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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75th Birth Anniversary Celebrations)

ON A PSEUDO FIBONACCI SEQUENCE

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Abstract

This article deals with a pseudo Fibonacci sequence and its properties. Some well known identities are obtained in terms of the identities of generalised Fibonacci sequence. Modular properties different from those of Fibonacci sequence are reported.
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1 Introduction

The Fibonacci sequence $\{F_n\}$ is defined by the recurrence relation

(1.1) $F_{n+2} = F_{n+1} + F_n, n \geq 0,$

with $F_0 = 0$ and $F_1 = 1$ [3, 10]. This sequence has been extended in many ways [See [2, 8] and references therein]. In [1], generalised Fibonacci sequence called B- Fibonacci sequence, defined by

(1.2) ${}^fB_{n+2} = a\,{}^fB_{n+1} + b\,{}^fB_n,$

with ${}^fB_0 = 0$, ${}^fB_1 = 1$, is discussed. In [4], Phadte - Pethe has introduced pseudo Fibonacci sequence $\{g_n\}$, defined by the non-homogeneous recurrence relation,

(1.3) $g_{n+2} = g_{n+1} + g_n + At^n, \ n \geq 0$

with $g_0 = 0$ and $g_1 = 1$. Here $A \neq 0$ is a constant and t is a real number such that $t \neq 0, \ \lambda_1, \ \lambda_2$ where λ_1, λ_2 are roots of the equation $\lambda^2 - \lambda - 1 = 0$. g_n is called the n^{th} pseudo Fibonacci number. First few pseudo Fibonacci numbers are:

$g_0 = 0, \ \ g_1 = 1, \ \ g_2 = 1 + A, \ \ g_3 = 2 + A + At \ \text{ and } \ g_4 = 3 + 2A + At + At^2.$

Observe that each pseudo Fibonacci number is such that its first term is a Fibonacci number and the remaining terms form a polynomial in t whose coefficients are A times Fibonacci numbers. More literature on pseudo Fibonacci sequence and its extensions can be seen in [5, 6, 7].

In this paper we shall consider pseudo Fibonacci sequence $\{G_n\}$ defined by the non-homogeneous recurrence relation

(1.4) $G_{n+2} = aG_{n+1} + b\,G_n + A(-1)^n, \ n \geq 0,$

with $G_0 = \omega, \ G_1 = 1 - \omega$ and study its properties. We assume that $a, b \in \mathbb{Z}$ and A be a constant such that $\omega = \dfrac{A}{1 + a - b} \in \mathbb{Z}$. Following is immediate.

***Theorem* 1.1** *The n^{th} term G_n of* (1.4) *is given by*

(1.5) $G_n = {}^f B_n + \omega(-1)^n,$

where ${}^fB_n$ is defined by (1.2).

We list below some identities for the sequence G_n. These idenetities can be obtained by using corresponding identities for ${}^fB_n$.[1]

***Theorem* 1.2** *G_n satisfies following identities*

i) $G_{n+1}G_{n-1} - G_n^2 = (-1)^n\, b^{n-1} - \omega(-1)^n(G_{n-1} + 2G_n + G_{n+1})$

ii) $\sum_{r=0}^{n} G_r = \dfrac{bG_n + G_{n+1} - \omega(-1)^n(b-1) - 1}{a+b-1} + \omega\epsilon_n$
where

$$\epsilon_n = \begin{cases} 0, if\ n\ is\ odd, \\ 1, if\ n\ is\ even. \end{cases}$$

iii) $G_{n+1}G_m - G_nG_{m+1}$
$= (-b)^n G_{m-n} + \omega\{(G_{n+1} + G_n)(-1)^m + (G_{m+1} - G_m)(-1)^n - ((-b)^n + 2(-1)^{m+n})\}.$

iv) $G_n^2 - G_{n+r}G_{n-r} = (-b)^{n-r}G_r^2 + \omega[2G_n - (-1)^{-r}G_{n+r} - (-1)^r G_{n-r}](-1)^n +$
$(-b)^{n-r}\omega^2 - 2\omega(-b)^{n-r}(-1)^r G_r.$

2 Modulo Properties

In this section we study some modulo properties of the sequence $\{G_n\}$. We have the following result.

