hg\, ({}^tB)_{h,g,n+i+1} + g^2 ({}^tB)_{h,g,n+i} \Big)$$

$$= h^2 ({}^tB)_{h,g,n+3m+2} + 2hg ({}^tB)_{h,g,n+3m+1} + g^2 ({}^tB)_{h,g,n+3m}$$

$$= ({}^tB)_{h,g,n+3m+3}.$$

Hence the result is true for $s = m + 1$.

Therefore, by mathematical induction on s, the result follows. □

Theorem 5.2.7. *For* $s \geq 1$,

$$\sum_{i=0}^{s-1} \Big(2h^{2s-1-2i}\, g\, ({}^tB)_{h,g,n+1+i} \;+\; h^{2s-2-2i}\, g^2 ({}^tB)_{h,g,n+i} \Big)$$

$$= ({}^tB)_{h,g,n+2+s} \;-\; h^{2s} ({}^tB)_{h,g,n+2}. \tag{5.11}$$

Proof. By induction on s. If $s = 1$, then (5.11) reduces to

$$2hg\, ({}^tB)_{h,g,n+1} + g^2 ({}^tB)_{h,g,n} = ({}^tB)_{h,g,n+3} - h^2\, ({}^tB)_{h,g,n+2},$$

which is true from (5.3). Hence (5.11) holds for $s = 1$.

Now let the result be true for $s \leq m$. We prove it for $s = m + 1$.

Consider, $\sum_{i=0}^{m} \Big(2h^{2m+1-2i} g ({}^tB)_{h,g,n+1+i} + h^{2m-2i} g^2 ({}^tB)_{h,g,n+i} \Big)$.

$$= \sum_{i=0}^{m-1} \Big(2h^{2m+1-2i} g ({}^tB)_{h,g,n+1+i} + h^{2m-2i} g^2 ({}^tB)_{h,g,n+i} \Big)$$

$$+ \Big(2hg ({}^tB)_{h,g,n+m+1} + g^2 ({}^tB)_{h,g,n+m} \Big)$$

$$= h^2 \Big(\sum_{i=0}^{m-1} \big(2h^{2m-1-2i} g ({}^tB)_{h,g,n+1+i} + h^{2m-2-2i} g^2 ({}^tB)_{h,g,n+i} \big) \Big)$$

$$+\Big(2hg({}^tB)_{h,g,n+m+1}+g^2({}^tB)_{h,g,n+m}\Big)$$

$$=h^2\Big(({}^tB)_{h,g,n+m+2}-h^{2m}({}^tB)_{h,g,n+2}\Big)$$

$$+2hg({}^tB)_{h,g,n+m+1}+g^2({}^tB)_{h,g,n+m}$$

$$=h^2({}^tB)_{h,g,n+m+2}-h^{2m+2}({}^tB)_{h,g,n+2}+2hg({}^tB)_{h,g,n+m+1}+g^2({}^tB)_{h,g,n+m}$$

$$=({}^tB)_{h,g,n+m+3}-h^{2m+2}({}^tB)_{h,g,n+2}, \text{ from (5.3).}$$

Hence the theorem is proved. □

Remark 5.2.8. *If $g(y)=1$, then all the identities listed above reduce to corresponding identities of (5.4) which are published in (P3).*

5.3 Generalized bivariate B-Tri Lucas polynomials

In this section, we define generalized bivariate B-Tri Lucas polynomials and study their various identities. We also prove the relation between generalized bivariate B-Tribonacci polynomials and generalized bivariate B-Tri Lucas polynomials.

Definition 5.3.1. *The generalized bivariate B-Tri Lucas polynomials are defined by*

$$({}^tL)_{h,g,n+2}(x,y)=h^2(x)({}^tL)_{h,g,n+1}(x,y)$$

$$+2h(x)g(y)({}^tL)_{h,g,n}(x,y)+g^2(y)({}^tL)_{h,g,n-1}(x,y), \forall n\in\mathbb{N}, \tag{5.12}$$

with $({}^tL)_{h,g,0}(x,y)=0$, $({}^tL)_{h,g,1}(x,y)=2$ and $({}^tL)_{h,g,2}(x,y)=h^2(x)$,

where the coefficients of the terms on the right hand side are the terms of the binomial expansion of $\Big(h(x)+g(y)\Big)^2$ and $({}^tL)_{h,g,n}(x,y)$ is the n^{th} polynomial.

For $0 \leq n \leq 5$, the terms of (5.12) are $({}^tL)_{h,g,0}(x,y) = 0$, $({}^tL)_{h,g,1}(x,y) = 2$, $({}^tL)_{h,g,2}(x,y) = h^2(x), ({}^tL)_{h,g,3}(x,y) = h^4(x) + 4h(x)g(y)$, $({}^tL)_{h,g,4}(x,y) = h^6(x) + 6h^3(x)g(y) + 2g^2(y)$ and $({}^tL)_{h,g,5}(x,y) = h^8(x) + 8h^5(x)g(y) + 11h^2(x)g^2(y)$.

In particular if $g(y) = 1$, then (5.12) with $({}^tL)_{h,1,n}(x,y)$ written as $({}^tL)_{h,n}(x)$ reduces to (3.1) of (P3), namely,

$$({}^tL)_{h,n+2}(x) = h^2(x)({}^tL)_{h,n+1} + 2h(x)({}^tL)_{h,n}(x) + ({}^tL)_{h,n-1}(x), \forall n \in \mathbb{N}, \tag{5.13}$$

$$\text{with } ({}^tL)_{h,0}(x) = 0, ({}^tL)_{h,1}(x) = 2 \text{ and } ({}^tL)_{h,2}(x) = h^2(x).$$

For $0 \leq n \leq 5$, the terms of (5.13) are $({}^tL)_{h,0}(x) = 0$, $({}^tL)_{h,1}(x) = 2$, $({}^tL)_{h,2}(x) = h^2(x), ({}^tL)_{h,3}(x) = h^4(x) + 4h(x)$, $({}^tL)_{h,4}(x) = h^6(x) + 6h^3(x) + 2$ and $({}^tL)_{h,5}(x) = h^8(x) + 8h^5(x) + 11h^2(x)$.

If $h(x) = 1$, then (5.13) reduces to B-Tri Lucas sequence defined by

$$({}^tL)_{1,n+2} = ({}^tL)_{1,n+1} + 2({}^tL)_{1,n} + ({}^tL)_{1,n-1}, \ \forall n \in \mathbb{N}, \tag{5.14}$$

$$\text{with } ({}^tL)_{1,0} = 0, ({}^tL)_{1,1} = 2 \text{ and } ({}^tL)_{1,2} = 1.$$

First few terms of (5.14) are $({}^tL)_{1,0} = 0, ({}^tL)_{1,1} = 2, ({}^tL)_{1,2} = 1$, $({}^tL)_{1,3} = 5$, $({}^tL)_{1,4} = 9$, $({}^tL)_{1,5} = 20, ({}^tL)_{1,6} = 43$ and $({}^tL)_{1,7} = 92$.

For simplicity, we use $({}^tL)_{h,g,n}(x,y) = ({}^tL)_{h,g,n}$, $({}^tL)_{h,n}(x) = ({}^tL)_{h,n}$, $h(x) = h$ and $g(y) = g$.

Following table show coefficients of $({}^tL)_{h,n}$ arranged in ascending order of h and also terms of sequence $({}^tL)_{1,n}$.

n	h^0	h^1	h^2	h^3	h^4	h^5	h^6	h^7	h^8	h^9	h^{10}	h^{11}	h^{12}	$({}^tL)_{1,n}$
0	0													0
1	2													2
2	0	0	1											1
3	0	4	0	0	1									5
4	2	0	0	6	0	0	1							9
5	0	0	11	0	0	8	0	0	1					20
6	0	8	0	0	24	0	0	10	0	0	1			43
7	2	0	0	36	0	0	41	0	0	12	0	0	1	92

Table 5.2: Coefficients of $({}^tL)_{h,n}$ and terms of $({}^tL)_{1,n}$.

In Table 5.2, the sum of the n^{th} row is the n^{th} term of $({}^tL)_{1,n}$. Also, for n $\geq$ 2, sum of the elements in the anti-diagonal of corresponding (2n-1)x(2n-1) matrix is 7 $\left(2^{2(n-2)}\right)$.

We state below theorems related to the n^{th} term $({}^tL)_{h,g,n}$, of B-Tri Lucas polynomials. These theorems can be proved using the procedure similar to that of theorems in Section 3 of Chapter 3 and hence omitted.

Theorem 5.3.2. *The n^{th} term $({}^tL)_{h,g,n}$ of* (5.12) *is given by*

$$({}^tL)_{h,g,n} = \frac{(\alpha-\beta)\gamma^n(2\gamma-h^2)-(\alpha-\gamma)\beta^n(2\beta-h^2)+(\beta-\gamma)\alpha^n(2\alpha-h^2)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}, \quad (5.15)$$

where α, β and γ are the distinct roots of the characteristics equation $\lambda^3 - h^2\lambda^2 - 2hg\lambda - g^2 = 0$ corresponding to (5.12).

Equation (5.15) is called Binet type formula for (5.12).

Theorem 5.3.3. *The n^{th} term $({}^tL)_{h,g,n}$ of (5.12) is given by*

$({}^tL)_{h,g,n}$

$$= \sum_{r=0}^{\left\lfloor \frac{2n-2}{3} \right\rfloor} \left(\frac{(2n-2)}{(2n-2-2r)} \frac{(2n-2-2r)^{\underline{r}}}{r!} - r(r-1) \frac{(2n-4-2r)^{\underline{r-2}}}{r!} \right) h^{2n-2-3r} g^r, \forall n \geq 2. \quad (5.16)$$

Theorem 5.3.4. *The sum of the first $n+1$ terms of $({}^tL)_{h,g,n}$ is*

$$\sum_{r=0}^{n}({}^tL)_{h,g,r} = \frac{({}^tL)_{h,g,n+1} + \big(2hg+g^2\big)({}^tL)_{h,g,n} + g^2({}^tL)_{h,g,n-1} + ({}^tL)_{h,g,2} - ({}^tL)_{h,g,1}}{(h+g)^2-1}, \tag{5.17}$$

provided $h+g \neq \pm 1$.

Theorem 5.3.5. *The generating function for $({}^tL)_{h,g,n}$ is given by*

$$({}^tG_{(L)})_{h,g}(z) = \frac{2-h^2z}{1-z\big(h+gz\big)^2}. \tag{5.18}$$

We have the following theorems on recurrence properties of generalized bivariate B-Tri Lucas polynomials.

Theorem 5.3.6.

$$({}^tL)_{h,g,n+1} = ({}^tB)_{h,g,n+2} + 2hg\ ({}^tB)_{h,g,n} + g^2({}^tB)_{h,g,n-1},\ \forall n \geq 1. \tag{5.19}$$

Proof. By induction on n. Since $({}^tL)_{h,g,2} = h^2, ({}^tB)_{h,g,3} = h^2, ({}^tB)_{h,g,1} = 0$ and $({}^tB)_{h,g,0} = 0$, (5.19) holds for $n = 1$.

Now assume that it holds for $n \leq m-1$ and consider (5.12),

$$\begin{aligned}
({}^tL)_{h,g,m+1} &= h^2\ ({}^tL)_{h,g,m} + 2hg\ ({}^tL)_{h,g,m-1} + g^2({}^tL)_{h,g,m-2}\\
&= h^2\Big(({}^tB)_{h,g,m+1} + 2hg\ ({}^tB)_{h,g,m-1} + g^2({}^tB)_{h,g,m-2}\Big)\\
&\quad +2hg\ \Big(({}^tB)_{h,g,m} + 2hg\ ({}^tB)_{h,g,m-2} + g^2({}^tB)_{h,g,m-3}\Big)\\
&\quad +g^2\ \Big(({}^tB)_{h,g,m-1} + 2hg\ ({}^tB)_{h,g,m-3} + g^2({}^tB)_{h,g,m-4}\Big),\ \text{by assumption.}\\
&= ({}^tB)_{h,g,m+2} + 2hg\ ({}^tB)_{h,g,m} + g^2({}^tB)_{h,g,m-1}.
\end{aligned}$$

Hence by mathematical induction the result is proved. □

Following Corollary can be deduced from (5.3) and (5.19).

Corollary 5.3.7.

$$({}^tL)_{h,g,n} = 2\ ({}^tB)_{h,g,n+1} - h^2\ ({}^tB)_{h,g,n},\ \forall n \geq 0. \tag{5.20}$$

Using the above Corollary, we can establish the following results.

Theorem 5.3.8.

$$({}^tL)_{h,g,n+3s} = \sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!}\ ({}^tL)_{h,g,n+i}\ h^i g^{2s-i},\ s \geq 1. \tag{5.21}$$

Proof. Since $({}^tL)_{h,g,n} = 2\ ({}^tB)_{h,g,n+1} - h^2({}^tB)_{h,g,n}$,

$$\sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!}\ ({}^tL)_{h,g,n+i}\ h^i g^{2s-i}$$

$$= \sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!} \left(2\ ({}^tB)_{h,g,n+1+i} - h^2({}^tB)_{h,g,n+i}\right) h^i g^{2s-i}$$

$$= 2\sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!}\ ({}^tB)_{h,g,n+1+i}\ h^i g^{2s-i} - h^2 \sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!}\ ({}^tB)_{h,g,n+i}\ h^i g^{2s-i}$$

$$= 2\ ({}^tB)_{h,g,n+1+3s} - h^2\ ({}^tB)_{h,g,n+3s},\ \text{ from (5.10).}$$

$$= ({}^tL)_{h,g,n+3s}.$$ □

Using (5.11), (5.20) and the procedure similar to that of Theorem 5.3.8, we get the following result.

Theorem 5.3.9.

$$\sum_{i=0}^{s-1} \left(2\ h^{2s-1-2i}\ g\ ({}^tL)_{h,g,n+1+i} + h^{2s-2-2i}\ g^2\ ({}^tL)_{h,g,n+i}\right) = ({}^tL)_{h,g,n+2+s} - h^{2s}({}^tL)_{h,g,n+2}. \tag{5.22}$$

Following identities involving partial derivatives of the polynomials $({}^tB)_{h,g,n}$ and $({}^tL)_{h,g,n}$ are extensions of some identities discussed in [11].

Let $({}^tB)^{(k,j)}_{h,g,n} = \frac{\partial^{k+j}}{\partial x^k \partial y^j}\big(({}^tB)_{h,g,n}\big)$, $({}^tL)^{(k,j)}_{h,g,n} = \frac{\partial^{k+j}}{\partial x^k \partial y^j}\big(({}^tL)_{h,g,n}\big)$, $h^{(k,0)} = \frac{d^k}{dx^k}(h(x))$ and $g^{(0,j)} = \frac{d^j}{dy^j}(g(y))$.

We have the following identities involving $({}^tB)^{(k,j)}_{h,g,n}$ and $({}^tL)^{(k,j)}_{h,g,n}$.

Theorem 5.3.10. .

(1) $({}^tL)^{(k,j)}_{h,g,n} = ({}^tB)^{(k,j)}_{h,g,n+1} + \sum_{r=1}^{2} \frac{2^r}{r!} \sum_{s=0}^{k} \sum_{i=0}^{j} \frac{k^{\underline{s}}}{s!} \frac{j^{\underline{i}}}{i!} (h^{2-r})^{(s,0)} (g^r)^{(0,i)} ({}^tB)^{(k-s,j-i)}_{h,g,n-r}$.

(2) $({}^tB)^{(k,j)}_{h,g,n} = \sum_{r=0}^{2} \frac{2^r}{r!} \sum_{s=0}^{k} \sum_{i=0}^{j} \frac{k^{\underline{s}}}{s!} \frac{j^{\underline{i}}}{i!} (h^{2-r})^{(s,0)} (g^r)^{(0,i)} ({}^tB)^{(k-s,j-i)}_{h,g,n-1-r}$.

(3) $({}^tL)^{(k,j)}_{h,g,n} = \sum_{r=0}^{2} \frac{2^r}{r!} \sum_{s=0}^{k} \sum_{i=0}^{j} \frac{k^{\underline{s}}}{s!} \frac{j^{\underline{i}}}{i!} (h^{2-r})^{(s,0)} (g^r)^{(0,i)} ({}^tL)^{(k-s,j-i)}_{h,g,n-1-r}$.

(4) $2(n-1) \sum_{s=0}^{k} \frac{k^{\underline{s}}}{s!} (h)^{(1+s,0)} ({}^tB)^{(k-s,j)}_{h,g,n+1}$

$$= 3 \sum_{i=0}^{j} \frac{j^{\underline{i}}}{i!} (g)^{(0,i)} ({}^tB)^{(k+1,j-i)}_{h,g,n} + \sum_{s=0}^{k} \frac{k^{\underline{s}}}{s!} (h)^{(s,0)} ({}^tB)^{(k+1-s,j)}_{h,g,n+1}.$$

(5) $2(n-2) \sum_{i=0}^{j} \frac{j^{\underline{i}}}{i!} (g)^{(0,1+i)} ({}^tB)^{(k,j-i)}_{h,g,n}$

$$= 3 \sum_{i=0}^{j} \frac{j^{\underline{i}}}{i!} (g)^{(0,i)} ({}^tB)^{(k,j+1-i)}_{h,g,n} + \sum_{s=0}^{k} \frac{k^{\underline{s}}}{s!} (h)^{(s,0)} ({}^tB)^{(k-s,j+1)}_{h,g,n+1}.$$

(6) $\sum_{i=0}^{j} \frac{j^{\underline{i}}}{i!} (g)^{(0,1+i)} ({}^tB)^{(k+1,j-i)}_{h,g,n} = \sum_{s=0}^{k} \frac{k^{\underline{s}}}{s!} (h)^{(1+s,0)} ({}^tB)^{(k-s,j)}_{h,g,n+1}$.

Proof.

(1) Equation (5.19) implies

$$({}^tL)_{h,g,n} = ({}^tB)_{h,g,n+1} + 2hg\ ({}^tB)_{h,g,n-1} + g^2 ({}^tB)_{h,g,n-2}$$

Differentiating both sides k times with respect to x and j times with respect to y and using Leibnitz theorem for derivatives, we get

$$({}^tL)^{(k,j)}_{h,g,n} = ({}^tB)^{(k,j)}_{h,g,n+1} + 2 \sum_{s=0}^{k} \sum_{i=0}^{j} \frac{k^{\underline{s}}}{s!} \frac{j^{\underline{i}}}{i!} h^{(s,0)} g^{(0,i)} ({}^tB)^{(k-s,j-i)}_{h,g,n-1}$$

$$+\sum_{i=0}^{j} \frac{j^{\underline{i}}}{i!}\ (g^2)^{(0,i)} ({}^tB)^{(k,j-i)}_{h,g,n-2}$$

$$= ({}^tB)^{(k,j)}_{h,g,n+1} + \sum_{r=1}^{2} \frac{2^{\underline{r}}}{r!}\ \sum_{s=0}^{k}\ \sum_{i=0}^{j}\ \frac{k^{\underline{s}}}{s!}\ \frac{j^{\underline{i}}}{i!}\ (h^{2-r})^{(s,0)} (g^r)^{(0,i)} ({}^tB)^{(k-s,j-i)}_{h,g,n-r}.$$

(2) Equation (5.3) implies

$$({}^tB)_{h,g,n} = h^2({}^tB)_{h,g,n-1} + 2hg({}^tB)_{h,g,n-2} + g^2({}^tB)_{h,g,n-3}.$$

Differentiating both sides k times with respect to x and j times with respect to y and using Leibnitz theorem for derivatives, we get

$$({}^tB)^{(k,j)}_{h,g,n} = \sum_{s=0}^{k} \frac{k^{\underline{s}}}{s!}\ (h^2)^{(s,0)} ({}^tB)^{(k-s,j)}_{h,g,n-1} + 2\sum_{s=0}^{k}\ \sum_{i=0}^{j}\ \frac{k^{\underline{s}}}{s!}\ \frac{j^{\underline{i}}}{i!}\ h^{(s,0)} g^{(0,i)} ({}^tB)^{(k-s,j-i)}_{h,g,n-2}$$

$$+\sum_{i=0}^{j} \frac{j^{\underline{i}}}{i!}\ (g^2)^{(0,i)} ({}^tB)^{(k,j-i)}_{h,g,n-3}$$

$$= \sum_{r=0}^{2} \frac{2^{\underline{r}}}{r!}\ \sum_{s=0}^{k}\ \sum_{i=0}^{j}\ \frac{k^{\underline{s}}}{s!}\ \frac{j^{\underline{i}}}{i!}\ (h^{2-r})^{(s,0)} (g^r)^{(0,i)} ({}^tB)^{(k-s,j-i)}_{h,g,n-1-r}.$$

(3) Equation (5.12) implies

$$({}^tL)_{h,g,n} = h^2({}^tL)_{h,g,n-1} + 2hg({}^tL)_{h,g,n-2} + g^2({}^tL)_{h,g,n-3}.$$

Hence identity (3) can be proved by a method similar to that used in identity (2) above.

(4) We first prove that $2(n-1)h^{(1,0)}({}^tB)_{h,g,n+1} = 3g\ ({}^tB)^{(1,0)}_{h,g,n} + h\ ({}^tB)^{(1,0)}_{h,g,n+1}$, using (5.7). For this purpose, we divide the proof in to three cases depending on n, i.e. $n = 3k,\ 3k+1,\ 3k+2$.

Case 1 : Let $n = 3k$. Consider,

$$3g\ ({}^tB)^{(1,0)}_{h,g,n} + h\ ({}^tB)^{(1,0)}_{h,g,n+1}$$

$$=3g\ \frac{\partial}{\partial x}(({}^tB)_{h,g,3k})\ + h\ \frac{\partial}{\partial x}(({}^tB)_{h,g,3k+1})$$

$$= 3g\frac{\partial}{\partial x}\Big(\sum_{r=0}^{2k-2} \frac{(6k-4-2r)^{r}}{r!} \ h^{6k-4-3r} \ g^{r}\Big) + h \ \frac{\partial}{\partial x}\Big(\sum_{r=0}^{2k-1} \frac{(6k-2-2r)^{r}}{r!} \ h^{6k-2-3r} \ g^{r}\Big)$$

$$= 3\Big(\sum_{r=0}^{2k-2} \frac{(6k-4-2r)^{r+1}}{r!} \ h^{6k-5-3r} \ h^{(1,0)} \ g^{r+1}\Big) + \Big(\sum_{r=0}^{2k-1} \frac{(6k-2-2r)^{r+1}}{r!} \ h^{6k-2-3r} \ h^{(1,0)} \ g^{r}\Big)$$

$$= (6k-2)h^{6k-2}h^{(1,0)} + \sum_{r=1}^{2k-1} \left(3r\frac{(6k-2-2r)^{r}}{r!} + \frac{(6k-2-2r)^{r+1}}{r!}\right) h^{6k-2-3r} \ h^{(1,0)} \ g^{r}$$

$$= (6k-2)h^{6k-2}h^{(1,0)} + \sum_{r=1}^{2k-1} \frac{(6k-2-2r)^{r}}{r!}\big(3r + (6k-2-3r)\big) \ h^{6k-2-3r} \ h^{(1,0)} \ g^{r}$$

$$= (6k-2)h^{6k-2}h^{(1,0)} + \sum_{r=1}^{2k-1} \frac{(6k-2-2r)^{r}}{r!}(6k-2) \ h^{6k-2-3r} \ h^{(1,0)} \ g^{r}$$

$$= \sum_{r=0}^{2k-1}(6k-2)\frac{(6k-2-2r)^{r}}{r!} \ h^{6k-2-3r} \ h^{(1,0)} \ g^{r}$$

$$= 2(3k-1)h^{(1,0)}({}^{t}B)_{h,g,3k+1}.$$

Hence the result is true for $n = 3k$.

Similarly, the result can be proved for $n = 3k+1$ and $n = 3k+2$.

Thus, we have, $2(n-1)h^{(1,0)}({}^{t}B)_{h,g,n+1} = 3g \ ({}^{t}B)^{(1,0)}_{h,g,n} + h \ ({}^{t}B)^{(1,0)}_{h,g,n+1}$.

Now differentiating both sides k times with respect to x and j times with respect to y and using Leibnitz theorem for derivatives, we get the required result.

(5) We first show that

$$2(n-2)g^{(0,1)}({}^{t}B)_{h,g,n} = 3g \ ({}^{t}B)^{(0,1)}_{h,g,n} + h \ ({}^{t}B)^{(0,1)}_{h,g,n+1}.$$

We consider 3 cases by taking $n = 3k$, $3k+1$, $3k+2$.

Case 1: Let $n = 3k$. Consider,

$$3g \ ({}^{t}B)^{(0,1)}_{h,g,n} + h \ ({}^{t}B)^{(0,1)}_{h,g,n+1}$$

$=3g\ \frac{\partial}{\partial y}(({}^tB)_{h,g,3k})\ + h\ \frac{\partial}{\partial y}(({}^tB)_{h,g,3k+1})$, from (5.7)

$$= 3g\frac{\partial}{\partial y}\big(\sum_{r=0}^{2k-2} \frac{(6k-4-2r)_{\underline{r}}}{r!}\ h^{6k-4-3r}\ g^r\big) + h\ \frac{\partial}{\partial y}\big(\sum_{r=0}^{2k-1} \frac{(6k-2-2r)_{\underline{r}}}{r!}\ h^{6k-2-3r}\ g^r\big)$$

$$= 3\big(\sum_{r=1}^{2k-2} \frac{(6k-4-2r)_{\underline{r}}}{(r-1)!}\ h^{6k-4-3r}\ g^r\ g^{(0,1)}\big)+ \big(\sum_{r=1}^{2k-1} \frac{(6k-2-2r)_{\underline{r}}}{(r-1)!}\ h^{6k-1-3r}\ g^{r-1}\ g^{(0,1)}\big)$$

$$= 3\big(\sum_{r=1}^{2k-2} \frac{(6k-4-2r)_{\underline{r}}}{(r-1)!}\ h^{6k-5-3r}\ g^{r-1}\ g^{(0,1)}\big)+ \big(\sum_{r=0}^{2k-2} \frac{(6k-4-2r)_{\underline{r+1}}}{r!}\ h^{6k-4-3r}\ g^r\ g^{(0,1)}\big)$$

$$= (6k-4)h^{6k-4}g^{(0,1)} + \sum_{r=1}^{2k-2} \frac{(6k-4-2r)_{\underline{r}}}{r!}\big(3r + (6k-4-3r)\big)\ h^{6k-4-3r}\ g^r\ g^{(0,1)}$$

$$= (6k-4)h^{6k-4}g^{(0,1)} + \sum_{r=1}^{2k-2} \frac{(6k-4-2r)_{\underline{r}}}{r!}(6k-4)\ h^{6k-4-3r}\ g^r\ g^{(0,1)}$$

$$= \sum_{r=0}^{2k-2}(6k-4)\frac{(6k-4-2r)_{\underline{r}}}{r!}\ h^{6k-4-3r}\ g^r\ g^{(0,1)}$$

$$= 2(3k-2)g^{(0,1)}({}^tB)_{h,g,3k}.$$

Hence the result is true for $n = 3k$.

Similarly, the result can be proved for $n = 3k + 1$ and $n = 3k + 2$.

Thus we have, $2(n-2)({}^tB)_{h,g,n}\ g^{(0,1)} = 3g\ ({}^tB)^{(0,1)}_{h,g,n} + h\ ({}^tB)^{(0,1)}_{h,g,n+1}$

Now differentiating above equation both sides k times with respect to x and j times with respect to y and using Leibnitz theorem for derivatives, we get the required result.

(6) We first show that $g^{(0,1)}\ ({}^tB)^{(1,0)}_{h,g,n} = h^{(1,0)}({}^tB)^{(0,1)}_{h,g,n+1}$.

We divide the proof in 3 cases, $n = 3k, 3k+1, 3k+2$.

Putting $n = 3k$ in (5.3) and differentiate it with respect to x, we get,

$$({}^tB)^{(1,0)}_{h,g,3k} = \frac{\partial}{\partial x}\Big(\sum_{r=0}^{2k-2} \frac{(6k-4-2r)_{\underline{r}}}{r!}\ h^{6k-4-3r}\ g^r\Big)$$

$$= \sum_{r=0}^{2k-2} \frac{(6k-4-2r)^{r+1}}{r!} h^{6k-5-3r} h^{(1,0)} g^r.$$

Therefore, $g^{(0,1)}({}^tB)^{(1,0)}_{h,g,3k} = \sum_{r=0}^{2k-2} \frac{(6k-4-2r)^{r+1}}{r!} h^{6k-5-3r} h^{(1,0)} g^r g^{(0,1)}$.

