SYMMETRY ANALYSIS OF SOME FUNCTIONAL DIFFERENTIAL EQUATIONS

A THESIS SUBMITTED FOR THE AWARD OF THE DEGREE OF

DOCTOR OF PHILOSOPHY
IN
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BY

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Declaration

This thesis entitled "Symmetry Analysis Of Some Functional Differential Equations" submitted by me to the Goa University for the award of the degree of *Doctor of Philosophy in Mathematics* is a research work carried out by me under the supervision and guidance of *Dr. Y. S. Valaulikar*, Associate Professor, Department of Mathematics, Goa University.

I hereby declare that this thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has not been submitted earlier either in part or full or in any other form to any university or institute, here or elsewhere, for the award of any degree or diploma.

Place: Goa University, Taleigao Plateau.

Jervin Zen Lobo
August 2020

Certificate

This is to certify that Mr. Jervin Zen Lobo has successfully completed this thesis entitled "Symmetry Analysis Of Some Functional Differential Equations" for the award of the degree of *Doctor of Philosophy in Mathematics* under my guidance during the period 2016-2020 and to the best of my knowledge, the research work embodied in it is original and has not been submitted earlier in part or full or in any other form to any university or institute, here or elsewhere, for the award of any degree or diploma.

Date: August, 2020.

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Dedicated to the memory of my Dad.

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Abstract

Differential equations play an important role in physical processes, chemical processes, biological processes and also social sciences as well as commerce and finance. A powerful tool in solving ordinary and partial differential equations is the method of symmetry. By this method many unfamiliar differential equation can be solved. This method was given by a Norwegian mathematician Sophus Lie. The method involves some fundamental ideas which can be easily employed to solve a differential equations.

Symmetries are transformations that leave an object unchanged or invariant. In [48] it is explained that symmetries are very useful in formulating and exploiting the laws of nature. The reproducibility of experiments at different times and places heavily rely on invariance laws. The existence of conservation laws in Physics and Mathematics is an important implication of symmetry. Nöether's theorem proved in [47] relates symmetries and conservation laws. The concept of symmetries has interested scientists from Kepler — in determining the orbits of planets to Newton — in studying the laws of mechanics as a symmetry principle. The motivation to study Lie groups is to model the continuous symmetries of differential equations, in much the same way as finite groups are used in Galois theory to model the discrete symmetries of algebraic equations.

However, in differential equations, the unknown function and its derivatives are all evaluated at the same instant t. More general types of differential equations, called functional differential equations, are ones in which the unknown function occurs with various different arguments. In Russian literature these are called "differential equations with deviating arguments ". The simplest of these are called "delay differential equations "(or "differential equations with retarded arguments "). This basically means expressing some derivative of the unknown function x at time t in terms of x and its lower derivatives, if any, at t and earlier instants. Functional differential equations are further classified as differential difference equations, integro-differential equations, delay differential equations, neutral differential equations, etc. Functional differential equations find a wide range of applications in traffic flow problems, signal processing, control systems, heat transfer problems, population models, evolution of species, prey-predator models, biological systems, population dynamics, networking problems, study of epidemics, rolling of ships, electrical engineering, etc [33]. The best known method to solve delay differential equations is the method of steps. Other methods in solving

functional differential equations include substitutions, numerical solutions and power series solutions [12]. The theory on delay differential equations can be found in [15, 21].

The main problems encountered in applying symmetry analysis to functional differential equations are that:

1. The presence of the delay term in functional differential equations make it seemingly difficult to solve the higher order and nonlinear equations.
2. Differential equations with deviating arguments do not possess any equivalent transformations related with the change of the variables – both dependent and independent. These equivalent transformations could be found for ordinary differential equations which could reduce them to separable equations, which in turn were easy to solve.
3. Symmetry analysis cannot be used to explicitly find solutions of many functional differential equations due to the presence of the delay term.

As there is no analytic method to solve functional differential equations, symmetry analysis is a powerful tool for studying the properties of the solutions of these functional differential equations. Such group classification of these functional differential equations are of great importance to Applied Mathematicians, Physicists, scientists and engineers in modeling the physical phenomenon under study which in many cases involve delay differential equations.

The research was carried out with the following objectives:

1. To find a new procedure to get the Lie type invariance condition of first and second order delay differential equations used in obtaining their equivalent symmetries.
2. To use the newly developed procedure to make a complete group classification of first and second order neutral differential equations for which there is no literature.
3. To identify if any alternate classification scheme exists. If yes, to develop the alternate scheme and assess its merits and demerits.
4. To develop a novel approach in obtaining the Lie type invariance condition for first order partial differential equations with delay. Having developed this, to classify the Inviscid Burgers' equation with delay, with respect to an arbitrary and special case of its differentiable functional. Having done this, to obtain a representation of its analytic solutions and the reduced equations from its symmetry.
5. To develop a novel approach in obtaining the Lie type invariance condition for second order partial differential equations with delay. Having developed this, to classify the wave equation with delay, with respect to an arbitrary and special

case of its differentiable functional and to obtain the reduced equations from its symmetries, along with obtaining a representation of its invariant solutions.

Subsequent to this abstract of the thesis, the ideas, terminologies, existing results and terminologies of group analysis for ordinary and partial differential equations is developed. The existing approach to classify delay differential equations is researched by defining an operator equivalent to the canonical Lie Bäcklund operator. This approach uses an invariant manifold theorem and results in terms with double delay when applied to higher order equations. We have used this approach to illustrate it only for first order delay differential equations with constant coefficients, for which there was no existing literature, in chapter 2. In our study, we have obtained an approach different from the existing one — using Taylor's theorem for a function of several variables. We obtain our determining equations and split them in a manner different from the existing approach for delay differential equations. In addition, our approach does not result into any terms with double delay, even when working with higher order equations. In this thesis, using the approach we have obtained (a Lie type invariance condition for functional differential equations using Taylor's theorem for a function of several variables), we have classified several linear and nonlinear functional differential equations with variable coefficients.

In chapter 3, a Lie type invariance condition for first order linear and nonlinear delay differential equations with the most general time delay $g(t)$ is developed. This condition is used to make a thorough group classification of the first order delay differential equation. Next, we choose the standard time delay of $t - r$ and classify the resulting delay differential equation. This change in the delay gives us different results. The classification is generalized in chapter 4 by obtaining a Lie type invariance condition and making a group classification of first order neutral differential equations with the most general and standard delay. We also show that if the derivative term with delay vanishes (that is the neutral differential equations reduces to a delay differential equation), our results obtained for neutral differential equations agree with our results obtained for delay differential equations. Examples in both chapters illustrate our theories.

In chapter 5, a Lie type invariance condition for second order linear delay differential equations with the most standard time delay is developed. The Taylor's theorem approach here does not give us any terms with double delay in the determining equations as seen in the existing literature. We also use certain results to simplify our existing delay differential equation. The developed condition is then used to make a thorough group classification of the second order delay differential equation. This classification is generalized in chapter 6 by obtaining a Lie type invariance condition and making a group classification of second order neutral differential equations with the most standard time delay. We also show that if the derivative term with delay vanishes (that is the neutral differential equations reduces to a delay differential equation), our results obtained for for neutral differential equations agree with our results obtained for delay differential equations. The results obtained herein are an improvement to several well established results for second order delay differential equations using Lie Bäcklund operators. We illustrate some practical examples in both chapters.

It may be noted that the drawback of the approach in chapters $2, 3, 4, 5, 6$ was that the inverse of the classification could not be found. We overcome this difficulty in chapter 7 and 8. Differential equations with deviating arguments do not possess any equivalent transformations related with the change of the variables – both dependent and independent. We consider the absence of such equivalent transformations to obtain a basis for the solvable Lie algebras of such functional differential equations. In chapter 7 we provide a basis for the Lie algebra given by the first order linear and nonlinear functional (delay and neutral) differential equations with constant coefficients, for which there is no existing literature. In chapter 8, with the aid of some existing results to simplify our equations, we extend our results to second order functional (delay and neutral) differential equations with constant coefficients. The approach to get to the determining equations in these two chapters, using Taylor's theorem is slightly different from those developed in the preceding chapters. The only drawback in this approach established is that if it is applied to functional differential equations with variable coefficients, then solving the resulting splitting equations require certain Computer Algebra Systems.

The theories developed so far was for ordinary functional differential equations. In chapters 9 and 10 symmetry analysis is applied to partial differential equations with delay. The procedure for establishing the invariance conditions and extended infinitesimals gets complicated for partial differential equations and requires certain local invertibility. In chapter 9, group analysis of first order partial differential equations with delay is discussed and used to obtain symmetries of the Inviscid Burgers' equation with delay, its kernel and extensions of the kernel. A Lie type invariance condition by using Taylor's theorem for a function of several variables is obtained. Further, representations of analytic solutions and the reduced equations from the symmetries are obtained. In chapter 10, we establish a Lie type invariance condition for second order partial differential equations with delay. The symmetries of the wave equation with delay, its kernel and extensions of the kernel have been found. We make a complete group classification of the wave equation containing an arbitrary differentiable functional with delay. Further, the complete set of invariant solutions led by this classification have been found.

Finally, we conclude the thesis with future scope led by this research work which can be continued by researchers interested in this area.

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Chapter 1

Introduction and Review of Literature

Part of the contents of this chapter are published in

Proceedings of International Conference on Applied Mathematics and Computational Sciences.

1.1 What is a Symmetry?

A symmetry is a transformation that leaves an object unchanged or "invariant". For example, if we start with an equilateral triangle with the vertices labeled 1, 2, and 3 (see Figure 1.1), then a reflection through any one of the three bisection axes (see Figure 1.2) or rotations through the angles of $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ (see Figure 1.3) leaves the triangle invariant.

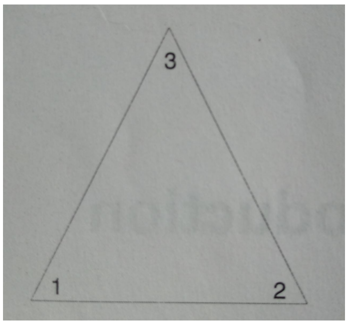

Figure 1.1: An equilateral triangle

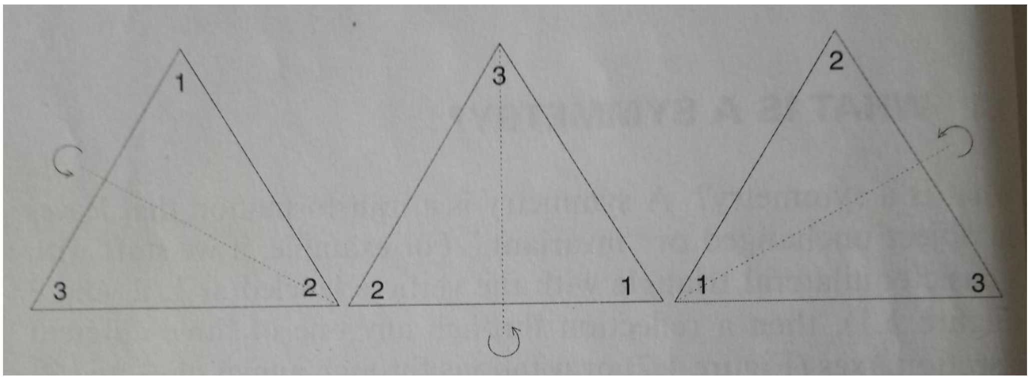

Figure 1.2: Reflections of an equilateral triangle

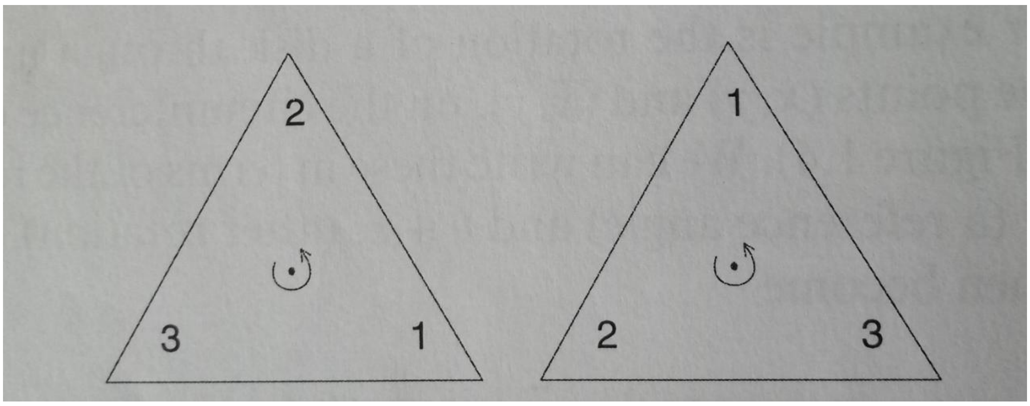

Figure 1.3: Rotations of an equilateral triangle through $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$

As another example, consider a disk, which is rotated by an angle δ. Let the points (t, x)

and $(\bar{t}, \bar{x})$ lie on the circumference of the circle of radius r. (see Figure 1.4). In terms of the radius and the angles θ (a reference angle) and $\theta+\delta$, (after rotation), we can write these as

$$t = r\cos\theta, \quad \bar{t} = r\cos(\theta+\delta),$$

$$x = r\sin\theta, \quad \bar{x} = r\sin(\theta+\delta),$$

which on elimination of θ gives,

$$\bar{t} = t\cos\delta - x\sin\delta, \quad \bar{x} = x\cos\delta + t\sin\delta. \tag{1.1}$$

We shall show the invariance of the circle under (1.1). That is, we shall show that $\bar{t}^2 + \bar{x}^2 = r^2$ if $t^2 + x^2 = r^2$. Thus,

$$\begin{aligned}\bar{t}^2 + \bar{x}^2 &= (t\cos\delta - x\sin\delta)^2 + (x\cos\delta + t\sin\delta)^2 \\ &= t^2\cos^2\delta + x^2\sin^2\delta - 2tx\sin\delta\cos\delta \\ &+ t^2\sin^2\delta + x^2\cos^2\delta + 2tx\sin\delta\cos\delta \\ &= t^2 + x^2 \\ &= r^2\end{aligned}$$

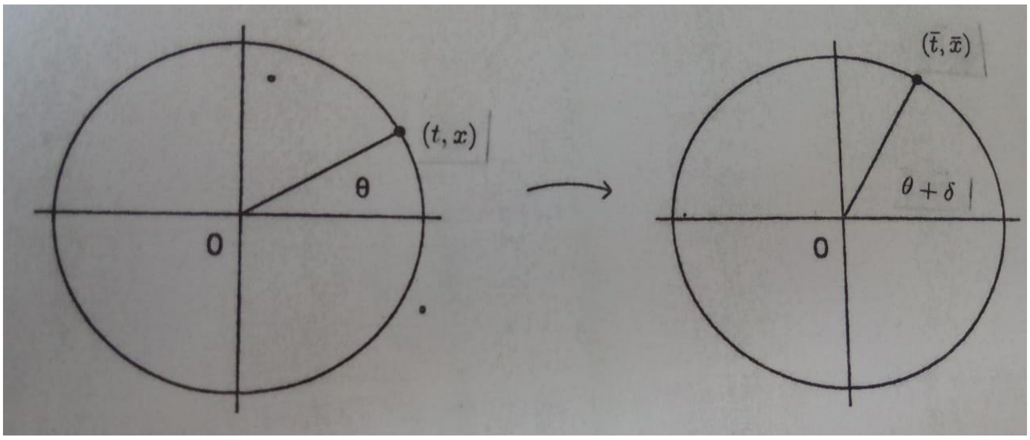

Figure 1.4: Rotations of a circle

As a final example, consider the line $x = \frac{1}{2}t$ and the transformation

$$\bar{t} = e^{\delta}t, \quad \bar{x} = e^{\delta}x. \tag{1.2}$$

The line is invariant under (1.2),
For if $\bar{x} = \frac{1}{2}\bar{t}$, then $e^{\delta}x = \frac{1}{2}e^{\delta}t$ if $x = \frac{1}{2}t$.
(See Figure 1.5).

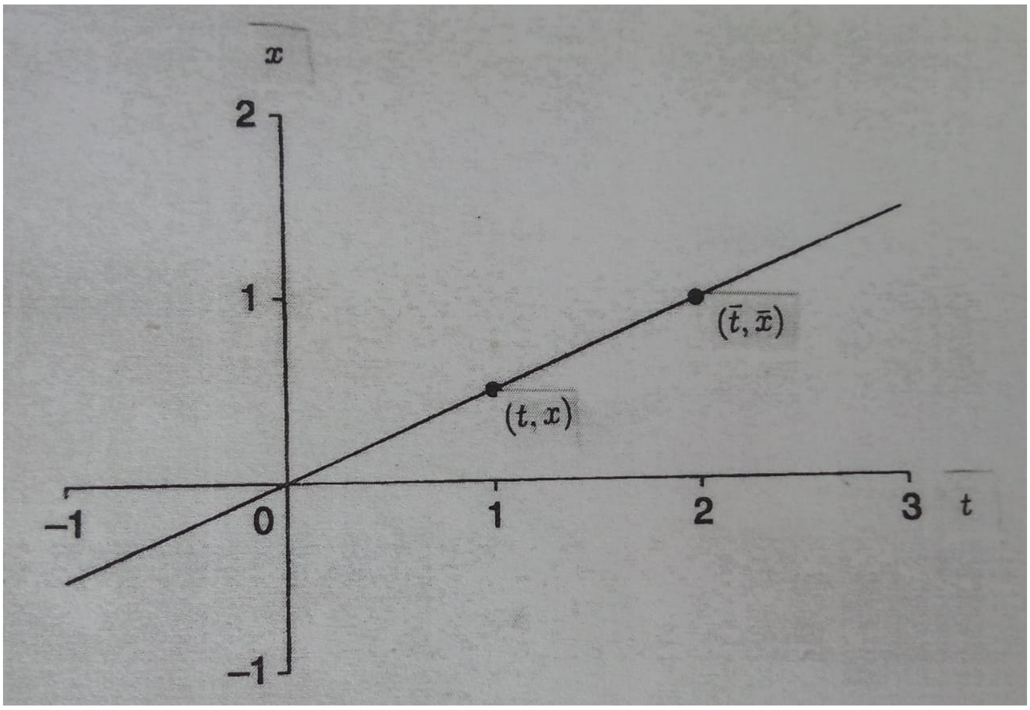

Figure 1.5: Invariance of the line $x = \frac{1}{2}t$

We shall now illustrate the invariance of an equation. Consider the equation

$$t^2x^2 - tx^2 + 2tx - x^2 - x + 1 = 0. \tag{1.3}$$

This equation is invariant under

$$\bar{t} = t + \delta, \quad \bar{x} = \frac{x}{1 - \delta x}. \tag{1.4}$$

It is easier to write (1.3) as

$$\left(t + \frac{1}{x}\right)^2 - \left(t + \frac{1}{x}\right) - 1 = 0. \tag{1.5}$$

Under the transformation (1.4), the term $t + \dfrac{1}{x}$ becomes

$$\bar{t} + \frac{1}{\bar{x}} = t + \delta + \frac{1 - \delta x}{x} = t + \delta + \frac{1}{x} - \delta = t + \frac{1}{x},$$

and the invariance of (1.5) readily follows.

Remark 1.1.1. Not all equations are invariant under all transformations.

Consider the line $x - 1 = 3(t - 1)$ and the transformation given by (1.2). If this were invariant, then,
$\bar{x} - 1 = 3(\bar{t} - 1)$ if $x - 1 = 3(t - 1)$.
On substituting, we get, $e^{\delta}x - 1 = 3(e^{\delta}t - 1)$, which is very clearly not the original line and hence not invariant under (1.2).

1.2 One-Parameter Group of Transformations

Lie, while investigating differential equations, found it necessary to distinguish between two approaches given below:

1. The natural approach which deals with the totality of solutions of a given differential equation.

2. Regarding the differential equation as a surface in the space of independent and dependent variables together with the derivatives involved in the given equation.

We explain these approaches in the subsequent sections. We formally define a one-parameter group of transformations as below:

Definition 1.2.1. Consider transformations given by, $\bar{t}_i = g_i(t_j, \delta), i, j = 1, 2, \cdots, n.$ where δ is the parameter and these transformations, depend continuously on δ.
Let for each i, g_i be a smooth function of the variables t_j having a convergent Taylor series in δ.
We say that this set of transformations form a one-parameter group of transformations if:

1. (*Closure*) The product of two transformations of the set is again a transformation of the set.
   That is, if $\bar{t}_i = g_i(t_j, \delta_1), i, j = 1, 2, \cdots, n.$, and $\hat{t}_i = g_i(\bar{t}_j, \delta_2), i, j = 1, 2, \cdots, n.$, are two transformations of the set corresponding to parameters δ_1 and δ_2 respectively, then there exists a parameter δ_3, such that $\hat{t}_i = g_i(t_j, \delta_3), i, j = 1, 2, \cdots, n.$

2. (*Identity*) Every transformation has an identity.
   That is, there is a value of the parameter, say $\delta = \delta_e$, such that
   $t_i = g_i(t_j, \delta_e), i, j = 1, 2, \cdots, n.$

3. (*Inverse*) Every transformation has an inverse.
   That is, there exists a parameter, say δ^{-1}, such that
   $t_i = g_i(\bar{t}_j, \delta^{-1}), i, j = 1, 2, \cdots, n.$

We have the following:

Definition 1.2.2. A real Lie group is a group that is also a finite-dimensional real smooth manifold, in which the group operations of multiplication and inversion are smooth maps. Smoothness of the group multiplication

$$\mu : G \times G \to G, \quad \mu(t, x) = tx,$$

means that μ is a smooth mapping of the product manifold $G \times G$ into G. These two requirements can be combined to the single requirement that the mapping $(t, x) \mapsto t^{-1}x$ be a smooth mapping of the product manifold into G.

Example 1.2.1. *The rotation matrices form a subgroup of $GL(2, \mathbb{R})$ which is the group (under multiplication) of 2×2 real invertible matrices, and is denoted by $SO(2, \mathbb{R})$. It is a Lie group in its own right. Using the rotation angle δ as the parameter, this group can be parametrized as follows:*

$$SO(2, \mathbb{R}) = \left\{ \begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix} : \delta \in \mathbb{R}/2\pi\mathbb{Z} \right\}.$$

Addition of the angles corresponds to multiplication of the elements of $SO(2, \mathbb{R})$, and taking the opposite angle corresponds to inversion. Thus both multiplication and inversion are differentiable maps.

Remark 1.2.1. The associativity law for groups follows from the *closure* property.

In general, the order in which we carry out the transformations matter. If the order of carrying out the transformations is immaterial, then the group is termed as **abelian**.

Definition 1.2.3. A symmetry group of a differential equation is a group that converts every solution of the equation under consideration into a solution of the same equation. That is, a symmetry group of a system of differential equations is a group of transformations mapping every solution to another solution of the same system.

Remark 1.2.2. The terms “groups admitted by differential equations”, “admitted group” and “symmetry groups” are used interchangeably in literature.

We provide a few examples to illustrate a Lie group:

Example 1.2.2. *Consider Figure 1.6.*

We shall show that the set of transformations given by $\bar{t} = at, \quad a \in \mathbb{R}\backslash\{0\}$ form a one-parameter group.

1. *Closure. If $\bar{t} = at$ and $\tilde{t} = b\bar{t}$, then $\tilde{t} = abt$. That is, the product of two transformations in the group result into another transformation of the group.*
2. *Identity. Clearly 1 is the identity, because when the value of the parameter becomes equal to 1, the source point t and the image point $\bar{t}$ coincide.*

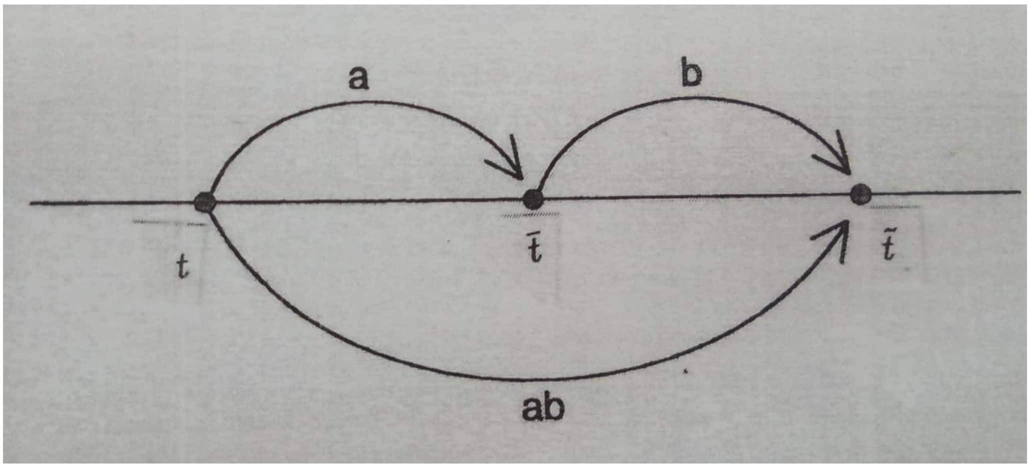

Figure 1.6: Scaling group $\bar{t} = at$

3. Inverse. As $t = \frac{1}{a}\bar{t}$, $\frac{1}{a}$ *characterizes the inverse of* a.

Remark 1.2.3. We note that if we reparametrize the group by letting $a = e^{\delta}$, then the group in Example 1.2.2 becomes a Lie group.