***Theorem* 2.1** *Let $\pi(m)$ be the period of G_n modulo m. Let $e \geq 1$ be given. Then*

i) For odd prime p, $\pi(p^e) = p^{e-e'}\pi(p)$, where $1 \leq e' \leq e$ is maximal so that $\pi(p^{e'}) = \pi(p)$.

ii) For $p = 2$ and $e \geq 2$, $\pi(2^e) = 2^{e-e'}\pi(4)$, where $2 \leq e' \leq e$ is maximal so that $\pi(2^{e'}) = \pi(4)$.

Proof. Let $\pi'(m)$ be the period of $\{^f B_n\}$ modulo m. $\pi'(m)$ is always even.
Now $G_0 = ^f B_0 + \omega = \omega$ and $G_1 = ^f B_1 - \omega = 1 - \omega$.
Hence $G_{\pi'(m)} = ^f B_{\pi'(m)} + \omega(-1)^{\pi'(m)} \equiv \omega(\mod m)$ and
$G_{\pi'(m)+1} = ^f B_{\pi'(m)+1} + \omega(-1)^{\pi'(m)+1} \equiv 1 - \omega(\mod m)$ so that the period
$\pi'(m)$ of $^f B_n$ and $\pi(m)$ of G_n are same. Now the result follows from Theorem 2 of [9].

***Remark* 2.1** *Note that if three consecutive values of G_n modulo m are same, then the remaining values repeat. This is different from Fibonacci sequence where two consecutive values of F_n modulo m are same then the remaining values repeat.*

We now consider a particular case of $\{G_n\}$ with $a = 1$, $b = 2$, and $A = 1$. For this, ***Table* 2.1** below gives $G_n(mod\ n)$.

Using ***Table* 2.1**, we can state the following results.

***Proposition* 2.1**

$$G(n) = \begin{cases} 0 \mod 3\ \ if\ n \equiv 0,5,6 \mod 8, \\ 1 \mod 3\ \ if\ n \equiv 1 \mod 8, \\ 2 \mod 3\ \ if\ n \equiv 2,3,4,7 \mod 8. \end{cases}$$

***Proposition* 2.2**

$$G(n) = n \mod 4.$$

***Proposition* 2.3**

$$G(n) = \begin{cases} 0 \mod 5\ \ if\ n \equiv 0,8,17,21,22 \mod 24, \\ 1 \mod 5\ \ if\ n \equiv 1,4,6,7,13,14,19 \mod 24, \\ 2 \mod 5\ \ if\ n \equiv 2,5,9,16,18 \mod 24, \\ 3 \mod 5\ \ if\ n \equiv 10,11,12,15,20 \mod 24, \\ 4 \mod 5\ \ if\ n \equiv 3,23 \mod 24. \end{cases}$$

***Proposition* 2.4**

$$G(n) = \begin{cases} 0 \mod 6\ \ if\ n \equiv 0,6 \mod 8, \\ 1 \mod 6\ \ if\ n \equiv 1 \mod 8, \\ 2 \mod 6\ \ if\ n \equiv 2,4 \mod 8, \\ 3 \mod 6\ \ if\ n \equiv 5 \mod 8, \\ 5 \mod 6\ \ if\ n \equiv 3,7 \mod 8. \end{cases}$$