Now consider,

$$({}^tB)^{(0,1)}_{h,g,3k+1} = \frac{\partial}{\partial y}(({}^tB)_{h,g,3k+1})$$

$$= \sum_{r=0}^{2k-1} \frac{(6k-2-2r)^r}{r!} h^{6k-2-3r} rg^{r-1} g^{(0,1)}$$

$$= \sum_{r=1}^{2k-1} \frac{(6k-2-2r)^r}{(r-1)!} h^{6k-2-3r} g^{r-1} g^{(0,1)}$$

$$= \sum_{r=0}^{2k-2} \frac{(6k-4-2r)^{r+1}}{r!} h^{6k-5-3r} g^r g^{(0,1)}$$

Thus, $h^{(1,0)}({}^tB)^{(0,1)}_{h,g,3k+1} = \sum_{r=0}^{2k-2} \frac{(6k-4-2r)^{r+1}}{r!} h^{6k-5-3r} h^{(1,0)} g^r g^{(0,1)}$.

Therefore, $({}^tB)^{(1,0)}_{h,g,n} h^{(1,0)} = ({}^tB)^{(0,1)}_{h,g,n+1} g^{(0,1)}$.

Differentiating both sides k times with respect to x and j times with respect to y and using Leibnitz theorem for derivatives, we get

$$\sum_{i=0}^{j} \frac{j^i}{i!} g^{(0,1+i)}({}^tB)^{(k+1,j-i)}_{h,g,n} = \sum_{s=0}^{k} \frac{k^s}{s!} h^{(1+s,0)}({}^tB)^{(k-s,j+1)}_{h,g,(n+1)}.$$

□

With $h(x) = x$ and $g(y) = y$, generalized bivariate B-Tribonacci polynomials and generalized bivariate B-Tri Lucas polynomials respectively reduce to

$$({}^tB)_{n+2}(x,y) = x^2({}^tB)_{n+1}(x,y) + 2xy({}^tB)_n(x,y) + y^2({}^tB)_{n-1}(x,y), \forall n \geq 1, \quad (5.23)$$

with $({}^tB)_0(x,y) = 0$, $({}^tB)_1(x,y) = 0, ({}^tB)_2(x,y) = 1$

and

$$({}^tL)_{n+2}(x,y) = x^2({}^tL)_{n+1}(x,y) + 2xy({}^tL)_n(x,y) + y^2({}^tL)_{n-1}(x,y), \forall n \geq 1, \quad (5.24)$$

with $({}^tL)_0(x,y) = 0$, $({}^tL)_1(x,y) = 2$ and $({}^tL)_2(x,y) = x^2$.

Following Corollary give the corresponding identities of (5.23) and (5.24).

Corollary 5.3.11. *For all $n \geq 2$,*

(1) $({}^tL)_n^{(k,j)} = ({}^tB)_{n+1}^{(k,j)} + \sum_{r=1}^{2} \frac{2^r}{r!} \sum_{s=0}^{2-r} \sum_{i=0}^{r} \frac{k^s}{s!} \frac{j^i}{i!} (x^{2-r})^{(s,0)} (y^r)^{(0,i)} ({}^tB)_{n-r}^{(k-s,j-i)}$

(2) $({}^tB)_n^{(k,j)} = \sum_{r=0}^{2} \frac{2^r}{r!} \sum_{s=0}^{2-r} \sum_{i=0}^{r} \frac{k^s}{s!} \frac{j^i}{i!} (x^{2-r})^{(s,0)} (y^r)^{(0,i)} ({}^tB)_{n-1-r}^{(k-s,j-i)}$

(3) $({}^tL)_n^{(k,j)} = \sum_{r=0}^{2} \frac{2^r}{r!} \sum_{s=0}^{2-r} \sum_{i=0}^{r} \frac{k^s}{s!} \frac{j^i}{i!} (x^{2-r})^{(s,0)} (y^r)^{(0,i)} ({}^tL)_{n-1-r}^{(k-s,j-i)}$

(4) $2(n-1)({}^tB)_{n+1}^{(k,j)} = 3\sum_{i=0}^{1} \frac{j^i}{i!} (y)^{(0,i)} ({}^tB)_n^{(k+1,j-i)} + \sum_{s=0}^{1} \frac{k^s}{s!} (x)^{(s,0)} ({}^tB)_{n+1}^{(k+1-s,j)}$

(5) $2(n-2)({}^tB)_n^{(k,j)} = 3\sum_{i=0}^{1} \frac{j^i}{i!} (y)^{(0,i)} ({}^tB)_n^{(k,j+1-i)} + \sum_{s=0}^{1} \frac{k^s}{s!} (x)^{(s,0)} ({}^tB)_{n+1}^{(k-s,j+1)}$

(6) $({}^tB)_n^{(k+1,j)} = ({}^tB)_{n+1}^{(k,j+1)}$

Theorem 5.3.12. (***Convolution property*** *for $({}^tB)_{h,g,n}$*)

$$({}^tB)_{h,g,n}^{(1,0)} = h^{(1,0)} \sum_{i=0}^{n} \Big(2h({}^tB)_{h,g,n+1-i} + 2g({}^tB)_{h,g,n-i}\Big)({}^tB)_{h,g,i}. \quad (5.25)$$

Proof. Equation (5.9) implies

$$\sum_{n=0}^{\infty} ({}^tB)_{h,g,n} z^{n-2} = \frac{1}{1 - z(h+gz)^2}$$

Differentiating both sides with respect to x we get,

$$\sum_{n=0}^{\infty} ({}^tB)_{h,g,n}^{(1,0)} z^{n-2}$$

$$= h^{(1,0)} \Big(\frac{2hz}{1-z(h+gz)^2} + \frac{2gz^2}{1-z(h+gz)^2} \Big)$$

$$= h^{(1,0)}\Big(2hz\big[\sum_{n=0}^{\infty}({}^tB)_{h,g,n}z^{n-2}\big]^2 + 2gz^2\big[\sum_{n=0}^{\infty}({}^tB)_{h,g,n}z^{n-2}\big]^2\Big)$$

$$= h^{(1,0)}\ \Big(2h\ z^{-3}\big[\sum_{n=0}^{\infty}({}^tB)_{h,g,n}z^{n}\big]^2 + 2gz^{-2}\big[\sum_{n=0}^{\infty}({}^tB)_{h,g,n}z^{n}\big]^2\Big)$$

Therefore, $\sum_{n=0}^{\infty}({}^tB)^{(1,0)}_{h,g,n}z^{n+1}$

$$= h^{(1,0)}\Big(2h\sum_{n=0}^{\infty}\big(\sum_{i=0}^{n}({}^tB)_{h,g,i}({}^tB)_{h,g,n-i}\big)z^n$$

$$+2g\sum_{n=0}^{\infty}\big(\sum_{i=0}^{n}({}^tB)_{h,g,i}({}^tB)_{h,g,n-i}\big)z^{n+1}\Big)$$

Comparing the coefficients of z^{n+1},

$$({}^tB)^{(1,0)}_{h,g,n} = h^{(1,0)}\Big(\sum_{i=0}^{n}\Big(2h\ ({}^tB)_{h,g,n+1-i} + 2g({}^tB)_{h,g,n-i}\Big)({}^tB)_{h,g,i}\Big).$$

□

Theorem 5.3.13. .(***Convolution property*** *for* $({}^tL)_{h,g,n}$)

$$({}^tL)^{(1,0)}_{h,g,n} = h^{(1,0)}\ \Big[\sum_{i=0}^{n}\Big(2h\ ({}^tL)_{h,g,n+1-i} + 2g\ ({}^tL)_{h,g,n-i}\Big)({}^tB)_{h,g,i} - 2h\ ({}^tB)_{h,g,n}\Big]. \tag{5.26}$$

Proof. Equation (5.20) implies

$$({}^tL)_{h,g,n} = 2\ ({}^tB)_{h,g,n+1} - h^2\ ({}^tB)_{h,g,n}$$

Differentiating both sides with respect to x, we get

$$({}^tL)^{(1,0)}_{h,g,n} = 2\ ({}^tB)^{(1,0)}_{h,g,n+1} - h^2({}^tB)^{(1,0)}_{h,g,n} - 2hh^{(1,0)}\ ({}^tB)_{h,g,n}$$

$$= 2h^{(1,0)}\sum_{i=0}^{n+1}\Big(2h\ ({}^tB)_{h,g,n+2-i} + 2g\ ({}^tB)_{h,g,n+1-i}\Big)({}^tB)_{h,g,i}$$

$$-h^{(1,0)}\Big[h^2 \sum_{i=0}^{n} \Big(2h\ ({}^tB)_{h,g,n+1-i}+2g\ ({}^tB)_{h,g,n-i}\Big)({}^tB)_{h,g,i}-2h({}^tB)_{h,g,n}\Big]$$

$$= h^{(1,0)}\Big[\sum_{i=0}^{n} \Big(2h\ \big(2({}^tB)_{h,g,n+2-i} - h^2({}^tB)_{h,g,n+1-i}\big)$$

$$+2g\ \big(2\ ({}^tB)_{h,g,n+1-i} - h^2({}^tB)_{h,g,n-i}\big)({}^tB)_{h,g,i}\Big) - 2h\ ({}^tB)_{h,g,n}\Big]$$

$$= h^{(1,0)}\Big[\sum_{i=0}^{n} \Big(2h\ ({}^tL)_{h,g,n+1-i} + 2g\ ({}^tL)_{h,g,n-i}\Big)({}^tB)_{h,g,i} - 2h\ ({}^tB)_{h,g,n}\Big].$$

□

5.4 Incomplete generalized bivariate B-Tribonacci polynomials

In this section, we define the incomplete generalized bivariate B-Tribonacci polynomials and obtain various identities related to these polynomials.

Definition 5.4.1. *The incomplete generalized bivariate B-Tribonacci polynomials are defined by*

$$({}^tB)^l_{h,g,n}(x,y) = \sum_{r=0}^{l} \frac{(2n-4-2r)^{\underline{r}}}{r!}\ h^{2n-4-3r}(x)g^r(y),\ \forall\ 0 \le l \le \lfloor \frac{2n-4}{3} \rfloor \text{ and } n \ge 2. \tag{5.27}$$

We list below terms of (5.27) for $2 \le n \le 5$.

$({}^tB)^0_{h,g,2}(x,y) = 1, ({}^tB)^0_{h,g,3}(x,y) = h^2(x), ({}^tB)^0_{h,g,4}(x,y) = h^4(x),$

$({}^tB)^1_{h,g,4}(x,y) = h^4(x) + 2h(x)g(y), ({}^tB)^0_{h,g,5}(x,y) = h^6(x),$

$({}^tB)^1_{h,g,5}(x,y) = h^6(x) + 4h^3(x)g(y)$ and $({}^tB)^2_{h,g,5}(x,y) = h^6(x) + 4h^3(x)g(y) + g^2(y).$

Note that $({}^tB)^{\lfloor \frac{2n-4}{3} \rfloor}_{h,g,n}(x,y) = ({}^tB)_{h,g,n}(x,y)$.

For simplicity, we use $({}^tB)^l_{h,g,n}(x,y) = ({}^tB)^l_{h,g,n}$, $({}^tB)_{h,g,n}(x,y) = ({}^tB)_{h,g,n}$, $h(x) = h$ and $g(y) = g$.

Following table shows terms of incomplete generalized bivariate B-Tribonacci polynomials.

$n \setminus l$	0	1	2	3
2	1			
3	h^2			
4	h^4	$h^4 + 2hg$		
5	h^6	$h^6 + 4h^3g$	$h^6 + 4h^3g + g^2$	
6	h^8	$h^8 + 6h^5g$	$h^8 + 6h^5g + 6h^2g^2$	
7	h^{10}	$h^{10} + 8h^7g$	$h^{10} + 8h^7g + 15h^4g^2$	$h^{10} + 8h^7g + 15h^4g^2 + 4hg^3$
8	h^{12}	$h^{12} + 10h^9g$	$h^{12} + 10h^9g + 28h^6g^2$	$h^{12} + 10h^9g + 28h^6g^2 + 20h^3g^3$
9	h^{14}	$h^{14} + 12h^{11}g$	$h^{14} + 12h^{11}g + 45h^8g^2$	$h^{14} + 12h^{11}g + 45h^8g^2 + 56h^5g^3$
10	h^{16}	$h^{16} + 14h^{13}g$	$h^{16} + 14h^{13}g + 66h^{10}g^2$	$h^{16} + 14h^{13}g + 66h^{10}g^2 + 120h^7g^3$

Table 5.3: Terms of $({}^tB)^l_{h,g,n}$, for $0 \le l \le 3$, $2 \le n \le 10$.

With $g(y) = 1$, the identities of (5.27) can be seen in (P2).

Next, we prove the recurrence properties of polynomials $({}^tB)^l_{h,g,n}$.

Theorem 5.4.2. *The recurrence relation of the incomplete generalized bivariate B-Tribonacci polynomials $({}^tB)^l_{h,g,n}$ is*

$$({}^tB)^{l+2}_{h,g,n+3} = h^2({}^tB)^{l+2}_{h,g,n+2} + 2hg({}^tB)^{l+1}_{h,g,n+1} + g^2({}^tB)^l_{h,g,n}, \tag{5.28}$$

$0 \le l \le \lfloor \frac{2n-6}{3} \rfloor$ *and* $n \ge 3$.

Proof. Consider, $h^2({}^tB)^{l+2}_{h,g,n+2} + 2hg({}^tB)^{l+1}_{h,g,n+1} + g^2({}^tB)^l_{h,g,n}$

$$= \sum_{r=0}^{l+2} \frac{(2n-2r)^{\underline{r}}}{r!} \; h^{2n+2-3r} g^r + 2 \sum_{r=0}^{l+1} \frac{(2n-2-2r)^{\underline{r}}}{r!} \; h^{2n-1-3r} g^{r+1}$$

$$+ \sum_{r=0}^{l} \frac{(2n-4-2r)^{\underline{r}}}{r!} \; h^{2n-4-3r} g^{r+2}$$

$= h^{2n+2} + \sum_{r=1}^{l+2} \left[\frac{(2n-2r)^{\underline{r}}}{r!} + \frac{(2n-2r)^{\underline{r-1}}}{(r-1)!}\right] h^{2n+2-3r} g^r$

$+h^{2n-1} + \sum_{r=1}^{l+1} \left[\frac{(2n-2-2r)^{\underline{r}}}{r!} + \frac{(2n-2-2r)^{\underline{r-1}}}{(r-1)!}\right] h^{2n-1-3r} g^r$

$= h^{2n+2} + \sum_{r=1}^{l+2} \frac{(2n+1-2r)^{\underline{r}}}{r!} h^{2n+2-3r} g^r + \sum_{r=0}^{l+1} \frac{(2n-1-2r)^{\underline{r}}}{r!} h^{2n-1-3r} g^r$

$= h^{2n+2} + \sum_{r=1}^{l+2} \left[\frac{(2n+1-2r)^{\underline{r}}}{r!} + \frac{(2n+1-2r)^{\underline{r-1}}}{(r-1)!}\right] h^{2n+2-3r} g^r$

$= \sum_{r=0}^{l+2} \frac{(2n+2-2r)^{\underline{r}}}{r!} \; h^{2n+2-3r} g^r$

$= ({}^tB)^{l+2}_{n+3}.$ □

Remark 5.4.3. *Using (5.27), equation (5.28) can be rewritten in terms of non-homogeneous recurrence relation as*

$$({}^tB)^{l}_{h,g,n+3} = h^2({}^tB)^{l}_{h,g,n+2} + 2hg({}^tB)^{l}_{h,g,n+1} + g^2({}^tB)^{l}_{h,g,n}$$

$$-\Big[\frac{(2n-4-2l)^{\underline{l}}}{l!} h^{2n-4-3l} g^{l+2} + \Big(2\,\frac{(2n-2-2l)^{\underline{l}}}{l!} + \frac{(2n-2-2l)^{\underline{l-1}}}{(l-1)!}\Big) h^{2n-1-3l} g^{l+1}\Big]. \tag{5.29}$$

Theorem 5.4.4. *For $s \geq 1$,*

$$\sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!} \; ({}^tB)^{l+i}_{h,g,n+i} \; h^i g^{2s-i} = ({}^tB)^{l+2s}_{h,g,n+3s}, \tag{5.30}$$

$0 \leq l \leq \lfloor \frac{2n-2s-4}{3} \rfloor$.

Proof. We prove (5.30) by mathematical induction on s.

Let $s = 1$. Then L.H.S. of (5.30) $= \sum_{i=0}^{2} \frac{2^{\underline{i}}}{i!} \; ({}^tB)^{l+i}_{h,g,n+i} \; h^i g^{2-i} = ({}^tB)^{l+2}_{h,g,n+3} = R.H.S.$

Thus, the theorem is true for $s = 1$. Assume that the result is true for all $s \leq m$.

Consider, $\sum_{i=0}^{2m+2} \frac{(2m+2)^{\underline{i}}}{i!} \; ({}^tB)^{l+i}_{h,g,n+i} h^i g^{2m+2-i}$

$$= \sum_{i=0}^{2m+2} \Big(\tfrac{(2m-2)^{\underline{i-2}}}{(i-2)!} ({}^tB)^{l+i}_{h,g,n+i} h^i g^{2m+2-i}$$

$$+2\ \tfrac{(2m-1)^{\underline{i-1}}}{(i-1)!} ({}^tB)^{l+i}_{h,g,n+i} h^i g^{2m+2-i} + \tfrac{(2m)^{\underline{i}}}{i!} ({}^tB)^{l+i}_{h,g,n+i} h^i g^{2m+2-i} \Big)$$

$$= \sum_{i=0}^{2m} \Big(\tfrac{(2m)^{\underline{i}}}{i!} ({}^tB)^{l+i+2}_{h,g,n+i+2} h^{i+2} g^{2m-i}$$

$$+2\ \tfrac{(2m)^{\underline{i}}}{i!} ({}^tB)^{l+i+1}_{h,g,n+i+1} h^{i+1} g^{2m-i+1} + \tfrac{(2m)^{\underline{i}}}{i!}\ ({}^tB)^{l+i}_{h,g,n+i} h^i g^{2m-i+2} \Big)$$

$$= h^2 ({}^tB)^{l+2m+2}_{h,g,n+3m+2} + 2hg ({}^tB)^{l+2m+1}_{h,g,n+3m+1} + g^2 ({}^tB)^{l+2m}_{h,g,n+3m}$$

$$= ({}^tB)^{l+2m+2}_{h,g,n+3m+3}.$$

Hence the result is true for $s = m + 1$.

Thus, by mathematical induction the theorem is proved. □

Theorem 5.4.5. *For $n \geq \lfloor \frac{3l+6}{2} \rfloor$ and $s \geq 1$,*

$$\sum_{i=0}^{s-1} \Big(2\ h^{2s-1-2i}\ g\ ({}^tB)^{l+1}_{h,g,n+1+i} + h^{2s-2-2i}\ g^2\ ({}^tB)^{l}_{h,g,n+i}\Big) = ({}^tB)^{l+2}_{h,g,n+2+s} - h^{2s} ({}^tB)^{l+2}_{h,g,n+2}. \tag{5.31}$$

Proof. By mathematical induction on s.

Note that (5.28) implies, (5.31) holds for $s = 1$. Now let the result be true for $s \leq m$.

We prove it for $s = m + 1$. Consider,

$$\sum_{i=0}^{m} \Big(2h^{2m+1-2i}\ g\ ({}^tB)^{l+1}_{h,g,n+1+i} + h^{2m-2i}\ g^2 ({}^tB)^{l}_{h,g,n+i}\Big)$$

$$= \sum_{i=0}^{m-1} \Big(2h^{2m+1-2i} g\ ({}^tB)^{l+1}_{h,g,n+1+i} + h^{2m-2i} g^2 ({}^tB)^{l}_{h,g,n+i}\Big)$$

$$+ \Big(2hg\ ({}^tB)^{l+1}_{h,g,n+1+m} + g^2 ({}^tB)^{l}_{h,g,n+m}\Big)$$

$$= h^2 \Big(\sum_{i=0}^{m-1} \big(2\ h^{2m-1-2i} g\ ({}^tB)^{l+1}_{h,g,n+1+i} + h^{2m-2-2i} g^2 ({}^tB)^{l}_{h,g,n+i}\big)\Big)$$

$$+\Big(2\ hg({}^tB)^{l+1}_{h,g,n+1+m} + g^2({}^tB)^{l}_{h,g,n+m}\Big)$$

$$= h^2\Big(({}^tB)^{l+2}_{h,g,n+2+m} - h^{2m}({}^tB)^{l+2}_{h,g,n+2}\Big)$$

$$+2hg\ ({}^tB)^{l+1}_{h,g,n+1+m} + g^2\ ({}^tB)^{l}_{h,g,n+m}, \text{ by induction assumption.}$$

$$= h^2({}^tB)^{l+2}_{h,g,n+2+m} - h^{2m+2}({}^tB)^{l+2}_{h,g,n+2} + 2hg\ ({}^tB)^{l+1}_{h,g,n+1+m} + g^2({}^tB)^{l}_{h,g,n+m}$$

$$= ({}^tB)^{l+2}_{h,g,n+3+m} - h^{2m+2}({}^tB)^{l+2}_{h,g,n+2}, \text{ from (5.28).}$$

Hence the theorem is proved. □

Lemma 5.4.6. *For all* $n \geq 2$,

$\sum_{r=0}^{\lfloor \frac{2n-4}{3} \rfloor} r \frac{(2n-4-2r)^{\underline{r}}}{r!}\ h^{2n-4-3r}\ g^r$

$$= \frac{2n-4}{3}\ ({}^tB)_{h,g,n} - \frac{h}{3}\sum_{i=0}^{n}\Big(2h({}^tB)_{h,g,n+1-i} + 2g({}^tB)_{h,g,n-i}\Big)({}^tB)_{h,g,i}. \qquad (5.32)$$

Proof. We use (5.7) to prove the result.

Consider, $({}^tB)_{h,g,n} = \sum_{r=0}^{\lfloor \frac{2n-4}{3} \rfloor} \frac{(2n-4-2r)^{\underline{r}}}{r!}\ h^{2n-4-3r}\ g^r,\ \forall n \geq 2.$

Differentiating both sides with respect to x, we get

$$({}^tB)^{(1,0)}_{h,g,n} = \sum_{r=0}^{\lfloor \frac{2n-4}{3} \rfloor} \frac{(2n-4-3r)\ (2n-4-2r)^{\underline{r}}}{r!}\ h^{2n-5-3r}\ h^{(1,0)}\ g^r$$

Therefore, $({}^tB)^{(1,0)}_{h,g,n}\ h = (2n-4)\ h^{(1,0)}\ \sum_{r=0}^{\lfloor \frac{2n-4}{3} \rfloor} \frac{(2n-4-2r)^{\underline{r}}}{r!}\ h^{2n-4-3r}\ g^r$

$$-3\ h^{(1,0)}\ \sum_{r=0}^{\lfloor \frac{2n-4}{3} \rfloor} r\ \frac{(2n-4-2r)^{\underline{r}}}{r!}\ h^{2n-4-3r} g^r.$$

Using Convolution property of $({}^tB)_{h,g,n}$, we get

$$\Big(h^{(1,0)}\ \sum_{i=0}^{n}\Big(2h({}^tB)_{h,g,n+1-i} + 2g({}^tB)_{h,g,n-i}\Big)({}^tB)_{h,g,i}\Big)\ h$$

$$= (2n-4)\ ({}^tB)_{h,g,n}\ h^{(1,0)}\ \ - 3h^{(1,0)}\ \sum_{r=0}^{\lfloor\frac{2n-4}{3}\rfloor} r\ \tfrac{(2n-4-2r)^{\underline{r}}}{r!}\ \ h^{2n-4-3r} g^r.$$

Thus, $h\sum_{i=0}^{n}\Big(2h({}^tB)_{h,g,n+1-i} + 2g({}^tB)_{h,g,n-i}\Big)({}^tB)_{h,g,i}$

$$= (2n-4)\ ({}^tB)_{h,g,n} - 3\sum_{r=0}^{\lfloor\frac{2n-4}{3}\rfloor} r\ \tfrac{(2n-4-2r)^{\underline{r}}}{r!}\ \ h^{2n-4-3r} g^r.$$

Therefore, $\sum_{r=0}^{\lfloor\frac{2n-4}{3}\rfloor} r\,\tfrac{(2n-4-2r)^{\underline{r}}}{r!}\ h^{2n-4-3r}\ g^r$

$$= \tfrac{2n-4}{3}\ ({}^tB)_{h,g,n} - \tfrac{h}{3}\sum_{i=0}^{n}\Big(2h({}^tB)_{h,g,n+1-i} + 2g({}^tB)_{h,g,n-i}\Big)({}^tB)_{h,g,i}.$$

Hence the lemma is proved. □

Theorem 5.4.7. *For all $n \geq 2$,*

$\sum_{l=0}^{\lfloor\frac{2n-4}{3}\rfloor}({}^tB)_{h,g,n}^{l}$

$$= \Big(\lfloor\frac{2n-4}{3}\rfloor - \frac{2n-7}{3}\Big)({}^tB)_{h,g,n} + \frac{h}{3}\ \sum_{i=0}^{n}\Big(2h({}^tB)_{h,g,n+1-i} + 2g({}^tB)_{h,g,n-i}\Big)({}^tB)_{h,g,i}. \tag{5.33}$$

Proof. $\sum_{l=0}^{\left\lfloor\frac{2n-4}{3}\right\rfloor}({}^tB)_{h,g,n}^{l} = ({}^tB)_{h,g,n}^{0} + ({}^tB)_{h,g,n}^{1} + \cdots + ({}^tB)_{h,g,n}^{r} + \cdots + ({}^tB)_{h,g,n}^{\lfloor\frac{2n-4}{3}\rfloor}$

$$= \tfrac{(2n-4-2r)^{\underline{0}}}{0!}\ h^{2n-4} + \Big[\tfrac{(2n-4)^{\underline{0}}}{0!}h^{2n-4} + \tfrac{(2n-4-2)^{\underline{1}}}{1!}h^{2n-4-3}g\Big] + \cdots$$

$$+\Big[\tfrac{(2n-4)^{\underline{0}}}{0!}\ h^{2n-4} + \cdots + \tfrac{(2n-4-2r)^{\underline{r}}}{r!}h^{2n-4-3r}g^r\Big] + \cdots$$

$$+\Big[\tfrac{(2n-4)^{\underline{0}}}{0!}\ h^{2n-4} + \cdots + \tfrac{(2n-4-2r)^{\underline{r}}}{r!}h^{2n-4-3r}g^r + \cdots$$

$$+\frac{(2n-4-2\lfloor\frac{2n-4}{3}\rfloor)^{\underline{\lfloor\frac{2n-4}{3}\rfloor}}}{(\lfloor\frac{2n-4}{3}\rfloor)!}\ h^{2n-4-\lfloor\frac{2n-4}{3}\rfloor}g^{\lfloor\frac{2n-4}{3}\rfloor}\Big]$$

$$= \Big(\lfloor\tfrac{2n-4}{3}\rfloor + 1\Big)\tfrac{(2n-4)^{\underline{0}}}{0!}\ h^{2n-4} + \Big(\lfloor\tfrac{2n-4}{3}\rfloor\Big)\tfrac{(2n-4-2)^{\underline{1}}}{1!}\ h^{2n-4-2}g + \cdots$$

$$+\Big(\lfloor\tfrac{2n-4}{3}\rfloor+1-r\Big)\tfrac{(2n-4-2r)^{\underline{r}}}{r!}\ h^{2n-4-3r}g^r+\cdots+\frac{(2n-4-2\lfloor\frac{2n-4}{3}\rfloor)^{\underline{\lfloor\frac{2n-4}{3}\rfloor}}}{(\lfloor\frac{2n-4}{3}\rfloor)!}\ h^{2n-4-\lfloor\frac{2n-4}{3}\rfloor}g^{\lfloor\frac{2n-4}{3}\rfloor}$$

$= \sum_{r=0}^{\lfloor\frac{2n-4}{3}\rfloor} \left(\lfloor\frac{2n-4}{3}\rfloor + 1 - r\right) \frac{(2n-4-2r)^{\underline{r}}}{r!}\ h^{2n-4-3r} g^r$

$= \left(\lfloor\frac{2n-4}{3}\rfloor + 1\right) \sum_{r=0}^{\lfloor\frac{2n-4}{3}\rfloor} \frac{(2n-4-2r)^{\underline{r}}}{r!}\ h^{2n-4-3r} g^r - \sum_{r=0}^{\lfloor\frac{2n-4}{3}\rfloor} r\ \frac{(2n-4-2r)^{\underline{r}}}{r!}\ h^{2n-4-3r} g^r$

$= \left(\lfloor\frac{2n-4}{3}\rfloor + 1\right) ({}^tB)_{h,g,n} - \sum_{r=0}^{\lfloor\frac{2n-4}{3}\rfloor} r\ \frac{(2n-4-2r)^{\underline{r}}}{r!}\ h^{2n-4-3r} g^r$

$= \left(\lfloor\frac{2n-4}{3}\rfloor + 1\right) ({}^tB)_{h,g,n} - \frac{2n-4}{3}\ ({}^tB)_{h,g,n}$

$+\frac{h}{3} \sum_{i=0}^{n} \left(2h\ ({}^tB)_{h,g,n+1-i} + 2\ g({}^tB)_{h,g,n-i}\right) ({}^tB)_{h,g,i}$, by Lemma 5.4.6.