Other standard examples of Lie groups include:

Example 1.2.3. $\bar{t}_1 = t_1,\ \bar{t}_2 = t_2 + \delta$ *is known as a* Translation group.

Example 1.2.4. *For any constant* a, $\bar{t}_1 = a^{\delta} t_1,\ \bar{t}_2 = a^{\delta} t_2$ *is known as a* Stretching group.

Example 1.2.5. $\bar{t}_1 = t_1 \cos\delta - t_2 \sin\delta,\ \bar{t}_2 = t_1 \sin\delta + t_2 \cos\delta$ *is known as a* Rotational group.

For each $i, j = 1, 2, \cdots, n$, the functions $g_i(t_j, \delta)$, are referred to as the *global form* of the group.
For two variables (the case for ordinary differential equations, one being dependent while the other being independent), we shall denote the variables by x and t. Thus, we consider the transformations

$$\bar{t} = f(t, x, \delta), \quad \bar{x} = g(t, x, \delta), \tag{1.6}$$

If we assume that δ is small, then we construct a Taylor series of equation (1.6) about $\delta = 0$. Therefore,

$$\bar{t} = t + \delta \left(\frac{d\bar{t}}{d\delta}\right)_{\delta=0} + O(\delta^2), \quad \bar{x} = x + \delta \left(\frac{d\bar{x}}{d\delta}\right)_{\delta=0} + O(\delta^2), \tag{1.7}$$

where $O(\delta^2)$ indicates terms involving only powers of δ greater than or equal to two. Let,

$$\left(\frac{d\bar{t}}{d\delta}\right)_{\delta=0} = \omega(t,x), \quad \left(\frac{d\bar{x}}{d\delta}\right)_{\delta=0} = \Upsilon(t,x). \tag{1.8}$$

Then we get,

$$\bar{t} = t + \delta\omega(t,x) + O(\delta^2), \quad \bar{x} = x + \delta\Upsilon(t,x) + O(\delta^2). \tag{1.9}$$

Equation (1.9) is referred to as the *infinitesimal form* of the group.
Further, ω and Υ are called *coefficients of the infinitesimal transformations* or simply *infinitesimals*.

Remark 1.2.4. The crucial property of one-parameter transformation groups is that given the infinitesimal form of the group we can deduce the global form by integrating the following autonomous system of differential equations,

$$\frac{d\bar{t}}{d\delta} = \omega(\bar{t},\bar{x}), \quad \frac{d\bar{x}}{d\delta} = \Upsilon(\bar{t},\bar{x}), \tag{1.10}$$

subject to the initial conditions $\bar{t} = t, \quad \bar{x} = x$, when $\delta = 0$.
A proof of this result can be found in [13].

We conclude this section by giving an example of a set of transformations that do not form a one-parameter transformation group.

Example 1.2.6. *Consider the transformations given by* $\bar{t} = \dfrac{1}{\delta}\log(1+\delta t), \quad \bar{x} = (1+\delta t)x.$
We see that $t = \dfrac{e^{\delta\bar{t}} - 1}{\delta}, \quad x = e^{-\delta\bar{t}}\bar{x}.$
Further,

$$\frac{d\bar{t}}{d\delta} = \frac{1 - \delta\bar{t} - e^{-\delta\bar{t}}}{\delta^2}, \quad \frac{d\bar{x}}{d\delta} = \frac{(1 - e^{-\delta\bar{t}})}{\delta}\bar{x},$$

and since δ occurs explicitly on the right-hand sides of these equations, the system is non-autonomous and therefore does not generate a one-parameter group of transformations. In addition, $-\delta$ does not characterize the inverse.

Remark 1.2.5. The infinitesimal transformations given by (1.9) is an Euler finite difference algorithm for solving the coupled differential equations namely,

$$\frac{dt}{\omega(t,x)} = \frac{dx}{\Upsilon(t,x)} = d\delta \tag{1.11}$$

1.3 Invariant Curves and Families of Curves

An invariant curve C, is one whose points, considered as source points, map into other points of curve C for all transformations of the group. Thus, C must either be an orbit or

a locus on which the infinitesimal coefficients $\omega(t,x)$ and $\Upsilon(t,x)$ simultaneously vanish. A one-parameter family of curves can be represented parametrically by the equation $\Phi(t,x) = c_2$, where Φ is the function defining the family and c_2 is a parameter that labels different curves of the family. The family is said to be invariant if the image of each curve of it is another curve of the family. The condition for this is that, for any fixed value of δ, the image points $(\bar{t},\bar{x})$ satisfy, $\Phi(\bar{t},\bar{x}) = c_3$ when the source points satisfy $\Phi(t,x) = c_2$. Here c_3 is different from c_2 and depends on c_2 and δ.
The representation of the family of curves $\Phi(t,x) = c_2$, is not unique and any other representation $\Omega(t,x) = c_3$ for which Ω is a function of Φ, that is, $\Omega = G(\Phi)$ is equivalent to $\Phi(t,x) = c_2$.
Now,

$$\begin{aligned}\omega\Omega_t + \Upsilon\Omega_x &= \omega\left[\frac{dG}{d\Phi}\Phi_t\right] + \Upsilon\left[\frac{dG}{d\Phi}\Phi_x\right]\\ &= (\omega\Phi_t + \Upsilon\Phi_x)\frac{dG}{d\Phi}\\ &= \frac{dG}{d\Phi}F(\Phi),\end{aligned}$$

where F is an arbitrary function.
Choosing $G(\Phi) = \int \frac{d\Phi}{F(\Phi)}$, the right hand side of the above equation becomes equal to 1. Therefore,

$$\omega\Omega_t + \Upsilon\Omega_x = 1 \tag{1.12}$$

1.4 Invariance of Differential Equations

Having seen in the previous examples that equations can be invariant under a Lie group, we now provide an example to illustrate the invariance of an ordinary differential equation under a Lie group.

Example 1.4.1. *We shall show that the differential equation given by*

$$\frac{dx}{dt} = \frac{(tx+1)^3}{t^5} + \frac{1}{t^2} \tag{1.13}$$

is invariant under $\bar{t} = \frac{t}{1+\delta t}, \quad \bar{x} = x - \delta.$
By chain rule,

$$\begin{aligned}\frac{d\bar{x}}{d\bar{t}} &= \frac{d\bar{x}}{dt}\bigg/\frac{d\bar{t}}{dt}\\ &= \frac{dx}{dt}(1+\delta t)^2\end{aligned} \tag{1.14}$$

The right hand side of equation (1.13) becomes

$$\begin{aligned}\frac{(\bar{t}\bar{x}+1)^3}{\bar{t}^5} &= \frac{\left(\dfrac{t}{1+\delta t}(x-\delta)+1\right)^3}{\left(\dfrac{t}{1+\delta t}\right)^5}\\ &= \frac{\left(\dfrac{tx+1}{1+\delta t}\right)^3}{\left(\dfrac{t}{1+\delta t}\right)^5}\\ &= \frac{(tx+1)^3}{t^5}(1+\delta t)^2.\end{aligned} \tag{1.15}$$

Hence, the right hand side of equation (1.13) becomes

$$\frac{(\bar{t}\bar{x}+1)^3}{\bar{t}^5}+\frac{1}{\bar{t}^2}=\frac{(tx+1)^3}{t^5}(1+\delta t)^2+\frac{(1+\delta t)^2}{t^2} \tag{1.16}$$

Using equations (1.15) and (1.16) we see that

$$\frac{d\bar{x}}{d\bar{t}}=\frac{(\bar{t}\bar{x}+1)^3}{\bar{t}^5}+\frac{1}{\bar{t}^2} \quad \textit{if} \quad \frac{dx}{dt}=\frac{(tx+1)^3}{t^5}+\frac{1}{t^2} \tag{1.17}$$

Consider a first order ordinary differential equation,[1]

$$F(t,x,\dot{x})=0. \tag{1.18}$$

In Lie's first approach mentioned in Section 1.2, the symmetry group of equation (1.18) is a one-parameter group of transformations given by equation (1.6) and is called *point transformations* (unlike contact transformations, where the transformed values also depend on the derivative $\dot{x}$.) Any solution $h(t)$ of equation (1.18) is converted into a solution of equation (1.18) in the following way. Consider the integral curve,

$$x=h(t). \tag{1.19}$$

Fix the parameter δ in equation (1.6) and apply the transformation given by equation (1.6) to the integral curve given by equation (1.19). This yields a curve given by,

$$\bar{t}=f(t,h(t),\delta), \quad \bar{x}=g(t,h(t),\delta), \tag{1.20}$$

which, according to the first approach, is an integral curve. After elimination of t from

[1]The notations $\dot{x}$ means $\dfrac{dx}{dt}$. Similarly the notation $\ddot{x}$ means $\dfrac{d^2x}{dt^2}$ and so on.

equation (1.20), this curve can be rewritten in the form,

$$\bar{x} = H(\bar{t}, \delta). \tag{1.21}$$

Because $\bar{t}$ is arbitrary we can again denote it by t. Hence, the original solution $h(t)$ of equation (1.18) is converted by the symmetry group into a one-parameter family of solutions $H(t, \delta)$ of equation (1.18).

We now turn to see why group methods is the only unified understanding as to why differential equations can be solved.

Consider the first order differential equation

$$\frac{dx}{dt} = \frac{t^2 + x^2}{tx}. \tag{1.22}$$

This is a homogeneous differential equation which can be made separable by the substitution $v(t, x) = \dfrac{x}{t}$, which can then be integrated to yield

$$\log t - \frac{1}{2}\left(\frac{x}{t}\right)^2 = c_1, \tag{1.23}$$

where c_1 denotes an arbitrary constant.

The substitution $v(t, x) = \dfrac{x}{t}$ leads to a separable equation for v, because $v(t, x)$ is an invariant of the Lie group

$$\bar{t} = e^{\delta} t, \quad \bar{x} = e^{\delta} x. \tag{1.24}$$

This is because $v(\bar{t}, \bar{x}) = \dfrac{\bar{x}}{\bar{t}} = \dfrac{x}{t} = v(t, x)$. It is this property that results in the simplification of equation (1.22).

In general, if a differential equation is invariant under a one-parameter group of transformations then the use of an invariant of the group results in a simplification of the differential equation. The differential equation becomes separable if it is of first order and the use of an invariant of the group reduces the order of a higher order differential equation by one.

From equations (1.23) and (1.24) we have,

$$\log \bar{t} - \frac{1}{2}\left(\frac{\bar{x}}{\bar{t}}\right)^2 = c_1 + \delta,$$

so that the degrees of freedom in the solution given by equation (1.23) resulting from the arbitrary constant c_1 is related to the invariance of the differential equation (1.22) under the Lie group given by equation (1.24) which is characterized by arbitrary parameter δ. That is, the Lie group given by equation (1.24) permutes the solution curves given by equation (1.23).

In general, for every one-parameter group in two variables there are functions $u(t, x)$ and

$v(t, x)$ such that the group becomes

$$u(\bar{t}, \bar{x}) = u(t, x), \quad v(\bar{t}, \bar{x}) = v(t, x) + \delta. \tag{1.25}$$

The function $u(t, x)$ is said to be an *invariant* of the group while together (u, v) are referred to as the *canonical coordinates* of the group. Moreover, every first order differential equation invariant under this group, takes the form in terms of these new variables u aand v as

$$\frac{dv}{du} = \phi(u),$$

and consequently has a solution of the form

$$v + \psi(u) = c_1$$

for appropriate functions $\phi(u)$ and $\psi(u)$.

Example 1.4.2. *We shall obtain the canonical coordinates of the one-parameter Lie group*

$$\bar{t} = \frac{t}{1 + \delta t}, \quad \bar{x} = (1 + \delta t)^2 x. \tag{1.26}$$

The infinitesimal form of the group is given by,

$$\bar{t} = t - \delta t^2 + O(\delta^2), \quad \bar{x} = x + 2\delta t x + O(\delta^2),$$

which from equation (1.9) shows that $\omega(t, x) = -t^2$ and $\Upsilon(t, x) = 2tx$. This can also be obtained from

$$\frac{d\bar{t}}{d\delta} = \bar{t}^2, \quad \frac{d\bar{x}}{d\delta} = 2\bar{t}\bar{x} \tag{1.27}$$

To obtain $u(t, x)$, we solve $\dfrac{d\bar{x}}{d\bar{t}} = \dfrac{\omega(\bar{t}, \bar{x})}{\Upsilon(\bar{t}, \bar{x})}$, to get,

$$\frac{d\bar{t}}{d\bar{x}} = -\frac{\bar{t}}{2\bar{x}},$$

which on integration yields $u(t, x) = t^2 x$ as an invariant. From the first equation of (1.27), we see that, $v(t, x) = \dfrac{1}{t}$ satisfies equation (1.25).

Remark 1.4.1. There are differential equations invariant under transformations which cannot be characterized as one-parameter groups.
For example, a differential equation arising in fluid dynamics is given by

$$\frac{d^2 x}{dt^2} = 2\frac{dx}{dt} + \frac{(5 + 3x)}{4x(1 + x)} \left(\frac{dx}{dt}\right)^2 + \frac{3x(1 - x)}{1 + x}. \tag{1.28}$$

If $x(t)$ is a solution of equation (1.28), then so is $x(t)^{-1}$. This is seen by putting $X(t) = x(t)^{-1}$ and using

$$\frac{dX}{dt} = -\frac{1}{x^2}\frac{dx}{dt}, \quad \frac{d^2X}{dt^2} = -\frac{1}{x^2}\frac{d^2x}{dt^2} + \frac{2}{x^3}\left(\frac{dx}{dt}\right)^2$$

to get,

$$\frac{d^2X}{dt^2} - 2\frac{dX}{dt} - \frac{5+3X}{4X(1+X)}\left(\frac{dX}{dt}\right)^2 - \frac{3X(1-X)}{1+X}$$
$$= -\frac{1}{x^2}\left\{\frac{d^2x}{dt^2} - 2\frac{dx}{dt} - \frac{5+3x}{4x(1+x)}\left(\frac{dx}{dt}\right)^2 - \frac{3x(1-x)}{1+x}\right\} = 0$$

Setting $y = \dfrac{dx}{dt}$, equation (1.28) becomes

$$y\frac{dy}{dx} = \frac{3x(1-x)}{1+x} + 2y + \frac{5+3x}{4x(1+x)}y^2, \tag{1.29}$$

which is an Abel equation of second kind. From the solution property of equation (1.28) we can deduce that equation (1.29) remains invariant under the transformation

$$\bar{x} = \frac{1}{x}, \quad \bar{y} = -\frac{y}{x^2},$$

which cannot be characterized by a one-parameter group.
We shall illustrate the procedure of finding the canonical coordinates.

We now obtain a relationship between the infinitesimals ω and Υ and canonical coordinates u and v.
On differentiating equation (1.25) with respect to δ, we get,

$$\frac{\partial\bar{u}}{\partial\bar{t}}\frac{d\bar{t}}{d\delta} + \frac{\partial\bar{u}}{\partial\bar{x}}\frac{d\bar{x}}{d\delta} = 0, \quad \frac{\partial\bar{v}}{\partial\bar{t}}\frac{d\bar{t}}{d\delta} + \frac{\partial\bar{v}}{\partial\bar{x}}\frac{d\bar{x}}{d\delta} = 1, \tag{1.30}$$

where $\bar{u}$ and $\bar{v}$ denote $u(\bar{t}, \bar{x})$ and $v(\bar{t}, \bar{x})$ respectively.
From equations (1.10) and (1.30), we have on replacing $(\bar{t}, \bar{x})$ by (t, x),

$$\frac{\partial u}{\partial t}\omega + \frac{\partial u}{\partial x}\Upsilon = 0, \quad \frac{\partial v}{\partial t}\omega + \frac{\partial v}{\partial x}\Upsilon = 1 \tag{1.31}$$

which can we solved for ω and Υ to deduce

$$\omega(t, x) = -\frac{\partial u}{\partial x}\bigg/\frac{\partial(u, v)}{\partial(t, x)}, \quad \Upsilon(t, x) = \frac{\partial u}{\partial t}\bigg/\frac{\partial(u, v)}{\partial(t, x)} \tag{1.32}$$

where the Jacobian is given by

$$\frac{\partial(u,v)}{\partial(t,x)} = \frac{\partial u}{\partial t}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial t}.$$

Our arguments upto equation (1.31) lead us to the following theorem:

Theorem 1.4.1. *A one-parameter group of transformations* $\bar{t} = f(t,x,\delta),\ \bar{x} = g(t,x,\delta)$ *can be reduced under a suitable change of variables, called canonical variables* $u = u(t,x),\ v = v(t,x)$*, to the translation group* $\bar{u} = u,\ \bar{v} = v + \delta$.

1.5 Transformations of Derivatives

1.5.1 The Extended Group

Since the one-parameter transformations for fixed δ determine the image curve C_1 of any curve C , it is possible to calculate the slope $\dot{\bar{x}} = \frac{d\bar{x}}{d\bar{t}}$ of the curve C_1 from the slope $\dot{x} = \frac{dx}{dt}$ of the curve C.
If $A:(t,x)$ and $B:(t+dt,x+dx)$ are neighboring points on the curve C, the coordinates of their images $\bar{A}:(\bar{t},\bar{x})$ and $\bar{B}:(\bar{t}+d\bar{t},\bar{x}+d\bar{x})$ on the curve C_1 are given by

$$\bar{t} = \alpha(t,x,\delta), \quad \bar{x} = \beta(t,x,\delta) \tag{1.33}$$

and,

$$\bar{t} + d\bar{t} = \alpha(t+dt,x+dx,\delta), \quad \bar{x} + d\bar{x} = \beta(t+dt,x+dx,\delta). \tag{1.34}$$

Equations (1.33) and (1.34) yield,

$$d\bar{t} = \alpha_t dt + \alpha_x dx, \quad d\bar{x} = \beta_t dt + \beta_x dx. \tag{1.35}$$

From equation (1.35),

$$\dot{\bar{x}} = \frac{d\bar{x}}{d\bar{t}} = \frac{\beta_t dt + \beta_x dx}{\alpha_t dt + \alpha_x dx} = \frac{\beta_t + \beta_x \dot{x}}{\alpha_t + \alpha_x \dot{x}}. \tag{1.36}$$

Equations (1.33) and (1.36) specify a set of extended transformations of the quantities t, x and $\dot{x}$. Geometrically speaking t, x and $\dot{x}$ define an infinitesimal line element at point (t,x) having slope $\dot{x}$. The set of extended transformations thus carry one such line element to another.
When the transformation law for $\bar{x}$ is equation (1.36), then these extended transformations form a group called *once-extended group* or the *first prolongation* of the group of point transformations.
The coefficients of the infinitesimal transformation of the first extended group corresponding to $\dot{\bar{x}} = \frac{d\bar{x}}{d\bar{t}}$ is the derivative $\left(\frac{\partial \dot{\bar{x}}}{\partial \delta}\right)_{\delta=0}$.

Noting that for the identity transformation corresponding to $\delta = 0$, equations (1.33) to (1.36) transform to,

$$t = \alpha(t, x, 0), \quad x = \beta(t, x, 0),$$

and,

$$t + dt = \alpha(t + dt, x + dx, 0), \quad x + dx = \beta(t + dt, x + dx, 0).$$

Hence,

$$dt = \alpha_t dt + \alpha_x dx, \quad dx = \beta_t dt + \beta_x dx.$$

Therefore,

$$\dot{x} = \frac{dx}{dt} = \frac{\beta_t + \beta_x \dot{x}}{\alpha_t + \alpha_x \dot{x}}.$$

Hence we get, $\beta_t = 0 = \alpha_x$ and $\beta_x = 1 = \alpha_t$.

$$\begin{aligned} \Upsilon_{[t]} &= \left(\frac{\partial \dot{\bar{x}}}{\partial \delta}\right)_{\delta=0} \\ &= \left[\frac{\partial}{\partial \delta}\left(\frac{\beta_t + \beta_x \dot{x}}{\alpha_t + \alpha_x \dot{x}}\right)\right]_{\delta=0} \\ &= \left[\frac{(\alpha_t + \alpha_x \dot{x})(\Upsilon_t + \Upsilon_x \dot{x}) - (\beta_t + \beta_x \dot{x})(\omega_t + \omega_x \dot{x})}{(\alpha_t + \alpha_x \dot{x})^2}\right]_{\delta=0} \end{aligned}$$

Since $\beta_t = 0 = \alpha_x$ and $\beta_x = 1 = \alpha_t$, we get,

$$\begin{aligned} \Upsilon_{[t]} &= (\Upsilon_t + \Upsilon_x \dot{x}) - \dot{x}(\omega_t + \omega_x \dot{x}) \\ &= \frac{d\Upsilon}{dt} - \dot{x}\frac{d\omega}{dt} \end{aligned} \tag{1.37}$$

The importance of equation (1.37) is that it is possible to find the coefficient $\Upsilon_{[t]}$ of the infinitesimal transformation

$$\dot{\bar{x}} = \dot{x} + \Upsilon(t, x, \dot{x})\delta + O(\delta^2) \tag{1.38}$$

directly from the coefficients $\omega$and Υ.

In Lie's second approach mentioned in Section 1.2, the differential equation is considered as a surface in the three-dimensional space of variables t, x, y given by,

$$F(t, x, y) = 0. \tag{1.39}$$

Here, t, x, y are considered to be three independent variables that transform as

$$\bar{t} = f(t, x, \delta), \quad \bar{x} = g(t, x, \delta), \bar{y} = Df/Dg, \tag{1.40}$$

where $D = \dfrac{\partial}{\partial t} + y\dfrac{\partial}{\partial x}$.

A symmetry group, in the sense of the second approach, is defined as the group of transformations such that its first prolongation leaves invariant the surface given by equation (1.39). The constraint on the transformation law for y that appears in equation (1.40) provides a connection with the first approach because the prolongation is consistent with the transformation law for first derivatives with the identification $y = \dot{x}$. This constraint provides the important fact of providing an algorithm for finding symmetry groups.

It is clear from the second approach that the symmetry group of equation (1.18) is identical to the invariance group for the surface given by equation (1.39) and does not depend on the existence of solutions of the differential equation. Because of this fundamental role played by the surface given by equation (1.39), it is called the *frame* of the differential equation.

In integrating differential equations, a decisive step is that of simplifying the frame. For this purpose, it suffices to "straighten out" the one-parameter symmetry group, that is, to reduce its action to a translation by a suitable change of the variables t and x. This automatically simplifies the equation by coverting its frame into a cylinder, that is, the explicit dependence on one of the variables t or eliminating x.

An invariant of the once-extended group is a function $h(t, x, \dot{x})$ of $t, x, \dot{x}$ whose value at any image point is the same as its value at a source point. That is,

$$h(\bar{t}, \bar{x}, \dot{\bar{x}}) = h(t, x, \dot{x}) \tag{1.41}$$

Differentiating equation (1.41) with respect to δ and setting $\delta = 0$, we obtain the first order linear partial differential equation for h, namely, $\omega h_t + \Upsilon h_x + \Upsilon_{[t]} h_{\dot{x}} = 0$.

The characteristic equations of which are,

$$\frac{dt}{\omega(t, x)} = \frac{dx}{\Upsilon(t, x)} = \frac{d\dot{x}}{\Upsilon_{[t]}(t, x, \dot{x})}. \tag{1.42}$$

These equations have two independent integrals and the most general solution for h is an arbitrary function of the two integrals.

Example 1.5.1. *Consider the Lie group,* $\bar{t} = \delta t, \quad \bar{x} = \delta^{\alpha} x$. *The coefficients of the infinitesimal transformation are given by,*

$$\omega(t, x) = t, \quad \Upsilon(t, x) = \alpha x, \quad \Upsilon_{[t]}(t, x, \dot{x}) = (\alpha - 1)\dot{x}.$$

The characteristic equation (1.42) gives,

$$\frac{dt}{t} = \frac{dx}{\alpha x} = \frac{d\dot{x}}{(\alpha - 1)\dot{x}}.$$

Solving this, we get the most general first order differential equation invariant under this Lie group is

$$\frac{\dot{x}}{t^{\alpha-1}} = F\left(\frac{t}{x^{\alpha}}\right),$$

where F is an arbitrary function.