***Table* 2.1:** G_n(mod n) for $a = 1, b = 2$ and $A = 1$

n	G_n	mod 3	mod 4	mod 5	mod 6	mod 7	mod 8	mod 9	mod10	mod15
0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2
3	-1	2	3	4	5	6	7	8	9	14
4	-4	2	0	1	2	3	4	5	6	11
5	-3	0	1	2	3	4	5	6	74	12
6	6	0	2	1	0	6	6	6	6	6
7	11	2	3	1	5	4	3	2	1	11
8	0	0	0	0	0	0	0	0	0	0
9	-23	1	1	2	1	5	1	4	7	7
10	-22	2	2	3	2	6	2	5	8	8
11	23	2	3	3	5	2	7	5	3	8
12	68	2	0	3	2	5	4	5	8	8
13	21	0	1	1	3	2	5	3	1	6
14	-114	0	2	1	0	5	6	3	6	6
15	-157	2	3	3	5	4	3	5	3	8
16	72	0	0	2	0	2	0	0	2	12
17	385	1	1	0	1	0	1	7	5	10
18	242	2	2	2	2	4	2	8	2	2
19	-529	2	3	1	5	3	7	2	1	11
20	-1012	2	0	3	2	3	4	5	8	8
21	45	0	1	0	3	3	5	5	0	0
22	2070	0	2	0	0	5	6	0	0	0
23	1979	2	3	4	5	5	3	8	9	14
24	-2160	0	0	0	0	3	0	0	0	0
25	-6119	1	1	1	1	6	1	1	1	1

***Proposition* 2.5**

$$G(n) = \begin{cases} n \mod 8 & if\ n \not\equiv 3,7 \mod 8, \\ 3 \mod 8 & if\ n \equiv 7 \mod 8, \\ 7 \mod 8 & if\ n \equiv 3 \mod 8. \end{cases}$$

***Proposition* 2.6**

$$G(n) = \begin{cases} 0 \mod 9 & if\ n \equiv 0, 8, 16 \mod 24, \\ 1 \mod 9 & if\ n \equiv 1 \mod 24, \\ 2 \mod 9 & if\ n \equiv 2, 7, 19 \mod 24, \\ 3 \mod 9 & if\ n \equiv 13, 14 \mod 24, \\ 4 \mod 9 & if\ n \equiv 9 \mod 24, \\ 5 \mod 9 & if\ n \equiv 4, 10, 11, 12, 15, 20 \mod 24, \\ 6 \mod 9 & if\ n \equiv 5, 6 \mod 24, \\ 7 \mod 9 & if\ n \equiv 17 \mod 24, \\ 8 \mod 9 & if\ n \equiv 3, 18, 23 \mod 24. \end{cases}$$

***Proposition* 2.7**

$$G(n)=\begin{cases} 0 \mod 10 \;\; if\ n\equiv 0,8,22 \mod 24,\\ 1 \mod 10 \;\; if\ n\equiv 1,7,13,19 \mod 24,\\ 2 \mod 10 \;\; if\ n\equiv 2,16,18 \mod 24,\\ 3 \mod 10 \;\; if\ n\equiv 11,15 \mod 24,\\ 5 \mod 10 \;\; if\ n\equiv 17,21 \mod 24,\\ 6 \mod 10 \;\; if\ n\equiv 4,6,14 \mod 24,\\ 7 \mod 10 \;\; if\ n\equiv 5,9 \mod 24,\\ 8 \mod 10 \;\; if\ n\equiv 10,12,20 \mod 24,\\ 9 \mod 10 \;\; if\ n\equiv 3,23 \mod 24. \end{cases}$$

***Proposition* 2.8**

$$G(n)=\begin{cases} 0 \mod 15 \;\; if\ n\equiv 0,8,21,22 \mod 24,\\ 1 \mod 15 \;\; if\ n\equiv 1 \mod 24,\\ 2 \mod 15 \;\; if\ n\equiv 2,18 \mod 24,\\ 6 \mod 15 \;\; if\ n\equiv 6,13,14 \mod 24,\\ 7 \mod 15 \;\; if\ n\equiv 9 \mod 24,\\ 8 \mod 15 \;\; if\ n\equiv 10,11,12,15,20 \mod 24,\\ 10 \mod 15 \;\; if\ n\equiv 17 \mod 24,\\ 11 \mod 15 \;\; if\ n\equiv 4,7,19 \mod 24,\\ 12 \mod 15 \;\; if\ n\equiv 5,16 \mod 24,\\ 14 \mod 15 \;\; if\ n\equiv 3,23 \mod 24. \end{cases}$$

3 Conclusion

A new pseudo Fibonacci sequence is studied whose modular properties are different from those of Fibonacci sequence.

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