$= \left(\lfloor\frac{2n-4}{3}\rfloor - \frac{2n-7}{3}\right) ({}^tB)_{h,g,n} + \frac{h}{3} \sum_{i=0}^{n} \left(2h\ ({}^tB)_{h,g,n+1-i} + 2\ ({}^tB)_{h,g,n-i}\right) ({}^tB)_{h,g,i}$. □

5.5 Incomplete generalized bivariate B-Tri Lucas polynomials

In this section, we introduce the incomplete generalized bivariate B-Tri Lucas polynomials and study some identities related to it. We also study its relation with the incomplete generalized bivariate B-Tri bonacci polynomials.

Definition 5.5.1. *The incomplete generalized bivariate B-Tri Lucas polynomials are defined by*

$({}^tL)^l_{h,g,n}(x, y)$

$$= \sum_{r=0}^{l} \left(\frac{(2n-2)}{(2n-2-2r)} \frac{(2n-2-2r)^{\underline{r}}}{r!} - \frac{(2n-4-2r)^{\underline{r-2}}}{(r-2)!} \right) h^{2n-2-3r}(x)\ g^r(y), \quad (5.34)$$

$$\forall\ \ 0 \le l \le \lfloor\frac{2n-2}{3}\rfloor \text{ and } n \ge 2.$$

Note that $({}^tL)^{\lfloor\frac{2n-2}{3}\rfloor}_{h,g,n}(x, y) = ({}^tL)_{h,g,n}(x, y)$.

For simplicity, we use $({}^{t}L)_{h,g,n}^{\lfloor\frac{2n-2}{3}\rfloor}(x,y) = ({}^{t}L)_{h,g,n}^{\lfloor\frac{2n-2}{3}\rfloor}$, $h(x) = h$ and $g(y) = g$.

Following table shows terms of incomplete generalized bivariate B-Tri Lucas polynomials.

$n \setminus l$	0	1	2	3
2	h^2			
3	h^4	$h^4 + 4hg$		
4	h^6	$h^6 + 6h^3g$	$h^6 + 6h^3g + 2g^2$	
5	h^8	$h^8 + 8h^5g$	$h^8 + 8h^3g + 11h^2g^2$	
6	h^{10}	$h^{10} + 10h^7g$	$h^{10} + 10h^7g + 24h^4g^2$	$h^{10} + 10h^7g + 24h^4g^2 + 8hg^3$
7	h^{12}	$h^{12} + 12h^9g$	$h^{12} + 12h^9g + 41h^6g^2$	$h^{12} + 12h^9g + 41h^6g^2 + 36h^3g^3$

Table 5.4: Terms of $({}^{t}L)_{h,g,n}^{l}$, for $0 \le l \le 3$ and $0 \le n \le 7$.

Following theorems give the relation between incomplete generalized bivariate B-Tribonacci and B-Tri Lucas polynomials.

Theorem 5.5.2.

$$({}^{t}L)_{h,g,n}^{l} = ({}^{t}B)_{h,g,n+1}^{l} + 2hg\ ({}^{t}B)_{h,g,n-1}^{l-1} + g^2\ ({}^{t}B)_{h,g,n-2}^{l-2}, \tag{5.35}$$

$2 \le l \le \lfloor\frac{2n-2}{3}\rfloor$.

Proof. From (5.27), we have

$$({}^{t}B)_{h,g,n+1}^{l} + 2hg\ ({}^{t}B)_{h,g,n-1}^{l-1} + g^2\ ({}^{t}B)_{h,g,n-2}^{l-2}$$

$$= \sum_{r=0}^{l} \frac{(2n-2-2r)^{\underline{r}}}{r!}\ h^{2n-2-3r}g^r + 2hg\ \sum_{r=0}^{l-1} \frac{(2n-6-2r)^{\underline{r}}}{r!}\ h^{2n-6-3r}g^r$$

$$+g^2\ \sum_{r=0}^{l-2} \frac{(2n-8-2r)^{\underline{r}}}{r!}\ \ h^{2n-8-3r}g^r$$

$= \sum_{r=0}^{l} \frac{(2n-2-2r)^{\underline{r}}}{r!} \; h^{2n-2-3r} g^r + 2 \sum_{r=1}^{l} \frac{(2n-4-2r)^{\underline{r-1}}}{(r-1)!} \; h^{2n-2-3r} g^r$

$+ \sum_{r=2}^{l} \frac{(2n-4-2r)^{\underline{r-2}}}{(r-2)!} \;\; h^{2n-2-3r} g^r$

$= \sum_{r=0}^{l} \left[\frac{(2n-2-2r)^{\underline{r}}}{r!} + 2\Big(\frac{(2n-4-2r)^{\underline{r-1}}}{(r-1)!} + \frac{(2n-4-2r)^{\underline{r-2}}}{(r-2)!}\Big) - \frac{(2n-4-2r)^{\underline{r-2}}}{(r-2)!} \right] h^{2n-2-3r} g^r$

$= \sum_{r=0}^{l} \left[\frac{(2n-2-2r)^{\underline{r}}}{r!} + 2\Big(\frac{(2n-3-2r)^{\underline{r-1}}}{(r-1)!}\Big) - \frac{(2n-4-2r)^{\underline{r-2}}}{(r-2)!} \right] h^{2n-2-3r} g^r$

$= \sum_{r=0}^{l} \left[\frac{(2n-2-2r)^{\underline{r}}}{r!} (1 + \frac{2\,r}{2n-2-2r}) - \frac{(2n-4-2r)^{\underline{r-2}}}{(r-2)!} \right] h^{2n-2-3r} g^r$

$= \sum_{r=0}^{l} \left[\frac{2n-2}{2n-2-2r} (\frac{(2n-2-2r)^{\underline{r}}}{r!}) - \frac{(2n-4-2r)^{\underline{r-2}}}{(r-2)!} \right] h^{2n-2-3r} g^r$

$= ({}^{t}L)^{l}_{h,g,,n}.$

Hence the theorem is proved. □

Using (5.28) and (5.35), following Corollary can be proved.

Corollary 5.5.3.

$$({}^{t}L)^{l}_{h,g,n} = 2\;({}^{t}B)^{l}_{h,g,n+1} - h^2\;({}^{t}B)^{l}_{h,g,n}, \tag{5.36}$$

$0 \leq l \leq \left\lfloor \frac{2n-2}{3} \right\rfloor$.

Theorem 5.5.4. *The recurrence relation of the incomplete generalized bivariate B-Tri Lucas sequence* $({}^{t}L)^{l}_{h,g,n}$ *is given by*

$$({}^{t}L)^{l+2}_{h,g,n+3} = h^2 ({}^{t}L)^{l+2}_{h,g,n+2} + 2hg({}^{t}L)^{l+1}_{h,g,n+1} + g^2({}^{t}L)^{l}_{h,g,n}, \tag{5.37}$$

$0 \leq l \leq \left\lfloor \frac{2n-2}{3} \right\rfloor$.

Proof. Equation (5.35) implies

$({}^{t}L)^{l+2}_{h,g,n+3} = ({}^{t}B)^{l+2}_{h,g,n+4} + 2hg({}^{t}B)^{l+1}_{h,g,n+2} + g^2({}^{t}B)^{l}_{h,g,n+1}$

$$= h^2({}^tB)^{l+2}_{h,g,n+3} + 2hg({}^tB)^{l+1}_{h,g,n+2} + g^2({}^tB)^{l}_{h,g,n+1}$$

$$+2hg(h^2({}^tB)^{l+1}_{h,g,n+1} + 2hg({}^tB)^{l}_{h,g,n} + g^2({}^tB)^{l-1}_{h,g,n-1})$$

$$+h^2({}^tB)^{l}_{h,g,n} + 2hg({}^tB)^{l-1}_{h,g,n-1} + g^2({}^tB)^{l-2}_{h,g,n-2}, \text{ from (5.28)}$$

$$= h^2({}^tL)^{l+2}_{h,g,n+2} + 2hg({}^tL)^{l+1}_{h,g,n+1} + g^2({}^tL)^{l}_{h,g,n}, \text{ from (5.35).} \qquad \square$$

Theorem 5.5.5. *For $s \geq 1$,*

$$({}^tL)^{l+2s}_{h,g,n+3s} = \sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!}\, ({}^tL)^{l+i}_{h,g,n+i}\, h^i g^{2s-i},\ 0 \leq l \leq \lfloor \frac{2n-2-2s}{3} \rfloor. \tag{5.38}$$

Proof. Equation (5.36) implies,

$\sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!}\, ({}^tL)^{l+i}_{h,g,n+i}\, h^i g^{2s-i}$

$$= \sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!} \Big(2({}^tB)^{l+i}_{h,g,n+1+i} - h^2({}^tB)^{l+i}_{h,g,n+i}\Big) h^i g^{2s-i}$$

$$= 2\sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!} ({}^tB)^{l+i}_{h,g,n+1+i} h^i g^{2s-i} - h^2 \sum_{i=0}^{2s} \frac{(2s)^{\underline{i}}}{i!} ({}^tB)^{l+i}_{h,g,n+i} h^i g^{2s-i}, \text{ from (5.10).}$$

$$= 2\ ({}^tB)^{l+2s}_{h,g,n+1+2s} - h^2({}^tB)^{l+2s}_{h,g,n+2s}$$

$$= ({}^tL)^{l+2s}_{h,g,n+2s}. \qquad \square$$

Similarly, using (5.36) following theorem can be proved.

Theorem 5.5.6. *For $s \geq 1$,*

$$({}^tL)^{l+2}_{n+2+s} - h^{2s}({}^tL)^{l+2}_{n+2} = \sum_{i=0}^{s-1} \Big(2\ h^{2s-1-2i}({}^tL)^{l+1}_{h,g,n+1+i} + h^{2s-2-2i}({}^tL)^{l}_{h,g,n+i}\Big), \tag{5.39}$$

$0 \leq l \leq \lfloor \frac{2n-6}{3} \rfloor$.

Lemma 5.5.7. *For all $n \geq 2$,*

$$\sum_{r=0}^{\lfloor\frac{2n-2}{3}\rfloor} r\Big(\tfrac{(2n-2)}{(2n-2-2r)}\tfrac{(2n-2-2r)^{\underline{r}}}{r!} - \tfrac{(2n-4-2r)^{\underline{r-2}}}{(r-2)!}\Big)\ h^{2n-2-3r}\ g^r$$

$$= \frac{(2n-2)}{3}({}^tL)_{h,g,n} - \frac{h}{3}\Big[\sum_{i=0}^{n}\Big(2h\ ({}^tL)_{h,g,n+1-i} + 2g\ ({}^tL)_{h,g,n-i}\Big)({}^tB)_{h,g,i} - 2h\ ({}^tB)_{h,g,n}\Big]. \tag{5.40}$$

Proof. Equation (5.21) implies

$$({}^tL)_{h,g,n} = \sum_{r=0}^{\left\lfloor\frac{2n-2}{3}\right\rfloor}\Big(\tfrac{(2n-2)}{(2n-2-2r)}\tfrac{(2n-2-2r)^{\underline{r}}}{r!} - r(r-1)\tfrac{(2n-4-2r)^{\underline{r-2}}}{r!}\Big)\ h^{2n-2-3r}g^r.$$

Differentiating both sides with respect to x.

$({}^tL)^{(1,0)}_{h,g,n}$

$$= \sum_{r=0}^{\lfloor\frac{2n-2}{3}\rfloor}(2n-2-3r)\Big(\tfrac{(2n-2)}{(2n-2-2r)}\tfrac{(2n-2-2r)^{\underline{r}}}{r!} - \tfrac{(2n-4-2r)^{\underline{r-2}}}{(r-2)!}\Big)\ h^{2n-3-3r}\ h^{(1,0)}\ g^r.$$

This implies, $h({}^tL)^{(1,0)}_{h,g,n} = (2n-2)h^{(1,0)}\ ({}^tL)_{h,g,n}$

$$-3h^{(1,0)}\sum_{r=0}^{\lfloor\frac{2n-2}{3}\rfloor} r\Big(\tfrac{(2n-2)}{(2n-2-2r)}\tfrac{(2n-2-2r)^{\underline{r}}}{r!} - \tfrac{(2n-4-2r)^{\underline{r-2}}}{(r-2)!}\Big)\ h^{2n-2-3r}\ g^r.$$

Thus, $h^{(1,0)}\sum_{r=0}^{\left\lfloor\frac{2n-2}{3}\right\rfloor} r\Big(\tfrac{(2n-2)}{(2n-2-2r)}\tfrac{(2n-2-2r)^{\underline{r}}}{r!} - \tfrac{(2n-4-2r)^{\underline{r-2}}}{(r-2)!}\Big)\ h^{2n-2-3r}\ g^r$

$$= \tfrac{(2n-2)}{3}({}^tL)_{h,g,n}h^{(1,0)} - \tfrac{h}{3}({}^tL)^{(1,0)}_{h,g,n}.$$

Therefore, $\sum_{r=0}^{\lfloor\frac{2n-2}{3}\rfloor} r\Big(\tfrac{(2n-2)}{(2n-2-2r)}\tfrac{(2n-2-2r)^{\underline{r}}}{r!} - \tfrac{(2n-4-2r)^{\underline{r-2}}}{(r-2)!}\Big)\ h^{2n-2-3r}\ g^r$

$$= \tfrac{(2n-2)}{3}({}^tL)_{h,g,n} - \tfrac{h}{3}\ \Big[\sum_{i=0}^{n}\Big(2h\ ({}^tL)_{h,g,n+1-i} + 2g\ ({}^tL)_{h,g,n-i}\Big)({}^tB)_{h,g,i}$$

$$-2h\ ({}^tB)_{h,g,n}\Big],\ \text{ from (5.26).}$$ □

Theorem 5.5.8. *For all $n \geq 2$,*

$$\sum_{l=0}^{\lfloor \frac{2n-2}{3} \rfloor} ({}^tL)^l_{h,g,n} = \Big(\lfloor \tfrac{2n-2}{3} \rfloor - \tfrac{2n-5}{3} \Big) ({}^tL)_{h,g,n}$$

$$+ \frac{h}{3} \Big[\sum_{i=0}^{n} \Big(2h \ ({}^tL)_{h,g,n+1-i} + 2g \ ({}^tL)_{h,g,n-i} \Big) ({}^tB)_{h,g,i} - 2h \ ({}^tB)_{h,g,n} \Big]. \qquad (5.41)$$

Proof. $\sum_{l=0}^{\lfloor \frac{2n-2}{3} \rfloor} ({}^tL)^l_{h,g,n} = ({}^tL)^0_{h,g,n} + ({}^tL)^1_{h,g,n} + \cdots + ({}^tL)^r_{h,g,n} + \cdots + ({}^tL)^{\lfloor \frac{2n-2}{3} \rfloor}_{h,g,n}$

$$= \Big(\tfrac{(2n-2)}{(2n-2)} \tfrac{(2n-2)^{\underline{0}}}{0!} \Big) \ h^{2n-2} \ g^0 + \Big[\Big(\tfrac{(2n-2)}{(2n-2)} \tfrac{(2n-2)^{\underline{0}}}{0!} \Big) \ h^{2n-2} \ g^0 + \Big(\tfrac{(2n-2)}{(2n-4)} \tfrac{(2n-4)^{\underline{1}}}{1!} \Big) \ h^{2n-5} \ g^1 \Big]$$

$$+ \Big[\Big(\tfrac{(2n-2)}{(2n-2)} \tfrac{(2n-2)^{\underline{0}}}{0!} \Big) \ h^{2n-2} \ g^0 + \Big(\tfrac{(2n-2)}{(2n-4)} \tfrac{(2n-4)^{\underline{1}}}{1!} \Big) \ h^{2n-5} \ g^1 + \Big(\tfrac{(2n-2)}{(2n-6)} \tfrac{(2n-6)^{\underline{2}}}{2!} - \tfrac{(2n-8)^{\underline{0}}}{0!} \Big) \ h^{2n-8} \ g^2 \Big]$$

$$+ \cdots$$

$$+ \Big[\Big(\tfrac{(2n-2)}{(2n-2)} \tfrac{(2n-2)^{\underline{0}}}{0!} \Big) \ h^{2n-2} \ g^0 + \Big(\tfrac{(2n-2)}{(2n-4)} \tfrac{(2n-4)^{\underline{1}}}{1!} \Big) \ h^{2n-5} \ g^1 + \Big(\tfrac{(2n-2)}{(2n-6)} \tfrac{(2n-6)^{\underline{2}}}{2!} - \tfrac{(2n-8)^{\underline{0}}}{0!} \Big) \ h^{2n-8} \ g^2$$

$$+ \cdots + \Big(\tfrac{(2n-2)}{(2n-2-2r)} \tfrac{(2n-2-2r)^{\underline{r}}}{r!} - \tfrac{(2n-4-2r)^{\underline{r-2}}}{(r-2)!} \Big) \ h^{2n-2-3r} \ g^r \Big] + \cdots$$

$$+ \Big[\Big(\tfrac{(2n-2)}{(2n-2)} \tfrac{(2n-2)^{\underline{0}}}{0!} \Big) \ h^{2n-2} \ g^0 + \Big(\tfrac{(2n-2)}{(2n-4)} \tfrac{(2n-4)^{\underline{1}}}{1!} \Big) \ h^{2n-5} \ g^1 + \Big(\tfrac{(2n-2)}{(2n-6)} \tfrac{(2n-6)^{\underline{2}}}{2!} - \tfrac{(2n-8)^{\underline{0}}}{0!} \Big) \ h^{2n-8} \ g^2$$

$$+ \cdots + \Big(\tfrac{(2n-2)}{(2n-2-2r)} \tfrac{(2n-2-2r)^{\underline{r}}}{r!} - \tfrac{(2n-4-2r)^{\underline{r-2}}}{(r-2)!} \Big) \ h^{2n-2-3r} \ g^r$$

$$+ \cdots + \Big(\tfrac{(2n-2)}{(2n-2-2\lfloor \frac{2n-2}{3} \rfloor)} \tfrac{(2n-2-2r)^{\underline{\lfloor \frac{2n-2}{3} \rfloor}}}{(\lfloor \frac{2n-2}{3} \rfloor)!} - \tfrac{(2n-4-2\lfloor \frac{2n-2}{3} \rfloor)^{\underline{\lfloor \frac{2n-2}{3} \rfloor - 2}}}{(\lfloor \frac{2n-2}{3} \rfloor - 2)!} \Big) \ h^{2n-2-3\lfloor \frac{2n-2}{3} \rfloor} \ g^{\lfloor \frac{2n-2}{3} \rfloor} \Big]$$

$$= \Big(\lfloor \tfrac{2n-2}{3} \rfloor + 1 \Big) \Big(\tfrac{(2n-2)}{(2n-2)} \tfrac{(2n-2)^{\underline{0}}}{0!} \Big) \ h^{2n-2} \ g^0 + \big(\lfloor \tfrac{2n-2}{3} \rfloor \big) \Big(\tfrac{(2n-2)}{(2n-4)} \tfrac{(2n-4)^{\underline{1}}}{1!} \Big) \ h^{2n-5} \ g^1 + \cdots$$

$$+ \Big(\lfloor \tfrac{2n-2}{3} \rfloor + 1 - r \Big) \Big(\tfrac{(2n-2)}{(2n-2-2r)} \tfrac{(2n-2-2r)^{\underline{r}}}{r!} - \tfrac{(2n-4-2r)^{\underline{r-2}}}{(r-2)!} \Big) \ h^{2n-2-3r} \ g^r + \cdots$$

$$+\Bigg(\frac{(2n-2)}{(2n-2-2\lfloor\frac{2n-2}{3}\rfloor)}\frac{(2n-2-2r)^{\lfloor\frac{2n-2}{3}\rfloor}}{(\lfloor\frac{2n-2}{3}\rfloor)!}-\frac{(2n-4-2\lfloor\frac{2n-2}{3}\rfloor)^{\lfloor\frac{2n-2}{3}\rfloor-2}}{(\lfloor\frac{2n-2}{3}\rfloor-2)!}\Bigg)\ h^{2n-2-3\lfloor\frac{2n-2}{3}\rfloor}\ g^{\lfloor\frac{2n-2}{3}\rfloor}\Bigg]$$

$$=\sum_{r=0}^{\lfloor\frac{2n-2}{3}\rfloor}\Big(\lfloor\tfrac{2n-2}{3}\rfloor+1-r\Big)\Big(\frac{(2n-2)}{(2n-2-2r)}\frac{(2n-2-2r)^{r}}{r!}-\frac{(2n-4-2r)^{r-2}}{(r-2)!}\Big)\ h^{2n-2-3r}\ g^{r}$$

$$=\Big(\lfloor\tfrac{2n-2}{3}\rfloor+1\Big)\sum_{r=0}^{\lfloor\frac{2n-2}{3}\rfloor}\Big(\frac{(2n-2)}{(2n-2-2r)}\frac{(2n-2-2r)^{r}}{r!}-\frac{(2n-4-2r)^{r-2}}{(r-2)!}\Big)\ h^{2n-2-3r}\ g^{r}$$

$$-\sum_{r=0}^{\lfloor\frac{2n-2}{3}\rfloor} r\ \Big(\frac{(2n-2)}{(2n-2-2r)}\frac{(2n-2-2r)^{r}}{r!}-\frac{(2n-4-2r)^{r-2}}{(r-2)!}\Big)\ h^{2n-2-3r}\ g^{r}$$

$$=\Big(\lfloor\tfrac{2n-2}{3}\rfloor+1-\tfrac{(2n-2)}{3}\Big)({}^tL)_{h,g,n}$$

$$+\frac{h}{3}\Big[\sum_{i=0}^{n}\Big(2h\ ({}^tL)_{h,g,n+1-i}+2g\ ({}^tL)_{h,g,n-i}\Big)({}^tB)_{h,g,i}-2h\ ({}^tB)_{h,g,n}\Big],\ \text{from (5.40)}$$

$$=\Big(\lfloor\tfrac{2n-2}{3}\rfloor-\tfrac{2n-5}{3}\Big)({}^tL)_{h,g,n}$$

$$+\frac{h}{3}\Big[\sum_{i=0}^{n}\Big(2h\ ({}^tL)_{h,g,n+1-i}+2g\ ({}^tL)_{h,g,n-i}\Big)({}^tB)_{h,g,i}-2h\ ({}^tB)_{h,g,n}\Big].$$

□

Let $\big(({}^tB)^l_{h,g,n}\big)^{(k,j)}=\frac{\partial^{k+j}}{\partial x^k\partial y^j}\big(({}^tB)^l_{h,g,n}\big)$ and $(({}^tL)^l_{h,g,n})^{(k,j)}=\frac{\partial^{k+j}}{\partial x^k\partial y^j}\big(({}^tL)^l_{h,g,n}\big)$.

We have the following identities involving $\big(({}^tB)^l_{h,g,n}\big)^{(k,j)}$ and $\big(({}^tL)^l_{h,g,n}\big)^{(k,j)}$.

Theorem 5.5.9. *For $n\geq 2$,*

(1) $\big(({}^tL)^l_{h,g,n}\big)^{(k,j)}=\big(({}^tB)^l_{h,g,n+1}\big)^{(k,j)}$

$$+\sum_{r=1}^{2}\frac{2^r}{r!}\ \sum_{s=0}^{k}\ \sum_{i=0}^{j}\ \frac{k^{\underline{s}}}{s!}\frac{j^{\underline{i}}}{i!}(h^{2-r})^{(s,0)}(g^r)^{(0,i)}\big(({}^tB)^{l-r}_{h,g,n-r}\big)^{(k-s,j-i)},$$

(2) $\big(({}^tB)^l_{h,g,n}\big)^{(k,j)}$

$$=\sum_{r=0}^{2}\frac{2^r}{r!}\ \sum_{s=0}^{k}\ \sum_{i=0}^{j}\ \frac{k^{\underline{s}}}{s!}\ \frac{j^{\underline{i}}}{i!}\ (h^{2-r})^{(s,0)}\ (g^r)^{(0,i)}\big(({}^tB)^{l-r}_{h,g,n-1-r}\big)^{(k-s,j-i)},$$

(3) $\left(({}^tL)^l_{h,g,n}\right)^{(k,j)}$

$$= \sum_{r=0}^{2} \tfrac{2^{\underline{r}}}{r!} \sum_{s=0}^{k} \sum_{i=0}^{j} \tfrac{k^{\underline{s}}}{s!} \tfrac{j^{\underline{i}}}{i!} (h^{2-r})^{(s,0)} (g^r)^{(0,i)} \left(({}^tL)^{l-r}_{h,g,n-1-r}\right)^{(k-s,j-i)}.$$

(4) $\sum_{s=0}^{k} \tfrac{k^{\underline{s}}}{s!} \left(({}^tB)^{l+1}_{h,g,n+1}\right)^{(k-s,j+1)} h^{(s+1,0)} = \sum_{i=0}^{j} \tfrac{j^{\underline{i}}}{i!} \left(({}^tB)^l_{h,g,n}\right)^{(k+1,j-i)} g^{(0,i+1)}.$

Proof. (1) Consider,

$$({}^tL)^l_{h,g,n} = ({}^tB)^l_{h,g,n+1} + 2hg({}^tB)^{l-1}_{h,g,n-1} + g^2({}^tB)^{l-2}_{h,g,n-2}.$$

Differentiating both sides k times with respect to x and j times with respect to y and using Leibnitz theorem for derivatives, we get

$$\left(({}^tL)^l_{h,g,n}\right)^{(k,j)} = (({}^tB)^l_{h,g,n+1})^{(k,j)} + 2\sum_{s=0}^{k} \sum_{i=0}^{j} \frac{k^{\underline{s}}}{s!} \frac{j^{\underline{i}}}{i!} h^{(s,0)} g^{(0,i)} \left(({}^tB)^{l-1}_{h,g,n-1}\right)^{(k-s,j-i)}$$

$$+ \sum_{i=0}^{j} \tfrac{j^{\underline{i}}}{i!} (g^2)^{(0,i)} \left(({}^tB)^{l-2}_{h,g,n-2}\right)^{(k,j-i)}$$

$$= \left(({}^tB)^l_{h,g,n+1}\right)^{(k,j)}$$

$$+ \sum_{r=1}^{2} \tfrac{2^{\underline{r}}}{r!} \sum_{s=0}^{k} \sum_{i=0}^{j} \tfrac{k^{\underline{s}}}{s!} \tfrac{j^{\underline{i}}}{i!} (h^{2-r})^{(s,0)} (g^r)^{(0,i)} \left(({}^tB)^{l-r}_{h,g,n-r}\right)^{(k-s,j-i)}.$$

(2) From (5.28), we have

$$({}^tB)^{l+2}_{h,g,n} = h^2({}^tB)^{l+2}_{h,g,n-1} + 2hg({}^tB)^{l+1}_{h,g,n-2} + g^2({}^tB)^l_{h,g,n-3}.$$

Differentiating both sides k times with respect to x and j times with respect to y and using Leibnitz theorem for derivatives, we get

$$\left(({}^tB)^l_{h,g,n}\right)^{(k,j)} = \sum_{s=0}^{k} \tfrac{k^{\underline{s}}}{s!} (h^2)^{(s,0)} \left(({}^tB)^l_{h,g,n-1}\right)^{(k-s,j)}$$

$$+2\sum_{s=0}^{k} \sum_{i=0}^{j} \tfrac{k^{\underline{s}}}{s!} \tfrac{j^{\underline{i}}}{i!} h^{(s,0)} g^{(0,i)} \left(({}^tB)^l_{h,g,n-2}\right)^{(k-s,j-i)}$$

$$+\sum_{i=0}^{j} \frac{j^{\underline{i}}}{i!}\ (g^2)^{(0,i)}\left(({}^tB)^l_{h,g,n-3}\right)^{(k,j-i)}$$

$$=\sum_{r=0}^{2} \frac{2^{\underline{r}}}{r!}\ \sum_{s=0}^{k}\ \sum_{i=0}^{j} \frac{k^{\underline{s}}}{s!}\ \frac{j^{\underline{i}}}{i!}\ (h^{2-r})^{(s,0)}(g^r)^{(0,i)}\left(({}^tB)^l_{h,g,n-1-r}\right)^{(k-s,j-i)}.$$

(3) Identity (3) can be proved by differentiating (5.37), k times with respect to x and j times with respect to y and using Leibnitz theorem for derivatives.