Remark 1.5.1. Equation (1.37) can be written as

$$\Upsilon_{[t]} = \left[\frac{\Upsilon_t + \Upsilon_x \dot{x}}{\alpha_t + \alpha_x \dot{x}} - \frac{(\beta_t + \beta_x \dot{x})(\omega_t + \omega_x \dot{x})}{(\alpha_t + \alpha_x \dot{x})^2}\right],$$

which is written as the total directional derivative in the direction whose slope is $\dot{x}$.

Since the transformations, for a fixed δ determine the image C_1 of the curve C, it must be possible to calculate the k^{th} derivative, $\bar{x}^{(k)} = \dfrac{d^k \bar{x}}{d\bar{t}^k}$ of C_1 from the k^{th} derivative $x^{(k)} = \dfrac{d^k x}{dt^k}$ of C.
Using equation (1.38) we can find the infinitesimal coefficient $\Upsilon_{[kt]}$, where $\Upsilon_{[kt]} = \Upsilon_{\underbrace{[ttt\cdots t]}_{\text{k-times}}}$ corresponding to $x^{(k)}$ as follows:

$$d\bar{x}^{(k)} = dx^{(k)} + d\Upsilon_{[tk]}\delta + O(\delta^2), \quad d\bar{t} = dt + d\omega\delta + O(\delta^2) \tag{1.43}$$

Thus, we get,

$$\bar{x}^{(k+1)} = x^{(k+1)} + \left(\frac{d\Upsilon_{[tk]}}{dt} - x^{(k+1)}\frac{d\omega}{dt}\right)\delta + O(\delta^2). \tag{1.44}$$

Hence, we can define,

$$\Upsilon_{[tk]} = \frac{d\Upsilon_{[tk]}}{dt} - x^{(k+1)}\frac{d\omega}{dt}. \tag{1.45}$$

Inductively, $\Upsilon_{[tk]}$ is a function of $t, x, \dot{x}, x^{(2)}, x^{(3)}, \cdots, x^{(k)}$. Therefore,

$$\frac{d\Upsilon_{[tk]}}{dt} = \frac{\partial\Upsilon_{[tk]}}{\partial t} + \dot{x}\frac{\partial\Upsilon_{[tk]}}{\partial x} + x^{(2)}\frac{\partial\Upsilon_{[tk]}}{\partial \dot{x}} + x^{(3)}\frac{\partial\Upsilon_{[tk]}}{\partial x^{(2)}} + \cdots + x^{(k+1)}\frac{\partial\Upsilon_{[tk]}}{\partial x^{(k)}}.$$

Remark 1.5.2. Due to the profusion of terms appearing in the total derivative, the expression for $\Upsilon_{[tk]}$ rapidly becomes more complicated as k increases.

Example 1.5.2. *For the rotation group given in Example 1.2.5, the coefficients of the infinitesimals are $\omega(t,x) = -x$, $\quad \Upsilon(t,x) = t$. So,*

$$\Upsilon_{[t]} = \frac{d\Upsilon}{dt} - \dot{x}\frac{d\omega}{dt} = 1 + \dot{x}^2.$$

1.5.2 Prolongations

By definition, groups of point transformations act only on the space (t,x) of $n+m$ variables. However, one needs the transformation of derivatives in order to apply these groups to differential equations. Therefore it becomes necessary to extend a group of point transformations acting on the (t,x) space to groups of point transformations acting on the $(t,x,\dot{x})$ space, $(t,x,\dot{x},x^{(2)})$ space, $\cdots$, $(t,x,\dot{x},x^{(2)},\cdots,x^{(s)})$ space, $s \geq 1$, for a given differential equation with order s. These groups are called *the first prolongation, the second prolongation,* $\cdots$, *the s-times prolongation group,* respectively, where, following the notations from [26], the transformations are of the form,

$$\begin{aligned}
\bar{t} &= \phi^t(t,x,\delta) = t + \omega(t,x)\delta + O(\delta^2),\\
\bar{x} &= \phi^x(t,x,\delta) = x + \Upsilon(t,x)\delta + O(\delta^2),\\
\dot{\bar{x}} &= \phi^{\dot{x}}(t,x,\dot{x},\delta) = \dot{x} + \Upsilon_{[t]}(t,x,\dot{x})\delta + O(\delta^2),\\
\bar{x}^{(2)} &= \phi^{x^{(2)}}(t,x,\dot{x},x^{(2)},\delta) = x^{(2)} + \Upsilon_{[tt]}(t,x,\dot{x},x^{(2)})\delta + O(\delta^2),\\
&\vdots\\
\bar{x}^{(s)} &= \phi^{x^{(s)}}(t,x,\dot{x},x^{(2)},\cdots,x^{(s)},\delta) = x^{(s)} + \Upsilon_{\underbrace{[ttt\cdots t]}_{\text{s-times}}}(t,x,\dot{x},x^{(2)},\cdots,x^{(s)})\delta + O(\delta^2).
\end{aligned}$$

The prolongation transformation formulae of the components $\{\bar{x}^\alpha_{,i}\}$ of $\dot{\bar{x}}$ are determined by (More details on the prolongation formulas can be found in [49]).
$\begin{bmatrix} \bar{x}^\alpha_{,1} \\ \bar{x}^\alpha_{,2} \\ \vdots \\ \bar{x}^\alpha_{,n} \end{bmatrix} = \begin{bmatrix} (\phi^{\dot{x}})^\alpha_1(t,x,\dot{x},\delta) \\ (\phi^{\dot{x}})^\alpha_2(t,x,\dot{x},\delta) \\ \vdots \\ (\phi^{\dot{x}})^\alpha_n(t,x,\dot{x},\delta) \end{bmatrix} = B^{-1} \begin{bmatrix} D_1\phi^x(t,x,\delta) \\ D_2\phi^x(t,x,\delta) \\ \cdots \\ D_1\phi^x(t,x,\delta) \end{bmatrix}$ where B^{-1} is the inverse (assumed to exist) of the matrix $B = \begin{bmatrix} D_1\phi^t_1 & D_1\phi^t_2\cdots D_1\phi^t_n \\ D_2\phi^t_1 & D_2\phi^t_2\cdots D_2\phi^t_n \\ \vdots & \vdots \;\cdots\; \vdots \\ D_n\phi^t_1 & D_n\phi^t_2\cdots D_n\phi^t_n \end{bmatrix}$ and the prolongation transformation

of the formulas of the components $\{\bar{x}^{\alpha}_{i_1}\cdots i_s\}$ of $\bar{x}^{(s)}$ are determined by,

$$\begin{bmatrix} \bar{x}^{\alpha}_{,i_1\cdots i_{s-1}1} \\ \bar{x}^{\alpha}_{,i_1\cdots i_{s-1}2} \\ \vdots \\ \bar{x}^{\alpha}_{,i_1\cdots i_{s-1}n} \end{bmatrix} = \begin{bmatrix} (\phi^{x^{(s)}})^{\alpha}_{i_1\cdots i_{s-1}1}(t,x,\dot{x},x^{(2)},\cdots x^{(s)},\delta) \\ (\phi^{x^{(s)}})^{\alpha}_{i_1\cdots i_{s-1}2}(t,x,\dot{x},x^{(2)},\cdots x^{(s)},\delta) \\ \vdots \\ (\phi^{x^{(s)}})^{\alpha}_{i_1\cdots i_{s-1}n}(t,x,\dot{x},x^{(2)},\cdots x^{(s)},\delta) \end{bmatrix}$$
$$= B^{-1}\begin{bmatrix} D_1[(\phi^{x^{(s-1)}})^{\alpha}_{i_1\cdots i_{s-1}}(t,x,\dot{x},x^{(2)},\cdots x^{(s-1)},\delta)] \\ D_2[(\phi^{x^{(s-1)}})^{\alpha}_{i_1\cdots i_{s-1}}(t,x,\dot{x},x^{(2)},\cdots x^{(s-1)},\delta)] \\ \vdots \\ D_n[(\phi^{x^{(s-1)}})^{\alpha}_{i_1\cdots i_{s-1}}(t,x,\dot{x},x^{(2)},\cdots x^{(s-1)},\delta)] \end{bmatrix}$$

The formulas of the coefficients, $\Upsilon^{\alpha}_{i},\cdots,\Upsilon^{\alpha}_{i_1\cdots i_s}$, of the infinitesimal generator are determined by

$$\begin{aligned} \Upsilon^{\alpha}_{i} &= D_i(\Upsilon^{\alpha}) - x^{\alpha}_{,j}D_i(\omega_j), \\ \Upsilon^{\alpha}_{i_1i_2} &= D_{i_2}(\Upsilon^{\alpha}_{i_1}) - x^{\alpha}_{,i_1j}D_{i_2}(\omega_j), \\ &\vdots \\ \Upsilon^{\alpha}_{i_1i_2\cdots i_s} &= D_{i_s}(\Upsilon^{\alpha}_{i_1\cdots i_{s-1}}) - x^{\alpha}_{,i_1\cdots i_{s-1}j}D_{i_s}(\omega_j). \end{aligned}$$

The first prolonged generator of the first order differential operator $\zeta^* = \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x}$, which is the tangent vector field, is given by,

$$\zeta^*_1 = \omega_i\frac{\partial}{\partial t_i} + \Upsilon^{\alpha}\frac{\partial}{\partial x^{\alpha}} + \Upsilon^{\alpha}_{i}\frac{\partial}{\partial x^{\alpha}_{,i}}.$$

and the s times prolonged generator is written recurrently as,

$$\zeta^*_s = \zeta^*_{s-1} + \Upsilon^{\alpha}_{,i_1\cdots i_s}.$$

1.5.3 Lie-Bäcklund Representation

Let $\mathfrak{D}$ denote the vector space (with respect to the usual addition of functions) of all differentiable functions of finite order. It is closed under differentiation given by, $D_i = \frac{\partial}{\partial t_i} + x^{\alpha}_{,i}\frac{\partial}{\partial x^{\alpha}} + x^{\alpha}_{,ij}\frac{\partial}{\partial x^{\alpha}_{,ij}} + \cdots$ Consider the operator having the form,

$$\zeta = \omega_i\frac{\partial}{\partial t_i} + \Upsilon^{\alpha}\frac{\partial}{\partial x^{\alpha}} + \Upsilon^{\alpha}_{i}\frac{\partial}{\partial x^{\alpha}_{,i}} + \Upsilon^{\alpha}_{i_1i_2}\frac{\partial}{\partial x^{\alpha}_{,i_1i_2}} + \cdots, \tag{1.46}$$

where $\omega_i, \Upsilon^\alpha \in \mathfrak{D}$ are infinitely differentiable functions and,

$$\begin{cases} \Upsilon_i^\alpha = D_i(\Upsilon^\alpha - \omega_j x^\alpha_{,j}) + \omega_j x^\alpha_{,ji}, \\ \Upsilon^\alpha_{i1} i_2 = D_{i_2} D_{i_1}(\Upsilon^\alpha - \omega_j x^\alpha_{,j}) + \omega_j x^\alpha_{,ji_1i_2}, \\ \cdots \end{cases} \tag{1.47}$$

The operator given by (1.46) with coefficients given by (1.47) is called a *Lie-Bäcklund operator.*

Remark 1.5.3. The operator given by (1.46) is the infinite prolongation of

$$\zeta^* = \omega_i \frac{\partial}{\partial t^i} + \Upsilon^\alpha \frac{\partial}{\partial x^\alpha}, \quad \omega_i, \Upsilon^\alpha \in \mathfrak{D}.$$

Lemma 1.5.1. *The Lie-Bäcklund operator satisfies the commutation relation*

$$\zeta^* D_i - D_i \zeta^* = -D_i(\omega_j) D_j.$$

The proof of this lemma follows by a straightforward computation.

Lemma 1.5.2. *Every operator*

$$\tilde{\zeta^*} = \tilde{\omega}_i D_i = \tilde{\omega}_i \frac{\partial}{\partial t_i} + \tilde{\omega}_i x^\alpha_{,j} \frac{\partial}{\partial x^\alpha} + \tilde{\omega}_i x^\alpha_{,jj_1} \frac{\partial}{\partial x^\alpha_{,j_1}} + \cdots \tag{1.48}$$

with arbitrary analytic coefficients $\tilde{\omega}_i$ is a Lie-Bäcklund operator.

Remark 1.5.4. Rather than working with the full algebra, it is more advantageous to work with the factor algebra of all Lie-Bäcklund operators by its ideal L^* of operators (1.48). In accordance with this two Lie-Bäcklund operators ζ_1 and ζ_2 are said to be *equivalent* whenever $\zeta_1 - \zeta_2 \in L^*$.
In particular, every operator (1.47) is equivalent to a Lie-Bäcklund operator with coordinates $\omega_i = 0$ $(i = 1, 2, \cdots, n)$; given by

$$\zeta_1 \equiv \zeta_2 = (\Upsilon^\alpha - \omega_i x_i^\alpha) \frac{\partial}{\partial x^\alpha} + \cdots$$

Definition 1.5.1. A Lie-Bäcklund operator given by equation (1.47) of the form

$$\zeta = \Upsilon^\beta \frac{\partial}{\partial x^\beta}, \quad \Upsilon^\beta \in \mathfrak{D} \tag{1.49}$$

is called a *canonical Lie-Bäcklund operator.*

For such operators, a simpler form for the prolongation formulas given by (1.46) are

$$\Upsilon^{\alpha}_{i_1 i_2 \cdots i_s} = D_{i_1 i_2 \cdots i_s}(\Upsilon^{\alpha}). \tag{1.50}$$

On account of Lemma 1.5.1, the canonical Lie-Bäcklund operators commute with the differentiation operators D_i. Also, conversely, the operator (with $\omega_i = 0$) given by equation (1.46) commutes with the differentiation operators D_i implying that equation (1.50) holds.

The canonical Lie-Bäcklund operator although being convenient leads to a loss of geometric transparency in some cases. This is true for all groups of point transformations. For example, the simplest one-parameter group, which is the translation group $\bar{t_i} = t_i + \delta,\ \bar{x_i} = x_i$ with infinitesimal generator $\zeta^* = \dfrac{\partial}{\partial t_i}$ is reduced to the canonical form, namely $\zeta = \Upsilon^{\alpha}_i \dfrac{\partial}{\partial x^{\alpha}} + \cdots$

1.6 The Determining and Splitting Equations

Most of the times it may not be possible to explicitly find a group under which a differential equation is invariant. There is an approach to this difficulty, but it involves extensive computation, especially for partial differential equations.

The invariants u of any group are the solutions of the partial differential equation given by, $\omega u_t + \Upsilon u_x = 0$, which we shall write as $\zeta^* u = 0$, where ζ^* is the first order differential operator which is the *tangent vector field*,

$$\zeta^* = \omega(t,x)\frac{\partial}{\partial t} + \Upsilon(t,x)\frac{\partial}{\partial x}. \tag{1.51}$$

Lie called the operator given by equation (1.51) a *symbol* of the infinitesimal transformation given by equation (1.9). The terms *infinitesimal operator*, *group operator*, *Lie operator* and *group generator* came into use later. All these terms are used interchangeably.

The first prolongation was given by Lie as,

$$\zeta^*_1 = \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon_{[t]}\frac{\partial}{\partial \dot{x}}, \tag{1.52}$$

and the second prolongation was given by Lie as,

$$\zeta^*_2 = \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon_{[t]}\frac{\partial}{\partial \dot{x}} + \Upsilon_{[tt]}\frac{\partial}{\partial \ddot{x}}. \tag{1.53}$$

We consider constructing the group admitted by a given second order differential equation $w(t,x,\dot{x},\ddot{x}) = 0$, for which the invariance condition given by $\omega w_t + \Upsilon w_x + \Upsilon_{[t]} w_{\dot{x}} + \Upsilon_{[tt]} w_{\ddot{x}} = 0$, at $w = 0$ is obtained by operating $\zeta^*_2 w \mid_{w=0} = 0$.

In treating differential equations of the form,

$$\ddot{x} = \tilde{w}(t, x, \dot{x}), \tag{1.54}$$

the determining equations becomes,

$$\begin{aligned}\Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})\dot{w} + (\Upsilon_{xx} - 2\omega_{tx})\dot{w}^2 - \dot{w}^3\omega_{xx} \\ + (\Upsilon_x - 2\omega_t - 3\dot{x}\omega_x)\tilde{w} - [\Upsilon_t + (\Upsilon_x - \omega_t)\dot{x} - \dot{x}^2\omega_x]\tilde{w}_{\dot{x}} - \omega\tilde{w}_t - \Upsilon\tilde{w}_x = 0.\end{aligned} \tag{1.55}$$

These determining equations can be split into several independent equations. As a result we obtain an overdetermined system of differential equations for ω and Υ. Solving this system of determining equations, we can find all operators admitted by equation (1.54). In this section, the basic ideas of the method are explained by working the example of the simple second order differential equation $\ddot{x} = 0$. For this differential equation, we have, $\Upsilon_{[tt]} = 0$.
Now using equations (1.37) and (1.45) with $k = 1$, we get, for $\ddot{x} = 0$,

$$\Upsilon_{[tt]} = \Upsilon_{tt} + \dot{x}(2\Upsilon_{tx} - \omega_{tt}) + \dot{x}^2(\Upsilon_{xx} - 2\omega_{tx}) - \dot{x}^3\omega_{xx} = 0. \tag{1.56}$$

We refer to equation (1.56) as an invariant equation. Since equation (1.56) is an identity in $t, x, \dot{x}$, and since ω and Υ are functions of t and x only, the various powers of $\dot{x}$ must vanish separately to give us the splitting equations namely,

$$\Upsilon_{tt} = 0, \quad \omega_{xx} = 0, \quad \omega_{tt} = 2\Upsilon_{tx}, \quad \Upsilon_{xx} = 2\omega_{tx}. \tag{1.57}$$

According to the first two splitting equations,

$$\omega = A(t)x + B(t) \quad \text{and} \quad \Upsilon = C(x)t + D(x), \tag{1.58}$$

where A, B, C, D are functions yet to be determined.
According to the last two splitting equations,

$$\ddot{A}(t)x + \ddot{B}(t) = 2\dot{C}(x) \quad \text{and} \quad \ddot{C}(x)t + \ddot{D}(x) = 2\dot{A}(t). \tag{1.59}$$

By differentiating the first equation of (1.59) partially with respect to x and the second equation of (1.59) partially with respect to t, we get,

$$\ddot{A}(t) = 2\ddot{C}(x), \quad \ddot{C}(x) = 2\ddot{A}(t).$$

These equations imply $\ddot{A}(t) = 0 = \ddot{C}(x)$.
Substituting this equation in equation (1.59), we get,

$$\ddot{B}(t) = 2\dot{C}(x) \quad \text{and} \quad \ddot{D}(x) = 2\dot{A}(t),$$

are both constants, say $\ddot{B}(t) = 2c_{14}$ and $\ddot{D}(x) = 2c_{15}$.
Solving these equations we get,

$$B(t) = c_{14}t^2 + c_{16}t + c_{18} \quad \text{and} \quad D(x) = c_{15}x^2 + c_{20}x + c_{21}.$$

Therefore, we get,

$$A(t) = c_{15}t + c_{16} \quad \text{and} \quad C(x) = c_{14}x + c_{19.}$$

With these values, from equation (1.58), we get,

$$\omega = c_{14}t^2 + c_{15}tx + c_{16}t + c_{17}x + c_{18},$$

and,

$$\Upsilon = c_{15}x^2 + c_{14}tx + c_{19}t + c_{20}x + c_{21},$$

where the coefficients c_{14} through c_{21} are constants. The coefficients of the infinitesimal transformation ω and Υ belong to an eight parameter group whose infinitesimal transformation given by equation (1.51), can be obtained by taking one of the coefficients c_{14} through c_{21} to be 1 and the rest 0.
The ordinary differential equation $\ddot{x} = 0$ admits the eight parameter group spanned by the generators

$$\zeta_1^* = \frac{\partial}{\partial t}, \quad \zeta_2^* = \frac{\partial}{\partial x}, \quad \zeta_3^* = t\frac{\partial}{\partial t}, \quad \zeta_4^* = x\frac{\partial}{\partial t}, \quad \zeta_5^* = x\frac{\partial}{\partial x},$$
$$\zeta_6^* = t\frac{\partial}{\partial x}, \quad \zeta_7^* = tx\frac{\partial}{\partial t} + x^2\frac{\partial}{\partial x}, \quad \zeta_8^* = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x}.$$

Remark 1.6.1. In order to work with this approach using the determining and splitting equations, it may not be possible to find the closed form of the group to which the differential equation is invariant. Hence, we assume our infinitesimals ω and Υ of a certain form and proceed. The most commonly used forms are

$$\omega = A(t), \quad \Upsilon = B(t)x + C(t) \text{ or } \omega = A(x), \quad \Upsilon = B(x)t + C(x)$$
$$\text{or } \omega = A(t), \quad \Upsilon = B(x) \text{ or } \omega = A(x), \quad \Upsilon = B(t).$$

1.7 Lie Algebras

The determining equation is a linear partial differential equation in $\omega(t, x)$ and $\Upsilon(t, x)$ and hence it follows that the set of all its solutions is a vector space. However, there is another property that is intrinsic to determining equations. A set of solutions of any determining equations forms what is called a *Lie algebra.* (This term was introduced by H. Weyl; Sophus Lie himself used the term *infinitesimal group.*)

Definition 1.7.1. Let $\mathbb{F}$ be a field. A Lie algebra over $\mathbb{F}$ is an $\mathbb{F}-$ vector space L, together with a bilinear map, the *Lie bracket*

$$L \times L \to L, \quad (S_1, S_2) \mapsto [S_1, S_2],$$

satisfying the following properties:

1. $[S_1, S_1] = 0, \ \forall S_1 \in L.$
2. (Jacobi identity) $[S_1, [S_2, S_3]] + [S_2, [S_3, S_1]] + [S_3, [S_1, S_2]] = 0, \ \forall S_1, S_2, S_3 \in L.$

The Lie bracket $[S_1, S_2]$ is referred to as a commutator of S_1 and S_2.
Condition (1) of the definition of Lie algebra implies that $[S_1, S_2] = -[S_2, S_1]$, $\forall \ S_1, S_2 \in L$, which is known as anti-symmetry.

Remark 1.7.1. The associative property, $[[S_1, S_2], S_3] = [S_1, [S_2, S_3]]$, holds if and only if $\forall \ S_1, S_2 \in L$, the commutator $[S_1, S_2]$ lies in the centre of of L which is defined as $Z(L) = \{S_i \in L \mid [S_i, S_j] = 0 \ \forall \ S_j \in L\}$.

Definition 1.7.2. (Commutator) Let $S_i = \omega_{i\alpha}(x)\dfrac{\partial}{\partial x_\alpha}$, $S_j = \omega_{j\beta}(x)\dfrac{\partial}{\partial x_\beta}$, where $x = (x_1, x_2, \cdots, x_n)$. Then the commutator of S_i and S_j is defined as:

$$[S_i, S_j] = S_iS_j - S_jS_i = \sum_{\alpha,\beta=1}^{n} \Big[\left(\omega_{i\alpha}(x)\frac{\partial}{\partial x_\alpha}\right)\left(\omega_{j\beta}(x)\frac{\partial}{\partial x_\beta}\right) - \left(\omega_{j\beta}(x)\frac{\partial}{\partial x_\beta}\right)\left(\omega_{i\alpha}(x)\frac{\partial}{\partial x_\alpha}\right)\Big] = \sum_{\beta=1}^{n} \varUpsilon_\beta(x)\frac{\partial}{\partial x_\beta},$$

where $\varUpsilon_\beta(x) = \sum_{\alpha=1}^{n}\left[\omega_{i\alpha}(x)\dfrac{\partial \omega_{j\beta}(x)}{\partial x_\alpha} - \omega_{j\beta}(x)\dfrac{\partial \omega_{i\beta}(x)}{\partial x_\beta}\right]$.

Remark 1.7.2. If we have the operators

$$S_i = \omega_i\frac{\partial}{\partial t} + \varUpsilon_i\frac{\partial}{\partial x}, \quad i = 1, 2.,$$

then we can define,

$$[S_1, S_2] = (S_1(\omega_2) - S_2(\omega_1))\frac{\partial}{\partial t} + (S_1(\varUpsilon_2) - S_2(\varUpsilon_1))\frac{\partial}{\partial x}.$$

Example 1.7.1. *Let* $S_1 = t\frac{\partial}{\partial t}, \quad S_2 = tx\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right)$*, then,*

$$\begin{aligned}
[S_1, S_2] &= S_1 S_2 - S_2 S_1 \\
&= \left(t\frac{\partial}{\partial t}\right)\left(tx\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right)\right) - \left(tx\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right)\right)\left(t\frac{\partial}{\partial t}\right) \\
&= tx\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right) \\
&= S_2.
\end{aligned}$$

Definition 1.7.3. (Basis for a Lie algebra) The basis for a Lie algebra is the basis of its underlying vector space. More precisely, a set B is a basis for a Lie algebra L if:

1. B is linearly independent, and
2. $\text{span}(B) = L$.

Definition 1.7.4. (Structure Constants) If L is a Lie algebra over a field $\mathbb{F}$ with a basis $\{S_1, S_2, \cdots, S_n\}$, we define scalars $c_{ijk} \in \mathbb{F}$ such that $[S_i, S_j] = \sum_{k=1}^{n} c_{ijk} S_k$. The scalars c_{ijk} are called the structure constants of L with respect to the basis.