(4) Differentiating (5.27) with respect to x, we get

$$(({}^tB)^l_{h,g,n})^{(1,0)} = \sum_{r=0}^{l} \frac{(2n-4-2r)^{\underline{r+1}}}{r!}\ h^{2n-5-3r}h^{(1,0)}g^r.$$

Also, (5.27) implies,

$$({}^tB)^{l+1}_{h,g,n+1} = \sum_{r=0}^{l+1} \frac{(2n-2-2r)^{\underline{r}}}{r!}\ h^{2n-2-3r}g^r.$$

Differentiating both sides with respect to y,

$$(({}^tB)^{l+1}_{h,g,n+1})^{(0,1)} = \sum_{r=0}^{l+1} \frac{(2n-2-2r)^{\underline{r}}}{r!}\ \ h^{2n-2-3r}\ r\ g^{r-1}g^{(0,1)}$$

$$= \sum_{r=1}^{l+1} \frac{(2n-2-2r)^{\underline{r}}}{r!}\ \ h^{2n-2-3r}\ r\ g^{r-1}g^{(0,1)}$$

$$= \sum_{r=0}^{l} \frac{(2n-4-2r)^{\underline{r+1}}}{r!}\ \ h^{2n-5-3r}\ \ g^r g^{(0,1)}$$

Therefore, $\left(({}^tB)^{l+1}_{h,g,n+1}\right)^{(0,1)}h^{(1,0)} = \left(({}^tB)^l_{h,g,n}\right)^{(1,0)}g^{(0,1)}$.

Differentiating k times both sides with respect to x and j times with respect to y and using Leibnitz theorem, we get

$$\sum_{s=0}^{k} \frac{k^{\underline{s}}}{s!}\left(({}^tB)^{l+1}_{h,g,n+1}\right)^{(k-s,j+1)}h^{(s+1,0)} = \sum_{i=0}^{j} \frac{j^{\underline{i}}}{i!}\left(({}^tB)^l_{h,g,n}\right)^{(k+1,j-i)}g^{(0,i+1)}.$$

□

5.6 Generalized bivariate B-q bonacci polynomials

In this section, we extend generalized bivariate B-Tribonacci polynomials to generalized B-q bonacci polynomials and state its identities. These identities are similar to the identities of B-q bonacci sequence defined by (4.1), studied in Section 2 of Chapter 4. Hence the proof of these results is omitted.

Definition 5.6.1. *Let $n \in \mathbb{N} \cup \{0\}$. The generalized B-q bonacci polynomials $({}^qB)_{h,g,n}(x,y)$, are defined by*

$$({}^qB)_{h,g,n+q-1}(x,y) = \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \; h^{q-1-r}(x)\; g^r(y)\; ({}^qB)_{h,g,n+q-2-r}(x,y), \forall\; n \geq 1, \tag{5.42}$$

with $({}^qB)_{h,g,i}(x,y) = 0,\; i = 0,1,2,3,\cdots q-2,$ and $({}^qB)_{h,g,q-1}(x,y) = 1,$

where $({}^qB)_{h,g,n}(x,y)$ is n^{th} B-q bonacci polynomial.

Few terms of (5.42) are $({}^qB)_{h,g,q}(x,y) = h^{q-1}(x)$

$({}^qB)_{h,g,q+1}(x,y) = h^{2(q-1)}(x) + (q-1)h^{q-2}(x)\; g(y),$

$({}^qB)_{h,g,q+2}(x,y) = h^{3(q-1)}(x) + \frac{(2(q-1))^{\underline{1}}}{1!}\; h^{2q-3}(x)\; g(y) + \frac{(q-1)^{\underline{2}}}{2!} h^{q-3}(x) g^2(y),$

$({}^qB)_{h,g,q+3}(x,y) = h^{4(q-1)}(x) + \frac{(3(q-1))^{\underline{1}}}{1!} h^{3q-4}(x) g(y) + \frac{(2(q-1))^{\underline{2}}}{2!} h^{2q-4}(x) g^2(y)$
$\qquad + \frac{(q-1)^{\underline{3}}}{3!} h^{q-4}(y) g^3(y),$

$({}^qB)_{h,g,q+4}(x,y) = h^{5(q-1)}(x) + \frac{(4(q-1))^{\underline{1}}}{1!}\; h^{4q-5}(x) g(y) + \frac{(3(q-1))^{\underline{2}}}{2!}\; h^{3q-5}(x) g^2(y)$
$\qquad + \frac{(2(q-1))^{\underline{3}}}{3!} h^{2q-5}(x) g^3(y) + \frac{(q-1)^{\underline{4}}}{4!} h^{q-5}(x)\; g^4(y).$

For simplicity, we write $({}^qB)_{h,g,n}(x,y) = ({}^qB)_{h,g,n}$ and $h(x) = h$ and $g(y) = g$.

We have following results for $({}^qB)_{h,g,n}$.

Theorem 5.6.2. *The n^{th} term of (5.42) is given by*

$$({}^qB)_{h,g,n} = \frac{\sum_{k=1}^{q} (-1)^{k+1} \prod_{1 \leq i < j \leq q, i,j \neq k} (\phi_i - \phi_j) \phi_k^n}{\prod_{1 \leq i < j \leq q} (\phi_i - \phi_j)}, \tag{5.43}$$

where $\phi_p, p = 1, 2, \cdots, q$ are q distinct roots of characteristic equation corresponding to (5.42).

Equation (5.43) is called Binet type formula for (5.42).

Theorem 5.6.3. *The n^{th} term $({}^qB)_{h,g,n}$ of (5.42) is given by*

$$({}^qB)_{h,g,n} = \sum_{r=0}^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor} \frac{\Big((q-1)(n-(q-1)-r)\Big)^{\underline{r}}}{r!} \; h^{(q-1)(n-(q-1)-r)-r} \; g^r, \quad (5.44)$$

for all $n \geq q-1$.

Theorem 5.6.4. *The sum of the first $n+1$ terms of (5.42) is given by*

$$\sum_{r=0}^{n} ({}^qB)_{h,g,r} = \frac{({}^qB)_{h,g,n+1} + \sum_{i=0}^{q-2} \sum_{r=1+i}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \; h^{q-1-r} \; g^r \; ({}^qB)_{h,g,n-i} - 1}{(h+g)^{q-1} - 1}, \qquad (5.45)$$

provided $\begin{cases} h+g \neq 1, & \text{if q is even;} \\ h+g \neq \pm 1, & \text{if q is odd.} \end{cases}$

Theorem 5.6.5. *The generating function for (5.42) is given by*

$$({}^qG_{(B)})_{h,g}(z) = \frac{1}{1 - z(h+gz)^{q-1}} \qquad (5.46)$$

The next two theorems are related to the recurrence properties of $({}^qB)_{h,g,n}$. Proof of these theorems is similar to the proof of Theorem 5.2.6 and Theorem 5.2.7 respectively.

Theorem 5.6.6. *For all $s \geq 1$,*

$$({}^qB)_{h,g,n+qs} = \sum_{i=0}^{(q-1)s} \frac{((q-1)s)^{\underline{i}}}{i!} \; ({}^qB)_{h,g,n+i} \; h^i g^{(q-1)s-i}. \qquad (5.47)$$

Theorem 5.6.7. *For all $s \geq 1$ and $q \geq 2$,*

$$({}^{q}B)_{h,g,n+(q-1)+s} - h^{(q-1)s}({}^{q}B)_{h,g,n+q-1} = \sum_{i=0}^{s-1}\sum_{j=1}^{q-1} \frac{(q-1)^{\underline{j}}}{j!}\, g^{j} h^{(q-1)(s-i)-j}({}^{q}B)_{h,g,n+(q-1)+i-j}. \tag{5.48}$$

We prove below the results related to first order partial derivative of $({}^{q}B)_{h,g,n}$ with respect to x and y.

Theorem 5.6.8. *For all $n \geq 0$,*

(1) $qg\, \frac{\partial}{\partial x}[({}^{q}B)_{h,g,n}] + h\, \frac{\partial}{\partial x}[({}^{q}B)_{h,g,n+1}] = (q-1)(n-(q-2))({}^{q}B)_{h,g,n+1}h^{(1,0)}.$

(2) $g^{(0,1)} \frac{\partial}{\partial x}[({}^{q}B)_{h,g,n}] = h^{(1,0)} \frac{\partial}{\partial y}[({}^{q}B)_{h,g,n+1}].$

(3) $qg\, \frac{\partial}{\partial y}[({}^{q}B)_{h,g,n}] + h\, \frac{\partial}{\partial y}[({}^{q}B)_{h,g,n+1}] = (q-1)(n-(q-1))({}^{q}B)_{h,g,n}g^{(0,1)}.$

(4) $qg\, \frac{\partial}{\partial y}[({}^{q}B)_{h,g,n}]\, h^{(1,0)} + h\, \frac{\partial}{\partial x}[({}^{q}B)_{h,g,n}]\, g^{(0,1)} = (q-1)(n-(q-1))({}^{q}B)_{h,g,n}h^{(1,0)}g^{(0,1)}.$

Proof. (1) Note that for $0 \leq n \leq q-2$, L.H.S.$=0=$ R.H.S.

Now let $n \geq q-1$ and take $n = qm$. Using (5.44) and L.H.S. of (1), we have

$$qg\, \frac{\partial}{\partial x}[({}^{q}B)_{h,g,qm}] + h\, \frac{\partial}{\partial x}[({}^{q}B)_{h,g,qm+1}]$$

$$= \Bigg[(q-1)\big(qm-(q-2)\big)h^{(q-1)(qm-(q-2))} + \sum_{r=1}^{(q-1)m-(q-2)} \Bigg[qr\frac{\big((q-1)(qm-(q-2)-r)\big)^{\underline{r}}}{r!} + \frac{\big((q-1)(qm-(q-2)-r)\big)^{\underline{r+1}}}{r!}\Bigg] h^{(q-1)(qm-(q-2)-r)-r}\, g^{r}\Bigg]\, h^{(1,0)}$$

$$= \Bigg[(q-1)\big(qm-(q-2)\big)\, h^{(q-1)(qm-(q-2))} + \sum_{r=1}^{(q-1)m-(q-2)} \frac{\big((q-1)(qm-(q-2)-r)\big)^{\underline{r}}}{(r-1)!} \Big[qr + (q-1)(qm-(q-2)-r) - r\Big] h^{(q-1)(qm-(q-2)-r)-r}\, g^{r}\Bigg]\, h^{(1,0)}$$

$$= (q-1)\big(qm-(q-2)\big) \sum_{r=0}^{(q-1)m-(q-2)} \frac{\big((q-1)(qm-(q-2)-r)\big)^{\underline{r}}}{r!}\, h^{(q-1)(qm-(q-2)-r)-r} g^{r} h^{(1,0)}$$

$= (q-1)(qm-(q-2))(^{q}B)_{h,g,qm+1}h^{(1,0)}.$

Therefore, the result is true for $n = qm$.

Similarly, the result can be proved for $n = qm+1, \cdots, qm+q-1$. Hence (1) is proved.

Identity (2) can be verified by differentiating $(^{q}B)_{h,g,n}$ and $(^{q}B)_{h,g,n+1}$ respectively with respect to x and with respect to y. Identity (3) can be proved using Identities (1) and (2). Identity (4) can be deduced from (2) and (3). □

5.7 Generalized bivariate B-q Lucas polynomials

In this section, we define generalized bivariate B-q Lucas polynomials and obtain some identities related to these polynomials.

Definition 5.7.1. *Let $n \in \mathbb{N} \cup \{0\}$. The generalized bivariate B-q Lucas polynomials $(^{q}L)_{h,g,n}(x,y)$ are defined by*

$$(^{q}L)_{h,g,n+q-1}(x,y) = \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\ h^{q-1-r}(x)\ g^{r}(y)\ (^{q}L)_{h,g,n+q-2-r}(x,y), \text{ for all n} \geq 1, \tag{5.49}$$

with $(^{q}L)_{h,g,i}(x,y) = 0,\ i = 0,1,2,3,\cdots q-3, (\forall q \geq 3),\ (^{q}L)_{h,g,q-2}(x,y) = 2$ and $(^{q}L)_{h,g,q-1}(x,y) = h^{q-1}(x)$, where $(^{q}L)_{h,g,n}(x,y)$ is n^{th} B-q Lucas polynomial.

For $q \geq 2$ and $q-1 \leq n \leq q+1$, the terms of (5.49) are $(^{q}L)_{h,g,q-1}(x,y) = h^{q-1}(x)$, $(^{q}L)_{h,g,q}(x,y) = h^{2(q-1)}(x) + 2(q-1)h^{q-2}(x)\ g(y)$ and
$(^{q}L)_{h,g,q+1}(x,y) = h^{3(q-1)}(x) + 3(q-1)h^{2q-3}(x)\ g(y) + (q-1)(q-2)h^{q-3}(x)g^{2}(y)$.

For simplicity, we use $(^{q}L)_{h,g,n}(x,y) = (^{q}L)_{h,g,n}$ and $h(x) = h$ and $g(y) = g$.
We state below identities related to $(^{q}L)_{h,g,n}$.

Theorem 5.7.2. *The n^{th} term of 5.49) is given by*

$$({}^{q}L)_{h,g,n} = \frac{\sum_{k=1}^{q}(-1)^{k+1}\prod_{1\le i<j\le q, i,j\neq k}(\phi_i-\phi_j)\phi_k^n(2\phi_k-h^{q-1})}{\prod_{1\le i<j\le q}(\phi_i-\phi_j)}, \tag{5.50}$$

where $\phi_p, p=1,2,\cdots,q$ are q distinct roots of characteristic equation corresponding to (5.49).

Equation (5.50) is called a Binet type formula for (5.49).

Theorem 5.7.3. *The n^{th} term $({}^{q}L)_{h,g,n}$ of (5.49) is given by*

$$\begin{aligned}({}^{q}L)_{h,g,n} &\\ = \sum_{r=0}^{\left\lfloor\frac{(q-1)(n-(q-2))}{q}\right\rfloor} &\Big[\frac{(q-1)\big(n-(q-2)\big)}{(q-1)\big(n-(q-2)-r\big)}\frac{\Big((q-1)\big(n-(q-2)-r\big)\Big)^{\underline{r}}}{r!}\Big]h^{(q-1)\big(n-(q-2)\big)-qr}g^r\\ - \sum_{r=2}^{\left\lfloor\frac{(q-1)(n-(q-2))}{q}\right\rfloor} &\Big[\sum_{s=1}^{q-1}(s-1)\frac{\Big((q-1)\big(n-(q-1)-r\big)+s-2\Big)^{\underline{r-2}}}{(r-2)!}\Big]h^{(q-1)\big(n-(q-2)\big)-qr}g^r,\end{aligned} \tag{5.51}$$

for all $n \geq q-1$.

Theorem 5.7.4. *The sum of the first $n+1$ terms of (5.49)*

$$\sum_{r=0}^{n}({}^{q}L)_{h,g,r} = \frac{({}^{q}L)_{h,g,n+1} + \sum_{i=0}^{q-2}\sum_{r=1+i}^{q-1}\frac{(q-1)^{\underline{r}}}{r!}\, h^{q-1-r}\, g^r\, ({}^{q}L)_{h,g,n-i} + ({}^{q}L)_{h,g,q-1} - ({}^{q}L)_{h,g,q-2}}{(h+g)^{q-1}-1}, \tag{5.52}$$

provided $\begin{cases} h+g\neq 1, & \text{if q is even;}\\ h+g\neq \pm 1, & \text{if q is odd.}\end{cases}$

Theorem 5.7.5. *The generating function for (5.49) is given by*

$$({}^{q}G_{(L)})_{h,g}(z) = \frac{2-h^{q-1}z}{1-z(h+gz)^{q-1}}. \tag{5.53}$$

Following theorem gives the relation between bivariate B-q bonacci and B-q Lucas polynomials.

Theorem 5.7.6. *For all $n \geq q-1$,*

$$({}^qL)_{h,g,n} = ({}^qB)_{h,g,n+1} + \sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ h^{q-1-r} \ g^r \ ({}^qB)_{h,g,n-r}. \tag{5.54}$$

Proof. We prove the theorem by mathematical induction on n.

Note that (5.54) is true for $n = q-1$. Assume now that the result is true for $n \leq m$.

Consider, $({}^qL)_{h,g,m+1} = \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ h^{q-1-r} \ g^r \ ({}^qL)_{h,g,m-r}$

$= \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ h^{q-1-r} \ g^r \left[({}^qB)_{h,g,m+1-r} + \sum_{s=1}^{q-1} \frac{(q-1)^{\underline{s}}}{s!} \ h^{q-1-s} \ g^s ({}^qB)_{h,g,m-r-s}\right]$

$= ({}^qB)_{h,g,m+2} + \sum_{s=1}^{q-1} \frac{(q-1)^{\underline{s}}}{s!} h^{q-1-s} \ g^s \ ({}^qB)_{h,g,m+1-s}.$

Hence the result follows. □

Following result follows immediately.

Corollary 5.7.7. *For all $n \geq q-2$,*

$$({}^qL)_{h,g,n} = 2 \ ({}^qB)_{h,g,n+1} - h^{q-1}({}^qB)_{h,g,n}. \tag{5.55}$$

Proof. Note that $2 \ ({}^qB)_{h,g,q-1} - h^{q-1}({}^qB)_{h,g,q-2} = 2 = ({}^qL)_{h,g,q-2}$.

Hence equation (5.55) is true for $n = q-2$.

For $n \geq q-1$, the result can be proved using equations (5.42) and (5.54). □

Next two theorems are related to the recurrence properties of $({}^qL)_{h,g,n}$.

Theorem 5.7.8. *For all $s \geq 1$,*

$$({}^qL)_{h,g,n+qs} = \sum_{i=0}^{(q-1)s} \frac{((q-1)s)^{\underline{i}}}{i!} \ ({}^qL)_{h,g,n+i} \ h^i g^{(q-1)s-i}. \tag{5.56}$$

Theorem 5.7.9. *For all $s \geq 1$ and $q \geq 2$,*

$$({}^{q}L)_{h,g,n+(q-1)+s} - h^{(q-1)s}({}^{q}L)_{h,g,n+q-1} = \sum_{i=0}^{s-1}\sum_{j=1}^{q-1} \frac{(q-1)^{\underline{j}}}{j!}\, g^{j} h^{(q-1)(s-i)-j}({}^{q}L)_{h,g,n+(q-1)+i-j}. \tag{5.57}$$

Next, we prove the identities related to first order partial derivatives of $({}^{q}L)_{h,g,n}$ with respect to x and y.

Theorem 5.7.10. *For all $n \geq 0$,*

(1) $qg\, \frac{\partial}{\partial x}[({}^{q}L)_{h,g,n}] + h\, \frac{\partial}{\partial x}[({}^{q}L)_{h,g,n+1}]$

$= h^{(1,0)}\Big((q-1)(n-(q-3))({}^{q}L)_{h,g,n+1} - q(q-1)h^{q-2}g({}^{q}B)_{h,g,n}\Big).$

(2) $g^{(0,1)}\, \frac{\partial}{\partial x}[({}^{q}L)_{h,g,n}] = h^{(1,0)}\Big(\frac{\partial}{\partial y}[({}^{q}L)_{h,g,n+1}] - (q-1)h^{q-2}({}^{q}B)_{h,g,n}\Big).$

(3) $qg\, \frac{\partial}{\partial y}[({}^{q}L)_{h,g,n}] + h\, \frac{\partial}{\partial y}[({}^{q}L)_{h,g,n+1}]$

$= g^{(0,1)}\Big((q-1)(n-(q-1))({}^{q}L)_{h,g,n} + 2\,(q-1)({}^{q}B)_{h,g,n+1}\Big).$

(4) $qg\, \frac{\partial}{\partial y}[({}^{q}L)_{h,g,n}]\, h^{(1,0)} + h\, \frac{\partial}{\partial x}[({}^{q}L)_{h,g,n}]g^{(0,1)} = (q-1)(n-(q-2))({}^{q}L)_{h,g,n}h^{(1,0)}g^{(0,1)}.$

Proof. Equation (5.55) implies,

$({}^{q}L)_{h,g,n} = 2\,({}^{q}B)_{h,g,n+1} - h^{(q-1)}({}^{q}B)_{h,g,n}.$

Differentiating both sides with respect to x, we get

$$\frac{\partial}{\partial x}[({}^{q}L)_{h,g,n}] = 2\,\frac{\partial}{\partial x}[({}^{q}B)_{h,g,n+1}] - h^{(q-1)}\frac{\partial}{\partial x}[({}^{q}B)_{h,g,n}] - (q-1)h^{q-2}({}^{q}B)_{h,g,n}$$

Also, $\frac{\partial}{\partial x}[({}^{q}L)_{h,g,n+1}] = 2\,\frac{\partial}{\partial x}[({}^{q}B)_{h,g,n+2}] - h^{(q-1)}\frac{\partial}{\partial x}[({}^{q}B)_{h,g,n+1}] - (q-1)h^{q-2}({}^{q}B)_{h,g,n+1}.$

Thus, using (1) of Theorem 5.6.8, we get

$$qy\ \tfrac{\partial}{\partial x}[({}^qL)_{h,g,n}]\ + x\ \tfrac{\partial}{\partial x}[({}^qL)_{h,g,n+1}]$$

$$= h^{(1,0)}\Big(2(q-1)(n-(q-3))({}^qB)_{h,g,n+2} - h^{q-1}(q-1)(n-(q-3))({}^qB)_{h,g,n+1}$$

$$-h^{q-1}(q-1)({}^qB)_{h,g,n+1} - (q-1)h^{q-2}\big(qg\ ({}^qB)_{h,g,n} + h\ ({}^qB)_{h,g,n+1}\big)\Big)$$

$$= h^{(1,0)}\Big((q-1)(n-(q-3)({}^qL)_{h,g,n+1} - q(q-1)g\ h^{q-2}({}^qB)_{h,g,n}\Big).$$

Similarly, other identities can be proved. □

We now prove some identities involving k^{th} order partial derivative with respect to x and j^{th} order partial derivative with respect y of bivariate polynomials $({}^qB)_{h,g,n}$ and $({}^qL)_{h,g,n}$ respectively, where $k, j \geq 0$. Let $({}^qB)^{(k,j)}_{h,g,n}$ and $({}^qL)^{(k,j)}_{h,g,n}$ denote the k^{th} order partial derivative with respect to x and j^{th} order partial derivative with respect y of $({}^qB)_{h,g,n}$ and $({}^qL)_{h,g,n}$ respectively. Let $(.)^{(s,0)}$ denote the s^{th} order derivative of $(.)$ with respect to x and $(.)^{(0,p)}$ denote the p^{th} order derivative of $(.)$ with respect to y. We have following identities.

Theorem 5.7.11. *For all $n \geq q-1$,*

(1) $({}^qL)^{(k,j)}_{h,g,n} = ({}^qB)^{(k,j)}_{h,g,n+1}$
$+\sum_{r=1}^{q-1} \frac{(q-1)^r}{r!} \sum_{s=0}^{q-1-r} \sum_{p=0}^{r} \frac{k^s}{s!} \frac{j^p}{p!} (h^{q-1-r})^{(s,0)} (g^r)^{(0,p)} ({}^qB)^{(k-s,j-p)}_{h,g,n-r}.$

(2) $({}^qB)^{(k,j)}_{h,g,n} = \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} \sum_{s=0}^{q-1-r} \sum_{p=0}^{r} \frac{k^s}{s!} \frac{j^t}{p!} (h^{q-1-r})^{(s,0)} (g^r)^{(0,p)} ({}^qB)^{(k-s,j-p)}_{h,g,n-1-r}.$

(3) $({}^qL)^{(k,j)}_{h,g,n} = \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} \sum_{s=0}^{q-1-r} \sum_{p=0}^{r} \frac{k^s}{s!} \frac{j^p}{p!} (h^{q-1-r})^{(s,0)} (g^r)^{(0,p)} ({}^qL)^{(k-s,j-p)}_{h,g,n-1-r}.$

Proof.

(1) Note that $({}^qL)_{h,g,n} = ({}^qB)_{h,g,n+1} + \sum_{r=1}^{q-1} \frac{(q-1)^r}{r!}\ h^{q-1-r}\ g^r\ ({}^qB)_{h,g,n-r}.$

Differentiating both sides k times with respect to x and j times with respect to y and using Leibnitz theorem for derivatives, we get

$$({}^qL)^{(k,j)}_{h,g,n} = ({}^qB)^{(k,j)}_{h,g,n+1} + \sum_{r=1}^{q-1} \frac{(q-1)^r}{r!} \frac{\partial^k}{\partial h^k}\left(h^{q-1-r} \sum_{p=0}^{r} \frac{j^p}{p!} (g^r)^{(0,p)} \, ({}^qB)^{(0,j-p)}_{h,g,n-r}\right)$$

$$= ({}^qB)^{(k,j)}_{h,g,n+1}$$

$$+ \sum_{r=1}^{q-1} \frac{(q-1)^r}{r!} \sum_{s-0}^{q-1-r} \frac{k^s}{s!} \left(h^{q-1-r}\right)^{(s,0)} \sum_{p=0}^{r} \frac{j^p}{p!} (g^r)^{(0,p)} \, ({}^qB)^{(k-s,j-p)}_{h,g,n-r}$$

$$= ({}^qB)^{(k,j)}_{h,g,n+1}$$

$$+ \sum_{r=1}^{q-1} \frac{(q-1)^r}{r!} \sum_{s=0}^{q-1-r} \sum_{p=0}^{r} \frac{k^s}{s!} \frac{j^p}{p!} \left(h^{q-1-r}\right)^{(s,0)} (g^r)^{(0,p)} \, ({}^qB)^{(k-s,j-p)}_{h,g,n-r}.$$

Hence (1) is proved.

(2) We have from (5.42), $({}^qB)_{h,g,n} = \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} \; h^{q-1-r} \, g^r \, ({}^qB)_{h,g,n-1-r}$.

Differentiating both sides k times with respect to x and j times with respect to y and using Leibnitz theorem for derivatives, we get

$$({}^qB)^{(k,j)}_{h,g,n} = \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} \frac{\partial^k}{\partial h^k}\left(h^{q-1-r} \sum_{p=0}^{r} \frac{j^p}{p!} (g^r)^{(0,p)} \, ({}^qB)^{(0,j-p)}_{h,g,n-1-r}\right)$$

$$= \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} \sum_{s=0}^{q-1-r} \frac{k^s}{s!} \left(h^{q-1-r}\right)^{(s,0)} \sum_{p=0}^{r} \frac{j^p}{p!} (g^r)^{(0,p)} \, ({}^qB)^{(k-s,j-p)}_{h,g,n-1-r}$$

$$= \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} \sum_{s-0}^{q-1-r} \sum_{p=0}^{r} \frac{k^s}{s!} \frac{j^p}{p!} \left(h^{q-1-r}\right)^{(s,0)} (g^r)^{(0,p)} \, ({}^qB)^{(k-s,j-p)}_{h,g,n-1-r}.$$

Hence (2) is proved. Similarly, we can prove the identity (3). □

Remark 5.7.12. *Using Leibnitz theorem for derivatives we can established similar type of identities using identities in Theorem 5.6.8 and Theorem 5.7.10.*

Theorem 5.7.13. ***(Convolution property for*** $({}^qB)_{h,g,n}$***)***

$$({}^qB)^{(1,0)}_{h,g,n} = (q-1)h^{(1,0)}\left(\sum_{r=0}^{q-2} \frac{(q-2)^r}{r!} \, h^{(q-2)-r} \, g^r \sum_{i=0}^{n+q-2-r} ({}^qB)_{h,g,i} \, ({}^qB)_{h,g,n+q-2-r-i}\right). \tag{5.58}$$

Proof. Equation (5.46) implies,

$$({}^qG_{(B)})(z) = \frac{1}{1 - z\,(h + gz)^{q-1}}.$$

Therefore,

$$\sum_{n=0}^{\infty} ({}^qB)_{h,g,n}\; z^{n-(q-1)} = \frac{1}{1 - z(h + gz)^{(q-1)}}.$$

Differentiating both sides with respect to x we get,

$\sum_{n=0}^{\infty} ({}^qB)_{h,g,n}^{(1,0)} z^{n-(q-1)}$

$$= \left(z(q-1)(h+gz)^{q-2}\, \frac{1}{[1-z(h+gz)^{q-1}]^2} \right) h^{(1,0)}$$

$$= \left((q-1)\; \sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!}\; h^{(q-2)-r} g^r z^{r+1} \left[\sum_{n=0}^{\infty} ({}^qB)_{h,g,n} z^{n-(q-1)} \right]^2 \right) h^{(1,0)}$$

$$= \left((q-1)\; \sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!}\; h^{(q-2)-r} g^r z^{-2(q-1)+r+1} \left[\sum_{n=0}^{\infty} ({}^qB)_{h,g,n} z^{n} \right]^2 \right) h^{(1,0)}$$

$$= (q-1) h^{(1,0)}\; \sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!}\; h^{(q-2)-r} g^r \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} ({}^qB)_{h,g,i} ({}^qB)_{h,g,n-i} z^{n-2(q-1)+r+1} \right).$$

Comparing the coefficients of $z^{n-(q-1)}$ we get,

$$({}^qB)_{h,g,n}^{(1,0)} = (q-1) h^{(1,0)} \left(\sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!}\; h^{(q-2)-r} g^r\; \sum_{i=0}^{n+q-2-r} ({}^qB)_{h,g,i} ({}^qB)_{h,g,n+q-2-r-i} \right).$$

□

Theorem 5.7.14. (***Convolution property for*** $({}^qL)_{h,g,n}$)

$({}^qL)_{h,g,n}^{(1,0)}$

$$= (q-1) h^{(1,0)} \left(\left(\sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!}\; h^{(q-2)-r}\; g^r \sum_{i=0}^{n+q-2-r} ({}^qB)_{h,g,i}\; ({}^qL)_{h,g,n+q-2-r-i} \right) - h^{q-2} ({}^qB)_{h,g,n} \right). \tag{5.59}$$

Proof. Equation (5.55) implies,

$$({}^qL)_{h,g,n} = 2\ ({}^qB)_{h,g,n+1} - h^{q-1}({}^qB)_{h,g,n}.$$

Differentiating both sides with respect to x and then using (5.58), we get

$$({}^qL)^{(1,0)}_{h,g,n} = 2\ ({}^qB)^{(1,0)}_{h,g,n+1} - h^{q-1}({}^qB)^{(1,0)}_{h,g,n} - (q-1)h^{q-2}h^{(1,0)}({}^qB)_{h,g,n}$$

$$= (q-1)h^{(1,0)}\Big[\ \textstyle\sum_{r=0}^{q-2}\frac{(q-2)^{\underline{r}}}{r!}\ h^{(q-2)-r}\ g^r\sum_{i=0}^{n+q-2-r}({}^qB)_{h,g,i}$$

$$\Big(2({}^qB)_{h,g,n+1+q-2-r-i} - h^{q-1}({}^qB)_{h,g,n+q-2-r-i}\Big)\Big] - (q-1)h^{q-2}h^{(1,0)}({}^qB)_{h,g,n}$$

$$= (q-1)h^{(1,0)}\Big[\textstyle\sum_{r=0}^{q-2}\frac{(q-2)^{\underline{r}}}{r!}\ h^{(q-2)-r}\ g^r\sum_{i=0}^{n+q-2-r}({}^qL)_{n+q-2-r-i}({}^qB)_{h,g,i}$$

$$-h^{q-2}({}^qB)_{h,g,n}\Big].$$

□

5.8 Incomplete generalized bivariate B-q bonacci polynomials

In this section, we introduce the extension of incomplete generalized bivariate B-Tribonacci polynomials (5.27) to q^{th} order incomplete generalized bivariate polynomials and call it, incomplete generalized bivariate B-q bonacci polynomials. We also study their various identities.

Definition 5.8.1. *The incomplete generalized bivariate B-q bonacci polynomials are defined by*

$$({}^qB)^l_{h,g,n}(x,y) = \sum_{r=0}^{l}\frac{\big((q-1)(n-(q-1)-r)\big)^{\underline{r}}}{r!}\ h^{(q-1)(n-(q-1)-r)-r}(x)g^r(y), \quad (5.60)$$

$$\forall\ 0 \le l \le \Big\lfloor\frac{(q-1)\big(n-(q-1)\big)}{q}\Big\rfloor \text{ and } n \ge q-1.$$

Note that $({}^{q}B)_{h,g,n}^{\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor}(x,y) = ({}^{q}B)_{h,g,n}(x,y)$.

For Simplicity, we use

$({}^{q}B)_{h,g,n}^{l}(x,y) = ({}^{q}B)_{h,g,n}^{l}, ({}^{q}B)_{h,g,n}(x,y) = ({}^{q}B)_{h,g,n}, h(x) = h$ and $g(y) = g$.

We prove identities related to recurrence relations of $({}^{q}B)_{h,g,n}^{l}(x,y)$.

Theorem 5.8.2. *The recurrence relation of* $({}^{q}B)_{h,g,n}^{l}$ *is given by*

$$({}^{q}B)_{h,g,n+q}^{l+q-1} = \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ h^{q-1-r} \ g^{r} \ ({}^{q}B)_{h,g,n+q-1-r}^{l+q-1-r}, \ \ 0 \le l \le \lfloor\frac{(q-1)(n-q)}{q}\rfloor, \tag{5.61}$$

$\forall n \geq q$.

Proof. Consider, $\sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ ({}^{q}B)_{h,g,n+q-1-r}^{l+q-1-r} h^{q-1-r} g^{r}$

$= \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ h^{q-1-r} g^{r}$

$\sum_{i=0}^{l+q-1-r} \frac{\left((q-1)(n+q-1-r-(q-1)-i)\right)^{\underline{i}}}{i!} \ h^{(q-1)(n+q-1-r-(q-1)-i)-i} g^{i}$

$= \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ h^{q-1-r} \ \sum_{i=0}^{l+q-1-r} \frac{\left((q-1)(n-r-i)\right)^{\underline{i}}}{i!} \ h^{(q-1)(n-r)-qi} g^{r+i}$

$= \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ \sum_{i=0}^{l+q-1-r} \frac{\left((q-1)(n-r-i)\right)^{\underline{i}}}{i!} \ h^{(q-1)(n+1)-qr-qi} g^{r+i}$

$= \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ \sum_{i=0}^{l+q-1-r} \frac{\left((q-1)(n-(r+i))\right)^{\underline{i}}}{i!} \ h^{(q-1)(n+1)-q(r+i)} g^{r+i}$.

Taking $j = i + r$, we get

$\sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ ({}^{q}B)_{h,g,n+q-1-r}^{l+q-1-r} h^{q-1-r} g^{r}$

$= \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ \sum_{j=r}^{l+q-1} \frac{\left((q-1)(n-j)\right)^{\underline{j-r}}}{(j-r)!} \ h^{(q-1)(n+1)-qj} g^{j}$

$= \sum_{j=0}^{l+q-1} \frac{\left((q-1)(n+1-j)\right)^{\underline{j}}}{j!} \ h^{(q-1)(n+1)-qj} g^j$

$= ({}^qB)_{h,g,n+q}^{l+q-1}.$ □

Theorem 5.8.3. $s \geq 1$,

$$({}^qB)_{h,g,n+qs}^{l+(q-1)s} = \sum_{i=0}^{(q-1)s} \frac{((q-1)s)^{\underline{i}}}{i!} \ ({}^qB)_{h,g,n+i}^{l+i} \ h^i g^{(q-1)s-i}, \tag{5.62}$$

$0 \leq l \leq \lfloor \frac{(q-1)(n-s-(q-1))}{q} \rfloor.$

Proof. By mathematical induction on s. Clearly (5.62) holds for $s = 1$.

Assume that the result holds for all $s \leq m$.

Consider, $\sum_{i=0}^{(q-1)(m+1)} \frac{\left((q-1)(m+1)\right)^{\underline{i}}}{i!} \ ({}^qB)_{h,g,n+i}^{l+i} \ h^i g^{(q-1)(m+1)-i}$

$= \sum_{i=0}^{(q-1)(m+1)} \ \sum_{r=0}^{q-1} \ \frac{(q-1)^{\underline{r}}}{r!} \frac{\left((q-1)m\right)^{\underline{i-r}}}{(i-r)!} \ ({}^qB)_{h,g,n+i}^{l+i} \ h^i g^{(q-1)(m+1)-i}$

$= \sum_{r=0}^{q-1} \ \sum_{i=r}^{(q-1)m} \ \frac{(q-1)^{\underline{r}}}{r!} \ \frac{\left((q-1)m\right)^{\underline{i-r}}}{(i-r)!} \ ({}^qB)_{h,g,n+i}^{l+i} \ h^i g^{(q-1)(m+1)-i}$

$= \sum_{r=0}^{q-1} \ \frac{(q-1)^{\underline{r}}}{r!} \sum_{j=0}^{(q-1)m-r} \ \frac{\left((q-1)m\right)^{\underline{j}}}{j!} \ ({}^qB)_{h,g,n+r+j}^{l+r+j} \ h^{j+r} \ g^{(q-1)(m+1)-(j+r)}$

$= \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ h^r \ g^{(q-1)-r} \sum_{j=0}^{(q-1)m-r} \ \frac{\left((q-1)m\right)^{\underline{j}}}{j!} \ ({}^qB)_{h,g,n+r+j}^{l+r+j} \ h^j \ g^{(q-1)m-j}$

$= \sum_{r=0}^{q-1} \ \frac{(q-1)^{\underline{r}}}{r!} \ ({}^qB)_{h,g,n+r+qm}^{l+r+(q-1)m} \ h^r \ g^{(q-1)-r}$

$= ({}^qB)_{h,g,n+q(m+1)}^{l+(q-1)(m+1)}.$

Hence the result is true for $s = m + 1$.

Thus, by mathematical induction, the theorem is proved. □

Theorem 5.8.4. *For* $n \geq \lfloor \frac{ql+2(q-1)}{q-1} \rfloor$,

$$({}^qB)^{l+(q-1)}_{h,g,n+(q-1)+s} - h^{(q-1)s}({}^qB)^{l+q-1}_{h,g,n+q-1}$$

$$= \sum_{i=0}^{s-1}\sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} h^{(q-1)s-(q-1)i-r} g^r ({}^qB)^{l+(q-1)-r}_{h,g,n+(q-1)+i-r}. \tag{5.63}$$

Proof. By mathematical induction on s. Note that (5.63) clearly holds for $s = 1$. Now let the result be true for $s \leq m$. We prove it for $s = m+1$.

Consider,

$$\sum_{i=0}^{m}\sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\, h^{(q-1)(m+1)-(q-1)-r} g^r ({}^qB)^{l+(q-1)-r}_{h,g,n+(q-1)+i-r}$$

$$= \sum_{i=0}^{m-1}\sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\, h^{(q-1)(m+1)-(q-1)i-r} g^r ({}^qB)^{l+(q-1)-r}_{h,g,n+(q-1)+i-r}$$

$$+ \sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\, h^{(q-1)(m+1)-(q-1)m-r} g^r ({}^qB)^{l+(q-1)-r}_{h,g,n+(q-1)+m+1-r}$$

$$= h^{q-1}\sum_{i=0}^{m-1}\sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\, h^{(q-1)m-(q-1)i-r} g^r ({}^qB)^{l+(q-1)-r}_{h,g,n+(q-1)+i-r}$$
$$+ \sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\, h^{(q-1)-r} g^r ({}^qB)^{l+(q-1)-r}_{h,g,n+q+m-r}$$

$$= ({}^qB)^{l+(q-1)}_{h,g,n+(q-1)+m+1} - h^{(q-1)(m+1)}({}^qB)^{l+q-1}_{h,g,n+q-1}.$$ □

Lemma 5.8.5. *For* $n \geq q-1$,

$$\sum_{r=0}^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor} r \frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!}\; h^{(q-1)(n-(q-1))-qr} g^r$$

$$= \frac{(q-1)(n-(q-1))}{q}\; ({}^qB)_{h,g,n}$$

$$-\frac{h}{q}\sum_{i=0}^{n}(q-1)\Big(\sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!}\; h^{(q-2)-r}\; g^r \sum_{i=0}^{n+q-2-r} ({}^qB)_{h,g,q-2-r+i}\; ({}^qB)_{h,g,n-i}\Big). \tag{5.64}$$

Proof. Equation (5.44) implies,

$$({}^{q}B)_{h,g,n} = \sum_{r=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} \frac{\Big((q-1)(n-(q-1)-r)\Big)^{\underline{r}}}{r!} \; h^{(q-1)(n-(q-1)-r)-r} \; g^r.$$

Differentiating both sides with respect to x, we get

$$({}^{q}B)^{(1,0)}_{h,g,n} \; h$$

$$= \sum_{r=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} \frac{\Big((q-1)\Big(n-(q-1)\Big)-qr\Big)\;\Big((q-1)(n-(q-1)-r)\Big)^{\underline{r}}}{r!} \; h^{(q-1)(n-(q-1))-qr} g^r \; h^{(1,0)}.$$

Therefore,

$$({}^{q}B)^{(1,0)}_{h,g,n} h = ((q-1)(n-(q-1))) \; ({}^{q}B)_{h,g,n} \; h^{(1,0)}$$

$$-q \; h^{(1,0)} \sum_{r=0}^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor} r \; \frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!} \; h^{(q-1)(n-(q-1))-qr} g^r.$$

Thus, $h^{(1,0)} \; \sum_{r=0}^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor} r \frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!} \; h^{(q-1)(n-(q-1))-qr} g^r$

$$= \frac{(q-1)(n-(q-1))}{q} \; ({}^{q}B)_{h,g,n} h^{(1,0)} - \frac{h}{q} \; ({}^{q}B)^{(1,0)}_{h,g,n}.$$

Hence, $\sum_{r=0}^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor} r \frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!} \; h^{(q-1)(n-(q-1))-qr} g^r$

$$= \frac{(q-1)(n-(q-1))}{q} \; ({}^{q}B)_{h,g,n}$$

$$-\frac{h}{q} \; (q-1)\Big(\; \sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!} \; h^{(q-2)-r} \; g^r \sum_{i=0}^{n+q-2-r} ({}^{q}B)_{h,g,i} \; ({}^{q}B)_{h,g,n+q-2-r-i}\Big).$$

□

Theorem 5.8.6. *For all $n \geq q-1$,*

$$\sum_{l=0}^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor} ({}^qB)^l_{h,g,n} = \Big(\lfloor \tfrac{(q-1)(n-(q-1))}{q} \rfloor + \tfrac{q-(q-1)(n-(q-1))}{q}\Big) ({}^qB)_{h,g,n}$$

$$+\frac{h}{q}\,(q-1)\Big(\sum_{r=0}^{q-2} \frac{(q-2)^r}{r!}\,h^{(q-2)-r}\,g^r \sum_{i=0}^{n+q-2-r} ({}^qB)_{h,g,i}\,({}^qB)_{h,g,n+q-2-r-i}\Big). \qquad (5.65)$$

Proof. $\sum_{l=0}^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor} ({}^qB)^l_{h,g,n}$

$$= ({}^qB)^0_{h,g,n} + ({}^qB)^1_{h,g,n} + \cdots + ({}^qB)^r_{h,g,n} + \cdots + ({}^qB)^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor}_{h,g,n}$$

$$= \tfrac{((q-1)(n-(q-1)))^0}{0!}\,h^{(q-1)(n-(q-1))}$$

$$+\Big[\tfrac{((q-1)(n-(q-1)))^0}{0!} h^{(q-1)(n-(q-1))} + \tfrac{(q-1)(n-(q-1)-1))^1}{1!} h^{(q-1)(n-(q-1))-q} g\Big] + \cdots$$

$$+\Big[\tfrac{((q-1)(n-(q-1)))^0}{0!}\,h^{(q-1)(n-(q-1))} + \cdots + \tfrac{((q-1)(n-(q-1)-r))^r}{r!} h^{(q-1)(n-(q-1))-qr} g^r\Big]$$

$$+\cdots$$

$$+\Big[\tfrac{((q-1)(n-(q-1)))^0}{0!}\,h^{(q-1)(n-(q-1))} + \cdots + \tfrac{((q-1)(n-(q-1)-r))^r}{r!} h^{(q-1)(n-(q-1))-qr} g^r$$

$$+\cdots$$

$$+\frac{\Big((q-1)\Big(n-(q-1)-\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor\Big)\Big)^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor}}{(\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor)!}\,h^{(q-1)(n-(q-1))-q\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor} g^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor}\Big]$$

$$= \Big(\lfloor \tfrac{(q-1)(n-(q-1))}{q} \rfloor + 1\Big) \tfrac{((q-1)(n-(q-1)))^0}{0!}\,h^{(q-1)(n-(q-1))}$$

$$+\left(\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor\right)\frac{((q-1)(n-(q-1)-1))^{1}}{1!}\ h^{(q-1)(n-(q-1))-q}g+\cdots$$

$$+\left(\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor+1-r\right)\frac{((q-1)(n-(q-1)-r))^{r}}{r!}\ h^{(q-1)(n-(q-1))-qr}g^{r}$$

$$+\cdots$$

$$+\frac{\left((q-1)\left(n-(q-1)-\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor\right)\right)^{\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor}}{(\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor)!}\ h^{(q-1)(n-(q-1))-q\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor}g^{\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor}$$

$$=\sum_{r=0}^{\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor}\left(\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor+1-r\right)\frac{((q-1)(n-(q-1)-r))^{r}}{r!}\ h^{(q-1)(n-(q-1))-qr}g^{r}$$

$$=\sum_{r=0}^{\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor}\left(\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor+1\right)\frac{((q-1)(n-(q-1)-r))^{r}}{r!}\ h^{(q-1)(n-(q-1))-qr}g^{r}$$

$$-\sum_{r=0}^{\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor}r\frac{((q-1)(n-(q-1)-r))^{r}}{r!}\ h^{(q-1)(n-(q-1))-qr}g^{r}$$

$$=\left(\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor+1-\frac{(q-1)(n-(q-1))}{q}\right)({}^{q}B)_{h,g,n}$$

$$+\frac{h}{q}\sum_{i=0}^{n}(q-1)\left(\ \sum_{r=0}^{q-2}\frac{(q-2)^{r}}{r!}\ h^{(q-2)-r}g^{r}\ \sum_{i=0}^{n+q-2-r}({}^{q}B)_{h,g,q-2-r+i}\ ({}^{q}B)_{h,g,n-i}\right)$$

$$=\left(\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor+\frac{q-(q-1)(n-(q-1))}{q}\right)({}^{q}B)_{h,g,n}$$

$$+\frac{h}{q}\sum_{i=0}^{n}(q-1)\left(\ \sum_{r=0}^{q-2}\frac{(q-2)^{r}}{r!}\ h^{(q-2)-r}g^{r}\ \sum_{i=0}^{n+q-2-r}({}^{q}B)_{h,g,q-2-r+i}\ ({}^{q}B)_{h,g,n-i}\right).$$

□

5.9 Incomplete generalized bivariate B-q Lucas polynomials

In this section we introduce the extension of incomplete generalized bivariate B-Tri Lucas polynomials (5.34) to q^{th} order incomplete generalized bivariate polynomials and call it incomplete generalized bivariate B-q Lucas polynomials. We also study their various identities.

Definition 5.9.1. *The incomplete generalized bivariate B-q Lucas polynomials are defined by*

$$({}^{q}L)^{l}_{h,g,n}(x,y)$$

$$= \sum_{r=0}^{l} \Big[\frac{(q-1)\big(n-(q-2)\big)}{(q-1)\big(n-(q-2)-r\big)} \frac{\Big((q-1)\big(n-(q-2)-r\big)\Big)^{\underline{r}}}{r!}\Big] h^{(q-1)\big(n-(q-2)\big)-qr}(x) g^{r}(y)$$

$$-\sum_{r=2}^{l} \Big[\sum_{s=1}^{q-1}(s-1)\frac{\Big((q-1)\big(n-(q-1)-r\big)+s-2\Big)^{\underline{r-2}}}{(r-2)!}\Big] h^{(q-1)\big(n-(q-2)\big)-qr}(x) g^{r}(y), \tag{5.66}$$

$\forall n \geq q$ *and* $0 \leq l \leq \lfloor \frac{(q-1)(n-(q-2))}{q} \rfloor$.

Next three theorems give results on recurrence properties of incomplete generalized bivariate B-q Lucas polynomials (5.72). Proof of these results can be obtained using a procedure similar to that used in the relative identities of incomplete generalized bivariate B-q bonacci sequence (5.60).

Theorem 5.9.2. *The recurrence relation for incomplete generalized bivariate B-q Lucas polynomials $({}^{q}L)^{l}_{h,g,n}$ is given by*

$$({}^{q}L)^{l+q-1}_{h,g,n+q} = \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \; ({}^{q}L)^{l+q-1-r}_{h,g,n+q-1-r} h^{q-1-r} g^{r}, \tag{5.67}$$

$\forall\ 0 \le l \le \left\lfloor \frac{(q-1)(n-(q-2))}{q} \right\rfloor$ and $n \ge q-2$.

Theorem 5.9.3. *For all* $0 \le l \le \left\lfloor \frac{(q-1)\left(n-(q-2)-s\right)}{q} \right\rfloor$,

$$({}^{q}L)^{l+(q-1)s}_{h,g,n+qs} = \sum_{i=0}^{(q-1)s} \frac{((q-1)s)^{\underline{i}}}{i!}\ ({}^{q}L)^{l+(q-1)s-i}_{h,g,n+(q-1)s-i}\ h^{(q-1)s-i} g^{i}. \tag{5.68}$$

Theorem 5.9.4. *For* $n \ge \left\lfloor \frac{ql}{q-1} + q - 2 \right\rfloor$,

$$({}^{q}L)^{l+(q-1)}_{h,g,n+(q-1)+s} - h^{(q-1)s} ({}^{q}L)^{l+(q-1)}_{h,g,n+(q-1)}$$

$$= \sum_{i=0}^{s-1} \sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \Big(h^{(q-1)s-r-(q-1)i} g^{r} ({}^{q}L)^{l+(q-1)-r}_{h,g,n+(q-1)+i-r} \Big). \tag{5.69}$$

Next two results gives the relation between nth term $({}^{q}B)^{l}_{h,g,n}$ and $({}^{q}L)^{l}_{h,g,n}$.

Theorem 5.9.5. *The relation between the* n^{th} *term* $({}^{q}L)^{l}_{h,g,n}$ *and* n^{th} *term* $({}^{q}B)^{l}_{h,g,n}$ *is given by*

$$({}^{q}L)^{l}_{h,g,n} = ({}^{q}B)^{l}_{h,g,n+1} + \sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!}\ h^{q-1-r} g^{r} ({}^{q}B)^{l-r}_{h,g,n-r},\quad 0 \le l \le \left\lfloor \frac{(q-1)(n-(q-2))}{q} \right\rfloor. \tag{5.70}$$

Proof of the Theorem 5.9.5 is similar to that of Theorem 5.5.2.

Corollary 5.9.6.

$$({}^{q}L)^{l}_{h,g,n} = 2\ ({}^{q}B)^{l}_{h,g,n+1} - h^{q-1}\ ({}^{q}B)^{l}_{h,g,n},\quad 0 \le l \le \left\lfloor \frac{(q-1)(n-(q-2))}{q} \right\rfloor. \tag{5.71}$$

Proof. Using equations (5.61) and (5.70), the Corollary can be proved. □

Lemma 5.9.7. *For all $n \geq q-2$,*

$$= \sum_{r=0}^{l} r\Big[\frac{(q-1)\big(n-(q-2)\big)}{(q-1)\big(n-(q-2)-r\big)} \frac{\Big((q-1)\big(n-(q-2)-r\big)\Big)^{\underline{r}}}{r!}\Big] h^{(q-1)\big(n-(q-2)\big)-qr} g^r$$

$$- \sum_{r=2}^{l} r\Big[\sum_{s=1}^{q-1}(s-1)\frac{\Big((q-1)\big(n-(q-1)-r\big)+s-2\Big)^{\underline{r-2}}}{(r-2)!}\Big] h^{(q-1)\big(n-(q-2)\big)-qr} g^r, \tag{5.72}$$

$$= \tfrac{(q-1)(n-(q-2))}{q} \; ({}^qL)_{h,g,n} - \tfrac{h}{q}(q-1)$$

$$\Bigg(\Big(\sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!} \; h^{(q-2)-r} \; g^r \sum_{i=0}^{n+q-2-r} ({}^qB)_{h,g,i} \; ({}^qL)_{h,g,n+q-2-r-i}\Big) - h^{q-2}({}^qB)_{h,g,n}\Bigg), \tag{5.73}$$

where $l = \lfloor \frac{(q-1)(n-(q-2))}{q} \rfloor$.

Theorem 5.9.8. *For all $n \geq q-2$,*

$$\textstyle\sum_{l=0}^{\left\lfloor \frac{(q-1)(n-(q-2))}{q} \right\rfloor} ({}^qL)_{h,g,n}^{l}$$

$$= \Big(\lfloor \tfrac{(q-1)(n-(q-2))}{q} \rfloor + \tfrac{q-\big((q-1)(n-(q-2)\big)}{q}\Big)({}^qL)_{h,g,n}$$

$$+\tfrac{h}{q}(q-1)\Bigg(\Big(\textstyle\sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!} \; h^{(q-2)-r} \; g^r$$

$$\sum_{i=0}^{n+q-2-r} ({}^qB)_{h,g,i} \; ({}^qL)_{h,g,n+q-2-r-i}\Big) - h^{q-2}({}^qB)_{h,g,n}\Bigg). \tag{5.74}$$

Proof. Use Lemma 5.9.7 and procedure similar to that of Theorem 5.5.8. □

Chapter 6

Hyers-Ulam Stability of Generalized Functional Equation

This Chapter include the content of published paper (P5).

Chapter 6

Hyers-Ulam Stability of Generalized Functional Equation

6.1 Introduction

Ulam first proposed the problem of stability of the linear functional equation $f(x+y) = f(x) + f(y)$. In 1941, Donald H. Hyers gave a partial affirmative answer to the question of Ulam in the context of Banach spaces [9]. Since then, the problem of Hyers-Ulam stability of functional equations has become very popular and studied by many mathematicians. In Section 2 of this chapter, the solution of generalized linear Tribonacci functional equation is established in terms of generalized Tribonacci sequence. In Section 3, the Hyers-Ulam stability of this functional equation has been obtained in the class of functions $f : \mathbb{R} \to X$, where X is a real (or complex) Banach space. This result is further extended to generalized linear q-bonacci functional equation. In Section 4, the solution of the generalized linear q-bonacci functional equation is obtained in terms of generalized q-bonacci sequence and its Hyers-Ulam stability in the class of functions $f : \mathbb{R} \to X$, where X is a real (or complex) Banach space is discussed in Section 5.

6.2 Generalized Tribonacci functional equation

In this section, we define the generalized Tribonacci sequence and Tribonacci functional equation and prove that the general solution of Tribonacci functional equation is associated with the generalized Tribonacci sequence.

Definition 6.2.1. *The generalized Tribonacci sequence is defined by*

$$T_{n+2} = a\,T_{n+1} + b\,T_n + c\,T_{n-1}, \tag{6.1}$$

$$T_0 = 0,\ T_1 = 0 \text{ and } T_2 = 1,\ \forall\, n \in \mathbb{Z},$$

where $a,\ b$ and c are non-zero fixed real numbers and T_n is the n^{th} term. T_n is also given by the Binet type formula,

$$T_n = \frac{(\alpha-\beta)\gamma^n - (\alpha-\gamma)\beta^n + (\beta-\gamma)\alpha^n}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)},\ \forall n \in \mathbb{Z}, \tag{6.2}$$

where α, β and γ are distinct roots of the characteristics equation

$$\lambda^3 - a\,\lambda^2 - b\,\lambda - c = 0 \tag{6.3}$$

corresponding to (6.1).

Definition 6.2.2. *Let X be a real (or complex) Banach space. A function $f : \mathbb{R} \to X$ is called a generalized Tribonacci function if it satisfies the generalized Tribonacci functional equation*

$$f(x) = a\,f(x-1) + b\,f(x-2) + c\,f(x-3), \forall x \in \mathbb{R}, \tag{6.4}$$

where $a,\ b$ and c are non-zero fixed real numbers.

We need the following lemma.

Lemma 6.2.3. *If α, β and γ are distinct roots of the characteristics equation (6.3), then the generalized Tribonacci function $f : \mathbb{R} \to X$ satisfies*

$$f(x) = T_{n+2} f(x-n) + \big(b\, T_{n+1} + c\, T_n\big) f(x-n-1) + c\, T_{n+1} f(x-n-2), \tag{6.5}$$

$\forall x \in \mathbb{R}$ and $\forall n \in \mathbb{Z}$, where T_n is given by (6.2).