Remark 1.7.3. 1. The structure constants depends on the choice of the basis of L.

2. By the definition of Lie algebra, it is sufficient to know the structure constants c_{ijk} for $1 \leq i < j \leq n$.

Theorem 1.7.1. *Let L_1 and L_2 be Lie algebras. Then L_1 is isomorphic to L_2 if and only if there is a basis B_1 of L_1 and a basis B_2 of L_2 such that the structure constants of L_1 with respect to B_1 are equal to the structure constants of L_2 with respect to B_2.*

Definition 1.7.5. (Subalgebra) Given a Lie algebra L, the vector subspace $M \subset L$ is called a subalgebra of L if $[S, T] \in M, \ \forall \ S, T \in \ M$.

Definition 1.7.6. (Ideal) An ideal of a Lie algebra L is a subspace I of L such that $[S_1, S_2] \ \in \mathrm{I}, \forall \ S_1 \ \in L, \ S_2 \ \in \mathrm{I}$.

Example 1.7.2. *The set of operators given by equation (1.48) is an ideal in the Lie algebra of all Lie-Bäcklund operators with the product as* $[S_1, S_2] = S_1 S_2 - S_2 S_1$.

Definition 1.7.7. (Quotient or Factor algebra) Let I be an ideal of L, then I is in particular a subspace of L, and so we may consider the cosets, $L/\mathrm{I} = \{z + \mathrm{I} \mid z \in L\}$. By defining the Lie bracket on L/I as:

$$[w + \mathrm{I}, z + \mathrm{I}] = [w, z] + \mathrm{I}, \ \forall \ w, z \in \ L,$$

L/I becomes a Lie algebra with this bracket and is called the quotient or factor algebra of L by I.

Definition 1.7.8. (Derived algebra) Let M and T be ideals in L. We define $[M, T]$ to be the span of the commutators of elements of M and T rather than just the set of such commutators. An important example of this construction occurs when we take $M = T = L$. We write L' for $[L, L]$, and L' is known as the derived algebra of L. (We note that L' is an ideal of L).

Definition 1.7.9. (Derived series) We define the derived series of L to be the series with terms

$$L^{(1)} = L', \quad \text{and} \quad L^{(k)} = [L^{(k-1)}, L^{(k-1)}], \ \text{for} \ k \geq 2.$$

Remark 1.7.4. As the product of ideals is an ideal, $L^{(k)}$ is an ideal of L (and not just $L^{(k-1)}$.)

Definition 1.7.10. (Solvable Lie algebra) A Lie algebra L is said to be solvable if for some $m \geq 1$ we have $L^{(m)} = 0$.

Theorem 1.7.2. *If L is a Lie algebra with ideals*

$$L = I_0 \supset I_1 \supset \cdots \supset I_{m-1} \supset I_m = \{0\},$$

such that I_{k-1}/I_k *is abelian for* $1 \leq k \leq m$, *then L is solvable.*

Theorem 1.7.3. *Let L be a Lie algebra.*

(a) If L is solvable, then every sub-algebra and every homomorphic image of L are solvable.

(b) Suppose that L has an ideal I such that I and L/I are solvable. Then L is solvable.

Remark 1.7.5. (i) If we can find a sequence

$$L = L_r \supset L_{r-1} \supset \cdots \supset L_1$$

of sub-algebras of dimension $r, r-1, \cdots, 1$, respectively such that $\forall\ s = 2, 3, \cdots, r,\ L_{s-1}$ is an ideal in L_s, then the Lie algebra L is solvable.

(ii) Since the derived algebra of any two dimensional Lie algebra is abelian, it follows that every two dimensional Lie algebra is solvable.

(iii) The vector space of all 2×2 matrices over $\mathbb{C}$, with trace zero and with the Lie bracket defined by $[x, y] = xy - yx$, where xy is the usual product of matrices x and y is denoted by $sl(2, \mathbb{C})$ and is known as the special linear algebra. For this three dimensional Lie algebra, we have $sl(2, \mathbb{C})' = sl(2, \mathbb{C})$. Consequently, this Lie algebra is not solvable. In fact, upto isomorphism, it is the only three dimensional Lie algebra L such that $L' = L$.

More details on Lie algebras can be found in [18].

1.8 Symmetry Analysis of First Order Partial Differential Equations

1.8.1 Construction of Infinitesimals for First Order Partial Differential Equations

The computations required in order to determine a Lie group for partial differential equations is lengthy. As such, we shall restrict ourselves only to a class of partial differential equations, in order to keep our calculations to the minimum.
Let $u = u(t, x)$. Then we consider transformations with h_1, h_2, h_3 as smooth functions in t, x, u having convergent Taylor series in δ which are of the form,

$$\begin{cases} \bar{t} = h_1(t, x, u, \delta) = t + \delta T(t, x, u) + O(\delta^2), \\ \bar{x} = h_2(t, x, u, \delta) = x + \delta X(t, x, u) + O(\delta^2), \\ \bar{u} = h_3(t, x, u, \delta) = u + \delta U(t, x, u) + O(\delta^2), \end{cases} \tag{1.60}$$

where $T(t, x, u) = \left.\dfrac{\partial h_1}{\partial \delta}\right|_{\delta=0}, \quad X(t, x, u) = \left.\dfrac{\partial h_2}{\partial \delta}\right|_{\delta=0}, \quad U(t, x, u) = \left.\dfrac{\partial h_3}{\partial \delta}\right|_{\delta=0}.$
In order to calculate the prolongation of a given transformation, we need to differentiate (1.60) with respect to each of the parameters t and x. To do this we introduce the following *total derivatives*:

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + \cdots, \tag{1.61}$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \cdots. \tag{1.62}$$

The first two equations of (1.60) may be inverted (locally) to give t and x in terms of $\bar{t}$ and $\bar{x}$, provided that the Jacobian is non-zero, that is,

$$\mathcal{J} = \begin{vmatrix} D_t\bar{t} & D_t\bar{x} \\ D_x\bar{t} & D_x\bar{x} \end{vmatrix} \neq 0, \quad \text{when} \quad u = u(x, t). \tag{1.63}$$

If equation (1.63) is satisfied, then the last equation of (1.60) can be rewritten as

$$\bar{u} = \bar{u}(\bar{t}, \bar{x}). \tag{1.64}$$

Applying the chain rule to equation (1.64), we obtain,

$$\begin{bmatrix} D_t\bar{u} \\ D_x\bar{u} \end{bmatrix} = \begin{bmatrix} D_t\bar{t} & D_t\bar{x} \\ D_x\bar{t} & D_x\bar{x} \end{bmatrix} \begin{bmatrix} \bar{u}_{\bar{t}} \\ \bar{u}_{\bar{x}} \end{bmatrix},$$

and therefore by Cramer's rule,

$$\bar{u}_{\bar{t}} = \frac{1}{\mathcal{J}}\begin{vmatrix} D_t\bar{u} & D_t\bar{x} \\ D_x\bar{u} & D_x\bar{x} \end{vmatrix}, \quad \bar{u}_{\bar{x}} = \frac{1}{\mathcal{J}}\begin{vmatrix} D_t\bar{t} & D_t\bar{u} \\ D_x\bar{t} & D_x\bar{u} \end{vmatrix} \tag{1.65}$$

Equation (1.65) can be simplified to get the extended infinitesimal representation,

$$\bar{u}_{\bar{t}} = u_t + \delta U_{[t]} + O(\delta^2), \quad \bar{u}_{\bar{x}} = u_x + \delta U_{[x]} + O(\delta^2), \tag{1.66}$$

where,

$$U_{[t]} = D_t(U) - u_x D_t(X) - u_t D_t(T), \tag{1.67}$$

$$U_{[x]} = D_x(U) - u_x D_x(X) - u_t D_x(T). \tag{1.68}$$

The explicit expression for equation (1.68) is

$$U_{[t]} = U_t - X_t u_x + (U_u - T_t)u_t - X_u u_x u_t - T_u u_t^2,$$

$$U_{[x]} = U_x + (U_u - X_x)u_x - T_x u_t - X_u u_x^2 - T_u u_x u_t.$$

The Lie invariance condition for the first order partial differential equation $\Delta(t, x, u, u_t, u_x) = 0$ is $\zeta^{(1)^*}\Delta\Big|_{\Delta=0} = 0$, where

$$\zeta^{(1)^*} = T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + U\frac{\partial}{\partial u} + U_{[t]}\frac{\partial}{\partial u_t} + U_{[x]}\frac{\partial}{\partial u_x}.$$

Remark 1.8.1. The infinitesimal generator (or tangent vector field) of the Lie group is given by,

$$\zeta^* = T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + U\frac{\partial}{\partial u}. \tag{1.69}$$

1.8.2 Group Analysis of a Hamilton–Jacobi Type Equation

Consider a Hamilton–Jacobi equation given by,

$$u_t = u_x^2. \tag{1.70}$$

The determining equation (1.76) for equation (1.70) after substituting equation (1.74) and eliminating u_t from equation (1.70) is,

$$U_t - (X_t + 2U_x)u_x + (2X_x - T_t - U_u)u_x^2 + (X_u + 2T_x)u_x^3 + T_u u_x^4 = 0.$$

Setting the various coefficients of u_x to zero, we get the splitting equations namely,

$$U_{tt} = 0, \quad T_u = 0, \quad X_t + 2U_x = 0, \quad X_u + 2T_x = 0, \quad 2X_x - T_t - U_u = 0.$$

Solving these equations, we get,

$$T(t, x, u) = A(t, x), \quad X(t, x, u) = -2A_x u + B(t, x),$$

$$U(t, x, u) = -2A_{xx}u^2 + (2B_x - A_t)u + C(t, x),$$

where $A(t, x), B(t, x), C(t, x)$ are arbitrary functions.
Substituting these equations in the splitting equations we get,

$$A_{xxx} = 0, \quad A_{tx} - B_{xx} = 0, \quad B_t + 2C_x = 0, \quad A_{txx} = 0, \quad A_{tt} - 2B_{tx} = 0, \quad C_t = 0.$$

These equations can be solved to give,

$$A = c_{36}x^2 + (2c_{37}t + c_{38})x + 4c_{39}t^2 + c_{40}t + c_{41},$$

$$B = c_{37}x^2 + (4c_{39}t + c_{42})x + 2c_{43}t + c_{44},$$

$$C = -c_{39}x^2 - c_{43}x + c_{45}.$$

This in turn gives the infinitesimals,

$$T(t, x, u) = c_{36}x^2 + (2c_{37}t + c_{38})x + 4c_{39}t^2 + c_{40}t + c_{41},$$

$$X(t, x, u) = -2(2c_{36}x + 2c_{37}t + c_{38})u + c_{37}x^2 + (4c_{39}t + c_{42})x + 2c_{43}t + c_{44},$$

$$U(t, x, u) = -4c_{36}u^2 + (2c_{37}x - c_{40} + 2c_{42})u - c_{39}x^2 - c_{43}x + c_{45},$$

where $c_{36}, \cdots, c_{45}$ are arbitrary constants.
The presence of ten arbitrary constants signifies that the Lie algebra is ten dimensional. It is generated by,

$$\zeta_1^* = x^2\frac{\partial}{\partial t} - 4xu\frac{\partial}{\partial x} - 4u^2\frac{\partial}{\partial u}, \quad \zeta_2^* = 2tx\frac{\partial}{\partial t} + (x^2 - 4tu + 2xu)\frac{\partial}{\partial x} + 2xu\frac{\partial}{\partial u},$$
$$\zeta_3^* = x\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial x}, \quad \zeta_4^* = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - x^2\frac{\partial}{\partial u}, \quad \zeta_5^* = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, \quad \zeta_6^* = \frac{\partial}{\partial t},$$
$$\zeta_7^* = x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}, \quad \zeta_8^* = 2t\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}, \quad \zeta_9^* = \frac{\partial}{\partial x}, \quad \zeta_{10}^* = \frac{\partial}{\partial u}.$$

We note that, for ordinary differential equations, we introduced new variables, u and v, called the canonical variables, such that the original equation reduced to a new ordinary differential equation that was separable and independent of the variable v.

Precisely, under a change of variables $(t, x) \rightarrow (u, v)$, and by using equation (1.31), equation (1.51) gives $\zeta^* = \dfrac{\partial}{\partial v}$, which represents that the new ordinary differential equation admits translation in the variable v.

We extend a similar idea to partial differential equations. In order to obtain solutions of equation (1.70), we introduce new independent variables r and s and a new dependent variable v such that,

$$r = r(t, x, u), \quad s = s(t, x, u), \quad v = v(t, x, u),$$

then by employing a change of variables $(t, x, u) \rightarrow (r, s, v)$, we get,

$$\begin{aligned}
\zeta^* &= T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + U\frac{\partial}{\partial u}, \\
&= T\left(r_t\frac{\partial}{\partial r} + s_t\frac{\partial}{\partial s} + v_t\frac{\partial}{\partial v}\right) + X\left(r_x\frac{\partial}{\partial r} + s_x\frac{\partial}{\partial s} + v_x\frac{\partial}{\partial v}\right) \\
&\quad + U\left(r_u\frac{\partial}{\partial r} + s_u\frac{\partial}{\partial s} + v_u\frac{\partial}{\partial v}\right) \\
&= (Tr_t + Xr_x + Ur_u)\frac{\partial}{\partial r} + (Ts_t + Xs_x + Us_u)\frac{\partial}{\partial s} \\
&\quad + (Tv_t + Xv_x + Uv_u)\frac{\partial}{\partial v},
\end{aligned}$$

which upon choosing,

$$Tr_t + Xr_x + Ur_u = 0,$$

$$Ts_t + Xs_x + Us_u = 1,$$

$$Tv_t + Xv_x + Uv_u = 0,$$

gives $\zeta^* = \dfrac{\partial}{\partial s}$.

Hence, after a change of variables, the original equation $F(t, x, u, u_t, u_x) = 0$, transforms into an equation independent of s and takes the form, $G(r, v, v_r, v_s) = 0$. However, the transformed equation is still a partial differential equation; but if we assume that we are only interested in solutions of the form $v = v(r)$, then the transformed equation becomes $K(r, v, v_r) = 0$, which is an ordinary differential equation!

Thus, just as in the case for ordinary differential equations, we find the canonical coordinates (r, s, v) for partial differential equations and note a similar result for partial differential equations below:

Theorem 1.8.1. *A one-parameter group of transformations* $\bar{t} = f(t, x, u, \delta)$, $\bar{x} = g(t, x, u, \delta)$, $\bar{u} = h(t, x, u, \delta)$ *can be reduced under a suitable change of variables, called canonical variables*
$r = r(t, x, u)$, $s = s(t, x, u)$, $v = v(t, x, u)$, *to the translation group* $\bar{r} = r$, $\bar{s} = s + \delta$, $\bar{v} = v$.

Having constructed the symmetries of equation (1.70), as a particular case with $c_{40} = 1 = c_{42}$, and the remaining constants zero, we get,

$$T(t,x,u) = t, \quad X(t,x,u) = x, \quad U(t,x,u) = u.$$

By using change of variables described above,

$$tr_t + xr_x + ur_u = 0,$$

$$ts_t + xs_x + us_u = 1,$$

$$tv_t + xv_x + uv_u = 0.$$

The solutions are given by,

$$r = R\left(\frac{x}{t}, \frac{u}{t}\right), \quad s = \ln t + S\left(\frac{x}{t}, \frac{u}{t}\right), \quad v = V\left(\frac{x}{t}, \frac{u}{t}\right),$$

where R, S, T are arbitrary functions of t, x, u.
If we choose these to get,

$$r = \frac{x}{t}, \quad s = \ln t, \quad v = \frac{u}{t},$$

then transforming the equation (1.70) will give,

$$v + v_s - rv_r = v_r^2.$$

Setting $v_s = 0$, and simplifying gives us the Clairaut equation,

$$v_r^2 + rv_r - v = 0,$$

whose general solution is,

$$v = -\frac{1}{4}r^2, \quad v = c_{46}r + c_{46}^2,$$

where c_{46} is an arbitrary constant. Passing through the transformation we get the exact solution given by,

$$u = -\frac{1}{4}\frac{x^2}{t}, \quad u = c_{46}x + c_{46}^2 t.$$

1.9 Symmetry Analysis of Second Order Partial Differential Equations

1.9.1 Construction of Infinitesimals for Second Order Partial Differential Equations

In continuation with Section 1.8.1, we can obtain higher-order prolongations. If $\bar{u}_\Omega$ is any derivative of $\bar{u}$ with respect to $\bar{t}$ and $\bar{x}$, then

$$\bar{u}_{\Omega\bar{t}} = \frac{\partial \bar{u}_\Omega}{\partial \bar{t}} = \frac{1}{\mathcal{J}} \begin{vmatrix} D_t\bar{x} & D_t\bar{u}_\Omega \\ D_x\bar{x} & D_x\bar{u}_\Omega \end{vmatrix} \tag{1.71}$$

$$\bar{u}_{\Omega\bar{x}} = \frac{\partial \bar{u}_\Omega}{\partial \bar{x}} = \frac{1}{\mathcal{J}} \begin{vmatrix} D_t\bar{u}_\Omega & D_t\bar{t} \\ D_x\bar{u}_\Omega & D_x\bar{t} \end{vmatrix} \tag{1.72}$$

In particular, the transformation of the second derivative is as follows

$$\bar{u}_{\bar{t}\bar{t}} = \frac{1}{\mathcal{J}} \begin{vmatrix} D_t\bar{x} & D_t\bar{u}_{\bar{t}} \\ D_x\bar{x} & D_x\bar{u}_{\bar{t}} \end{vmatrix}, \quad \bar{u}_{\bar{x}\bar{x}} = \frac{1}{\mathcal{J}} \begin{vmatrix} D_t\bar{u}_{\bar{x}} & D_t\bar{t} \\ D_x\bar{u}_{\bar{x}} & D_x\bar{t} \end{vmatrix}, \tag{1.73}$$

$$\bar{u}_{\bar{x}\bar{t}} = \frac{1}{\mathcal{J}} \begin{vmatrix} D_t\bar{x} & D_t\bar{u}_{\bar{x}} \\ D_x\bar{x} & D_x\bar{u}_{\bar{x}} \end{vmatrix} = \frac{1}{\mathcal{J}} \begin{vmatrix} D_t\bar{u}_{\bar{t}} & D_t\bar{t} \\ D_x\bar{u}_{\bar{t}} & D_x\bar{t} \end{vmatrix} \tag{1.74}$$

On simplifying (1.74) we get the extended infinitesimal representations, namely

$$\bar{u}_{\bar{t}\bar{t}} = u_{tt}+\delta U_{[tt]}+O(\delta^2), \quad \bar{u}_{\bar{x}\bar{x}} = u_{xx}+\delta U_{[xx]}+O(\delta^2), \quad \bar{u}_{\bar{t}\bar{x}} = u_{tx}+\delta U_{[tx]}+O(\delta^2) \tag{1.75}$$

where,

$$U_{[tt]} = D_t(U_{[t]}) - u_{tx}D_t(X) - u_{tt}D_t(T), \quad U_{[xx]} = D_x(U_{[x]}) - u_{xx}D_x(X) - u_{tx}D_x(T), \tag{1.76}$$

and,

$$\begin{aligned} U_{[tx]} &= D_t(U_{[x]}) - u_{xx}D_t(X) - u_{tx}D_t(T). \\ &= D_x(U_{[t]}) - u_{tx}D_x(X) - u_{tt}D_x(T). \end{aligned} \tag{1.77}$$

The explicit expressions for $U_{[tt]}, U_{[xx]}, U_{[tx]}$ given by equations (1.76) and (1.77) are

$$\begin{aligned} U_{[tt]} = U_{tt} &- X_{tt}u_x + (2U_{tu} - T_{tt})u_t - 2X_{tu}u_xu_t + (U_{uu} - 2T_{tu})u_t^2 - X_{uu}u_xu_t^2 \\ &- T_{uu}u_t^3 - 2X_tu_{xt} - 2X_uu_tu_{xt} + (U_u - 2T_t)u_{tt} - X_uu_xu_{tt} - 3T_uu_tu_{tt}, \end{aligned} \tag{1.78}$$

$$
\begin{aligned}
U_{[xx]} = {} & U_{xx} + (2U_{xu} - X_{xx})u_x - T_{xx}u_t + (U_{uu} - 2X_{xu})u_x^2 - 2T_{xu}u_xu_t - X_{uu}u_x^3 \\
& - T_{uu}u_x^2u_t + (U_u - 2X_x)u_{xx} - 2T_xu_{xt} - 3X_uu_xu_{xx} - T_uu_tu_{xx} - 2T_uu_xu_{xt}, \quad (1.79)
\end{aligned}
$$

$$
\begin{aligned}
U_{[xt]} = {} & U_{xt} + (U_{tu} - X_{xt})u_x + (U_{xu} - T_{xt})u_t - X_{tu}u_x^2 + (U_{uu} - X_{xu} - T_{tu})u_xu_t \\
& - T_{xu}u_t^2 - X_{uu}u_x^2u_t - T_{uu}u_xu_t^2 - X_tu_{xx} - X_uu_tu_{xx} + (U_u - X_x - T_t)u_{xt} \\
& - 2X_uu_xu_{xt} - 2T_uu_tu_{xt} - T_xu_{tt} - T_uu_xu_{tt}. \quad (1.80)
\end{aligned}
$$

If the second order partial differential equation is $K(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0$, then the determining equation is given by

$$
T\frac{\partial K}{\partial t} + X\frac{\partial K}{\partial x} + U\frac{\partial K}{\partial u} + U_{[t]}\frac{\partial K}{\partial u_t} + U_{[x]}\frac{\partial K}{\partial u_x} + U_{[tt]}\frac{\partial K}{\partial u_{tt}} + U_{[tx]}\frac{\partial K}{\partial u_{tx}} + U_{[xx]}\frac{\partial K}{\partial u_{xx}}\bigg|_{K=0} = 0. \quad (1.81)
$$

Remark 1.9.1. The infinitesimal generator of the Lie group is given by equation (1.69).

1.9.2 Group Methods for the One-Dimensional Wave Equation

The one-dimensional wave equation with constant speed is given by $u_{tt} = c^2u_{xx}$, where $u = u(x, t)$ denotes displacement from initial position, t denotes time, x denotes position and c denotes velocity of propagation.
The determining equation (1.81) gives $U_{[tt]} = c^2U_{[xx]}$, where $U_{[tt]}$ and $U_{[xx]}$ are defined by equations (1.78) and (1.79). Hence, the determining equations become,

$$
\begin{aligned}
& U_{tt} - c^2U_{xx} - (X_{tt} + c^2(2U_{xu} - X_{xx}))u_x + (2U_{tu} - T_{tt} + c^2T_{xx})u_t - c^2(U_{uu} - 2X_{xu})u_x^2 \\
& - 2(X_{tu} + T_{xu})u_xu_t + (U_{uu} - 2T_{ut})u_t^2 + c^2X_{uu}u_x^3 + c^2T_{uu}u_x^2u_t - X_{uu}u_xu_t^2 - T_{uu}u_t^3 \\
& - c^2(U_u - 2X_x)u_{xx} + (2c^2T_x - 2X_t)u_{xt} + c^2(U_u - 2T_t)u_{xx} + 3c^2X_uu_xu_{xx} + c^2T_uu_tu_{xx} \\
& + 2c^2T_uu_xu_{tx} - 2X_uu_tu_{xt} - c^2X_uu_xu_{xx} - 3c^2T_uu_tu_{xx} = 0.
\end{aligned}
$$

Splitting the determining equations with respect to $u_x, u_t, u_x^2, u_xu_t, u_t^2, u_x^3, u_x^2u_t, u_xu_t^2, u_t^3, u_{xx}$, $u_{xt}, u_xu_{xx}, u_tu_{xx}, u_xu_{tx}, u_tu_{xt}$, we get,

$$
U_{tt} - c^2U_{xx} = 0, \quad X_{tt} + c^2(2U_{xu} - X_{xx}) = 0, \quad 2U_{tu} - T_{tt} + c^2T_{xx} = 0,
$$

$$
U_{uu} - 2X_{xu} = 0, \quad X_{tu} + T_{xu} = 0, \quad U_{uu} - 2T_{ut} = 0, \quad X_{uu} = 0,
$$

$$
T_{uu} = 0, \quad X_x - T_t = 0, \quad c^2T_x - X_t = 0, \quad X_u = 0, \quad T_u = 0.
$$

Solving these equations, we get,

$$
T(t, x, u) = A(t, x), \quad X(t, x, u) = B(t, x), \quad U(t, x, u) = P(t, x)u + Q(t, x),
$$

where, $P_t = 0, \quad P_x = 0 \Rightarrow P(t,x) = c_{47}$, a constant.
Hence, $U = c_{47}u + Q(t,x)$, where A, B are arbitrary functions of t and x, and Q satisfies the wave equation.
The wave equation admits a four dimensional Lie algebra with generators,

$$\zeta_1^* = A(t,x)\frac{\partial}{\partial t}, \quad \zeta_2^* = B(t,x)\frac{\partial}{\partial x}, \quad \zeta_3^* = u\frac{\partial}{\partial u}, \quad \zeta_4^* = Q(t,x)\frac{\partial}{\partial u}.$$

We now seek a solution of the wave equation by making a special choice of the infinitesimals namely, $A = x, \quad B = t, \quad Q = 0$.
The associated invariant surface condition is
$xu_x + tu_t = c_{47}u$, which is solved to get $u = x^p F\left(\frac{t}{x}\right)$, where $p = c_{47}$ and F is an arbitrary function.
Substituting in the one-dimension wave equation, we get,

$$(c^2r^2 - 1)F''(r) - 2c^2r(p-1)F'(r) + c^2p(p-1)F(r) = 0 \quad \text{where} \quad r = \frac{t}{x}. \tag{1.82}$$

This can be integrated easily giving,

$$F(r) = \begin{cases} \dfrac{c_1}{c_2} log\left(\dfrac{rc-1}{rc+1}\right) & \text{if } p = 0 \\ c_1\left(r - \dfrac{1}{c}\right)^p + c_2\left(r + \dfrac{1}{c}\right)^p & \text{if } p \neq 0 \end{cases}$$

If $p = 0$, exact solution is $u(x,t) = \frac{c_1}{c^2} log\left(\frac{ct-x}{ct+x}\right) + c_2$.
If $p \neq 0$, exact solution is $u(x,t) = x^p\left[c_1F_1\left(\frac{t}{x}\right) + c_2F_2\left(\frac{t}{x}\right)\right]$,
where $F_i, i = 1,2$ are solutions of equation (1.82).
The solutions of equation (1.82) for some integer values of p are presented in the table below:

p	F_1	F_2
1	1	r
2	$\frac{c^2r^2-2cr+1}{c^2}$	$\frac{c^2r^2+2cr+1}{c^2}$
3	$\frac{c^3r^3-3c^2r^2+3cr-1}{c^3}$	$\frac{c^3r^3+3c^2r^2+3cr+1}{c^3}$
4	$\frac{c^4r^4-4c^3r^3+6c^2r^2-4cr+1}{c^4}$	$\frac{c^4r^4+4c^3r^3+6c^2r^2+4cr+1}{c^4}$
-1	$\frac{c}{cr-1}$	$\frac{c}{cr+1}$
-2	$\frac{c^2}{c^2r^2-2cr+1}$	$\frac{c^2}{c^2r^2+2cr+1}$
-3	$\frac{c^3}{c^3r^3-3c^2r^2+3cr-1}$	$\frac{c^3}{c^3r^3+3c^2r^2+3cr+1}$
-4	$\frac{c^4}{c^4r^4-4c^3r^3+6c^2r^2-4cr+1}$	$\frac{c^4}{c^4r^4+4c^3r^3+6c^2r^2+4cr+1}$

.

We shall conclude the analysis for the wave equation by obtaining a particular solution of the wave equation with unit speed.
To obtain a particular solution, consider, $A = t, \quad B = x, \quad P = Q = 0.$
The associated invariant surface condition is $tu_x + xu_t = 0$
The solution of this linear partial differential equation is $u(x,t) = F(t^2 - x^2)$, where F is arbitrary.
To find F, we use the wave equation $u_{tt} = u_{xx}$, and get $rF''(r) + F'(r) = 0$, where $r = t^2 - x^2$.
The solution of this Clairaut differential equation is
$F(r) = c_{48}ln(r) + c_{49}$, where c_{48}, c_{49} are arbitrary constants.
Re substituting r, we get, $u(x,t) = c_{48}ln(t^2 - x^2) + c_{49}$, which is the exact solution of the one-dimensional unit speed wave equation.
The analysis of the uni speed one-dimensional wave equation can be found in [5].

1.10 Admitted Generator For Functional Differential Equations Using Lie-Bäcklund Operators

In this section we shall present the existing literature found in [40] on obtaining the admitted generator for (second-order) delay and neutral differential equations. This approach uses the Lie-Bäcklund operator and an invariant manifold theorem.
We shall assume the infinitesimal generator of the Lie group admitted by the functional

differential equation[2]

$$x'' = f(t, x(t), x(t-r), x'(t), x'(t-r)), \tag{1.83}$$

is given by equation (1.51).
The corresponding Lie-Bäcklund operator has the form

$$\zeta^* = \eta(t, x, x')\frac{\partial}{\partial x}, \tag{1.84}$$

where $\eta = \Upsilon - x'\omega$. In order to obtain the determining equations for functional differential equations (second-order described here), one has to prolong the canonical Lie-Bäcklund operator to the six-dimensional space of variables $(t, x(t), x(t-r), x'(t), x'(t-r), x''(t))$:

$$\zeta_{\mathcal{B}} = \eta^x\frac{\partial}{\partial x} + \eta^{x_r}\frac{\partial}{\partial x(t-r)} + \eta_{x'}\frac{\partial}{\partial x'} + \eta^{x'_r}\frac{\partial}{\partial x'(t-r)} + \eta^{x''}\frac{\partial}{\partial x''}. \tag{1.85}$$

where

$$\eta^x(t, x, x') = \Upsilon(t, x) - x'\omega(t, x),$$

$$\eta^{x_r}(t, x_r, x'_r) = \eta^x(t-r, x(t-r), x'(t-r)) = \eta(t-r, x(t-r)) - x'(t-r)\omega(t-r, x(t-r)),$$

$$\eta^{x'}(t, x, x', x'') = D(\eta^x) = \Upsilon_t(t, x) + [\Upsilon_x(t, x) - \omega_t(t, x)]x' - \omega_x(t, x)(x')^2 - \omega(t, x)x'',$$

$$\begin{aligned}\eta^{x'_r}(t, x(t-r), x'(t-r), x''(t-r)) &= \eta^{x'}(t-r, x(t-r), x'(t-r), x''(t-r))\\ &= \Upsilon_t(t-r, x(t-r)) + [\Upsilon_x(t-r, x(t-r))\\ &- \omega_t(t-r, x(t-r))]x'(t-r) - \omega_x(t-r, x(t-r))\\ &(x'(t-r))^2 - \omega(t-r, x(t-r))x''(t-r),\end{aligned}$$

$$\begin{aligned}\eta^{x''}(t, x, x', x'', x''') &= D(\eta^{x'}) = \Upsilon_{tt}(t, x) + [2\Upsilon_{tx}(t, x) - \omega_{tt}(t, x)]x'\\ &= [\Upsilon_{xx}(t, x) - 2\omega_{tx}(t, x)](x')^2 - \omega_{xx}(t, x)(x')^3\\ &+ [\Upsilon_x(t, x) - 2\omega_t(t, x)]x'' - 3\omega_x(t, x)x'x'' - \omega(t, x)x''',\end{aligned}$$

where D is the total derivative operator with respect to t given by, $D = \dfrac{\partial}{\partial t} + x'\dfrac{\partial}{\partial x} + \cdots$.
The determining equation for the functional differential equation (1.83) is given by,

$$\zeta_{\mathcal{B}}\left(x'' - f(t, x(t), x(t-r), x'(t), x'(t-r))\right)|_{(1.83)} = 0. \tag{1.86}$$

Equation (1.86) has to be satisfied by any solution of equation (1.83).
Substituting $x''' = f_t + x'f_x + x'_r f_{x_r} + x''f_{x'} + x''_r f_{x'_r} + x''_r f_{x'_r}$, $\quad x'' = f$ and $x''_r = f_r$, the

[2]For functional differential equations the notation x' means $\dfrac{dx}{dt}$. Similarly the notation x'' means $\dfrac{d^2x}{dt^2}$ and so on.

determining equation (1.86) can be written as

$$\begin{aligned}&-\omega_{xx}(x')^3+(\Upsilon_{xx}-2\omega_{tx}+\omega_x f_{x'})(x')^2+\omega^r_{x_r}f_{x'_r}(x'_r)^2+(2\Upsilon_{tx}-\omega_{tt}x')\\&+(\omega_t+\Upsilon_x)f_{x'}x'-3\omega_x f x'+\Upsilon_{tt}-\Upsilon_t f_{x'}+(\Upsilon_x-2\omega_t)f-\Upsilon^r_t f_{x'_r}+(\omega^r_t-\Upsilon^r_{x_r})f_{x'_r}x'_r\\&-f_t\omega-f_x\Upsilon-\Upsilon^r f_{x_r}+(\omega^r-\omega)f_{x_r}x'_r+(\omega^r-\omega)f_r f_{x'_r}=0,\end{aligned}\quad (1.87)$$

where $f_r=f(t-r,x(t-r),x(t-2r),x'(t-r),x'(t-2r))$.
By virtue of the Cauchy problem, one can account the variables[3] t,x,x_r,x',x'_r,x_{2r} and x'_{2r} where $x_{2r}=x(t-2r)$ and $x'_{2r}=x'(t-2r)$ in equation (1.87) as arbitrary variables.
If $f_{x'_r}\neq 0$, then splitting the determining equation (1.87) with respect to x'_{2r}, we get $\omega=\omega^r$.
If $f_{x'_r}=0$, then $f_{x_r}\neq 0$. Then splitting the determining equation (1.87) with respect to x'_r, we again get $\omega=\omega^r$.
This shows the periodic property of ω, that is,

$$\omega(t,x)=\omega(t-r,x(t-r)). \quad (1.88)$$

As this property is satisfied for any solution of the Cauchy problem, equation (1.88) implies that ω does not depend on x, that is, $\omega_x=0$. Moreover, the property (1.88) allows us to write the determining equation (1.87) as

$$\tilde{\zeta}^*_2\left(x''-f(t,x,x_r,x',x'_r)\right)|_{(1.83)}=0, \quad (1.89)$$

where $\tilde{\zeta}^*_2=\zeta_{\mathcal{B}}+\omega D=\omega\dfrac{\partial}{\partial t}+\Upsilon\dfrac{\partial}{\partial x}+\Upsilon_{[t]}\dfrac{\partial}{\partial x'}+\Upsilon^r\dfrac{\partial}{\partial x_r}+\Upsilon^r_{[t]}\dfrac{\partial}{\partial x'_r}+\Upsilon_{[tt]}\dfrac{\partial}{\partial x''}$,
where all symbols have their usual meaning and expressions as seen in the earlier sections.
In addition,

$$\Upsilon^r=\Upsilon(t-r,x(t-r)),$$

and,

$$\Upsilon^r_{[t]}=\Upsilon_t(t-r,x(t-r))+[\Upsilon_x(t-r,x(t-r))-\omega_t(t-r,x(t-r))]x'_r-\omega_x(t-r,x(t-r))(x'_r)^2.$$

The generator $\tilde{\zeta}^*_2$ acts in the space of variables (t,x,x_r,x',x'_r,x''), whereas the coefficients of the operator $\zeta_{\mathcal{B}}$ include the derivatives x''_r and x'''.
Equation (1.89) means the manifold defined by equation (1.83) is an invariant manifold of the generator $\tilde{\zeta}^*_2$. As a result of the invariant manifold theorem, any invariant manifold can be represented through invariants of the generator $\tilde{\zeta}^*_2$. Hence, for describing equations admitting the generator ζ^*, one needs to find all invariants of the generator $\tilde{\zeta}^*_2$.
Direct calculations show that if two generators X_1 and X_2 are admitted by equation (1.83),

[3] The notations x_r,x'_r,x''_r means $x(t-r),x'(t-r),x''(t-r)$ respectively. Similarly the notations ω^r means $\omega(t-r,x(t-r))$.

then their commutator $[X_1, X_2]$ is also admitted by equation (1.83). This property allows stating that the set of infinitesimal generators admitted by equation (1.83) composes a Lie algebra on the real plane.

The above can be summarized as the following theorem:

Theorem 1.10.1. *The second order functional differential equation given by (1.83), which contains the infinitesimal generator ω, obeys the periodic property for ω. That is,*

$$\omega(t, x) = \omega(t - r, x(t - r)).$$

Remark 1.10.1. The admitted group for second order functional (neutral and delay) differential equations is described above. The same procedure can also be used to get the admitted generator for first order functional differential equations.

A great detail of literature on group methods for ordinary and partial differential equations can be found in [2, 3, 8, 11, 14, 17, 24, 25, 26, 31, 50, 57].

Chapter 2

Symmetry Analysis of First Order Delay Differential Equations With Constant Coefficients

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2.1 Introduction

The symmetry approach developed for differential equations, cannot be applied to equations with nonlocal terms, as in delay differential equations. The nonlocality does not allow the manifold's approach for defining an admitted Lie group (i.e mapping any solution of equations into a solution of the same equation), given by Lie. In the sense of applications of group analysis for constructing equations, this definition of an admitted Lie group is more appropriate: it excludes the possibilities where an equation admits a Lie group, but the equation has no solution [22]. This definition was applied to study integro differential equations [19, 20, 32]. Some retarded equations being of the type of integro differential equations, we got a similar idea to use the approach, particularly for delay differential equations.

The definition of an admitted Lie group allows us to construct the determining equations. The determining equations are then split with respect to arbitrary elements. In the case of differential equations the arbitrary elements are parametrical derivatives. In the case of analytical systems the parametrical derivatives are dictated by the Cauchy-Kovalevskaya theorem for a Cauchy type systems and by the Cartan-Khäler theorem for involutive systems. For other types of equations an answer to the question about arbitrary elements is obtained on the basis of a theorem of existence of the Cauchy problem.

In this chapter, we state an approach developed in [34] where a delay differential equation is replaced by an underdetermined system of differential equations for which classical group analysis is applied. The reduction to underdetermined system widens a class of equations admitted by a given Lie group. But an extension of equations narrows a set of admitted Lie groups.

This chapter is devoted to studying symmetry analysis to first order delay differential equations with constant coefficients.

2.2 Construction of Determining equations

As delay differential equations have terms with delay (nonlocal terms), this serves as a major hindrance to applying symmetry analysis. To overcome this hindrance, we adopt the following method. We describe the method for functional differential equations with one independent variable

$$S \equiv x'(t) - F(t, x_t) = 0. \tag{2.1}$$

If $\chi(t)$ is a function defined at least on $[t-r,t]$, then one defines a new function $\chi_t : [-r,0] \to D$ by

$$\chi_t(s) = \chi(t+s), \quad s \in [-r,0]$$

where D is an open subset in $\mathbb{R}^n$, J is some interval in $\mathbb{R}$, F is a functional. Here we have used notations accepted in literature given in [15, 21, 43]. For delay differential equations

$$F(t,x_t) \equiv f(t,\chi(g_1(t)),\cdots,\chi(g_m(t))),$$

where $f : [t_0,\beta) \times D^m \to \mathbb{R}^n$, and $g_j(t) \leq t$ for $t_0 \leq t \leq \beta$ for each $j = 1,2,\cdots,m$. A continuous function $\chi(t), t \in [t_0-r, t_0+\beta)$ is called a solution of the delay differential equation if it is differentiable in the interval (t_0,β) and satisfies equation (2.1) in the interval (t_0,β). The value $\chi'(t_0)$ is understood as the right hand derivative.
As done for differential equations, let the symmetry group G of transformations f_δ :

$$\bar{t} = f_1(t,x;\delta), \quad \bar{x} = f_2(t,x;\delta),$$

where f_1 and f_2 are smooth functions in t and x having a convergent Taylor series in δ, with $t = f_1(t,x;0)$ and $x = f_2(t,x;0)$, map solutions of equation (2.1) to solutions of the same equation. We usually consider, the following infinitesimal generator instead of a Lie group,

$$X = \omega(t,x)\frac{\partial}{\partial t} + \Upsilon(t,x)\frac{\partial}{\partial x},$$

where,

$$\omega(t,x) = \frac{\partial f_1}{\partial \delta}(t,x;0), \quad \Upsilon(t,x) = \frac{\partial f_2}{\partial \delta}(t,x;0).$$

Let $x = \chi(t)$ be a solution. A parametrical representation of the transformed function $\chi_\delta(\bar{t})$ is given by the equations $\bar{t} = f_1(t,\chi(t);\delta), \bar{x} = f_2(t,\chi(t);\delta)$. In order to find $\chi_\delta(\bar{t})$, one has to define

$$t = \Psi(\bar{t};\delta), \tag{2.2}$$

from the equation $\bar{t} = f_1(t,\chi(t);\delta)$. For differential equations this is guaranteed by local inverse function theorem. For delay differential equations (2.1), one has to define the function Ψ not only in a neighbourhood of the point t, but also in the interval $[t-r,t]$ and in a right-hand neighbourhood of t. For obtaining this it is not enough only a local inverse function theorem. Assume that the given Lie group possess this property. Hence, the function $\chi_\delta(\bar{t})$ is defined by $\chi_\delta(\bar{t}) = f_2(\Psi(\bar{t};\delta),\chi(\Psi(\bar{t};\delta));\delta)$, and then,

$$\frac{d\bar{\chi}(\bar{t})}{d\bar{t}} = (f_{1,1} + f_{1,2}\chi'(\Psi(\bar{t};\delta)))\frac{\partial \Psi(\bar{t};\delta)}{\partial \bar{t}},$$

where $f_{1,i}$ means the partial derivative of f with respect to the i^{th} argument. Thus

$$F(\bar{t},\bar{\chi}_{\bar{t}}) = f(t,f_{11},\cdots,f_{1m}),$$

where $f_{1i} = f_1(\Psi(g_i(\bar{t});\delta), \chi(\Psi(g_i(\bar{t});\delta));\delta), \quad i = 1, 2, \cdots, m.$
Let $\Delta(t, x_t, x') = x' - F(t, x_t)$. Since $\chi(t)$ and $\chi_\delta(\bar{t})$ are solutions, then,

$$\Delta(t, x_t, x') \equiv 0. \tag{2.3}$$

for $x = \chi(t)$ and $x = \chi_\delta(\bar{t})$. Differentiating the functions $\Delta(t, x_t, x')$, where $x = \chi_\delta(\bar{t})$, with respect to the group parameter δ, then setting $\delta = 0$, we get,

$$\frac{\partial \Delta(t, x_t, x')}{\partial \delta} = 0. \tag{2.4}$$

The left side of these equations is expressed only through the coefficients of the infinitesimal generator X, their derivatives, the function $\chi(t)$ and its derivatives. This we denote by

$$S(\chi, \omega, \Upsilon) \equiv \frac{\partial \Delta(t, x_t, x')}{\partial \delta}, \text{ at } \delta = 0.$$

Thus, equation (2.4) becomes

$$S(\chi, \omega, \Upsilon) = 0.$$

where $\chi(t)$ is an arbitrary solution of (2.1).
Note that equation (2.4) coincides with the equations obtained in the result of an action onto (2.1) by the prolonged canonical Lie-Bäcklund operator equivalent to the generator X:

$$\hat{X} = (\Upsilon(t, x) - \omega(t, x)x')\frac{\partial}{\partial x} + \cdots$$

For functional differential equations the action of the derivative $\dfrac{\partial}{\partial x}$ has to be considered in the sense of Frechet derivative; this is the difference in applying the canonical operator as applied for ordinary differential equations. A Lie group satisfying,
At $[S]$,

$$S(\chi, \omega, \Upsilon) = \hat{X}(S) = 0. \tag{2.5}$$

for any solution of (2.1), is called an admitted Lie group. Equation (2.5) is called the determining equation. The notation $[S]$ in (2.5) means that equation (2.5) has to be satisfied for any solution of equation (2.1).
The definition of admitted Lie group admits the following features:

1. They must be satisfied for any solution of equation (2.1).
2. The definition is free from the requirement for the admitted Lie group to have (2.2) globally.
3. In the sense of Lie Bäcklund representation, the definition coincides with one of the classical definitions of an admitted Lie group in the case of differential equations.
4. Was applied for integro-differential equations.

5. Can be applied for finding an equivalence group, contact and Lie-Bäcklund transformations for functional differential equations.

The determining equations are then split with respect to arbitrary elements. Since arbitrary elements of delay differential equations are contained in determining equations by similar ways, as for differential equations, the process of solving determining equations for delay differential equations is similar to obtaining solutions of determining equations for ordinary differential equations.

2.3 Infinitesimal Generator Admitted by the First Order Delay Differential Equation With Constant Coefficients

In this section we extend group analysis to delay differential equations of the form,

$$x'(t) = ax(t) + bx(t-r). \tag{2.6}$$

We define the operator,

$$\zeta = (\Upsilon - x'\omega)\frac{\partial}{\partial x} + (\Upsilon^r - x'^r\omega^r)\frac{\partial}{\partial x^r} + \Upsilon_{[t]}\frac{\partial}{\partial x'} = 0. \tag{2.7}$$

Let,

$$\Delta = x'(t) - ax(t) - bx(t-r). \tag{2.8}$$

Then by Lie's Invariance condition, at $\Delta = 0, \zeta\Delta = 0$, using the notations $x^r = x(t-r),\ \omega^r = \omega(t-r, x(t-r)),\ \Upsilon^r = \Upsilon(t-r, x(t-r))$, we get,
$(\Upsilon - x'\omega)\frac{\partial}{\partial x}(x'(t) - ax(t) - bx(t-r)) + (\Upsilon^r - x'^r\omega^r)\frac{\partial}{\partial x^r}(x'(t) - ax(t) - bx(t-r)) + \Upsilon_{[t]}\frac{\partial}{\partial x'}(x'(t) - ax(t) - bx(t-r)) = 0.$
Therefore, at (t_0, x_0), we get,
$\Phi(t_0, x_0, x_1, x_2, x_3) = -a\Upsilon(t_0, x_0) + ax'(t_0 - r)\omega(t_0, x_0) - b\Upsilon(t_0 - r, x_0 - r) + bx'(t_0 - r)\omega(t_0 - r, x_0 - r) + \Upsilon_{[t]} = 0.$
Let, $x'(t_0) = x_1, x_0 = \psi(t_0), x_2 = \psi(t_0 - r), x_3 = \psi'(t_0 - r)$.
Now,

$$\begin{aligned}\Upsilon_{[t]} &= \Upsilon_t + x'\Upsilon_x - x'\omega_t - x'^2\omega_x \\ &= \Upsilon_t + (ax((t) + bx(t-r))\Upsilon_x - (ax(t) + bx(t-r)\omega_t - (ax(t) + bx(t-r))^2\omega_x.\end{aligned}$$

Therefore, at (t_0, x_0),
$\Upsilon_{[t]} = (a\Upsilon_x(t_0, x_0) - a\omega_t(t_0, x_0)x_0 + (b\Upsilon_x(t_0, x_0) - b\omega_t(t_0, x_0)x_2 + \Upsilon_t(t_0, x_0) - a^2\omega_x(t_0, x_0)x_0^2 - 2ab\omega_x(t_0, x_0)x_0x_2 + b^2\omega_x(t_0, x_0)x_2^2.$
Hence, we get,
$\Phi(t_0, x_0, x_1, x_2, x_3) = -a\Upsilon(t_0, x_0) + (a\omega(t_0, x_0) + b\omega(t_0 - r, x_0 - r)x_3 - b\Upsilon(t_0 - r, x_0 - r) +$

$(a\Upsilon_x(t_0, x_0) - a\omega_t(t_0, x_0)x_0 + (b\Upsilon_x(t_0, x_0) - b\omega_t(t_0, x_0)x_2 + \Upsilon_t(t_0, x_0) - a^2\omega_x(t_0, x_0)x_0^2 - 2ab\omega_x(t_0, x_0)x_0x_2 + b^2\omega_x(t_0, x_0)x_2^2.$

By splitting this determining equation, we get,

$$a\Upsilon_x - a\omega_t = 0, \tag{2.9}$$

$$b\Upsilon_x - b\omega_t = 0, \tag{2.10}$$

$$a^2\omega_x = 0, \tag{2.11}$$

$$2ab\omega_x = 0, \tag{2.12}$$

$$b^2\omega_x = 0, \tag{2.13}$$

$$a\omega(t_0, x_0) + b\omega(t_0 - r, x_0 - r) = 0, \tag{2.14}$$

$$\Upsilon_t - b\Upsilon(t_0 - r, x_0 - r) - a\Upsilon = 0. \tag{2.15}$$

From equations (2.11), (2.12) and (2.13), $\omega_x = 0 \Rightarrow \omega = \omega(t)$.
From equations (2.9) and (2.10), $\Upsilon(t, x) = \omega'(t)x + \alpha(t)$, where $\alpha(t)$ is an arbitrary function of t. From equations (2.14), (2.15), if $a \neq -b, \quad \omega = 0$, hence, $\Upsilon = \alpha(t)$, where $\alpha(t)$ is an arbitrary solution of equation (2.6)
The generator is given by $X = \alpha(t)\dfrac{\partial}{\partial x}$, where $\alpha(t)$ is an arbitrary solution of equation (2.6).
If $a = -b$, from equations (2.9)-(2.13), $\omega = c$, an arbitrary constant, $\Upsilon = \beta(t)$, where $\beta(t)$ is an arbitrary solution of equation (2.6).
The generator in this case is $X = c\dfrac{\partial}{\partial t} + \beta(t)\dfrac{\partial}{\partial x}$, where c is an arbitrary constant and $\beta(t)$ is an arbitrary solution of equation (2.6).