Proof. Since α, β and γ are distinct roots of (6.3), we get
$a = \alpha + \beta + \gamma$, $b = -(\alpha\gamma + \beta\gamma + \alpha\beta)$ and $c = \alpha\beta\gamma$. Substituting a, b and c in (6.4), we have
$f(x) = (\alpha+\beta+\gamma) f(x-1) - (\alpha\beta + \alpha\gamma + \beta\gamma) f(x-2) + (\alpha\beta\gamma) f(x-3)$, which implies

$$f(x) - (\alpha+\beta) f(x-1) + (\alpha\beta) f(x-2) = \gamma\Big(f(x-1) - (\alpha+\beta) f(x-2) + (\alpha\beta) f(x-3)\Big). \tag{6.6}$$

Replacing x by $x-1$ in (6.6), we get

$$f(x-1) - (\alpha+\beta) f(x-2) + (\alpha\beta) f(x-3) = \gamma\Big(f(x-2) - (\alpha+\beta) f(x-3) + (\alpha\beta) f(x-4)\Big)$$

and (6.6) yields

$$f(x) - (\alpha+\beta) f(x-1) + (\alpha\beta) f(x-2) = \gamma^2\Big(f(x-2) - (\alpha+\beta) f(x-3) + (\alpha\beta) f(x-4)\Big).$$

Hence, by induction on n, we get

$$f(x) - (\alpha+\beta) f(x-1) + (\alpha\beta) f(x-2) = \gamma^n\Big(f(x-n) - (\alpha+\beta) f(x-n-1) + (\alpha\beta) f(x-n-2)\Big). \tag{6.7}$$

Similarly, we have

$$f(x) - (\alpha+\gamma) f(x-1) + (\alpha\gamma) f(x-2) = \beta^n\Big(f(x-n) - (\alpha+\gamma) f(x-n-1) + (\alpha\gamma) f(x-n-2)\Big) \tag{6.8}$$

and

$$f(x)-(\beta+\gamma)f(x-1)+(\beta\gamma)f(x-2)=\alpha^n\Big(f(x-n)-(\beta+\gamma)f(x-n-1)+(\beta\gamma)f(x-n-2)\Big), \tag{6.9}$$

$\forall x\in\mathbb{R}$ and $\forall n\in\mathbb{N}\cup\{0\}$.

Now replacing x by $x+1$ in (6.6), we get

$$f(x+1)-(\alpha+\beta)f(x)+(\alpha\beta)f(x-1)=\gamma\Big(f(x)-(\alpha+\beta)f(x-1)+(\alpha\beta)f(x-2)\Big).$$

Therefore,

$$f(x)-(\alpha+\beta)f(x-1)+(\alpha\beta)f(x-2)=\gamma^{-1}\Big(f(x+1)-(\alpha+\beta)f(x)+(\alpha\beta)f(x-1)\Big),$$

since c is non-zero and $c=\alpha\beta\gamma$, $\gamma\neq 0$. Thus, by induction on n, we get

$$f(x)-(\alpha+\beta)f(x-1)+(\alpha\beta)f(x-2)=\gamma^{-n}\Big(f(x+n)-(\alpha+\beta)f(x+n-1)+(\alpha\beta)f(x+n-2)\Big).$$

Similarly, we have

$$f(x)-(\alpha+\gamma)f(x-1)+(\alpha\gamma)f(x-2)=\beta^{-n}\Big(f(x+n)-(\alpha+\gamma)f(x+n-1)+(\alpha\gamma)f(x+n-2)\Big)$$

and

$$f(x)-(\beta+\gamma)f(x-1)+(\beta\gamma)f(x-2)=\alpha^{-n}\Big(f(x+n)-(\beta+\gamma)f(x+n-1)+(\beta\gamma)f(x+n-2)\Big),$$

$\forall x\in\mathbb{R}$ and $\forall n\in\mathbb{N}\cup\{0\}$.
Therefore, equations (6.7),(6.8) and (6.9) are true $\forall x\in\mathbb{R}$ and $\forall n\in\mathbb{Z}$.

Now multiplying equations (6.7),(6.8) and (6.9) by $\gamma^2(\alpha-\beta), -\beta^2(\alpha-\gamma)$, $\alpha^2(\beta-\gamma)$ respectively and adding, we get

$$f(x) = \Big(\tfrac{\gamma^{n+2}(\alpha-\beta)-\beta^{n+2}(\alpha-\gamma)+\alpha^{n+2}(\beta-\gamma)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\Big) f(x-n)$$

$$+\Big(b\,\tfrac{\gamma^{n+1}(\alpha-\beta)-\beta^{n+1}(\alpha-\gamma)+\alpha^{n+1}(\beta-\gamma)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)} + c\tfrac{\gamma^{n}(\alpha-\beta)-\beta^{n}(\alpha-\gamma)+\alpha^{n}(\beta-\gamma)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\Big) f(x-n-1)$$

$$+c\Big(\tfrac{\gamma^{n+1}(\alpha-\beta)-\beta^{n+1}(\alpha-\gamma)+\alpha^{n+1}(\beta-\gamma)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\Big) f(x-n-2).$$

Using (6.2), this gives

$$f(x) = T_{n+2}\, f(x-n) + \big(bT_{n+1} + cT_n\big)\, f(x-n-1) + c\, T_{n+1}\, f(x-n-2),$$

$\forall x \in \mathbb{R}$ and $\forall n \in \mathbb{Z}$. □

We use Lemma 6.2.3 to prove the following result.

Theorem 6.2.4. *A function $f : \mathbb{R} \to X$ is a solution of functional equation* (6.4) *if and only if there exists a function $h : [-2,1) \to X$ such that*

$$f(x) = T_{\lfloor x\rfloor+2}\, h(x-\lfloor x\rfloor)+(bT_{\lfloor x\rfloor+1}+cT_{\lfloor x\rfloor})h(x-\lfloor x\rfloor-1)+cT_{\lfloor x\rfloor+1}h(x-\lfloor x\rfloor-2), \quad (6.10)$$

$\forall x \in \mathbb{R}$, *where T_n is given by* (6.2).

Proof. If $f(x)$ is a solution of (6.4), then by Lemma 6.2.3, $f(x)$ satisfies (6.5). Putting $n = \lfloor x\rfloor$ in (6.5), we get

$$f(x) = T_{\lfloor x\rfloor+2} f(x-\lfloor x\rfloor) + \big(b\, T_{\lfloor x\rfloor+1} + c\, T_{\lfloor x\rfloor}\big) f(x-\lfloor x\rfloor-1) + c\, T_{\lfloor x\rfloor+1} f(x-\lfloor x\rfloor-2).$$

Since $0 \le x-\lfloor x\rfloor < 1, -1 \le x-\lfloor x\rfloor-1 < 0$ and $-2 \le x-\lfloor x\rfloor-2 < -1$, we define a function $h : [-2,1) \to X$ by $h := f\,|_{[-2,1)}$, then $f(x)$ is of the form (6.10).

Now we assume that $f(x)$ is a function of the form (6.10) and prove that it is a solution of (6.4).

Consider, $f(x) - a\, f(x-1) - b\, f(x-2) - c\, f(x-3)$

$$= \Big(T_{\lfloor x\rfloor+2} - a\, T_{\lfloor x\rfloor+1} - b\, T_{\lfloor x\rfloor} - c\, T_{\lfloor x\rfloor-1}\Big) h(x - \lfloor x\rfloor)$$

$$+b\, \left(T_{\lfloor x\rfloor+1} - a\, T_{\lfloor x\rfloor} - b\, T_{\lfloor x\rfloor-1} - c\, T_{\lfloor x\rfloor-2}\right)\, h(x - \lfloor x\rfloor - 1)$$

$$+c\big(T_{\lfloor x\rfloor} - a\, T_{\lfloor x\rfloor-1} - b\, T_{\lfloor x\rfloor-2} - c\, T_{\lfloor x\rfloor-3}\big) h(x - \lfloor x\rfloor - 1)$$

$$+c\, \Big(T_{\lfloor x\rfloor+1} - a\, T_{\lfloor x\rfloor} - b\, T_{\lfloor x\rfloor-1} - c\, T_{\lfloor x\rfloor-2}\Big) h(x - \lfloor x\rfloor - 2)$$

$= 0$, $\forall x \in \mathbb{R}$ and arbitrary function $h : [-2, 1) \to X$, from (6.1).

Therefore, (6.10) is a solution of (6.4). Hence the theorem is proved. □

The above result is illustrated by the following.

Example 6.2.5. *Consider the functional equation*

$$f(x) = \frac{23}{4}\, f(x-1) - \frac{31}{8} f(x-2) + \frac{5}{8} f(x-3). \tag{6.11}$$

Define the function $h : [-2, 1) \to X$ *by*

$$h(x) = x^3 - \frac{23}{4}\, x^2 + \frac{31}{8} x - \frac{5}{8}. \tag{6.12}$$

Note that $\frac{1}{2}, \frac{1}{4},$ *and* 5 *are distinct roots of the characteristic equation* $\lambda^3 - \frac{23}{4}\, \lambda^2 + \frac{31}{8}\, \lambda - \frac{5}{8} = 0$ *corresponding to the recurrence relation*

$$T_{n+2} = \frac{23}{4}\, T_{n+1} - \frac{31}{8}\, T_n + \frac{5}{8}\, T_{n-1},$$

with $T_0 = 0, T_1 = 0,$ *and* $T_2 = 1, \forall n \in \mathbb{Z}.$

Therefore (6.10) implies

$$f(x) = T_{\lfloor x \rfloor+2}\, h(x-\lfloor x \rfloor)+(-\frac{31}{8}\, T_{\lfloor x \rfloor+1}+\frac{5}{8}\, T_{\lfloor x \rfloor})h(x-\lfloor x \rfloor-1)+\frac{5}{8}\, T_{\lfloor x \rfloor+1}h(x-\lfloor x \rfloor-2),$$

is a solution of (6.11), where $h(x)$ is given by (6.12).

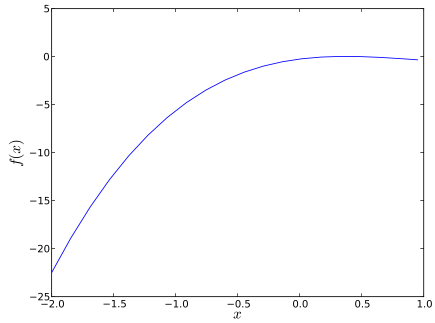

Figure 6-1: Graph showing solution of (6.11).

6.3 Hyers-Ulam stability of generalized Tribonacci functional equation

In this section, we prove the Hyers-Ulam stability of functional equation (6.4) by assuming that roots α, β and γ are distinct and $0 < |\alpha|, |\gamma| < 1, |\beta| > 1$.

We first prove the lemma required for this purpose.

Lemma 6.3.1. *If a function $f : \mathbb{R} \to X$ satisfies,*

$$\Big\| f(x) - a\, f(x-1) - b\, f(x-2) - c\, f(x-3) \Big\| \leq \epsilon, \forall x \in \mathbb{R}, \tag{6.13}$$

for some $\epsilon \geq 0$ and α, β and γ are distinct roots of (6.3) such that $0 < |\alpha|, |\gamma| < 1$, $|\beta| > 1$, then there exist Tribonacci functions $F_1, F_2, F_3 : \mathbb{R} \to X$ defined by

$F_1(x) = \lim_{n \longrightarrow \infty} \gamma^n \big[f(x-n) - (\alpha+\beta)f(x-n-1) + (\alpha\beta)f(x-n-2)\big]$,

$F_2(x) = \lim_{n \longrightarrow \infty} \alpha^n \big[f(x-n) - (\beta+\gamma)f(x-n-1) + (\beta\gamma)f(x-n-2)\big]$ and

$F_3(x) = \lim_{n\longrightarrow\infty} \beta^{-n}\big[f(x+n) - (\alpha+\gamma)f(x+n-1) + (\alpha\gamma)f(x+n-2)\big]$,

such that

$$\Big\|f(x) - (\alpha+\beta)f(x-1) + (\alpha\beta)f(x-2) - F_1(x)\Big\| \le \frac{\epsilon}{1-|\gamma|}, \tag{6.14}$$

$$\Big\|f(x) - (\beta+\gamma)f(x-1) + (\beta\gamma)f(x-2) - F_2(x)\Big\| \le \frac{\epsilon}{1-|\alpha|} \tag{6.15}$$

and

$$\Big\|F_3(x) - [f(x) - (\alpha+\gamma)f(x-1) + (\alpha\gamma)f(x-2)]\Big\| \le \frac{\epsilon}{|\beta|-1}, \tag{6.16}$$

$\forall x \in \mathbb{R}$.

Proof. Using (6.7) with $n = 1$ and (6.13), we have

$$\Big\|f(x) - (\alpha+\beta)f(x-1) + (\alpha\beta)f(x-2)$$

$$-\gamma\big[f(x-1) - (\alpha+\beta)f(x-2) + (\alpha\beta)f(x-3)\big]\Big\| \le \epsilon.$$

Replacing x by $x-k$, we have

$$\Big\|f(x-k) - (\alpha+\beta)f(x-k-1) + (\alpha\beta)f(x-k-2)$$

$$-\gamma\big[f(x-k-1) - (\alpha+\beta)f(x-k-2) + (\alpha\beta)f(x-k-3)\big]\Big\| \le \epsilon.$$

Multiplying both side by $|\gamma|^k$,

$$\Big\|\gamma^k\big[f(x-k) - (\alpha+\beta)f(x-k-1) + (\alpha\beta)f(x-k-2)\big]$$

$$-\gamma^{k+1}\big[f(x-k-1) - (\alpha+\beta)f(x-k-2) + (\alpha\beta)f(x-k-3)\big]\Big\|$$

$$\le |\gamma|^k\epsilon, \tag{6.17}$$

$\forall x \in \mathbb{R}, k \in \mathbb{Z}$.

Further, $\Big\| f(x) - (\alpha+\beta)f(x-1) + (\alpha\beta)f(x-2)$

$$-\gamma^n\big[f(x-n) - (\alpha+\beta)f(x-n-1) + (\alpha\beta)f(x-n-2)\big]\Big\|$$

$\leq \sum_{k=0}^{n-1} \Big\| \gamma^k\big[f(x-k) - (\alpha+\beta)f(x-k-1) + (\alpha\beta)f(x-k-2)\big]$

$$-\gamma^{k+1}\big[f(x-k-1) - (\alpha+\beta)f(x-k-2) + (\alpha\beta)f(x-k-3)\big]\Big\|$$

$\leq \sum_{k=0}^{n-1} \mid \gamma \mid^k \epsilon, \forall x \in \mathbb{R}$ and $\forall n \in \mathbb{N}$.

Therefore,

$\Big\| f(x) - (\alpha+\beta)f(x-1) + (\alpha\beta)f(x-2) - \gamma^n\big[f(x-n) - (\alpha+\beta)f(x-n-1)$

$$+(\alpha\beta)f(x-n-2)\big]\Big\| \leq \sum_{k=0}^{n-1} \mid \gamma \mid^k \epsilon, \tag{6.18}$$

$\forall x \in \mathbb{R}, \forall k \in \mathbb{Z}$ and $\forall n \in \mathbb{N}$.

Since $0 < |\gamma| < 1$, for any $x \in \mathbb{R}$, (6.17) implies that the sequence $\Big\{\gamma^n\big[f(x-n) - (\alpha+\beta)f(x-n-1) + (\alpha\beta)f(x-n-2)\big]\Big\}$ is a Cauchy sequence. Therefore, since X is Banach space, we can define a function $F_1 : \mathbb{R} \to X$ by

$F_1(x) = \lim_{n\longrightarrow\infty} \gamma^n\big[f(x-n) - (\alpha+\beta)f(x-n-1) + (\alpha\beta)f(x-n-2)\big]$

We now prove that $F_1(x)$ satisfies (6.4).

Consider, $a\, F_1(x-1) + b\, F_1(x-2) + c\, F_1(x-3)$

$= a\gamma^{-1} \lim_{n\longrightarrow\infty} \gamma^{n+1}\big[f(x-(n+1)) - (\alpha+\beta)f(x-(n+2)) + (\alpha\beta)f(x-(n+3))\big]$

$+b\, \gamma^{-2} \lim_{n\longrightarrow\infty} \gamma^{n+2}\big[f(x-(n+2)) - (\alpha+\beta)f(x-(n+3)) + (\alpha\beta)f(x-(n+4))\big]$

$+c\, \gamma^{-3} \lim_{n\longrightarrow\infty} \gamma^{n+3}\big[f(x-(n+3)) - (\alpha+\beta)f(x-(n+4)) + (\alpha\beta)f(x-(n+5))\big]$

$= F_1(x)\big(a\gamma^{-1} + b\gamma^{-2} + c\gamma^{-3}\big)$

$= F_1(x), \forall x \in \mathbb{R}$, since γ satisfies (6.3).

Hence $F_1(x)$ is a Tribonacci function.

Now taking $n \to \infty$, (6.18) implies

$$\Big\| f(x) - (\alpha+\beta)f(x-1) + (\alpha\beta)f(x-2) - F_1(x) \Big\| \leq \lim_{n\longrightarrow\infty} \sum_{k=0}^{n-1} \mid \gamma \mid^k \epsilon = \frac{\epsilon}{1-|\gamma|}.$$

Similarly since $0 < |\alpha| < 1$, using equation (6.9) with $n = 1$ and (6.13), we can prove that $\Big\{\alpha^n\big[f(x-n) - (\beta+\gamma)f(x-n-1) + (\beta\gamma)f(x-n-2)\big]\Big\}$ is a Cauchy sequence and since X is a Banach space, there exists a Tribonacci function $F_2 : \mathbb{R} \to X$ given by

$F_2(x) = \lim_{n\longrightarrow\infty} \alpha^n\big[f(x-n) - (\beta+\gamma)f(x-n-1) + (\beta\gamma)f(x-n-2)\big]$

such that

$$\Big\| f(x) - (\beta+\gamma)f(x-1) + (\beta\gamma)f(x-2) - F_2(x) \Big\| \leq \lim_{n\longrightarrow\infty} \sum_{k=0}^{n-1} \mid \alpha \mid^k \epsilon = \frac{\epsilon}{1-|\alpha|}.$$

Again from equation (6.8) with $n = 1$ and (6.13), it follows that

$$\Big\| f(x) - (\alpha+\gamma)f(x-1) + \alpha\gamma f(x-2) - \beta[f(x-1) - (\alpha+\gamma)f(x-2) + \alpha\gamma f(x-3)] \Big\| \leq \epsilon.$$

Since $|\beta| \neq 0$, replacing x by $x+k$ and multiplying both side by $|\beta|^{-k}$, we get

$$\Big\| \beta^{-k} f(x+k) - (\alpha+\gamma)f(x+k-1) + \alpha\gamma f(x+k-2)$$

$$-\beta^{-k+1}[f(x+k-1) - (\alpha+\gamma)f(x+k-2) + \alpha\gamma f(x+k-3)] \Big\| \leq |\beta|^{-k}\epsilon, \quad (6.19)$$

$\forall x \in \mathbb{R}$ and $\forall k \in \mathbb{Z}$.

Therefore, for $n \in \mathbb{N}$, $\Big\| \beta^{-n}\big[f(x+n) - (\alpha+\gamma)f(x+n-1) + \alpha\gamma f(x+n-2)\big]$

$$-\big[f(x) - (\alpha+\gamma)f(x-1) + \alpha\gamma f(x-2)\big] \Big\|$$

$\leq \sum_{k=1}^{n} \Big\| \beta^{-k}\big[f(x+k)-(\alpha+\gamma)f(x+k-1)+\alpha\gamma f(x+k-2)\big]-\beta^{-k+1}\big[f(x+k-1)$

$-(\alpha+\gamma)f(x+k-2)+\alpha\gamma f(x+k-3)\big]\Big\| \leq \sum_{k=1}^{n} |\beta|^{-k}\epsilon.$

Thus, for all $x \in \mathbb{R}$ and $\forall n \in \mathbb{N}$,

$$\Big\|\beta^{-n}\big[f(x+n)-(\alpha+\gamma)f(x+n-1)+\alpha\gamma f(x+n-2)\big]$$

$$-\big[f(x)-(\alpha+\gamma)f(x-1)+\alpha\gamma f(x-2)\big]\Big\| \leq \sum_{k=1}^{n} |\beta|^{-k}\epsilon. \tag{6.20}$$

Since $0 \leq |\beta| < 1$, Equation (6.19) implies that

$\Big\{\beta^{-n}\big[f(x+n)-(\alpha+\gamma)f(x+n-1)+(\alpha\gamma)f(x+n-2)\big]\Big\}$

is a Cauchy sequence for all $x \in \mathbb{R}$. Since X is a Banach space, we can define a function $F_3 : \mathbb{R} \to X$ by

$F_3(x) = \lim_{n\longrightarrow\infty} \beta^{-n}\big[f(x+n)-(\alpha+\gamma)f(x+n-1)+(\alpha\gamma)f(x+n-2)\big].$

Now Consider, $a\ F_3(x-1)+b\ F_3(x-2)+c\ F_3(x-3)$

$= a\ \beta^{-1}\lim_{n\longrightarrow\infty} \beta^{-n+1}\big[f(x+(n-1))-(\alpha+\gamma)f(x+(n-2))+(\alpha\gamma)f(x+(n-3))\big]$

$+b\ \beta^{-2}\lim_{n\longrightarrow\infty} \beta^{n+2}\big[f(x+(n-2))-(\alpha+\gamma)f(x+(n-3))+(\alpha\gamma)f(x+(n-4))\big]$

$+c\ \beta^{-3}\lim_{n\longrightarrow\infty} \beta^{n+3}\big[f(x+(n-3))-(\alpha+\gamma)f(x+(n-4))+(\alpha\gamma)f(x+(n-5))\big]$

$= F_3(x)(a\beta^{-1}+b\beta^{-2}+c\beta^{-3})$

$= F_3(x), \forall x \in \mathbb{R}.$

Therefore, $F_3(x)$ is a Tribonacci function. So as $n \to \infty$, we have

$$\Big\|F_3(x)-\Big(f(x)-(\alpha+\gamma)f(x-1)+(\alpha\gamma)f(x-2)\Big)\Big\| \leq \lim_{n\longrightarrow\infty}\sum_{k=1}^{n-1} |\beta|^{-k}\epsilon = \frac{\epsilon}{|\beta|-1},$$

$\forall x \in \mathbb{R}.$ □

Next we prove the following theorem.

Theorem 6.3.2. *If a function $f:\mathbb{R}\to X$ satisfies the inequality*

$$\Big\|f(x)-a\,f(x-1)-b\,f(x-2)-c\,f(x-3)\Big\|\leq\epsilon,\forall x\in\mathbb{R},$$

for some $\epsilon\geq 0$ and α,β and γ are distinct roots of (6.3), then there exists a unique solution function $F:\mathbb{R}\to X$ of the functional equation (6.4) such that

$$\Big\|f(x)-F(x)\Big\|$$
$$\leq\frac{\epsilon}{|\alpha-\beta|\,|\beta-\gamma|\,|\alpha-\gamma|}\Big(\frac{(|\alpha|-|\beta|)|\gamma|^2}{1-|\gamma|}+\frac{(|\alpha|-|\gamma|)|\beta|^2}{|\beta|-1}+\frac{(|\beta|-|\gamma|)|\alpha|^2}{1-|\alpha|}\Big),\forall x\in\mathbb{R}.$$

Proof. Since $(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)=(\alpha-\beta)\gamma^2-(\alpha-\gamma)\beta^2+(\beta-\gamma)\alpha^2$,

$(\alpha^2-\beta^2)\gamma^2-(\alpha^2-\gamma^2)\beta^2+(\beta^2-\gamma^2)\alpha^2=0$ and

$(\alpha\beta)(\alpha-\beta)\gamma^2-(\alpha\gamma)(\alpha-\gamma)\beta^2+(\beta-\gamma)(\beta\gamma)\alpha^2=0,$

$$\Big\|f(x)-\frac{(\alpha-\beta)\gamma^2F_1(x)-(\alpha-\gamma)\beta^2F_3(x)+(\beta-\gamma)\alpha^2F_2(x)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\Big\|$$

$$=\Big\|\frac{(\alpha-\beta)\gamma^2\Big(f(x)-F_1(x)\Big)-(\alpha-\gamma)\beta^2\Big(f(x)-F_3(x)\Big)+(\beta-\gamma)\alpha^2\Big(f(x)-F_2(x)\Big)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\Big\|$$

$$\leq\frac{1}{|\alpha-\beta|\,|\beta-\gamma|\,|\alpha-\gamma|}\Big\|((\alpha-\beta)\gamma^2\Big(f(x)-(\alpha+\beta)f(x-1)+(\alpha\beta)f(x-2)-F_1(x)\Big)\Big\|$$

$$+\frac{1}{|\alpha-\beta|\,|\beta-\gamma|\,|\alpha-\gamma|}\Big\|(\beta-\gamma)\alpha^2\Big(f(x)-(\beta+\gamma)f(x-1)+(\beta\gamma)f(x-2)-F_2(x)\Big)\Big\|$$

$$+\frac{1}{|\alpha-\beta|\,|\beta-\gamma|\,|\alpha-\gamma|}\Big\|(\alpha-\gamma)\beta^2\Big(F_3(x)-\big[f(x)-(\alpha+\gamma)f(x-1)+(\alpha\gamma)f(x-2)\big]\Big)\Big\|,$$

$$\leq\frac{\epsilon}{|\alpha-\beta|\,|\beta-\gamma|\,|\alpha-\gamma|}\ \Big(\frac{|\alpha-\beta|\,|\gamma|^2}{1-|\gamma|}+\frac{|\beta-\gamma|\,|\alpha|^2}{1-|\alpha|}+\frac{|\alpha-\gamma|\,|\beta|^2}{|\beta|-1}\Big),\text{ from (6.14),(6.15) and (6.16).}$$

We now define a function $F:\mathbb{R}\to X$ by

$$F(x)=\frac{(\alpha-\beta)\gamma^2F_1(x)-(\alpha-\gamma)\beta^2F_3(x)+(\beta-\gamma)\alpha^2F_2(x)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)},\ \forall x\in\mathbb{R}.$$

Consider, $a\, F(x-1) + b\, F(x-2) + c\, F(x-3)$

$$= a\ \frac{(\alpha-\beta)\gamma^2 F_1(x-1) - (\alpha-\gamma)\beta^2 F_3(x-1) + (\beta-\gamma)\alpha^2 F_2(x-1)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}$$

$$+ b \frac{(\alpha-\beta)\gamma^2 F_1(x-2) - (\alpha-\gamma)\beta^2 F_3(x-2) + (\beta-\gamma)\alpha^2 F_2(x-2)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}$$

$$+ c\ \frac{(\alpha-\beta)\gamma^2 F_1(x-3) - (\alpha-\gamma)\beta^2 F_3(x-3) + (\beta-\gamma)\alpha^2 F_2(x-3)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}$$

$$= \frac{(\alpha-\beta)\gamma^2 F_1(x) - (\alpha-\gamma)\beta^2 F_3(x) + (\beta-\gamma)\alpha^2 F_2(x)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}$$

$$= F(x), \forall x \in \mathbb{R}.$$

Therefore, $F(x)$ is a solution of (6.4). Now we prove the uniqueness of $F(x)$.

Assume that $F, \widehat{F} : \mathbb{R} \to X$ are solutions of (6.4) and that there exist positive constants C_1 and C_2 such that $\left\| f(x) - F(x) \right\| \le C_1$ and $\left\| f(x) - \widehat{F}(x) \right\| \le C_2, \forall x \in \mathbb{R}$.

Therefore, by Theorem 6.2.4, there exist $h, g : [-2, 1) \to X$ such that for all $x \in \mathbb{R}$,

$$F(x) = T_{\lfloor x \rfloor + 2}\ h(x - \lfloor x \rfloor) + (bT_{\lfloor x \rfloor + 1} + cT_{\lfloor x \rfloor})h(x - \lfloor x \rfloor - 1) + (cT_{\lfloor x \rfloor + 1})h(x - \lfloor x \rfloor - 2) \tag{6.21}$$

and

$$\widehat{F}(x) = T_{\lfloor x \rfloor + 2}\ g(x - \lfloor x \rfloor) + (bT_{\lfloor x \rfloor + 1} + cT_{\lfloor x \rfloor})g(x - \lfloor x \rfloor - 1) + (cT_{\lfloor x \rfloor + 1})g(x - \lfloor x \rfloor - 2). \tag{6.22}$$

Fix t with $0 \le t < 1$ and take $\lfloor x \rfloor = n$. It then follows from (6.21) and (6.22) that

$$\left\| T_{n+2}(h(t) - g(t)) + (bT_{n+1} + cT_n)(h(t-1) - g(t-1)) + (c\, T_{n+1})\ (h(t-2) - g(t-2)) \right\|$$

$$= \left\| F(n) - \widehat{F}(n) \right\|$$

$$\le \left\| F(n+t) - f(n+t) \right\| + \left\| f(n+t) - \widehat{F}(n+t) \right\|$$

$\le C_1 + C_2$, for each $n \in \mathbb{Z}$.