2.4 Summary

In this chapter, a Lie-Bäcklund operator for first order delay differential equations is defined. We have discussed its construction and properties. Further, using this operator, we have obtained symmetries of a first order linear delay differential equation with constant coefficients.

Chapter 3

Symmetry Analysis of First Order Delay Differential Equations

3.1 Introduction

Delay differential equations are shown in [33] to have a wide range of applications in control theory, signal processing, heat transfer problems, developing population models, biosciences (blood flow and disease related problems), ecology, evolution of species, electrical networking, physics, study of epidemics, etc.

In [59] we are shown the construction of an equivalent Lie-Bäcklund operator discussed by [25, 26] which is applied to delay differential equations to obtain symmetries. Linchuk in [34] suggests a group method to research functional differential equations based on a search of symmetries of underdetermined differential equations by methods of classical and modern group analysis, using the principle of factorization. His method encompasses the use of a basis of invariants consisting of universal and differential invariants. In the previous chapter, an admitted Lie group has been defined for first order delay differential equations with constant coefficients, using Lie Bäcklund operators, a method different from the one contained herein; the results of which are seen in the previous chapter.

In this chapter, we establish group methods to delay differential equations of the type

$$x'(t) = f(t, x, x(g(t))), \text{where } g(t) \text{ is a differentiable function such that } g(t) < t, \quad (3.1)$$

and where f is a real valued function defined on $I \times D^2$, with I as an open interval in $\mathbb{R}$ and D an open set in $\mathbb{R}$. We also assume $\dfrac{\partial f}{\partial x(g(t))} \neq 0$. The process of applying group methods involve some steps. We first need to find a group under which the delay differential equation is invariant. We can then use this group for obtaining symmetries of the delay differential equation. We call this the admitted Lie group, for which we shall mean that each transformation carries a solution of the differential equation to a solution of the same equation. We do not define any equivalent Lie-Bäcklund operator, but obtain a Lie type invariance condition, using which we define certain operators required in obtaining the desired symmetries. We shall be using Taylor's theorem to do this. The equations we obtain by acting our operator on the delay differential equation, will be called as determining equations. We shall then split these equations with respect to the independent variables to obtain an overdetermined system of partial differential equations which we shall call splitting equations. We shall finally solve this system to obtain the symmetry algebra of the delay differential equation. We shall make a complete group classification of the first order differential equation containing variable coefficients, and with the most general and most standard time delay.

The rest of the chapter is organized as followed: In the following section, the ideas of group methods for ordinary differential equations have been extended to first order delay differential equations, in a manner different from existing literature for delay differential equations. In the subsequent Section 3.3 and Section 3.4, symmetry analysis has been applied for first order linear delay differential equations with most general and most standard time delay, for which no literature has been found. The section that follows illustrates the classification and invariance under a Lie group, a delay differential equation of importance in some practical models. The next section demonstrates group classification and invariance under a Lie group, a nonlinear delay differential equation extensively studied by [28, 30] in researching population growth models.

3.2 Lie Type Invariance Condition for First Order Delay Differential Equations

Formally, a first order delay differential equation is defined as follows:

Definition 3.2.1. (First Order Delay Differential Equation)
Let J be an interval in $\mathbb{R}$, and let D be an open set in $\mathbb{R}$. Sometimes J will be $[t_0, \beta)$, and sometimes it will be (α, β), where $\alpha \leq t_0 \leq \beta$. Let $f : J \times D^3 \to \mathbb{R}$. Conveniently, a first order delay differential equation is expressed as

$$x'(t) = f(t, x(g_1(t)), x(g_2(t))), \tag{3.2}$$

where x and f are real valued functions, and each $g_j(t)$ is a retarded argument i.e. $g_j(t) \leq t$. Often $g_1(t) \equiv t$.
We consider equation (3.2) for $t_0 \leq t \leq \beta$ together with the initial function

$$x(t) = \theta(t), \text{for} \quad \gamma \leq t \leq t_0, \tag{3.3}$$

where $\gamma \in \mathbb{R}, \quad \gamma < t_0$ and θ is a given initial function mapping $[\gamma, t_0] \to D$.

Definition 3.2.2. (Solution of a First Order Delay Differential Equation)
By a solution of delay differential equations (3.2) and (3.3) we mean a continuous function $x : [\gamma, \beta_1) \to D$, for some $\beta_1 \in (t_0, \beta]$, such that,

1. $x(t) = \theta(t)$ for $\gamma \leq t \leq t_0$.
2. $x(t)$ reduces equation (3.2) to an identity on $t_0 \leq t \leq \beta_1$.

We understand $x'(t_0)$ to mean the right-hand derivative.

In this section we extend the results to delay differential equations of type equation (3.1). To determine the delay differential equation completely, we need to specify the delay term, where the delayed function is specified, otherwise the problem is not fully determined.

We obtain a Lie type invariance for delay differential equations using Taylor's theorem for a function of several variables:

Theorem 3.2.1. *For the first order delay differential equation*

$$\frac{dx}{dt} = F(t, x, g(t), x(g(t))), \tag{3.4}$$

defined on $I_1 \times D_1 \times I_2 \times D_2$*, where* I_1, I_2 *are open intervals in* $\mathbb{R}$ *and* D_1, D_2 *are open sets in* $\mathbb{R}$*, the Lie invariance condition is given by*

$$\omega F_t + \Upsilon F_x + \omega^{g(t)} F_{g(t)} + \Upsilon^{g(t)} F_{x(g(t))} = \Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2,$$

where the notations $\omega^{g(t)}$ *and* $\Upsilon^{g(t)}$ *mean* $\omega(g(t), x(g(t)))$ *and* $\Upsilon(g(t), x(g(t)))$ *respectively.*

Proof. Let the delay differential equation be invariant under the Lie group
$\bar{t} = t + \delta\omega(t, x) + O(\delta^2)$,
$\bar{x} = x + \delta\Upsilon(t, x) + O(\delta^2)$.
We then naturally define,
$\overline{g(t)} = g(t) + \delta\omega(g(t), x(g(t))) + O(\delta^2)$,
$\overline{x(g(t))} = x(g(t)) + \delta\Upsilon(g(t), x(g(t))) + O(\delta^2)$.
Then,

$$\begin{aligned}
\frac{d\bar{x}}{d\bar{t}} &= \frac{\frac{d\bar{x}}{dt}}{\frac{d\bar{t}}{dt}} \\
&= \left[\frac{dx}{dt} + (\Upsilon_t + \Upsilon_x x')\delta + O(\delta^2)\right][1 - (\omega_t + \omega_x x')\delta + O(\delta^2)] \\
&= \frac{dx}{dt} + [\Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2]\delta + O(\delta^2)
\end{aligned}$$

For invariance, $\frac{d\bar{x}}{d\bar{t}} = F(\bar{t}, \bar{x}, \overline{g(t)}, \overline{(x(g(t)))})$.
With the notations,
$\omega^{g(t)} = \omega(g(t), x(g(t))), \Upsilon^{g(t)} = \Upsilon(g(t), x(g(t)))$, we get,

$$\begin{aligned}
\frac{dx}{dt} + [\Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2]\delta + O(\delta^2) &= F(t + \delta\omega + O(\delta^2), x + \delta\Upsilon + O(\delta^2), \\
&\quad g(t) + \delta\omega^{g(t)} + O(\delta^2), x(g(t)) + \delta\Upsilon^{g(t)} \\
&\quad + O(\delta^2)) \\
&= F(t, x, g(t), x(g(t))) + \\
&\quad (\omega F_t + \Upsilon F_x + \omega^{g(t)} F_{g(t)} + \Upsilon^{g(t)} F_{x(g(t))})\delta \\
&\quad + O(\delta^2).
\end{aligned}$$

Comparing the coefficient of δ, we get

$$\omega F_t + \Upsilon F_x + \omega^{g(t)} F_{g(t)} + \Upsilon^{g(t)} F_{x(g(t))} = \Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2. \tag{3.5}$$

The above equation (3.5) obtained is a Lie type invariance condition. □

Similar to the case of ordinary differential equations, we can define a prolonged operator, for delay differential equations as:
$\zeta = \omega\frac{\partial}{\partial t} + \omega^{g(t)}\frac{\partial}{\partial g(t)} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^{g(t)}\frac{\partial}{\partial x(g(t))}$
With the notation,
$D_t = \frac{\partial}{\partial t} + x'\frac{\partial}{\partial x}$
We can write,

$$\begin{aligned}\frac{d\bar{x}}{d\bar{t}} &= \frac{dx}{dt} + (D_t(\Upsilon) - x' D_t(\omega))\delta + O(\delta^2)\\ &= \frac{dx}{dt} + \Upsilon_{[t]}\delta + O(\delta^2),\end{aligned}$$

where $\Upsilon_{[t]} = D_t(\Upsilon) - x' D_t(\omega)$.
We then define the extended operator as:

$$\zeta^{(1)} = \omega\frac{\partial}{\partial t} + \omega^{g(t)}\frac{\partial}{\partial g(t)} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^{g(t)}\frac{\partial}{\partial x(g(t))} + \Upsilon_{[t]}\frac{\partial}{\partial x'}. \tag{3.6}$$

Defining, $\Delta = x'(t) - F(t, x(t), g(t), x(g(t))) = 0$, we get,

$$\zeta^{(1)}\Delta = \Upsilon_{[t]} - \omega F_t - \Upsilon F_x - \omega^{g(t)} F_{g(t)} - \Upsilon^{g(t)} F_{x(g(t))}. \tag{3.7}$$

Comparing equation (3.7) and equation (3.5), we get,
$\Upsilon_{[t]} = \Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2$.
On substituting $x' = F$ into $\zeta^{(1)}\Delta = 0$, we get an invariance condition for the delay differential equation which is $\zeta^{(1)}\Delta\mid_{\Delta=0} = 0$, from which we shall obtain the determining equations.

3.3 Symmetries of First Order Non-homogeneous Differential Equations With Most General Time Delay

We establish the following result:

Theorem 3.3.1. *Consider the delay differential equation*

$$x'(t) = p(t)x(t) + q(t)x(g(t)) + s(t). \tag{3.8}$$

Then the symmetries of the non-homogeneous delay differential equation

$x'(t) = p(t)x(t) + q(t)x(g(t)) + s(t)$ where g is a sufficiently smooth non constant function such that $g(t) < t$, $p(t)$, $q(t)$ and $s(t)$ are sufficiently smooth functions satisfying $q(t)$ not identically zero, admit a symmetry algebra of infinite dimension due to the linear superposition principle, and is represented by the vector fields, $x_2(t)\frac{\partial}{\partial x}$ and $(x - x_1(t))\frac{\partial}{\partial x}$. Further, if the delay differential equation is homogeneous, then it admits a symmetry algebra which is again of infinite dimension, given by the vector fields $x_2(t)\frac{\partial}{\partial x}$ and $x\frac{\partial}{\partial x}$, where $x_1(t)$ is an arbitrary solution of (3.8) and $x_2(t)$ is the general solution of the associated homogeneous delay differential equation.

Proof. Let,

$$g(t) = h(t). \tag{3.9}$$

We look out for a coefficient of the infinitesimal generator ω of the form (that is we assume) $\omega(t, x) = \omega(t)$. Then applying equation (3.6) to the delay equation given by (3.9), we get,

$$\omega(h(t)) = h'(t)\omega(t). \tag{3.10}$$

Applying operator $\zeta^{(1)}$ defined by equation (3.6) to equation (3.8), we get,

$$\begin{aligned} \Upsilon_t(t, x) + (\Upsilon_x(t, x) - \omega'(t))(p(t)x(t) + q(t)x(g(t)) + s(t)) = \\ \omega(t)(p'(t)x(t) + q'(t)x(g(t)) + s'(t)) + p(t)\Upsilon(t, x) + q(t)\Upsilon^{g(t)}. \end{aligned} \tag{3.11}$$

Differentiating the above equation with respect to $x(g(t))$ twice, we get,

$$\Upsilon(t, x) = M(t)x + N(t). \tag{3.12}$$

Substituting equation (3.12) in equation (3.11), and splitting the equation with respect to $x(t)$, $x(g(t))$, and the constant term, we get

$$M(t) = p(t)\omega(t) + M_0, \quad M_0 = \text{constant}. \tag{3.13}$$

$$q(t)\omega'(t) + \omega(t)q'(t) = q(t)(M(t) - M(g(t))). \tag{3.14}$$

$$N'(t) = p(t)N(t) + q(t)N(g(t)) + s(t)\omega'(t) - s(t)M(t). \tag{3.15}$$

Substitute (3.13) in (3.14)
$q(t)\omega'(t) + \omega(t)q'(t) = q(t)(p(t)\omega(t) - p(g(t))\omega(g(t)))$.
Therefore,

$$\begin{aligned} \omega'(t) &= (p(t)\omega(t) - p(g(t))\omega(g(t))) - \frac{q'(t)\omega(t)}{q(t)} \\ &= \left(p(t) - g'(t)p(g(t)) - \frac{q'(t)}{q(t)}\right)\omega(t). \end{aligned} \tag{3.16}$$

Therefore, $\omega'(t) = \xi(t)\omega(t)$, where,

$$\xi(t) = p(t) - g'(t)p(g(t)) - \frac{q'(t)}{q(t)}. \tag{3.17}$$

Differentiate equation (3.10) with respect to t, we get,

$$\omega'(h(t))h'(t) = h''(t)\omega(t) + h'(t)\omega'(t). \tag{3.18}$$

Using equation (3.10) and (3.17), we get, $\xi(h(t))(h'(t))^2\omega(t) = h''(t)\omega(t) + h'(t)\xi(t)\omega(t)$.
The above equation is the compatibility condition for $\omega(t)$.
For a general $p(t), q(t), s(t)$, equations (3.10) and (3.16), have only one solution, namely, $\omega(t,x) = 0$.
Thus, we obtain the symmetries,
$\omega(t,x) = 0, \quad \Upsilon(t,x) = M_0 x + N(t), \quad M_0 = \text{constant},$
where $N(t)$ solves, $N'(t) = p(t)N(t) + q(t)N(g(t)) - M_0 s(t)$.
Thus, the general solution of the determining equations corresponds to the generator,
$\zeta^* = M_0 x \dfrac{\partial}{\partial x} + N(t)\dfrac{\partial}{\partial x}$. □

3.4 Symmetries of First Order Differential Equations With The Most Standard Time Delay

In the previous section, we have obtained the equivalent symmetries of (3.8). In (3.8), we have considered the most general time delay given by $g(t)$ where $g(t) < t$.
In this section, we shall make a group classification of the first order delay differential equation given by,

$$x'(t) = p(t)x(t) + q(t)x(t-r) + s(t). \tag{3.19}$$

It may be noted that our delay here is chosen by setting $g(t) = t - r$ in Section 3.3. However, it is pointed out here that for the differential equation with most general time delay given by (3.8), we have seen that one of the coefficients of the infinitesimal generator was 0. It will be seen in this section, that by choosing $g(t) = t - r$, both of our coefficients of the infinitesimal generator will be non-trivial.
Equation (3.19) is of paramount importance in modeling physical phenomenon arising in fluid mechanics, physics, ecology, biological processes, etc.

Proposition 3.4.1. *If $x_1(t)$ is an arbitrary solution of equation (3.19), then by employing the change of variables $\bar{t} = t, \quad \bar{x} = x - x_1(t)$, the delay differential equation given by equation (3.19), gets transformed a homogeneous delay differential equation, namely,*

$$x'(t) = p(t)x(t) + q(t)x(t-r). \tag{3.20}$$

Proof. The proposition easily follows by substituting $t = \bar{t}$ and $x(t) = \bar{x} + x_1(\bar{t})$ in (3.19), and by noting that $x_1'(t) = p(t)x_1(t) + q(t)x_1(t-r) + s(t)$. $\square$

We establish the following theorem for equation (3.19)

Theorem 3.4.1. *The delay differential equation given by equation (3.19) for which*

1. $p(t) \neq p(t-r)$ *admits the three dimensional group generated by*

$$\zeta_1^* = e^{\int \nu(t) - \frac{q'(t)}{q(t)} \mathrm{d}t} \frac{\partial}{\partial t} + xp(t)e^{\int \nu(t) - \frac{q'(t)}{q(t)} \mathrm{d}t} \frac{\partial}{\partial x}, \quad \zeta_2^* = x\frac{\partial}{\partial x}, \quad \zeta_3^* = x_3(t)\frac{\partial}{\partial x},$$

where $x_3(t)$ *is the general solution of the associated homogeneous delay differential equation and* $\nu(t) = p(t) - p(t-r)$.

2. $p(t) = p(t-r)$ *admits the three dimensional group generated by*

$$\zeta_1^* = \frac{1}{q(t)}\frac{\partial}{\partial t} + x\frac{p(t)}{q(t)}\frac{\partial}{\partial x}, \quad \zeta_2^* = x\frac{\partial}{\partial x}, \quad \zeta_3^* = x_4(t)\frac{\partial}{\partial x},$$

where $x_4(t)$ *is the general solution of the associated homogeneous delay differential equation.*

Proof. We first will obtain the equivalent symmetries of (3.20). We assume that
$p(t) \neq p(t-r)$.
With the notations,
$\omega^r = \omega(t-r, x(t-r))$,
$\Upsilon^r = \Upsilon(t-r, x(t-r))$,
it follows that (3.6) can be rewritten as,

$$\zeta^{(1)} = \omega\frac{\partial}{\partial t} + \omega^r\frac{\partial}{\partial(t-r)} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x(t-r)} + \Upsilon_{[t]}\frac{\partial}{\partial x'}. \tag{3.21}$$

Applying the operator defined by equation (3.21), to the delay equation $g(t) = t - r$, we get

$$\omega(t, x) = \omega(t-r, x(t-r)). \tag{3.22}$$

Again applying the operator (3.21) to equation (3.20), we get,

$$\Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2 = p(t)\Upsilon + q(t)\Upsilon^r + \omega[p'(t)x(t) + q'(t)x(t-r)]. \tag{3.23}$$

Differentiating (3.23) with respect to $x(t-r)$ twice, we get,
$\Upsilon^r_{x(t-r)x(t-r)} = 0$, which is solved to get,

$$\Upsilon(t, x) = \alpha(t)x + \beta(t). \tag{3.24}$$

Substituting equation (3.24) in equation (3.23), we get,

$$\alpha'(t)x + \beta'(t) + (\alpha(t) - \omega_t)(p(t)x(t) + q(t)x(t-r)) - \omega_x(p^2(t)x^2(t) + q^2(t)x^2(t-r)$$
$$+ 2p(t)q(t)x(t)x(t-r)) = p(t)(\alpha(t)x + \beta(t)) + q(t)(\alpha(t-r)x(t-r) + \beta(t-r))$$
$$+ \omega(p'(t)x(t) + q'(t)x(t-r)) \quad (3.25)$$

Splitting (3.25) with respect to $x^2(t)$ or $x^2(t-r)$ or $x(t)x(t-r)$ we get $\omega_x = 0$, which gives,

$$\omega(t, x) = \mu(t). \quad (3.26)$$

Since $\omega = \omega^r$, we get, $\mu(t) = \mu(t-r)$. Splitting (3.25) with respect to $x(t)$ and solving it gives,

$$\alpha(t) = p(t)\mu(t) + c_1, \quad (3.27)$$

where c_1 is an arbitrary constant.
Splitting (3.25) with respect to $x(t-r)$ gives,

$$q(t)(\alpha(t) - \alpha(t-r)) = \mu(t)q'(t) + q(t)\mu'(t). \quad (3.28)$$

Splitting (3.25) with respect to the constant term, we get,

$$\beta'(t) = p(t)\beta(t) + q(t)\beta(t-r). \quad (3.29)$$

That is, $\beta(t)$ solves equation (3.20).
Substituting equation (3.27) in equation (3.28), and using the fact that $\mu(t) = \mu(t-r)$ we get,

$$\mu'(t)q(t) - (q(t)\nu(t) - q'(t))\mu(t) = 0, \quad (3.30)$$

where $\nu(t) = p(t) - p(t-r)$. Equation (3.30) can be solved to give,

$$\mu(t) = c_2 e^{\int \nu(t) - \frac{q'(t)}{q(t)} \mathrm{d}t}. \quad (3.31)$$

where c_2 is an arbitrary constant.
Substituting equation (3.31) in equation (3.27) we get,

$$\alpha(t) = p(t)\left(c_2 e^{\int \nu(t) - \frac{q'(t)}{q(t)} \mathrm{d}t}\right) + c_1. \quad (3.32)$$

Thus, we obtain the coefficients of the infinitesimal transformation as,

$$\omega(t,x) = c_2 e^{\displaystyle\int \nu(t) - \frac{q'(t)}{q(t)} \mathrm{d}t}, \quad \Upsilon(t,x) = \left[c_2 p(t) e^{\displaystyle\int \nu(t) - \frac{q'(t)}{q(t)} \mathrm{d}t} + c_1 \right] x + \beta(t).$$

provided $c_2 \neq 0$.

Hence we get the infinitesimal generator of the Lie group as,

$$\zeta^* = c_2 \left[e^{\displaystyle\int \nu(t) - \frac{q'(t)}{q(t)} \mathrm{d}t} \frac{\partial}{\partial t} + x p(t) e^{\displaystyle\int \nu(t) - \frac{q'(t)}{q(t)} \mathrm{d}t} \frac{\partial}{\partial x} \right] + c_1 x \frac{\partial}{\partial x} + \beta(t) \frac{\partial}{\partial x}. \tag{3.33}$$

provided $c_2 \neq 0$.

If $c_2 = 0$, then, $\omega(t,x) = 0, \quad \Upsilon(t,x) = c_1 x + \beta(t)$.

In this case, we get the infinitesimal generator of the Lie group as,

$$\zeta^* = c_1 x \frac{\partial}{\partial x} + \beta(t) \frac{\partial}{\partial x}. \tag{3.34}$$

As a special case, we tend to see what happens when $p(t) = p(t-r)$. That is, we study and make a group classification of the special cases, where $p(t)$ satisfies a periodic property (this case includes the possibilities when $p(t)$ is any constant).

Following similar analysis as done above, we get from equation (3.30),

$$\mu(t) q(t) = c_3, \tag{3.35}$$

where c_3 is an arbitrary constant.

In this case, the coefficients of the infinitesimal generator in this case are given by,

$$\omega(t,x) = \frac{c_3}{q(t)}, \quad \Upsilon(t,x) = \left(c_3 \frac{p(t)}{q(t)} + c_1 \right) x + \rho(t),$$

provided $c_3 \neq 0$.

where $\rho(t)$ solves equation (3.20) with $p(t) = p(t-r)$.

The infinitesimal generator in this case is given by,

$$\zeta^* = c_3 \left(\frac{1}{q(t)} \frac{\partial}{\partial t} + x \frac{p(t)}{q(t)} \frac{\partial}{\partial x} \right) + c_1 x \frac{\partial}{\partial x} + \rho(t) \frac{\partial}{\partial x}. \tag{3.36}$$

provided $c_3 \neq 0$.

If $c_3 = 0$, then $\omega(t,x) = 0, \quad \Upsilon(t,x) = c_1 x + \rho(t)$, and the infinitesimal generator is given by,

$$\zeta^* = c_1 x \frac{\partial}{\partial x} + \rho(t) \frac{\partial}{\partial x}. \tag{3.37}$$

where $\rho(t)$ solves equation (3.20), with $p(t) = p(t-r)$. □

The following is an example. In this example, we shall perform symmetry analysis of a first order delay differential equation arising in models for mixing of liquids. We shall construct a Lie group under which this delay differential equation is invariant. We shall first quickly introduce how this delay differential equation comes into our model.

Example 3.4.1. *Let us consider a tub with U litres of glucose solution. Assume that fresh water flows in at the top of the tub at V litres per minute. The glucose solution is continually stirred, and the mixed solution flows out through the bottom, at the same rate of V litres per minute. Let us assume that this mixing cannot occur instantaneously throughout the tub. Then, if $x(t)$ is the amount of glucose in the tub at time t, the concentration of the solution leaving the tank at time t will equal the average concentration at some earlier instant say $t-r$. We shall assume that $r = \frac{\pi}{2}$, is a positive constant. Then the delay differential equation describing this model is given by,*

$$x'(t) = -\frac{Vx(t-r)}{U}.$$

As an example, further assuming, $U = V$, our delay differential equation becomes $x'(t) = -x(t - \frac{\pi}{2})$, whose solution is given by $x(t) = \sin t$.
Following the procedure outlined in this section, and using the same notations as in this section, we see that $p(t) = s(t) = 0$ and $q(t) = -1$.
Hence, $\omega = 0, \quad \Upsilon = c_1 x + \sin t$.
Solving the system, $\frac{d\bar{t}}{d\delta} = \omega(\bar{t}, \bar{x}) = 0$, $\frac{d\bar{x}}{d\delta} = \Upsilon(\bar{t}, \bar{x}) = c_1\bar{x} + \sin\bar{t}$, subject to the conditions $\bar{t} = t, \quad \bar{x} = x$ when $\delta = 0$, we get the delay differential equation invariant under the Lie group given by,

$$\bar{t} = t, \quad \bar{x} = \frac{1}{c_1}[e^{c_1\delta}(c_1 x + \sin t) - \sin t].$$

The infinitesimal generator of this delay differential equation arising in models involving mixing of liquids is given by,

$$\zeta^* = c_1 x\frac{\partial}{\partial x} + \sin t\frac{\partial}{\partial x}.$$

3.5 Symmetries of a Delay Differential Equation Arising in a Population Growth Model: A Nonlinear Case

If $x(t)$ is the population of any isolated species at time t, then the most naive model for the growth of population is $x'(t) = kx(t)$, where k is a positive constant. A more realistic model is obtained if we consider that the growth rate k will diminish as $x(t)$ grows due to overcrowding and shortage of food. This leads us to the differential equation

$x'(t) = k\left[1 - \frac{x(t)}{P}\right]x(t)$, where k and P are both positive constants.
Now suppose that the biological self-regulatory reaction is not instantaneous, but responds only after a time lag say $r > 0$, then we have the following nonlinear delay differential equation which aptly describes the population growth model:

$$x'(t) = k\left[1 - \frac{x(t-r)}{P}\right]x(t). \tag{3.38}$$

We shall perform symmetry analysis of equation (3.38). We have the following result for equation (3.38):

Theorem 3.5.1. *The delay differential equation given by (3.38) arising in population growth models admits the four dimensional group generated by*

$$\zeta_1^* = \frac{\partial}{\partial t}, \quad \zeta_2^* = x\frac{\partial}{\partial x}, \quad \zeta_3^* = e^{-kt}\frac{\partial}{\partial t}, \quad \zeta_4^* = xe^{-kt}\frac{\partial}{\partial x}.$$

Proof. Following the procedure given in the previous section, and applying the operator defined by equation (3.21) to equation (3.38), we get,

$$\Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2 = k\Upsilon - \frac{k}{P}\Upsilon x(t-r) - \frac{k}{P}\Upsilon^r x(t). \tag{3.39}$$

Differentiating equation (3.39) with respect to $x(t-r)$ twice and splitting with respect to $x(t)$ we get,

$$\Upsilon(t,x) = \phi(t)x + \psi(t). \tag{3.40}$$

Substituting equation (3.40) in equation (3.39) and using equation (3.38), we get,

$$\phi'(t)x + \psi'(t) + (\phi(t) - \omega_t)(kx(t) - \frac{k}{P}x(t)x(t-r)) - \omega_x(k^2x^2(t) - 2\frac{k^2}{P}x^2(t)x(t-r) + \frac{k^2}{P^2}x^2(t)x^2(t-r)) = k(\phi(t)x + \psi(t)) - \frac{k}{P}(\phi(t)x + \psi(t))x(t-r) - \frac{k}{P}(\phi(t-r)x(t-r) + \psi(t-r))x(t). \tag{3.41}$$

Splitting equation (3.41) with respect to either $x^2(t)$ or $x^2(t)x^2(t-r)$ or $x^2(t)x(t-r)$, we get,

$$\omega(t,x) = \Phi(t). \tag{3.42}$$

Since $\omega = \omega^r$, it follows that $\Phi(t) = \Phi(t-r)$.
Splitting equation (3.41) with respect to $x(t-r)$, we get,

$$\psi(t) = 0. \tag{3.43}$$

Splitting equation (3.41) with respect to $x(t)$, and using equations (3.42) and (3.43) we get, after solving,

$$\phi(t) = k\Phi(t) + c_4, \tag{3.44}$$

where c_4 is an arbitrary constant. Since $\Phi(t) = \Phi(t-r)$, we get, $\phi(t) = \phi(t-r)$. Splitting equation (3.41) with respect to $x(t)x(t-r)$ and using equation (3.44), and the periodic property of $\psi(t)$, we can solve it to get,

$$\Phi(t) = \frac{c_4}{k} + c_5 e^{-kt}, \tag{3.45}$$

where c_5 is an arbitrary constant.
Hence, we get the coefficients of the infinitesimal transformation as,

$$\omega(t,x) = \frac{c_4}{k} + c_5 e^{-kt}, \quad \Upsilon(t,x) = (2c_4 + c_5 k e^{-kt})x.$$

Thus the infinitesimal generator is given by,

$$\zeta^* = c_4 \left[\frac{1}{k}\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x}\right] + c_5 \left[e^{-kt}\frac{\partial}{\partial t} + kxe^{-kt}\frac{\partial}{\partial x}\right]. \tag{3.46}$$

□

Let us also find the Lie group under which this nonlinear delay differential equation is invariant. To do this, we need to solve the system, $\frac{d\bar{t}}{d\delta} = \omega(\bar{t},\bar{x}) = \frac{c_4}{k} + c_5 e^{-k\bar{t}}$, $\frac{d\bar{x}}{d\delta} = \Upsilon(\bar{t},\bar{x}) = (2c_4 + c_5 k e^{-k\bar{t}})\bar{x}$, subject to the conditions $\bar{t} = t, \quad \bar{x} = x$ when $\delta = 0$. This system can be solved to obtain the Lie group which is,

$$\bar{t} = \frac{1}{k}\ln\left(\frac{k}{c_4}\left[e^{c_4\delta}\left(c_5 + \frac{c_4}{k}e^{kt}\right) - c_5\right]\right),$$

$$\bar{x} = xe^{\delta\left(2c_4 + c_7\left[e^{c_4\delta}\left(c_5 + \frac{c_4}{k}e^{kt}\right) - c_5\right]^{-1}\right)},$$

where $c_7 = \frac{c_4 c_5}{k}$ is an arbitrary constant.

3.6 Summary

1. We have obtained the symmetries of the first order non-homogeneous delay ordinary differential equation, with a general delay. If $x_2(t)$ is the general solution of the associated homogeneous delay differential equation, then the non-homogeneous delay ordinary differential equation admits a symmetry algebra of infinite dimension, due to the linear superposition principle, and is represented by the vector fields, $x_2(t)\frac{\partial}{\partial x}$ and $(x - x_1(t))\frac{\partial}{\partial x}$. Further, if the delay differential equation is homogeneous, then it admits a symmetry algebra again, of infinite dimension, given

by the vector fields $x_2(t)\dfrac{\partial}{\partial x}$ and $x\dfrac{\partial}{\partial x}$.

2. For the delay differential equation given by $x'(t) = p(t)x(t)+q(t)x(t-r)+s(t)$, such that $p(t) \neq p(t-r)$, if $x_3(t)$ is the general solution of the associated homogeneous delay differential equation, then the infinitesimal generator of this delay differential equation is given by

$$\zeta^* = c_2\left[e^{\int \nu(t) - \frac{q'(t)}{q(t)}\mathrm{d}t}\frac{\partial}{\partial t} + xp(t)e^{\int \nu(t) - \frac{q'(t)}{q(t)}\mathrm{d}t}\frac{\partial}{\partial x}\right] + c_1 x\frac{\partial}{\partial x} + x_3(t)\frac{\partial}{\partial x}.$$

However, if $p(t) = p(t-r)$, and if $x_4(t)$ is the general solution of the associated homogeneous delay differential equation, then the infinitesimal generator of this delay differential equation is given by

$$\zeta^* = c_3\left(\frac{1}{q(t)}\frac{\partial}{\partial t} + x\frac{p(t)}{q(t)}\frac{\partial}{\partial x}\right) + c_1 x\frac{\partial}{\partial x} + x_4(t)\frac{\partial}{\partial x}.$$

3. On performing symmetry analysis of $x'(t) = -x(t-\frac{\pi}{2})$, a delay differential equation obtained in modeling mixing of liquids, we found its infinitesimal generator to be $\zeta^* = c_1 x\dfrac{\partial}{\partial x} + \sin t\dfrac{\partial}{\partial x}$. Further, this delay differential equation is invariant under the Lie group given by $\bar{t} = t, \quad \bar{x} = \dfrac{1}{c_1}[e^{c_1\delta}(c_1 x + \sin t) - \sin t]$.

4. We demonstrated the application of group methods to $x'(t) = k\left[1 - \dfrac{x(t-r)}{P}\right]x(t)$, a non linear delay differential equation arising in population growth models and found that the infinitesimal generator corresponding to this non linear delay differential equation given by $\zeta^* = c_4\left[\dfrac{1}{k}\dfrac{\partial}{\partial t} + 2x\dfrac{\partial}{\partial x}\right] + c_5\left[e^{-kt}\dfrac{\partial}{\partial t} + kxe^{-kt}\dfrac{\partial}{\partial x}\right]$. Further, this non linear delay differential equation is invariant under the Lie group given by $\bar{t} = \dfrac{1}{k}\ln\left(\dfrac{k}{c_4}\left[e^{c_4\delta}\left(c_5 + \dfrac{c_4}{k}e^{kt}\right) - c_5\right]\right)$ and $\bar{x} = xe^{\delta\left(2c_4 + c_7\left[e^{c_4\delta}\left(c_5+\frac{c_4}{k}e^{kt}\right)-c_5\right]^{-1}\right)}$.

where $c_i, i = 1, 2, 3, 4, 5, 7$ are arbitrary constants.

The results can be summarized in Table 3.1 below:

Table 3.1: Group Classification of First Order Delay Differential Equations

Type of First order Delay Differential Equation	Generators
$x'(t) = p(t)x(t) + q(t)x(g(t)) + s(t)$, $g(t) < t$.	$\zeta_1^* = x\frac{\partial}{\partial x}$, $\zeta_2^* = N(t)\frac{\partial}{\partial x}$ $N(t)$ solves $N'(t) = p(t)N(t) + q(t)N(g(t)) - M_0 s(t)$ where M_0 is an arbitrary constant.
$x'(t) = p(t)x(t) + q(t)x(t-r) + s(t)$. $p(t) \neq p(t-r)$.	With $\nu(t) = p(t) - p(t-r)$, $\zeta_1^* = e^{\int \nu(t) - \frac{q'(t)}{q(t)}\mathrm{d}t}\frac{\partial}{\partial t}$ $+xp(t)e^{\int \nu(t) - \frac{q'(t)}{q(t)}\mathrm{d}t}\frac{\partial}{\partial x}$, $\zeta_2^* = x\frac{\partial}{\partial x}$, $\zeta_3^* = \beta(t)\frac{\partial}{\partial x}$.
$x'(t) = p(t)x(t) + q(t)x(t-r) + s(t)$. $p(t) = p(t-r)$.	$\zeta_1^* = \frac{1}{q(t)}\frac{\partial}{\partial t} + x\frac{p(t)}{q(t)}\frac{\partial}{\partial x}$, $\zeta_2^* = x\frac{\partial}{\partial x}$, $\zeta_3^* = \rho(t)\frac{\partial}{\partial x}$.
$x'(t) = -x(t - \frac{\pi}{2})$. (*This delay differential equation arises in models involving mixing of liquids.*)	$\zeta_1^* = x\frac{\partial}{\partial x}$, $\zeta_2^* = \sin t\frac{\partial}{\partial x}$.
$x'(t) = k\left[1 - \frac{x(t-r)}{P}\right]x(t)$. (*This non linear delay differential equation arises in modeling population growth.*)	$\zeta_1^* = \frac{\partial}{\partial t}$, $\zeta_2^* = x\frac{\partial}{\partial x}$, $\zeta_3^* = e^{-kt}\frac{\partial}{\partial t}$, $\zeta_4^* = xe^{-kt}\frac{\partial}{\partial x}$.

Chapter 4

Lie Symmetries of First Order Neutral Differential Equations

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4.1 Introduction

In this chapter, we restrict our attention to neutral differential equations. Neutral differential equations are differential equations in which the unknown function and the derivative appear with time delays. Such equations are of importance in models involving flip-flop circuit [53], compartmental systems [61], etc. A lot of research has been dedicated to obtaining solutions of neutral differential equations. In [54], neutral differential equations are solved using multistep block method. Other methods of solution include implicit block method [27], and analysing discontinuities of the derivatives as studied in [1]. Our focus is to obtain symmetries and the corresponding generators of the Lie group admitted by neutral differential equations. It is noteworthy to mention here that the concept of symmetry analysis has been recently used by [45] to obtain Lie symmetries of fractional ordinary differential equations with neutral delay.

In the first part of this chapter, we study the first order neutral differential equation

$$x'(t) = F(t, x(t), x(t_1), x'(t_1)), \tag{4.1}$$

where F is defined on $I \times D^3$, where I is an open interval in $\mathbb{R}$ and D is an open set in $\mathbb{R}$. The notations $x(t_1)$ mean $x(g(t)), g(t) < t$ and $x'(t_1)$ mean $\frac{dx}{dt}(g(t))$. We further assume, $\frac{\partial F}{\partial x(t_1)} \neq 0$ and $\frac{\partial F}{\partial x'(t_1)} \neq 0$. To determine the problem completely, we specify the delay point t_1 by $t_1 = g(t)$, where $g(t) < t$, is the most general kind of delay. We assume that the delay function $g(t)$ is sufficiently smooth in some interval. We shall first need to find a group under which this neutral differential equation is invariant. We call this the admitted Lie group by which we mean that one solution curve is carried to another solution curve of the same equation. We then use this group to obtain the desired equivalent symmetries.

In the second part of this chapter, we study the neutral differential equation with most standard time delay $t - r$, which is given by

$$\psi(t, x(t), x(t-r), x'(t), x'(t-r)) = 0, \tag{4.2}$$

where ψ is a real valued function defined on $I \times D^4$ with I as an open interval in $\mathbb{R}$ and D as an open set in $\mathbb{R}$. We assume that ψ is not independent of $x'(t-r)$. We establish a result to obtain the determining equations by obtaining a Lie type invariance condition using Taylor's theorem for a function of several variables. In addition we make a group classification of the linear and a nonlinear first order neutral differential equation with the most standard time delay $t - r$. We also analyze the case for which equation (4.2) is independent of $x'(t-r)$, thus becoming a first order delay differential equation and perform symmetry analysis for the same. By choosing this widely used delay, we shall see that we get different results from the ones obtained by choosing the most general time delay.

4.2 Lie Type Invariance Condition for First Order Neutral Differential Equations With Most General Time Delay

Definition 4.2.1. (First Order Neutral Differential Equation)
Let J be an interval in $\mathbb{R}$, and let D be an open set in $\mathbb{R}$. Sometimes J will be $[t_0, \beta)$, and sometimes it will be (α, β), where $\alpha \leq t_0 \leq \beta$. Let $f: J \times D^3 \to \mathbb{R}$. Conveniently, a first order neutral differential equation is expressed as

$$x'(t) = f(t, x(t), x(g(t)), x'(g(t))), \tag{4.3}$$

where x and f are real valued functions, and $g(t)$ is a retarded argument i.e. $g(t) \leq t$. We consider equation (4.3) for $t_0 \leq t \leq \beta$ together with the initial function

$$x(t) = \theta(t),\ \gamma \leq t \leq t_0, \tag{4.4}$$

where θ is a given initial function mapping $[\gamma, t_0] \to D$.

Definition 4.2.2. (Solution of a First Order Neutral Differential Equation)
By a solution of the neutral differential equation (4.3) satisfying (4.4), we mean a differentiable function $x: [\gamma, \beta_1) \to D$, for some $\beta_1 \in (t_0, \beta]$, such that
1. $x(t) = \theta(t)$, for $\gamma \leq t \leq t_0$, and
2. $x(t)$ reduces equation (4.3) to an identity on $t_0 \leq t \leq \beta_1$.
We understand $x'(t_0)$ to mean the right-hand derivative.

In this section, we extend the results of ordinary differential equations to neutral differential equations given by equation (4.1). In order to determine the neutral differential equation completely, we need to specify the delay term, where the delayed function is specified, otherwise the problem is not fully determined.

Let a function F be defined on a 5-dimensional space. We extend our results to

$$\frac{dx}{dt} = F(t, x, g(t), x(g(t)), x'(g(t))). \tag{4.5}$$

Let the neutral differential equation be invariant under the Lie group

$$\bar{t} = t + \delta\omega(t,x) + O(\delta^2), \quad \bar{x} = x + \delta\Upsilon(t,x) + O(\delta^2).$$

We then naturally define $\overline{g(t)} = g(t) + \delta\omega(g(t), x(g(t))) + O(\delta^2)$ and $\overline{x(g(t))} = x(g(t)) + \delta\Upsilon(g(t), x(g(t))) + O(\delta^2)$.

With the notations, $\omega_1 = \omega(g(t), x(g(t)))$, and $\Upsilon_1 = \Upsilon(g(t), x(g(t))))$, it follows that,

$$\begin{aligned} \overline{x}'(t_1) &= \tfrac{d\bar{x}}{dt}(\overline{g(t)}) \\ &= x'(t_1) + ((\Upsilon_1)_{t_1} + ((\Upsilon_1)_{x(t_1)} - (\omega_1)_{t_1})x'(t_1) - (x'(t_1))^2(\omega_1)_{x(t_1)})\delta + O(\delta^2). \end{aligned} \tag{4.6}$$

For invariance, $\frac{d\bar{x}}{dt} = F(\bar{t}, \bar{x}, \overline{g(t)}, \overline{(x(g(t)))}, \overline{x'(g(t))})$.
This gives,

$$\begin{aligned} &\tfrac{dx}{dt} + [\Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2]\delta + O(\delta^2) \\ = \ & F(t + \delta\omega + O(\delta^2),\ x + \delta\Upsilon + O(\delta^2), g(t) + \delta\omega_1 + O(\delta^2),\ x(t_1) + \delta\Upsilon_1 + O(\delta^2), \\ & x'(t_1) + ((\Upsilon_1)_{t_1} + ((\Upsilon_1)_{x(t_1)} - (\omega_1)_{t_1})x'(t_1) - (x'(t_1))^2(\omega_1)_{x(t_1)})\delta + O(\delta^2)) \\ = \ & F(t, x, g(t), x(g(t)), x'(g(t))) + (\omega F_t + \Upsilon F_x + \omega_1 F_{t_1} + \Upsilon_1 F_{x(t_1)} \\ & + \Upsilon_{1[t]} F_{x'(t_1)})\delta + O(\delta^2), \end{aligned} \tag{4.7}$$

where $\Upsilon_{1[t]} = (\Upsilon_1)_{t_1} + ((\Upsilon_1)_{x(t_1)} - (\omega_1)_{t_1})x'(t_1) - (x'(t_1))^2(\omega_1)_{x(t_1)}$.
Comparing the coefficient of δ, we get

$$\omega F_t + \Upsilon F_x + \omega_1 F_{t_1} + \Upsilon_1 F_{x(t_1)} + \Upsilon_{1[t]} F_{x'(t_1)} = \Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2. \tag{4.8}$$

The above obtained equation (4.8) is a Lie type invariance condition.
Similar to the case of ordinary differential equations, we can define a prolonged operator for neutral differential equation as:

$$\zeta = \omega\frac{\partial}{\partial t} + \omega_1\frac{\partial}{\partial t_1} + \Upsilon\frac{\partial}{\partial x} + \Upsilon_1\frac{\partial}{\partial x(t_1)}.$$

With the notation $D_t = \frac{\partial}{\partial t} + x'\frac{\partial}{\partial x}$, we can write,

$$\begin{aligned} \tfrac{d\bar{x}}{dt} &= \tfrac{dx}{dt} + (D_t(\Upsilon) - x'D_t(\omega))\delta + O(\delta^2). \\ &= \tfrac{dx}{dt} + \Upsilon_{[t]}\delta + O(\delta^2), \end{aligned} \tag{4.9}$$

where $\Upsilon_{[t]} = D_t(\Upsilon) - x'D_t(\omega)$. We then define the extended operator as:

$$\zeta^{(1)} = \omega\frac{\partial}{\partial t} + \omega_1\frac{\partial}{\partial t_1} + \Upsilon\frac{\partial}{\partial x} + \Upsilon_1\frac{\partial}{\partial x(t_1)} + \Upsilon_{[t]}\frac{\partial}{\partial x'} + \Upsilon_{1[t]}\frac{\partial}{\partial x'(t_1)}. \tag{4.10}$$

Defining $\Delta = x'(t) - F(t, x(t), g(t), x(g(t)), x'(g(t))) = 0$, we get

$$\zeta^{(1)}\Delta = \Upsilon_{[t]} - \omega F_t + \Upsilon F_x + \omega_1 F_{t_1} + \Upsilon_1 F_{x(t_1)} + \Upsilon_{1[t]} F_{x'(t_1)}. \tag{4.11}$$

Comparing equations (4.8) and (4.11), we get
$\Upsilon_{[t]} = \Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2$.
On substituting $x' = F$ into $\zeta^{(1)}\Delta = 0$, we get an invariance condition for the neutral differential equation which is $\zeta^{(1)}\Delta \mid_{\Delta=0}= 0$, from which we shall obtain the determining equations.
We point out here that equations (4.9)-(4.11) is an easy way of working with higher order differential equations as compared to equations (4.6)-(4.8) which is simpler to use for lower order differential equations.

4.3 Symmetries of Non-homogeneous Neutral Differential Equation of First Order With Most General Time Delay

Consider the neutral differential equation, with once differentiable variable coefficients $\alpha(t), \beta(t), \gamma(t)$ and $\rho(t)$ given by,

$$x'(t) = \alpha(t)x(t) + \beta(t)x(g(t)) + \gamma(t) + \rho(t)x'(g(t)). \tag{4.12}$$

We obtain symmetries of the non-homogeneous neutral differential equation (4.12), where g is a sufficiently smooth function with $g(t) < t$. Also $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\rho(t)$ are sufficiently smooth functions satisfying $\beta^2(t) + \rho^2(t)$ not identically zero and $g(t)$ is non constant.
We seek our coefficient of the infinitesimal transformation ω of the form, (that is we assume) $\omega(t, x) = \omega(t)$. Then applying the operator defined by (4.11), to the delay equation $t_1 = g(t)$, we get,

$$\omega_1 = g'(t)\omega(t). \tag{4.13}$$

Applying operator $\zeta^{(1)}$ defined by (4.10) to equation (4.12), we get

$$\begin{aligned} &\Upsilon_t(t,x) + (\Upsilon_x(t,x) - \omega'(t))(\alpha(t)x(t) + \beta(t)x(g(t)) + \gamma(t) + \rho(t)x'(g(t)) = \\ &\omega(t)(\alpha'(t)x(t) + \beta'(t)x(g(t)) + \gamma'(t) + \rho'(t)x'(g(t))) + \alpha(t)\Upsilon(t,x) + \beta(t)\Upsilon_1 \\ &+\rho(t)((\Upsilon_1)_{t_1} + ((\Upsilon_1)_{x(t_1)} - \omega_1')(x'(g(t)))). \end{aligned} \tag{4.14}$$

Differentiating with respect to $x(t_1)$ twice, we get
$\beta(t)(\Upsilon_1)_{x(t_1)x(t_1)} + \rho(t)[(\Upsilon_1)_{t_1 x(t_1)x(t_1)} + (\Upsilon_1)_{x(t_1)x(t_1)x(t_1)}x'(g(t))] = 0.$

Splitting the equation with respect to $x'(g(t))$, we get $\rho(t)(\Upsilon_1)_{x(t_1)x(t_1)x(t_1)} = 0$, which is solved to give

$$\Upsilon(t, x) = \frac{1}{2}A(t)x^2 + B(t)x + C(t). \tag{4.15}$$

Substituting equation (4.15) into the determining equation (4.14), we get

$$\begin{aligned}
&\tfrac{1}{2}A'(t)x^2 + B'(t)x + C'(t) + (A(t)x + B(t) - \omega'(t))(\alpha(t)x \\
&+\beta(t)x(g(t)) + \gamma(t) + \rho(t)x'(g(t))) \\
&= \omega(t)(\alpha'(t)x + \beta'(t)x(g(t)) + \gamma'(t) + \rho'(t)x'(g(t))) + \alpha(t)(\tfrac{1}{2}A(t)x^2 \\
&+B(t)x + C(t)) + \beta(t)(\tfrac{1}{2}A(t_1)x^2(t_1) + B(t_1)x(t_1) + C(t_1)) \\
&+\rho(t)[(\tfrac{1}{2}A'(t_1)x^2(t_1) + B'(t_1)x(t_1) + C'(t_1)) + (A(t_1)x(t_1) + B(t_1) \\
&-\omega_1'(t_1))x'(g(t))].
\end{aligned} \tag{4.16}$$

Splitting equation (4.16) with respect to x^2, we get

$$A(t) = \exp(-\int^t \alpha(s)ds) + A_0, \tag{4.17}$$

where $A_0 =$ constant.
Similarly splitting equation (4.16) with respect to x, $x(g(t))$, $x^2(g(t))$, $x'(g(t))$, $x(t)x(g(t))$, $x(t)x'(g(t))$ and with respect to constant term , we get

$$B'(t) + A(t)\gamma(t) = \omega'(t)\alpha(t) + \alpha'(t)\omega(t), \tag{4.18}$$

$$\rho(t)B'(t_1) + \beta(t)[B(t_1) - B(t)] + \omega(t)\beta'(t) + \omega'(t)\beta(t) = 0, \tag{4.19}$$

$$A(t_1)\beta(t) + \rho(t)A'(t_1) = 0, \tag{4.20}$$

$$B(t)\rho(t) = \omega'(t)\rho(t) + \omega(t)\rho'(t), \tag{4.21}$$

$$A(t)\beta(t) = 0, \tag{4.22}$$

$$A(t)\rho(t) = 0, \tag{4.23}$$

and

$$C'(t) + B(t)\gamma(t) - \omega'(t)\gamma(t) = \omega(t)\gamma'(t) + \alpha(t)C(t) + \beta(t)C(t_1) + \rho(t)C'(t_1), \tag{4.24}$$

respectively.
For a general $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\rho(t)$ and $g(t)$, equations (4.13), (4.18), (4.19), (4.21) and (4.24) have only one solution namely, $\omega(t, x) = 0$.
Equations (4.22) and (4.23), give $A(t) = 0$.