Therefore, $\Big\|T_{n+2}\big(h(t)-g(t)\big)+\big(bT_{n+1}+cT_n\big)\big(h(t-1)-g(t-1)\big)$

$+cT_{n+1}\big(h(t-2)-g(t-2)\big)\Big\| \leq C_1+C_2.$

This implies

$$\Big\|\Big(\frac{(\alpha-\beta)\gamma^{n+2}-(\alpha-\gamma)\beta^{n+2}+(\beta-\gamma)\alpha^{n+2}}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\Big)\big(h(t)-g(t)\big)$$

$$+\Big(b\frac{(\alpha-\beta)\gamma^{n+1}-(\alpha-\gamma)\beta^{n+1}+(\beta-\gamma)\alpha^{n+1}}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}$$

$$+c\ \frac{(\alpha-\beta)\gamma^{n}-(\alpha-\gamma)\beta^{n}+(\beta-\gamma)\alpha^{n}}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\Big)\big(h(t-1)-g(t-1)\big)$$

$$+\Big(c\frac{(\alpha-\beta)\gamma^{n+1}-(\alpha-\gamma)\beta^{n+1}+(\beta-\gamma)\alpha^{n+1}}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\Big)\big(h(t-2)-g(t-2)\big)\Big\|$$

$$\leq C_1+C_2. \tag{6.23}$$

Dividing both sides by $|\beta|^n$ and by letting $n\to\infty$, we obtain

$$\Big\|-(\alpha-\gamma)\beta^2(h(t)-g(t))-\Big(b\ (\alpha-\gamma)\beta+c\ (\alpha-\gamma)\Big)(h(t-1)-g(t-1))$$

$$-c\ (\alpha-\gamma)\beta(h(t-2)-g(t-2))\Big\|=0$$

Therefore,

$$\Big\|\beta^2(h(t)-g(t))+(b\ \beta+c)(h(t-1)-g(t-1))+c\ \beta(h(t-2)-g(t-2))\Big\|=0. \tag{6.24}$$

Analogously, if we divide both sides of by $|\alpha|^n$ and $|\gamma|^n$ and let $n\to-\infty$, then we get respectively,

$$\Big\|\alpha^2(h(t)-g(t))+(b\ \alpha+c)(h(t-1)-g(t-1))+c\ \alpha(h(t-2)-g(t-2))\Big\|=0 \tag{6.25}$$

and

$$\Big\|\gamma^2(h(t)-g(t))+(b\ \gamma+c)(h(t-1)-g(t-1))+c\ \gamma(h(t-2)-g(t-2))\Big\|=0. \tag{6.26}$$

Rewriting equations (6.24), (6.25) and (6.26) in matrix form, we get

$$\begin{bmatrix} \gamma^2 & b\,\gamma + c & c\,\gamma \\ \alpha^2 & b\,\alpha + c & c\,\alpha \\ \beta^2 & b\,\beta + c & c\,\beta \end{bmatrix} \begin{bmatrix} h(t) - g(t) \\ h(t-1) - g(t-1) \\ h(t-2) - g(t-2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{6.27}$$

Note that since $c \neq 0$ and α, β, γ are distinct roots,

$$\begin{vmatrix} \gamma^2 & b\,\gamma + c & c\,\gamma \\ \alpha^2 & b\,\alpha + c & c\,\alpha \\ \beta^2 & b\,\beta + c & c\,\beta \end{vmatrix} = c^2 \begin{vmatrix} \gamma^2 & 1 & \gamma \\ \alpha^2 & 1 & \alpha \\ \beta^2 & 1 & \beta \end{vmatrix}$$

$$= c^2(\alpha - \gamma)(\beta - \gamma)(\alpha - \beta) \neq 0.$$

Therefore, (6.27) has only trivial solution and we have, $h(t) = g(t)$, $h(t-1) = g(t-1), h(t-2) = g(t-2), \forall t \in [0, 1)$.

That is, $h(x) = g(x)$ for all $x \in [-2, 1)$.

Therefore, we conclude that $F(x) = \widehat{F}(x), \forall x \in \mathbb{R}$. □

We illustrate this result.

Example 6.3.3. *Consider the functional equation*

$$f(x) = \frac{23}{4}\, f(x-1) - \frac{31}{8} f(x-2) + \frac{5}{8} f(x-3) \tag{6.28}$$

and Tribonacci recurrence relation associated to it.

$$T_{n+2} = \frac{23}{4}\, T_{n+1} - \frac{31}{8}\, T_n + \frac{5}{8}\, T_{n-1}, \tag{6.29}$$

with $T_0 = 0, T_1 = 0, T_2 = 1, \forall n \in Z$.

Roots of the characteristic equation $\lambda^3 - \frac{23}{4}\lambda^2 + \frac{31}{8}\lambda - \frac{5}{8} = 0$ *corresponding to (6.29) are* $\frac{1}{2}, \frac{1}{4}$ *and* 5.

Let $\alpha = \frac{1}{2}, \beta = 5$ *and* $\gamma = \frac{1}{4}$. *Note that roots* α, β, γ *are distinct and* $|\alpha| < 1, |\gamma| < 1$ *and* $|\beta| > 1$.

Hence the solution is given by

$F(x) = \frac{(\alpha-\beta)\gamma^2 F_1(x) - (\alpha-\gamma)\beta^2 F_3(x) + (\beta-\gamma)\alpha^2 F_2(x)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}$, *where*

$$F_1(x) = \lim_{n\longrightarrow\infty}(\tfrac{1}{4})^n \Big[f(x-n) - (\alpha+\beta)f(x-n-1) + (\alpha\beta)f(x-n-2)\Big]$$

$$F_2(x) = \lim_{n\longrightarrow\infty}(\tfrac{1}{2})^n \Big[f(x-n) - (\beta+\gamma)f(x-n-1) + (\beta\gamma)f(x-n-2)\Big]$$

$$F_3(x) = \lim_{n\longrightarrow\infty}(5)^{-n} \Big[f(x+n) - (\alpha+\gamma)f(x+n-1) + (\alpha\gamma)f(x+n-2)\Big],$$

Therefore,

$$\begin{aligned} F(x) &= \frac{(\frac{-9}{2} \mathrm{x} \frac{1}{16})F_1(x) - (\frac{1}{4}\mathrm{x}25)F_3(x) + (\frac{19}{4}\mathrm{x}\frac{1}{4})F_2(x)}{(\frac{-9}{2})(\frac{19}{4})(\frac{1}{4})} \\ &= \frac{9F_1(x) + 200F_3(x) - 38F_2(x)}{171} \end{aligned}$$

and

$$\Big\|f(x) - F(x)\Big\| \leq \frac{114}{171}\,\epsilon.$$

In the next section, we extend this result to generalized q-bonacci functional equation, where $q \in \mathbb{N}$ and $q \geq 2$. Through out this section we denote $a_i, i = 1, 2, \cdots, q$, by any fixed real numbers.

6.4 Generalized q-bonacci functional equation

In this section, we show that the solution of generalized q-bonacci functional equation is associated with the generalized q-bonacci sequence and prove its Hyer-Ulam stability in the class of functions $f : \mathbb{R} \to X$, where X is a real or complex Banach space.

Definition 6.4.1. *Let $q \in \mathbb{N}$ and $q \geq 2$. The generalized q-bonacci sequence defined by*

$$Q_{n+q-1} = \sum_{i=1}^{q} a_i \, Q_{n+q-1-i}, \tag{6.30}$$

with $Q_i = 0, i = 0, 1, 2, \cdots q-2$ and $Q_{q-1} = 1, \forall n \in \mathbb{Z}$.

The nth term of (6.30) is given by the Binet type formula

$$Q_n = \frac{\sum_{k=1}^{q} (-1)^{k+1} \prod_{1 \leq i < j \leq q, i,j \neq k} (\alpha_i - \alpha_j) \alpha_k^n}{\prod_{1 \leq i < j \leq q} (\alpha_i - \alpha_j)}, \forall n \in \mathbb{Z}, \tag{6.31}$$

where $\alpha_i, i = 1, 2, 3 \cdots q$ are the distinct roots of the characteristic equation

$$\lambda^q - \sum_{i=1}^{q} a_i \lambda^{q-i} = 0, \tag{6.32}$$

corresponding to (6.30).

Definition 6.4.2. *Let X be a real (or complex) Banach space. A function $f : \mathbb{R} \to X$ defined by*

$$f(x) = \sum_{i=1}^{q} a_i f(x-i), \tag{6.33}$$

is called a generalized q-bonacci functional equation.

We have the following lemma.

Lemma 6.4.3. *If $\alpha_i, i = 1, 2, \cdots, q$ are distinct roots of the characteristic equation (6.32), then the q-bonacci function $f : \mathbb{R} \to X$ defined by (6.33) satisfies*

$$f(x) = Q_{n+q-1} f(x-n) + \sum_{p=1}^{q-1} \sum_{s=0}^{q-1-p} a_{s+p+1} \, Q_{n+q-2-s} \, f(x-n-p), \tag{6.34}$$

where Q_n is given by (6.30).

Proof. Since $\alpha_i, i = 1, 2, 3, \cdots q$ are the q distinct roots of (6.32), we have

$f(x) = \sum_{p=1}^{q} a_p\, f(x-p)$, where $a_p = (-1)^{p+1} \sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q} \prod_{k=1}^{p} \alpha_{i_k}$.

Therefore, $f(x) - \sum_{p=1}^{q} \Big((-1)^{p+1} \sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q} \prod_{k=1}^{p} \alpha_{i_k} \Big) f(x-p) = 0$.

This implies,

$f(x) - \sum_{p=1}^{q-1} \Big((-1)^{p+1} \sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q, i_k\ne i_m} \prod_{k=1}^{p} \alpha_{i_k} \Big) f(x-p)$

$$= \alpha_{i_m} \Big[f(x-1) - \sum_{p=1}^{q-1} \Big((-1)^{p+1} \sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q, i_k\ne i_m} \prod_{k=1}^{p} \alpha_{i_k} \Big) f(x-1-p) \Big], \quad (6.35)$$

$\forall x \in \mathbb{R}$ and $i_m, m = 1, 2, \cdots, q$.

Replacing x by $x-1$, in (6.35), we get

$f(x-1) - \sum_{p=1}^{q-1} \Big((-1)^{p+1} \sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q, i_k\ne i_m} \prod_{k=1}^{p} \alpha_{i_k} \Big) f(x-1-p)$

$$= \alpha_{i_m} \Big[f(x-2) - \sum_{p=1}^{q-1} \Big((-1)^{p+1} \sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q, i_k\ne i_m} \prod_{k=1}^{p} \alpha_{i_k} \Big) f(x-2-p) \Big]. \quad (6.36)$$

Therefore, (6.35) and (6.36) implies

$f(x) - \sum_{p=1}^{q-1} \Big((-1)^{p+1} \sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q, i_k\ne i_m} \prod_{k=1}^{p} \alpha_{i_k} \Big) f(x-p)$

$$= \alpha_{i_m}^2 \Big[f(x-2) - \sum_{p=1}^{q-1} \Big((-1)^{p+1} \sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q, i_k\ne i_m} \prod_{k=1}^{p} \alpha_{i_k} \Big) f(x-2-p) \Big],$$

$\forall x \in \mathbb{R}$ and $i_m, m = 1, 2, \cdots, q$.

Thus by induction on n, we have

$f(x) - \sum_{p=1}^{q-1} \Big((-1)^{p+1} \sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q, i_k\ne i_m} \prod_{k=1}^{p} \alpha_{i_k} \Big) f(x-p)$

$$= \alpha_{i_m}^n \Big[f(x-n) - \sum_{p=1}^{q-1} \Big((-1)^{p+1} \sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q, i_k\ne i_m} \prod_{k=1}^{p} \alpha_{i_k} \Big) f(x-n-p) \Big], \quad (6.37)$$

$\forall x \in \mathbb{R}$ and $i_m, m = 1, 2, \cdots, q$.

Now replacing x by $x+1$ in (6.35), we get

$f(x+1) - \sum_{p=1}^{q-1} \Big((-1)^{p+1} \sum_{1 \le i_1 < \cdots < i_k < \cdots < i_p \le q, i_k \ne i_m} \prod_{k=1}^{p} \alpha_{i_k} \Big) f(x+1-p)$
$= \alpha_{i_m} \Big[f(x) - \sum_{p=1}^{q-1} \Big((-1)^{p+1} \sum_{1 \le i_1 < \cdots < i_k < \cdots < i_p \le q, i_k \ne i_m} \prod_{k=1}^{p} \alpha_{i_k} \Big) f(x-p) \Big], \forall\, x \in \mathbb{R}$
and $i_m, m = 1, 2, \cdots, q$.

Also, since $a_q \ne 0, \prod_{i_m=1}^{q} \alpha_{i_m}$ is non-zero, hence $\alpha_{i_m} \ne 0$, and we have
$\Big[f(x) - \sum_{p=1}^{q-1} \Big((-1)^{p+1} \sum_{1 \le i_1 < \cdots < i_k < \cdots < i_p \le q, i_k \ne i_m} \prod_{k=1}^{p} \alpha_{i_k} \Big) f(x-p) \Big]$

$$= \alpha_{i_m}^{-1} \Big[f(x+1) - \sum_{p=1}^{q-1} \Big((-1)^{p+1} \sum_{1 \le i_1 < \cdots < i_k < \cdots < i_p \le q, i_k \ne i_m} \prod_{k=1}^{p} \alpha_{i_k} \Big) f(x+1-p) \Big],$$

Hence by induction on n, we get

$\Big[f(x) - \sum_{p=1}^{q-1} \Big((-1)^{p+1} \sum_{1 \le i_1 < \cdots < i_k < \cdots < i_p \le q, i_k \ne i_m} \prod_{k=1}^{p} \alpha_{i_k} \Big) f(x-p) \Big]$

$$= \alpha_{i_m}^{-n} \Big[f(x+n) - \sum_{p=1}^{q-1} \Big((-1)^{p+1} \sum_{1 \le i_1 < \cdots < i_k < \cdots < i_p \le q, i_k \ne i_m} \prod_{k=1}^{p} \alpha_{i_k} \Big) f(x+n-p) \Big],$$

$\forall x \in \mathbb{R}$ and $i_m, m = 1, 2, \cdots, q$.

Thus, (6.37) is true for all $n \in \mathbb{Z}$. Note that corresponding to q values of m, there are q equations.

For simplicity, we write i_m as m, i_j as j and i_k as k. Multiplying the m^{th} equation by $(-1)^{m+1}\alpha_m^{q-1} \prod_{1 \le j < k \le q} (\alpha_j - \alpha_k), j, k \ne m$, for each $m = 1, 2, \cdots, q$ and then adding these equations, we get

$f(x) = Q_{n+q-1} f(x-n) + \sum_{p=1}^{q-1} \Big(\sum_{s=0}^{q-1-p} a_{s+p+1} Q_{n+q-2-s} \Big) f(x-n-p),$
$\forall x \in \mathbb{R}$ and $n \in \mathbb{Z}$. □

Theorem 6.4.4. *A function $f : \mathbb{R} \to X$ is a solution of the functional equation (6.33) if and only if there exists a function $h : [-(q-1), 1) \to X$ such that*

$$f(x) = Q_{\lfloor x \rfloor + q - 1} h(x - \lfloor x \rfloor) + \sum_{p=1}^{q-1} \Big(\sum_{s=0}^{q-1-p} a_{s+p+1}\, Q_{\lfloor x \rfloor + q - 2 - s} \Big) h(x - \lfloor x \rfloor - p), \quad (6.38)$$

$\forall x \in \mathbb{R}$, where Q_n is the n^{th} term of (6.31).

Proof. We use Lemma 6.4.3 to prove the theorem. Let $f(x)$ be a solution of (6.33), then replacing n by $\lfloor x \rfloor$ in (6.34), we get

$$f(x) = Q_{\lfloor x \rfloor + q-1} f(x - \lfloor x \rfloor) + \sum_{p=1}^{q-1} \Big(\sum_{s=0}^{q-1-s} a_{s+p+1} Q_{\lfloor x \rfloor + q-2-s} \Big) f(x - \lfloor x \rfloor - p), \ \forall x \in \mathbb{R}.$$

Also, since $0 \le x - \lfloor x \rfloor < 1$, we have $-1 \le x - \lfloor x \rfloor - 1 < 0, \cdots,$
$-(q-1) \le x - \lfloor x \rfloor - (q-1) < -(q-2)$.
So, if we define a function $h : \big[-(q-1), 1 \big) \to X$, by $h := f_{|[-(q-1),1)}$, then we see that $f(x)$ is a function of the form (6.38).
Now, we assume that $f(x)$ is a function of the form (6.38) where $h : \big[-(q-1), 1\big) \to X$, is an arbitrary function.
Then, it follows from (6.38) that

$$f(x) = Q_{\lfloor x \rfloor + q-1} h(x - \lfloor x \rfloor) + \sum_{p=1}^{q-1} \Big(\sum_{s=0}^{q-1-p} a_{s+p+1} \, Q_{\lfloor x \rfloor + q-2-s} \Big) \, h(x - \lfloor x \rfloor - p)$$

Therefore, for all $i = 1, 2, \cdots, q$,

$$f(x-i) = Q_{\lfloor x \rfloor + q-1-i} \, h(x - \lfloor x \rfloor) + \sum_{p=1}^{q-1} \Big(\sum_{s=0}^{q-1-p} a_{s+p+1} \, Q_{\lfloor x \rfloor + q-2-s-i} \Big) \, h(x - \lfloor x \rfloor - p).$$

Thus, we have

$$\begin{aligned}
& f(x) - \textstyle\sum_{i=1}^{q} a_i f(x-i) \\
& = \big(Q_{\lfloor x \rfloor + q-1} - \textstyle\sum_{i=1}^{q} a_i Q_{\lfloor x \rfloor + q-1-i} \big) \, h(x - \lfloor x \rfloor) \\
& \ + \textstyle\sum_{p=1}^{q-1} \sum_{s=0}^{q-1-p} \ a_{s+p+1} \Big(Q_{\lfloor x \rfloor + q-2-s} - \textstyle\sum_{i=1}^{q} a_i \, Q_{\lfloor x \rfloor + q-2-s-i} \Big) h(x - \lfloor x \rfloor - p) \\
& = 0, \text{ from (6.30).}
\end{aligned}$$

Hence $f(x)$ is a solution of (6.33). □

6.5 Hyers-Ulam stability of generalized q-bonacci functional equation

In this section, we assume that $\alpha_i, i = 1, 2, \cdots, q$ are the distinct roots of characteristics equation (6.32), $0 < |\alpha_{2l-1}| < 1,\ l = 1, 2, \cdots, \lfloor \frac{q+1}{2} \rfloor, |\alpha_{2l}| > 1,\ l = 1, 2, \cdots, \lfloor \frac{q}{2} \rfloor$. We now prove the Hyers-Ulam stability of the functional equation (6.33).

Lemma 6.5.1. *If $\alpha_i, i = 1, 2, \cdots, q$ are the distinct roots of (6.32) such that $0 \leq |\alpha_{2l-1}| \leq 1,\ l = 1, 2, \cdots, \lfloor \frac{q+1}{2} \rfloor,\ |\alpha_{2l}| \geq 1, l = 1, 2, \cdots, \lfloor \frac{q}{2} \rfloor$ and a function $f : \mathbb{R} \to X$ defined by (6.33) satisfies the inequality,*

$$\left\| f(x) - \sum_{i=1}^{q} a_i f(x-i) \right\| \leq \epsilon, \tag{6.39}$$

for some $\epsilon \geq 0$ and $\forall x \in \mathbb{R}$, then there exists q-bonacci functions $F_i(x) : \mathbb{R} \to X$, $i = 1, 2, \cdots, q$ of the functional equation (6.33) defined by

$$F_{2l-1}(x) = \lim_{n\to\infty} \alpha_{2l-1}^n \Big[f(x-n) - \sum_{p=1}^{q-1} (-1)^{p+1} \Big(\sum_{1\leq i_1 < \cdots < i_k < \cdots < i_p \leq q, i_k \neq i_{2l-1}} \prod_{k=1}^{p} \alpha_{i_k} \Big) f(x-n-p) \Big],$$

$l = 1, 2, \cdots, \lfloor \frac{q+1}{2} \rfloor$ and

$$F_{2l}(x) = \lim_{n\to\infty} \alpha_{2l}^n \Big[f(x+n) - \sum_{p=1}^{q-1} (-1)^{p} \Big(\sum_{1\leq i_1 < \cdots < i_k < \cdots < i_p \leq q, i_k \neq i_{2l}} \prod_{k=1}^{p} \alpha_{i_k} \Big) f(x-n-p) \Big],$$

$l = 1, 2, \cdots, \lfloor \frac{q}{2} \rfloor$ such that

$$\Big\| f(x) - \sum_{p=1}^{q-1} (-1)^{p+1} \Big(\sum_{1\leq i_1 < \cdots < i_k < \cdots < i_p \leq q, i_k \neq 2l-1} \prod_{k=1}^{p} \alpha_{i_k} \Big) f(x-p) - F_{2l-1}(x) \Big\| \leq \frac{\epsilon}{1- \mid \alpha_{2l-1} \mid} \tag{6.40}$$

and

$$\Big\|F_{2l}(x) - \Big[f(x) - \sum_{p=1}^{q-1}(-1)^{p+1}\Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q, i_k\ne 2l}\prod_{k=1}^{p}\alpha_{i_k}\Big)f(x-p)\Big]\Big\|$$

$$\le \frac{\epsilon}{\mid \alpha_{2l} \mid -1}, \tag{6.41}$$

$\forall x \in \mathbb{R}$.

Proof. Since $\alpha_i, i = 1, 2, \cdots, q$ are the distinct roots of characteristics equation (6.32), we have $a_p = (-1)^{p+1}\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q}\prod_{k=1}^{p}\alpha_{i_k}$.

Therefore, equation (6.39) implies

$$\|f(x) - \sum_{p=1}^{q-1}(-1)^{p+1}\Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q, i_k\ne 2l-1}\prod_{k=1}^{p}\alpha_{i_k}\Big)f(x-p)$$

$$-\alpha_{2l-1}\Big[f(x-1)-\sum_{p=1}^{q-1}(-1)^{p+1}(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q, i_k\ne 2l-1}\prod_{k=1}^{p}\alpha_{i_k})f(x-1-p)\Big]\|$$

$$\le \epsilon, \forall x \in \mathbb{R} \text{ and } l = 1, 2, \ldots, \lfloor \tfrac{q+1}{2} \rfloor.$$

If we replace x by $x - m_1$ in the last inequality, we get

$$\|f(x-m_1) - \sum_{p=1}^{q-1}(-1)^{p+1}\Big(\sum_{1\le i_1<i_2<\cdots<i_p\le q}\prod_{k=1}^{p}\alpha_{i_k}\Big)f(x-m_1-p)$$

$$-\alpha_{2l-1}\Big[f(x-m_1-1)-\sum_{p=1}^{q-1}(-1)^{p+1}\Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q}\prod_{k=1}^{p}\alpha_{i_k}\Big)f(x-m_1-1-p)\Big]\| \le \epsilon, \tag{6.42}$$

$\forall x \in \mathbb{R}, i_k \ne 2l-1$ and $l = 1, 2, \ldots, \lfloor \frac{q+1}{2} \rfloor$.

Multiplying both sides by $\mid \alpha_{2l-1} \mid^{m_1}$, we get

$$\|\alpha_{2l-1}^{m_1}\Big(f(x-m_1)-\sum_{p=1}^{q-1}(-1)^{p+1}\Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q}\prod_{k=1}^{p}\alpha_{i_k}\Big)f(x-m_1-p))\Big)$$

$$-\alpha_{2l-1}^{m_1+1}\Big[f(x-m_1-1)$$

$$-\sum_{p=1}^{q-1}(-1)^{p+1}\Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q}\prod_{k=1}^{p}\alpha_{i_k}\Big)f(x-m_1-1-p)\Big]\|$$

$$\le|\alpha_{2l-1}|^{m_1}\epsilon, \forall x\in\mathbb{R}, i_k\neq 2l-1 \text{ and } m_1\in\mathbb{Z}. \tag{6.43}$$

Thus, $\|f(x)-\sum_{p=1}^{q-1}(-1)^{p+1}\Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q}\prod_{k=1}^{p}\alpha_{i_k}\Big)f(x-p)$

$$-\alpha_{2l-1}^{n}\Big[f(x-n)-\sum_{p=1}^{q-1}(-1)^{p+1}\Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q}\prod_{k=1}^{p}\alpha_{i_k}\Big)f(x-n-p)\Big]\|$$

$$\le\sum_{m_1=0}^{n-1}\|\alpha_{2l-1}^{m_1}\big(f(x-m_1)$$

$$-\sum_{p=1}^{q-1}(-1)^{p+1}\Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q}\prod_{k=1}^{p}\alpha_{i_k}\Big)f(x-m_1-p)$$

$$-\alpha_{2l-1}^{m_1+1}\Big[f(x-m_1-1)$$

$$-\sum_{p=1}^{q-1}(-1)^{p+1}\Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q}\prod_{k=1}^{p}\alpha_{i_k}\Big)f(x-m_1-1-p)\Big]\|$$

$$\le\sum_{m_1=0}^{n-1}|\alpha_{2l-1}|^{m_1}\epsilon, \forall x\in\mathbb{R}, i_k\neq 2l-1 \text{ and } n\in\mathbb{N}. \tag{6.44}$$

Since $0<|\alpha_{2l-1}|<1$, for any $x\in\mathbb{R}$, equation (6.43) implies that

$$\Big\{\alpha_{2l-1}^{n}\Big[f(x-n)-\sum_{p=1}^{q-1}(-1)^{p+1}\Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q, i_k\neq 2l-1}\prod_{k=1}^{p}\alpha_{i_k}\Big)f(x-n-p)\Big]\Big\}$$

is a Cauchy sequence. Therefore, since X is a Banach space, for each $l=1,2,\cdots,\lfloor\frac{q+1}{2}\rfloor$, we can define a function $F_{2l-1}:\mathbb{R}\to X$ by

$$F_{2l-1}=\lim_{n\to\infty}\alpha_{2l-1}^{n}\Big[f(x-n)-\sum_{p=1}^{q}(-1)^{p+1}$$
$$\Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q, i_k\neq 2l-1}\prod_{k=1}^{p}\alpha_{i_k}\Big)f(x-n-p)\Big].$$

We prove that $F_{2l-1}(x)$ satisfies (6.33).

Consider, $\sum_{j=1}^{q} a_j F_{2l-1}(x-j)$

$$= \sum_{j=1}^{q} a_j \lim_{n\to\infty} \alpha_{2l-1}^{n} \big[f(x-(j+n))$$

$$- \sum_{p=1}^{q-1} (-1)^{p+1} \Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q} \prod_{k=1, i_k\neq 2l-1}^{p} \alpha_{i_k} \Big) f(x-(j+n)-p) \big]$$

$$= \sum_{j=1}^{q} a_j \alpha_{2l-1}^{-j} \lim_{n\to\infty} \alpha_{2l-1}^{j+n} \big[f(x-(j+n))$$

$$- \sum_{p=1}^{q-1} (-1)^{p+1} \Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q} \prod_{k=1, i_k\neq 2l-1}^{p} \alpha_{i_k} \Big) f(x-(j+n)-p) \big]$$

$$= \sum_{j=1}^{q} a_j \alpha_{2l-1}^{-j} F_{2l-1}(x)$$

$$= F_{2l-1}(x) \sum_{j=1}^{q} a_j \alpha_{2l-1}^{-j}$$

$$= F_{2l-1}(x) \left(\frac{a_1 \ \alpha_{2l-1}^{q-1} + a_2 \ \alpha_{2l-1}^{q-2} + \cdots + a_q}{\alpha_{2l-1}^{q}} \right)$$

$$= F_{2l-1}(x) \left(\frac{\alpha_{2l-1}^{q}}{\alpha_{2l-1}^{q}} \right), \text{ since } \alpha_{2l-1} \text{ satisfies (6.32).}$$

$$= F_{2l-1}(x).$$

Therefore, $F_{2l-1}(x)$ is a q-bonacci function, for each $l = 1, 2, \cdots, \lfloor \frac{q+1}{2} \rfloor$.

If $n \to \infty$, then (6.44) implies that

$$\Big\| f(x) - \sum_{p=1}^{q-1} (-1)^{p+1} \Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q} \prod_{k=1}^{p} \alpha_{i_k} \Big) f(x-p) - F_{2l-1} \Big\| \le \frac{\epsilon}{1 - \mid \alpha_{2l-1} \mid},$$

$\forall x \in \mathbb{R}, , i_k \neq 2l-1$ and $l = 1, 2, \cdots, \lfloor \frac{q+1}{2} \rfloor$.

On the other hand, it also follows from (6.39) and the fact that

$a_p = (-1)^{p+1} \sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q} \prod_{k=1}^{p} \alpha_{i_k}$,

$$\Big\|f(x)-\sum_{p=1}^{q-1}(-1)^{p+1}\Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q}\prod_{k=1}^{p}\ \alpha_{i_k}\Big)f(x-p)$$

$$-\alpha_{2l}\Big[f(x-1)-\sum_{p=1}^{q-1}(-1)^{p+1}\Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q}\prod_{k=1}^{p}\alpha_{i_k}\Big)f(x-p-1)\Big]\Big\|\le\epsilon,$$

$\forall x\in\mathbb{R}, i_k\ne 2l$ and $l=1,2,\cdots,\lfloor\frac{q}{2}\rfloor$.

If we replace x by $x+m_1$ in the last inequality, we get

$$\Big\|f(x+m_1)-\sum_{p=1}^{q-1}(-1)^{p+1}\Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q}\prod_{k=1}^{p}\ \alpha_{i_k}\Big)f(x+m_1-p)$$

$$-\alpha_{2l}\Big[f(x+m_1-1)$$

$$-\sum_{p=1}^{q-1}(-1)^{p+1}\Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q}\prod_{k=1}^{p}\alpha_{i_k}\Big)f(x+m_1-p-1)\Big]\Big\|\le\epsilon,$$

$\forall x\in\mathbb{R}, i_k\ne 2l$ and $l=1,2,\cdots,\lfloor\frac{q}{2}\rfloor$.

Since $a_q\ne 0, |\alpha_{2l}|\ne 0$. Therefore dividing both sides by $\mid\alpha_{2l}\mid^{m_1}$, we get

$$\Big\|\alpha_{2l}^{-m}\big(f(x+m_1)-\sum_{p=1}^{q-1}(-1)^{p+1}\Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q}\prod_{k=1}^{p}\ \alpha_{i_k}\Big)f(x+m_1-p)$$

$$-\alpha_{2l}^{-m_1+1}\Big[f(x+m_1-1)$$

$$-\sum_{p=1}^{q-1}(-1)^{p+1}\Big(\sum_{1\le i_1<i_2<\cdots<i_p\le q}\prod_{k=1}^{p}\ \alpha_{i_k}\Big)f(x+m_1-p-1)\Big]\Big\|\le\mid\alpha_{2l}\mid^{-m_1}\epsilon, \quad (6.45)$$

$\forall x\in\mathbb{R}$, $m_1\in\mathbb{Z}$ and $i_k\ne 2l$.

Therefore, $\Big\|\alpha_{2l}^{-n}\Big[f(x+n)$

$$-\sum_{p=1}^{q-1}(-1)^{p+1}\Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q}\prod_{k=1}^{p}\ \alpha_{i_k}\Big)f(x+n-p)\Big]$$

$$-\Big[f(x)-\sum_{p=1}^{q-1}(-1)^{p+1}\Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q}\prod_{k=1}^{p}\ \alpha_{i_k}\Big)f(x-p)\Big]\Big\|$$

$\leq \sum_{m_1=1}^{n} \Big\| \alpha_{2l}^{-m_1} \Big[f(x+m_1)$

$- \sum_{p=1}^{q-1} (-1)^{p+1} \Big(\sum_{1\leq i_1<\cdots<i_k<\cdots<i_p\leq q} \prod_{k=1}^{p} \ \alpha_{i_k} \Big) f(x+m_1-p) \Big]$

$-\alpha_{2l}^{-m_1+1} \Big[f(x+m_1-1)$

$- \sum_{p=1}^{q-1} (-1)^{p+1} \Big(\sum_{1\leq i_1<\cdots<i_k<\cdots<i_p\leq q} \prod_{k=1}^{p} \ \alpha_{i_k} \Big) f(x+m_1-p-1) \Big] \Big\|$

$\leq \sum_{m_1=1}^{n} \mid \alpha_{2l} \mid^{-m_1} \epsilon$, for $x \in \mathbb{R}, i_k \neq 2l$ and $n \in \mathbb{N}$.

Hence, we have

$$\Big\| \alpha_{2l}^{-n} \Big[f(x+n) - \sum_{p=1}^{q-1} (-1)^{p+1} \Big(\sum_{1\leq i_1<\cdots<i_k<\cdots<i_p\leq q} \prod_{k=1}^{p} \ \alpha_{i_k} \Big) f(x+n-p) \Big] - \Big[f(x) - \sum_{p=1}^{q-1} (-1)^{p+1} \Big(\sum_{1\leq i_1<\cdots<i_k<\cdots<i_p\leq q} \prod_{k=1}^{p} \ \alpha_{i_k} \Big) f(x-p) \Big] \Big\| \leq \sum_{m_1=1}^{n} \mid \alpha_{2l} \mid^{-m_1} \epsilon, \tag{6.46}$$

$\forall x \in \mathbb{R}, , i_k \neq 2l, l = 1, 2, \cdots, \lfloor \frac{q}{2} \rfloor$ and $n \in \mathbb{N}$.

Since $|\alpha_{2l}| > 1$, for any $x \in \mathbb{R}$ (6.45) implies that
$\Big\{ \alpha_{2l}^{-n} \Big[f(x+n) - \sum_{p=1}^{q-1} (-1)^{p+1} \sum_{1\leq i_1<\cdots<i_k<\cdots<i_p\leq q} \prod_{k=1, i_k\neq 2l}^{p} \ \alpha_{i_k} f(x+n-p) \Big] \Big\}$
is a Cauchy sequence.

Thus, since X is a Banach space, for each $l = 1, 2, \cdots, \lfloor \frac{q}{2} \rfloor$, we can define a function $F_{2l} : \mathbb{R} \to X$ by

$F_{2l}(x) = \lim_{n\to\infty} \alpha_{2l}^{-n} \Big[f(x+n)$
$- \sum_{p=1}^{q-1} (-1)^{p+1} \Big(\sum_{1\leq i_1<\cdots<i_k<\cdots<i_p\leq q} \prod_{k=1, i_k\neq 2l}^{p} \ \alpha_{i_k} \Big) f(x+n-p) \Big].$

We show that $F_{2l}(x)$ satisfies (6.33).

Consider, $\sum_{j=1}^{q} a_j F_{2l}(x-j)$

$= \sum_{j=1}^{q} a_j \lim_{n\to\infty} \alpha_{2l}^{-n}\big[f(x-(j-n))$

$- \sum_{p=1}^{q-1}(-1)^{p+1}\Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q, i_k\neq 2l}\prod_{k=1}^{p} \ \alpha_{i_k}\Big) f(x-(j-n)-p)\big]$

$= \sum_{j=1}^{q} a_j \alpha_{2l}^{-j} \lim_{n\to\infty} \alpha_{2l}^{j-n}\big[f(x-(j-n))$

$- \sum_{p=1}^{q-1}(-1)^{p+1}\Big(\sum_{1\le i_1<\cdots<i_k<\cdots<i_p\le q, i_k\neq 2l}\prod_{k=1}^{p} \ \alpha_{i_k}\Big) f(x-(j-n)-p)\big]$

$= \sum_{j=1}^{q} a_j \alpha_{2l}^{-j} F_{2l}(x)$

$= \Big(\frac{a_1\alpha_{2l}^{q-1}+a_2\alpha_{2l}^{q-2}+\cdots+a_q}{\alpha_{2l}^{q}}\Big) F_{2l}(x)$

$= F_{2l}(x)$, since α_{2l} satisfies (6.33).

If $n\to\infty$, then (6.46) implies that

$$\Big\|F_{2l} - \big[f(x) + \sum_{p=1}^{q-1}(-1)^p\Big(\sum_{1\le i_1\cdots<i_k<\cdots<i_p\le q}\prod_{k=1}^{p} \alpha_{i_k}\Big) f(x-p)\big]\Big\| \le \frac{\epsilon}{\mid \alpha_{2l}\mid -1}$$

for all $x\in\mathbb{R}, i_k\neq 2l$ and each $l = 1, 2, \cdots, \lfloor \frac{q}{2} \rfloor$. □

Theorem 6.5.2. *If a function $f:\mathbb{R}\to X$ defined by (6.33) satisfies the inequality,*

$$\Big\|f(x) - \sum_{i=1}^{q} a_i f(x-i)\Big\| \le \epsilon, \tag{6.47}$$

for some $\epsilon \ge 0$ and $\forall x\in\mathbb{R}$, then there exists a unique solution function $F:\mathbb{R}\to X$ of the functional equation (6.33) such that

$$\Big\|f(x) - F(x)\Big\| \le \frac{\epsilon}{\prod_{1\le j<k\le q}|\alpha_j-\alpha_k|}\sum_{p=1}^{q-1}\prod_{1\le j<k\le q}\Big(\frac{|\alpha_j|-|\alpha_k|}{1-|\alpha_m|}\Big)|\alpha_m|^n, \tag{6.48}$$

$\forall x\in\mathbb{R}, j,k\neq m$, for each $m = 1, 2, \cdots, q$.

Proof. From (6.40) and(6.41), we have

$$\left\| f(x) - \frac{\sum_{m=1}^{q}(-1)^{m+1}\prod_{1\leq j<k\leq q,\ j,k\neq\ m}(\alpha_j-\alpha_k)\alpha_m^{q-1}Q_m(x)}{\prod_{1\leq j,k\leq q}(\alpha_j-\alpha_k)} \right\|$$

$$= \left\| \frac{\prod_{1\leq j,k\leq q}(\alpha_j-\alpha_k)f(x)-\sum_{m=1}^{q}(-1)^{m+1}\prod_{1\leq j<k\leq q,\ j,k\neq\ m}(\alpha_j-\alpha_k)\alpha_m^{q-1}Q_m(x)}{\prod_{1\leq j,k\leq q}(\alpha_j-\alpha_k)} \right\|$$

$$= \left\| \frac{\sum_{m=1}^{q}(-1)^{m+1}\prod_{1\leq j<k\leq q,\ j,k\neq\ m}(\alpha_j-\alpha_k)\alpha_m^{q-1}\Big(f(x)-Q_m(x)\Big)}{\prod_{1\leq j,k\leq q}(\alpha_j-\alpha_k)} \right\|,$$

since $\prod_{1\leq j,k\leq q}(\alpha_j-\alpha_k) = \sum_{m=1}^{q}(-1)^{m+1}\prod_{1\leq j<k\leq q,\ j,k\neq\ m}(\alpha_j-\alpha_k)\alpha_m^{q-1}$.

Therefore, $\left\| f(x) - \frac{\sum_{m=1}^{q}(-1)^{m+1}\prod_{1\leq j<k\leq q,\ j,k\neq\ m}(\alpha_j-\alpha_k)\alpha_m^{q-1}Q_m(x)}{\prod_{1\leq j,k\leq q}(\alpha_j-\alpha_k)} \right\|,$

$$\leq \frac{1}{\prod_{1\leq j,k\leq q}|\alpha_j-\alpha_k|}\sum_{m=1}^{q}(-1)^{m+1}\prod_{1\leq j<k\leq q,\ j,k\neq\ m}|(\alpha_j-\alpha_k)\alpha_m^{q-1}$$

$$\left(\left\| f(x) - \sum_{p=1}^{q-1}(-1)^{p+1}\sum_{1\leq i_1<\cdots<i_k<\cdots<i_p\leq q}\prod_{k=1,i_k\neq m}^{p}\alpha_{i_k}f(x-p) - Q_m(x)\right\|\right)|$$

$$\leq \frac{\epsilon}{\prod_{1\leq j<k\leq q}|\alpha_j-\alpha_k|}\sum_{m=1}^{q}(-1)^{m+1}\prod_{1\leq j<k\leq q,\ j,k\neq\ m}\frac{|\alpha_j-\alpha_k|\ |\alpha_m|^{q-1}}{1-\ |\alpha_m|}, \forall x\in\mathbb{R}.$$

We now define a function $F:\mathbb{R}\to X$ by

$$F(x) = \frac{\sum_{m=1}^{q}(-1)^{m+1}\prod_{1\leq j<k\leq q,\ j,k\neq\ m}(\alpha_j-\alpha_k)\alpha_m^{q-1}Q_m(x)}{\prod_{1\leq j,k\leq q}(\alpha_j-\alpha_k)}, \ \forall x\in\mathbb{R}.$$

Consider, $\sum_{i=1}^{q} a_i\ f(x-i)$

$$= \sum_{i=1}^{q} a_i\ \frac{\sum_{m=1}^{q}(-1)^{m+1}\prod_{1\leq j<k\leq q,\ j,k\neq\ m}(\alpha_j-\alpha_k)\alpha_m^{q-1}F_m(x-i)}{\prod_{1\leq j,k\leq q}(\alpha_j-\alpha_k)}$$

$= F(x)$ for each $x\in\mathbb{R}$.

This implies $F(x)$ is a solution of (6.33).

Now, we prove the uniqueness of $F(x)$.

Assume that $F, \widehat{F} : \mathbb{R} \to X$ are solutions of (6.32) and that there exist positive constants C_1 and C_2 with $\left\|f(x) - F(x)\right\| \leq C_1$ and $\left\|f(x) - \widehat{F}(x)\right\| \leq C_2, \forall x \in \mathbb{R}$. According to Theorem 6.4.4, there exist functions $h, g : [-(q-1), 1) \to X$ such that

$$F(x) = Q_{\lfloor x \rfloor + q - 1} h(x - \lfloor x \rfloor) + \sum_{p=1}^{q-1} \sum_{s=0}^{q-1-p} a_{s+p+1}\, Q_{\lfloor x \rfloor + q - 2 - s}\, h(x - \lfloor x \rfloor - p) \qquad (6.49)$$

and

$$\widehat{F}(x) = Q_{\lfloor x \rfloor + q - 1} g(x - \lfloor x \rfloor) + \sum_{p=1}^{q-1} \sum_{s=0}^{q-1-p} a_{s+p+1}\, Q_{\lfloor x \rfloor + q - 2 - s}\, g(x - \lfloor x \rfloor - p) \qquad (6.50)$$

for any $x \in \mathbb{R}$.

Fix $t \in \mathbb{R}$ with $0 \leq t < 1$ and take $\lfloor x \rfloor = n$. It then follows from (6.49) and (6.50) that

$$\left\|Q_{n+q-1}\Big(h(t) - g(t)\Big) + \sum_{p=1}^{q-1} \sum_{s=0}^{q-1-p} a_{s+p+1}\, Q_{n+q-2-s}\, \Big(h(t-p) - g(t-p)\Big)\right\|$$

$$= \left\| \frac{\sum_{m=1}^{q} (-1)^{m+1} \prod_{1 \leq j < k \leq q, j,k \neq m} (\alpha_j - \alpha_k) \alpha_m^{n+q-1}}{\prod_{1 \leq j < k \leq q} (\alpha_j - \alpha_k)} \Big(h(t) - g(t)\Big) \right.$$

$$+ \sum_{p=1}^{q-1}$$

$$\left. \sum_{s=0}^{q-1-p} a_{s+p+1} \frac{\sum_{m=1}^{q} (-1)^{m+1} \prod_{1 \leq j < k \leq q, j,k \neq m} (\alpha_j - \alpha_k) \alpha_m^{n+q-2-s}}{\prod_{1 \leq j < k \leq q} (\alpha_j - \alpha_k)} \Big(h(t-p) - g(t-p)\Big) \right\|$$

$$= \left\|F(n) - \widehat{F}(n)\right\|$$

$$\leq \left\|F(n+t) - f(n+t)\right\| + \left\|f(n+t) - \widehat{F}(n+t)\right\|$$

$$\leq C_1 + C_2, \text{ (from assumption.)}$$

Dividing both sides by $|\alpha_{2l}|^n, l = 1, 2, 3, \cdots, \lfloor \frac{q}{2} \rfloor$ and letting $n \to \infty$, we obtain

$$\Big\| \frac{\sum_{m=1}^{q}(-1)^{m+1}\prod_{1\le j<k\le q, j,k\ne m}(\alpha_j-\alpha_k)\alpha_{2l}^{q-1}}{\prod_{1\le j<k\le q}(\alpha_j-\alpha_k)}\Big(h(t)-g(t)\Big)$$

$$+\sum_{p=1}^{q-1}\sum_{i=0}^{q-1-p} a_{i+p+1}\frac{\sum_{m=1}^{q}(-1)^{m+1}\prod_{1\le j<k\le q, j,k\ne m}(\alpha_j-\alpha_k)\alpha_{2l}^{q-2-i}}{\prod_{1\le j<k\le q}(\alpha_j-\alpha_k)}\Big(h(t-p)-g(t-p)\Big)\Big\|$$

$= 0$, since $0 < |\alpha_{2l-1}| < 1$ and $|\alpha_{2l}| > 1$, hence $0 < |\frac{\alpha_{2l-1}}{\alpha_{2l}}| < 1$.

Also, dividing both sides by $|\alpha_{2l-1}|^n, l = 1, 2, 3, \cdots, \lfloor \frac{q+1}{2} \rfloor$ and letting $n \to -\infty$, we obtain

$$\Big\| \frac{\sum_{m=1}^{q}(-1)^{m+1}\prod_{1\le j<k\le q, j,k\ne m}(\alpha_j-\alpha_k)\alpha_{2l-1}^{q-1}}{\prod_{1\le j<k\le q}(\alpha_j-\alpha_k)}\Big(h(t)-g(t)\Big)$$

$$+\sum_{p=1}^{q-1}\sum_{i=0}^{q-1-p} a_{i+p+1}\frac{\sum_{m=1}^{q}(-1)^{m+1}\prod_{1\le j<k\le q, j,k\ne m}(\alpha_j-\alpha_k)\alpha_{2l-1}^{q-2-i}}{\prod_{1\le j<k\le q}(\alpha_j-\alpha_k)}\Big(h(t-p)-g(t-p)\Big)\Big\|$$

$= 0$, since $0 < |\alpha_{2l-1}| < 1$ and $|\alpha_{2l}| > 1$, hence $|\frac{\alpha_{2l}}{\alpha_{2l-1}}| > 1$.

Thus, corresponding to q values of m, we have q equations. In matrix form this equations are represented by

$$\begin{bmatrix} \alpha_1^{q-1} & \cdots & \sum_{i=1+s}^{q} a_i\alpha_1^{q-i}\cdots & a_q\alpha_1^{q-2} \\ \alpha_2^{q-1} & \cdots & \sum_{i=1+s}^{q} a_i\alpha_2^{q-i}\cdots & a_q\alpha_2^{q-2} \\ \alpha_3^{q-1} & \cdots & \sum_{i=1+s}^{q} a_i\alpha_3^{q-i}\cdots & a_q\alpha_3^{q-2} \\ \cdots & & & \\ \alpha_q^{q-1} & \cdots & \sum_{i=1+s}^{q} a_i\alpha_q^{q-i}\cdots & a_q\alpha_q^{q-2} \end{bmatrix} \begin{bmatrix} h(t)-g(t) \\ h(t-1)-g(t-1) \\ h(t-2)-g(t-2) \\ \cdots \\ h(t-q-1)-g(t-q-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix},$$

where $s = 1, 2, \cdots, q$.

Note that

$$\begin{vmatrix} \alpha_1^{q-1} & \cdots & \sum_{i=1+s}^{q} a_i\alpha_1^{q-i} \cdots & a_q\alpha_1^{q-2} \\ \alpha_2^{q-1} & \cdots & \sum_{i=1+s}^{q} a_i\alpha_2^{q-i} \cdots & a_q\alpha_2^{q-2} \\ \alpha_3^{q-1} & \cdots & \sum_{i=1+s}^{q} a_i\alpha_3^{q-i} \cdots & a_q\alpha_3^{q-2} \\ \cdots & & & \\ \alpha_q^{q-1} & \cdots & \sum_{i=1+s}^{q} a_i\alpha_q^{q-i} \cdots & a_q\alpha_q^{q-2} \end{vmatrix},$$

where $s = 1, 2, \cdots, q$ and $\alpha_i, i = 1, 2, \cdots q$ are distinct roots.

$$= a_q^{q-1} \begin{vmatrix} \alpha_1^{q-1} & 1 & \alpha_1 & \cdots & \alpha_1^{q-2} \\ \alpha_2^{q-1} & 1 & \alpha_2 & \cdots & \alpha_2^{q-2} \\ \alpha_3^{q-1} & 1 & \alpha_3 & \cdots & \alpha_3^{q-2} \\ \cdots & & & & \\ \alpha_q^{q-1} & 1 & \alpha_q & \cdots & \alpha_q^{q-2} \end{vmatrix}$$

$$= a_q^{q-1} \prod_{1\le j<k\le q}(\alpha_j - \alpha_k) \neq 0,$$

Therefore, $h(t) = g(t), h(t-1) = g(t-1), \cdots, h(t-(q-1)) = g(t-(q-1))$, $\forall t \in [-(q-1), 1)$. Thus, $h(t) = g(t)$, $\forall t \in [-(q-1), 1)$.

Hence we conclude that $F(x) = \widehat{F}(x)$, for all $x \in \mathbb{R}$.

Thus, the theorem is proved. □

Summary

In our thesis, we look at the generalized Fibonacci sequence, $F_{n+1} = aF_n + bF_{n-1}$ with $F_0 = 0$ and $F_1 = 1$ in a new way. The coefficients a and b are considered as the terms of the binomial expansion of $(a + b)^1$. It's related generalized Lucas sequence is defined by $L_{n+1} = aL_n + bL_{n-1}$ with $L_0 = 2$ and $L_1 = a$. We call theses sequences by B-Fibonacci sequence and B-Lucas sequence respectively. In Chapter 3, we extend the B-Fibonacci sequence to B-Tribonacci sequence defined by $({}^tB)_{n+2} = a^2({}^tB)_{n+1} + 2ab({}^tB)_n + b^2({}^tB)_{n-1}, \forall n \in \mathbb{Z}$ with $({}^tB)_0 = 0, ({}^tB)_1 = 0$ and $({}^tB)_2 = 1$. Various identities of the B-Fibonacci sequence are extended to B-Tribonacci sequence. Some of these include Honsberger type identity, General Trilinear identity, d'Ocagne type identity and Catalan type identity. We also discuss incomplete B-Tribonacci and B-Tri Lucas sequences and their identities. In Chapter 4, we extend these B-Tri sequences to the q^{th} order sequences. These are called B-q bonaacci sequences. For example, the n^{th} term of this sequence is calculated by adding the preceding q terms having the coefficients as the terms of the binomial expansion of $(a + b)^{q-1}$. The identities of B-Tribonacci sequence and other sequences discussed in Chapter 3 are extended to B-q bonacci sequences.

Another way of looking at the generalized Fibonacci sequence is its associated Fibonacci polynomials. In Chapter 5, we study the generalized bivariate B-Tribonacci, B-Tri Lucas, B-q bonacci and B-q Lucas polynomials and some identities related to these polynomials. Besides these identities, we have obtained identities involving partial derivatives of these polynomials. We have also included Convolution property of these polynomials. In the same Chapter, we also study the incomplete generalized bivariate B-Tribonacci, B-Tri Lucas, B-q bonacci and B-q Lucas polynomials, and their various identities. In Chapter 6, we show that the solution of generalized linear Tribonacci functional equation is associated with generalized Tribonacci sequence

and also obtain its stability in the class of functions $f : X \to \mathbb{R}$ where X is a real (or complex) Banach space. This result is further extended to the generalized linear q-bonacci functional equation.

Problems for further studies:

There are many interesting identities of generalized Fibonacci and Lucas sequences and polynomials which can be extended to the sequences and polynomials that we have introduced in our thesis. In addition to this one can look for the applications of these sequences and polynomials in the area of Electrical Network Theory, Combinatorics, forecasting the stock market and many other areas in which the famous Fibonacci sequence is used.

Appendix

Content of the Appendix is published in (E2).

Appendix

The classical Fibonacci sequence is a unique and fascinating string of numbers with interesting properties which are obtained by using various Mathematical techniques. In (P1), B-Tribonacci sequence and its identities are discussed. We give here some Python programming codes which are used for verifying the identities obtained.

Python code for generating the terms of

$({}^tB)_{n+2} = a^2({}^tB)_{n+1} + 2ab\ ({}^tB)_n + b^2({}^tB)_{n-1}$.

Python Code 1.

```
from sympy import *
from pylab import *
a=Symbol('a')
b=Symbol('b')
def B(n):
        if n == 0:
            return 0
        elif n == 1:
            return 0
        elif n==2:
            return 1
        elif n<=0:
            return expand(1/b**2)*((B(n+3)-a**2*B(n+2)-2*a*b*B(n+1)))
        else :
            return expand(a**2*B(n-1)+2*a*b*B(n-2)+b**2*B(n-3))
  for i in range (0,15):
        print 'B(',i,')=',B(i)
```

Python code for generating the graph of

$({}^tB)_{n+2} = (\frac{1}{2})^2({}^tB)_{n+1} + 2(\frac{1}{2})(\frac{1}{2})\ ({}^tB)_n + (\frac{1}{2})^2({}^tB)_{n-1}.$

Python Code 2.

```
from sympy import *
from pylab import *
a=Symbol('a')
b=Symbol('b')
a=1/2.0
b=1/2.0
def B(n):
        if n == 0:
            return 0
        elif n == 1:
            return 0
        elif n==2:
            return 1
        elif n<=0:
            return expand(1/b**2)*((B(n+3)-a**2*B(n+2)-2*a*b*B(n+1)))
        else :
            return expand(a**2*B(n-1)+2*a*b*B(n-2)+b**2*B(n-3))
    for i in range (0,15):
        scatter(i,float64(B(i)))
        grid(True)
        xlabel(r'$n$', fontsize=18)
        ylabel(r'$B_n$', fontsize=18)
    show()
```

Python code for generating the terms of B-q bonacci sequence for $q \geq 2$ and $n \geq 0$.

Python Code 3.

```
from numpy import *
from math import *
from pylab import *
from sympy import *
a=Symbol('a')
b=Symbol('b')
q=input('Enter q')
def B(n):
        if n <= q-2 :
          return 0
        elif n == q-1:
          return 1
        elif n > q-1 :
          sum=0
          for r in range (q):
            sum=sum+ expand(binomial(q-1,r)*a**(q-1-r)*b**r*B(n-1-r))
          return sum
        else :
          print 'Exit'
```

List of Publications

(A) Papers Published

(P1) S. Arolkar and Y.S. Valaulikar, *On an extension of Fibonacci Sequence,* Bulletin of the Marathwada Mathematical Society, 17(1)(2016), 1-8.

(P2) S. Arolkar and Y.S. Valaulikar, *Incomplete* $h(x)$*-*B*-Tribonacci Polynomials*, Turkish Journal of Analysis and Number Theory, 4(6)(2016), 155-158.

(P3) S. Arolkar and Y.S. Valaulikar, $h(x)$*-*B*-Tribonacci and* $h(x)$*-*B*-Tri Lucas Polynomials*, Kyungpook Mathematical Journal, 56(4)(2016), 1125-1133.

(P4) S. Arolkar and Y.S. Valaulikar, *On a* B-q *bonacci Sequence*, International Journal of Advances in Mathematics, 1(2017), 1-8, 2017.

(P5) S. Arolkar and Y.S. Valaulikar, *Hyers-Ulam Stability of Generalized Tribonacci Functional Equation*, Turkish Journal of Analysis and Number Theory, 5(3)(2017), 80-85.

(B) Papers Communicated

(C1) S. Arolkar and Y.S. Valaulikar, *Identities Involving Partial Derivatives of Bivariate* B*-q bonacci and* B*-q Lucas Polynomials.*

(C2) S. Arolkar and Y.S. Valaulikar, *Hyers-Ulam Stability of Generalized q-bonacci Functional Equation.*

(C) Papers presented and published in conference proceedings

(E1) Attended and presented a paper entitled '*Generalized Bivariate B-Tribonacci and B-Tri-Lucas Polynomials*'at National conference on '*Innovative Research in Chemical, Physical, Mathematical Sciences, Applied Statistics and Environmental Dynamics (CPMSED-2015)*' held on 28th Nov, 2015 at held at Jawaharlal Nehru University, New Delhi. This paper is published in the Conference Proceedings with ISBN: 978-93-85822-07-0, (pg 10-13).

(E2) Attended and presented the paper entitled *'Python Programming Language Codes For Some Properties Of Fibonacci Sequence Extensions'* at the 2 day UGC sponsored Conference on '*Applied Mathematics : Numerical Analysis, Algebra and Computational Mathematics*' held on 30th and 31st january,2015, at Dept. of Mathematics K.L.E. society's Gudleppa Hallikeri College, Havery Karnataka. This paper is published in the Conference Proceedings with ISBN: 978-81-930850-2-8, (pg 85-90).

(D) Papers presented at the National conferences

(1) Attended and presented the paper entitled '*On an Extension of Fibonacci sequence*' at the National conference on '*Recent Advances in Mathematics*' held during 23-25 the Dec, 2014 at Dept. of Mathematics Deogiri College, Aurangabad. This is a published paper (P1).

(2) Attended and presented a paper entitled '*Hyers-Ulam Stability of Generalized Tribonacci Functional Equation* $f(x) = af(x-1)+bf(x-2)+cf(x-3)$ at the National conference on '*Emerging Trends in Mathematics and Mathematical Sciences*' held during 17-19 th Dec, 2015 at Calcutta Mathematical Society, Kolkata. This is a published paper (P5).

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