With $\omega(t,x) = 0$ and $A(t) = 0$, equation (4.18) gives

$$B(t) = B_1, \tag{4.25}$$

a constant.
With this, equation (4.21) gives

$$B(t) = 0. \tag{4.26}$$

From equation (4.24), we get

$$C'(t) = \alpha(t)C(t) + \beta(t)C(g(t)) + \rho(t)C'(g(t)). \tag{4.27}$$

That is, $C(t)$ satisfies the corresponding homogeneous neutral differential equation. Thus, we obtain the coefficients of the symmetries as

$$\omega(t,x) = 0,\ \Upsilon(t,x) = C(t).$$

Hence the most general solution of the determining equations corresponds to the infinitesimal generator $\zeta^* = C(t)\frac{\partial}{\partial x}$, where $C(t)$ solves the corresponding homogeneous neutral differential equation.

Remark 4.3.1. In obtainining equivalent symmetries of the neutral differential equation given by equation (4.12), we had assumed that, $\beta^2(t) + \rho^2(t)$ is not identically zero. However, we remark here that, if $\rho(t) = 0, \beta(t) \neq 0$, then equation (4.12) reduces to a first order ordinary delay differential equation. From equations (4.18) and (4.21), we get, $B(t) = B_0$, a constant. From equation (4.24), we get
$C'(t) = \alpha(t)x(t)+\beta(t)x(g(t))-B_0\gamma(t)$. Hence, the infinitesimal generator of the admitted Lie group in this case is given by, $\zeta^* = (B_0x + E(t))\frac{\partial}{\partial x}$, where $E(t)$ is the solution of the delay differential equation $x'(t) = \alpha(t)x(t) + \beta(t)x(g(t)) - B_0\gamma(t)$.

Remark 4.3.2. If in equation (4.12), $\rho(t) \neq 0, \beta(t) = 0$, then from equation (4.24), we get $C'(t) = \alpha(t)x(t) + \rho(t)C'(g(t))$. Hence the generator in this case is given by, $\zeta^* = G(t)\frac{\partial}{\partial x}$, where $G(t)$ is the solution of equation $x'(t) = \alpha(t)x(t) + \rho(t)x'(g(t))$.

Remark 4.3.3. Further, if $\rho(t) = 0,\ \beta(t) = 0$, then equation (4.12), reduces to a first order ordinary differential equation. Again, From equation (4.25), we get, $B(t) = B_2$, a constant. Hence, the infinitesimal generator of the admitted Lie group in this case is given by, $\zeta^* = (B_2x + C_0\exp(\int^t \alpha(s)ds))\frac{\partial}{\partial x}$, where C_0 is an arbitrary constant.

4.4 Example

We give an example to illustrate the construction of a Lie group under which a neutral differential equation is invariant.
Consider the neutral differential equation

$$x'(t) + x'(g(t)) = 0. \tag{4.28}$$

Compared with (4.12), we get $\alpha(t) = \beta(t) = \gamma(t) = 0$, and $\rho(t) = 1$.
For a smooth $g(t)$ satisfying $g(t) < t$, $x(t) = K$, a constant, is a solution of the equation (4.28). Following the procedure in Section 4.3, we see that, for a nonzero B_0,

$$\omega(t, x) = 0 \text{ and } \Upsilon(t, x) = B_0 x + K.$$

This yields $\frac{d\bar{t}}{d\delta} = \omega(\bar{t}, \bar{x}) = 0$ and $\frac{d\bar{x}}{d\delta} = \Upsilon(\bar{t}, \bar{x}) = B_0\bar{x} + K$.
On solving these equations with conditions $\bar{x} = x$ and $\bar{t} = t$, when $\delta = 0$, we get

$$\bar{t} = t \text{ and } \bar{x} = \tfrac{1}{B_0}[(B_0 x + K)e^{B_0\delta} - K]$$

which is the Lie group under which neutral differential equation (4.28) is invariant.

This completes the first half of this chapter. In the next half, we consider the most standard time delay and obtain the Lie symmetries of the corresponding first order neutral differential equation.

4.5 Lie Type Invariance Condition for First Order Neutral Differential Equations With Most Standard Time Delay

In this section, we extend the Lie invariance condition for ordinary differential equations to first order neutral differential equations of type equation (4.2). To determine the neutral differential equation completely, we need to specify the delay term, where the delayed function is specified, otherwise the problem is not fully determined.

We obtain a Lie type invariance for neutral differential equations using Taylor's theorem for a function of several variables:

Theorem 4.5.1. *Consider the first order neutral differential equation*

$$\frac{dx}{dt} = F(t, x, t-r, x(t-r), x'(t-r)), \tag{4.29}$$

defined on $I \times D \times I - r \times D^2$, *where* I *is an open interval in* $\mathbb{R}$, D *is an open set in* $\mathbb{R}$, *and* $I - r = \{y - r : y \in I\}$. *Then with* $\omega(t-r, x(t-r)) = \omega^r$, $\Upsilon(t-r, x(t-r)) = \Upsilon^r$, *the Lie invariance condition is given by*

$$\omega F_t + \Upsilon F_x + \omega^r F_{t-r} + \Upsilon^r F_{x(t-r)} + \Big(\Upsilon^r_{t-r} + (\Upsilon^r_{x(t-r)} - \omega^r_{t-r})x'(t-r) \\ - \omega^r_{x(t-r)}(x'(t-r))^2\Big)F_{x'(t-r)} = \Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2. \quad (4.30)$$

Proof. Let the neutral differential equation be invariant under the Lie group
$\bar{t} = t + \delta\omega(t,x) + O(\delta^2)$,
$\bar{x} = x + \delta\Upsilon(t,x) + O(\delta^2)$.
We then naturally define,
$\overline{t-r} = t - r + \delta\omega(t-r, x(t-r)) + O(\delta^2)$,
$\overline{x(t-r)} = x(t-r) + \delta\Upsilon(t-r, x(t-r)) + O(\delta^2)$.
Then,

$$\begin{aligned}\frac{d\bar{x}}{d\bar{t}} &= \frac{\frac{d\bar{x}}{dt}}{\frac{d\bar{t}}{dt}} \\ &= \left[\frac{dx}{dt} + (\Upsilon_t + \Upsilon_x x')\delta + O(\delta^2)\right]\left[1 - (\omega_t + \omega_x x')\delta + O(\delta^2)\right] \\ &= \frac{dx}{dt} + [\Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2]\delta + O(\delta^2).\end{aligned}$$

For invariance, we must have, $\frac{d\bar{x}}{d\bar{t}} = F(\bar{t}, \bar{x}, \overline{t-r}, \overline{x(t-r)}, \overline{x't-r})$.
With the notations,
$\omega^r = \omega(t-r, x(t-r)), \Upsilon^r = \Upsilon(t-r, x(t-r))$, we get,

$$\begin{aligned}\frac{dx}{dt} + [\Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2]\delta + O(\delta^2) &= F(t + \delta\omega + O(\delta^2), x + \delta\Upsilon + O(\delta^2), \\ &\quad t - r + \delta\omega^r + O(\delta^2), x(t-r) + \delta\Upsilon^r \\ &\quad + O(\delta^2), \\ &\quad x'(t-r) + \delta(\Upsilon^r_{t-r} + (\Upsilon^r_{x(t-r)} - \omega^r_{t-r}) \\ &\quad x'(t-r) - \omega^r_{x(t-r)}(x'(t-r))^2) + O(\delta^2)) \\ &= F(t, x, t-r, x(t-r), x'(t-r)) + \\ &\quad (\omega F_t + \Upsilon F_x + \omega^r F_{t-r} + \Upsilon^r F_{x(t-r)} \\ &\quad + (\Upsilon^r_{t-r} + (\Upsilon^r_{x(t-r)} - \omega^r_{t-r})x'(t-r) \\ &\quad - \omega^r_{x(t-r)}(x'(t-r))^2)\delta + O(\delta^2).\end{aligned}$$

Comparing the coefficient of δ, we get equation (4.30) which is a Lie type invariance condition for first order neutral differential equations. □

Similar to the case of ordinary differential equations, we can define extended operator for the neutral differential equation as:

$$\zeta^{(1)} = \omega\frac{\partial}{\partial t} + \omega^r\frac{\partial}{\partial t-r} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x(t-r)} + \Upsilon_{[t]}\frac{\partial}{\partial x'} + \Upsilon^r_{[t]}\frac{\partial}{\partial x'(t-r)}, \tag{4.31}$$

where with the notation $D_t = \frac{\partial}{\partial t} + x'\frac{\partial}{\partial x}$,
$\Upsilon_{[t]} = D_t(\Upsilon) - x'D_t(\omega)$, and

$$\Upsilon^r_{[t]} = \Upsilon^r_{t-r} + (\Upsilon^r_{x(t-r)} - \omega^r_{t-r})x'(t-r) - \omega^r_{x(t-r)}(x'(t-r))^2.$$

Defining, $\Delta = x'(t) - F(t, x(t), t-r, x(t-r), x'(t-r)) = 0$, we get,

$$\zeta^{(1)}\Delta = \Upsilon_{[t]} - \omega F_t - \Upsilon F_x - \omega^r F_{t-r} - \Upsilon^r F_{x(t-r)} - \Upsilon^r_{[t]} F_{x'(t-r)}. \tag{4.32}$$

Comparing equation (4.32) and equation (4.30), we get,
$\Upsilon_{[t]} = \Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2$.
On substituting $x' = F$ into $\zeta^{(1)}\Delta = 0$, we get an invariance condition for the neutral differential equation which is $\zeta^{(1)}\Delta\ |_{\Delta=0} = 0$, from which we shall obtain the determining equations.

4.6 Symmetries of First Order Linear Neutral Differential Equations With Most Standard Time Delay

We shall obtain symmetries and make a group classification of the first order neutral differential equation with twice differentiable variable coefficients, namely

$$x'(t) = \alpha(t)x(t) + \beta(t)x(t-r) + h(t) + \rho(t)x'(t-r), \tag{4.33}$$

where $\alpha(t), \beta(t), h(t), \rho(t)$ are twice differentiable functions in t.
We shall employ a proposition to convert the non-homogeneous equation (4.33) to it's corresponding homogeneous one. This change does not alter the group classification of (4.33).

Proposition 4.6.1. *If $x_1(t)$ is an arbitrary solution of equation (4.33), then by employing the change of variables $\bar{t} = t, \quad \bar{x} = x - x_1(t)$, the neutral differential equation given by equation (4.33), gets transformed into a homogeneous neutral differential equation, namely,*

$$x'(t) = \alpha(t)x(t) + \beta(t)x(t-r) + \rho(t)x'(t-r). \tag{4.34}$$

Proof. The proposition easily follows by substituting $t = \bar{t}$ and $x(t) = \bar{x} + x_1(\bar{t})$ in (4.33), and by noting that
$x_1'(t) = \alpha(t)x_1(t) + \beta(t)x_1(t-r) + h(t)x_1(t) + \rho(t)x_1'(t-r) = h(t)$. □

We shall obtain equivalent symmetries of equation (4.34).
Applying the operator defined by equation (4.30) to the delay term $g(t) = t - r$, we get $\omega(t, x) = \omega^r$.
Applying the operator defined by equation (4.30) to equation (4.34) we get

$$\Upsilon_{[t]} = \omega[\alpha'(t)x(t) + \beta'(t)x(t-r) + \rho'(t)x'(t-r)] + \Upsilon\alpha(t) + \Upsilon^r\beta(t) + \Upsilon^r_{[t]}\rho(t).$$

Substituting the values of $\Upsilon_{[t]}$ and $\Upsilon^r_{[t]}$ obtained before and then substituting for $x'(t)$ from equation (4.34), we get,

$$\begin{aligned}&\Upsilon_t + (\Upsilon_x - \omega_t)(\alpha(t)x(t) + \beta(t)x(t-r) + \rho(t)x'(t-r)) - \omega_x\Big[\alpha^2(t)x^2(t)\\&+\beta^2(t)(x(t-r))^2 + \rho^2(t)(x'(t-r))^2 + 2\alpha(t)\beta(t)x(t)x(t-r) + 2\alpha(t)\rho(t)x(t)x'(t-r)\\&+2\beta(t)\rho(t)x(t-r)x'(t-r)\Big] = \omega[\alpha'(x)x(t) + \beta'(t)x(t-r) + \rho'(t)x'(t-r)] + \Upsilon\alpha(t) + \Upsilon^r\beta(t)\\&+\Big[\Upsilon^r_{t-r} + (\Upsilon^r_{x(t-r)} - \omega^r_{t-r})x'(t-r) - \omega^r_{x(t-r)}(x'(t-r))^2\Big]\rho(t). \quad (4.35)\end{aligned}$$

Differentiating equation (4.35) with respect to x twice, we get $\omega_x = 0$ which implies $\omega = \omega(t)$.
With this, equation (4.35) becomes

$$\begin{aligned}&\Upsilon_t + (\Upsilon_x - \omega_t)(\alpha(t)x(t) + \beta(t)x(t-r) + \rho(t)x'(t-r)) = \omega\Big[\alpha'(x)x(t) + \beta'(t)x(t-r)\\&+\rho'(t)x'(t-r)\Big] + \Upsilon\alpha(t) + \Upsilon^r\beta(t) + \Big[\Upsilon^r_{t-r} + (\Upsilon^r_{x(t-r)} - \omega^r_{t-r})x'(t-r)\Big]\rho(t). \quad (4.36)\end{aligned}$$

Differentiate equation (4.36) with respect to $x(t-r)$ twice, we get,

$$\beta(t)\Upsilon^r_{x(t-r)x(t-r)} + \rho(t)\Big[\Upsilon^r_{(t-r)x(t-r)x(t-r)} + \Big(\Upsilon^r_{x(t-r)x(t-r)x(t-r)} - \omega^r_{(t-r)x(t-r)x(t-r)}\Big)\Big] = 0.$$

Splitting the above equation with respect to $x'(t-r)$ we get $\rho(t)\Upsilon^r_{x(t-r)x(t-r)x(t-r)} = 0$, which can be solved to give

$$\Upsilon(t, x) = \frac{1}{2}A(t)x^2 + B(t)x + C(t).$$

Substituting this in equation (4.35) we get,

$$\begin{aligned}&\frac{1}{2}A'(t)x^2 + B'(t)x + C'(t) + (A(t)x + B(t) - \omega'(t))(\alpha(t)x + \beta(t)x(t-r) + \rho(t)x'(t-r))\\&= \omega(t)[\alpha'(t)x + \beta'(t)x(t-r) + \rho'(t)x'(t-r)] + \alpha(t)\left[\frac{1}{2}A(t)x^2 + B(t)x + C(t)\right]\\&+\beta(t)\left[\frac{1}{2}A(t-r)x^2(t-r) + B(t-r)x(t-r) + C(t-r)\right] + \rho(t)\Big[\frac{1}{2}A'(t-r)x^2(t-r)\\&+B'(t-r)x(t-r) + C'(t-r) + (A(t-r)x(t-r) + B(t-r) - \omega'(t))x'(t-r)\Big] = 0. \quad (4.37)\end{aligned}$$

Splitting equation (4.37) with respect to $xx(t-r)$, and solving it for an arbitrary $\beta(t)$ we get, $A(t) = 0$.
Substituting $A(t) = 0$ in equation (4.37), we get,

$$\begin{aligned} B'(t)x + C'(t) + (B(t) - \omega'(t))(\alpha(t)x + \beta(t)x(t-r) + \rho(t)x'(t-r)) = \omega(t)[\alpha'(t)x \\ + \beta'(t)x(t-r) + \rho'(t)x'(t-r)] + \alpha(t)[B(t)x + C(t)] + \beta(t)[B(t-r)x(t-r) + C(t-r)] \\ + \rho(t)\Big[B'(t-r)x(t-r) + C'(t-r) + (B(t-r) - \omega'(t))x'(t-r)\Big] = 0. \quad (4.38) \end{aligned}$$

Splitting equation (4.38) with respect to x and solving it we get,

$$B(t) = \alpha(t)\omega(t) + c_1, \tag{4.39}$$

where c_1 is an arbitrary constant.
Since $\omega(t) = \omega(t-r)$, we get

$$B(t) - B(t-r) = [\alpha(t) - \alpha(t-r)]\omega(t). \tag{4.40}$$

Splitting equation (4.38) with respect to the constant term we get,

$$C'(t) = \alpha(t)C(t) + \beta(t)C(t-r) + \rho(t)C'(t-r). \tag{4.41}$$

That is $C(t)$ solves equation (4.34).
Splitting equation (4.38) with respect to $x(t-r)$, we get,

$$[B(t) - B(t-r)]\beta(t) = \omega(t)\beta'(t) + \omega'(t)\beta(t) + \rho(t)B'(t-r). \tag{4.42}$$

Splitting equation (4.38) with respect to $x'(t-r)$, we get,

$$[B(t) - B(t-r)]\rho(t) = \omega(t)\rho'(t). \tag{4.43}$$

Based on the symmetry analysis we have performed so far, we establish the following results:

Theorem 4.6.1. *The first order neutral differential equation given by equation (4.34), for which $\alpha(t) \neq \alpha(t-r)$, and $\alpha'(t-r) \neq -\dfrac{\beta(t)}{\rho(t)}$, admits a three dimensional group generated by*

$$\zeta_1^* = e^{\displaystyle\int \frac{\mu(t)\beta(t) - \beta'(t)}{\beta(t) + \rho(t)\alpha'(t-r)}dt} \frac{\partial}{\partial t} + x\alpha(t)e^{\displaystyle\int \frac{\mu(t)\beta(t) - \beta'(t)}{\beta(t) + \rho(t)\alpha'(t-r)}dt} \frac{\partial}{\partial x},$$

$$\zeta_2^* = x\frac{\partial}{\partial x}, \quad \zeta_3^* = C(t)\frac{\partial}{\partial x},$$

where $\mu(t) = \alpha(t) - \alpha(t-r)$ and $C(t)$ solves equation (4.34).

Proof. Using equation (4.40) in equation (4.42), with the notation $\mu(t) = \alpha(t) - \alpha(t-r)$, we get,

$$[\beta(t) + \rho(t)\alpha'(t-r)]\omega'(t) + [\beta'(t) - \mu(t)\beta(t)]\omega(t) = 0,$$

which can be solved to give

$$\omega(t) = c_2 e^{\displaystyle\int \frac{\mu(t)\beta(t) - \beta'(t)}{\beta(t) + \rho(t)\alpha'(t-r)} dt}, \tag{4.44}$$

where c_2 is an arbitrary constant.
Substituting equation (4.44) into equation (4.39), we get,

$$B(t) = c_2\alpha(t) e^{\displaystyle\int \frac{\mu(t)\beta(t) - \beta'(t)}{\beta(t) + \rho(t)\alpha'(t-r)} dt} + c_1.$$

Consequently,

$$\Upsilon(t,x) = \left[c_2\alpha(t) e^{\displaystyle\int \frac{\mu(t)\beta(t) - \beta'(t)}{\beta(t) + \rho(t)\alpha'(t-r)} dt} + c_1 \right] x + C(t).$$

Thus, the most general infinitesimal generator of the Lie group is given by

$$\begin{aligned} \zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} \\ &= c_2\left[e^{\displaystyle\int \frac{\mu(t)\beta(t) - \beta'(t)}{\beta(t) + \rho(t)\alpha'(t-r)} dt} \frac{\partial}{\partial t} + x\alpha(t) e^{\displaystyle\int \frac{\mu(t)\beta(t) - \beta'(t)}{\beta(t) + \rho(t)\alpha'(t-r)} dt} \frac{\partial}{\partial x} \right] \\ &\quad + c_1 x\frac{\partial}{\partial x} + C(t)\frac{\partial}{\partial x}. \end{aligned} \tag{4.45}$$

If $c_2 \neq 0$, the substituting equation (4.40) in equation (4.43), we get $[\alpha(t) - \alpha(t-r)]\omega(t)\rho(t) = \omega(t)\rho'(t)$, which can be solved to give $\rho(t) = c_3 e^{\int \mu(t)dt}$, where c_3 is an arbitrary constant.
If $c_2 = 0$, then $\omega(t) = 0$. Consequently, $B(t) = c_1$ and $\Upsilon(t,x) = c_1 x + C(t)$. The most general infinitesimal generator of the Lie group in this case is given by

$$\begin{aligned} \zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} \\ &= c_1 x\frac{\partial}{\partial x} + C(t)\frac{\partial}{\partial x}. \end{aligned} \tag{4.46}$$

□

Let us turn to see what happens when $\alpha(t) = \alpha(t-r)$. We establish the following result for this case:

Theorem 4.6.2. *The neutral differential equation given by equation (4.34), satisfying $\alpha(t) = \alpha(t-r)$ admits either*

1. *A two dimensional group generated by*

$$\zeta_1^* = (x - x_1(t))\frac{\partial}{\partial x}, \quad \zeta_2^* = x_1(t)\frac{\partial}{\partial x},$$

Or,

2. *A three dimensional group generated by*

$$\zeta_1^* = \frac{1}{\beta(t)+\rho(t)\alpha(t)}\frac{\partial}{\partial t} + \frac{x\alpha(t)}{\beta(t)+\rho(t)\alpha(t)}\frac{\partial}{\partial x}, \quad \zeta_2^* = x\frac{\partial}{\partial x}, \quad \zeta_3^* = x_1(t)(t)\frac{\partial}{\partial x},$$

provided $\alpha(t) \neq -\dfrac{\beta(t)}{\rho(t)}$,

depending on $\omega(t)$.

Proof. Suppose $\alpha(t) = \alpha(t-r)$, then from equation (4.40) we get $B(t) = B(t-r)$, and hence from equation (4.43) we get

$$\omega(t)\rho'(t) = 0. \tag{4.47}$$

If $\omega(t) = 0$, then consequently $B(t) = c_1$, and hence $\Upsilon(t,x) = c_1 x + C(t)$.
Thus the infinitesimal generator of the admitted Lie group is given by equation (4.46).
From equation (4.47), if $\omega(t) \neq 0$, then we must have $\rho(t) = c_4$ an arbitrary constant.
Substituing this value of $\rho(t)$ in equation (4.42), we get

$$\omega(t)\beta'(t) + \omega'(t)\beta(t) + c_4 B'(t) = 0.$$

Using equation (4.39), we get

$$\omega(t)\beta'(t) + \omega'(t)\beta(t) + c_4[\alpha'(t)\omega(t) + \omega'(t)\alpha(t)] = 0,$$

which can be solved to give

$$\omega(t) = \frac{c_5}{\beta(t)+\rho(t)\alpha(t)}, \tag{4.48}$$

where c_5 is an arbitrary constant.
Using equation (4.48) in equation (4.39), we get, $B(t) = \dfrac{c_5\alpha(t)}{\beta(t)+\rho(t)\alpha(t)} + c_1$.
Consequently,

$$\Upsilon(t,x) = \left[\frac{c_5\alpha(t)}{\beta(t)+\rho(t)\alpha(t)} + c_{1.}\right] x + C(t). \tag{4.49}$$

The infinitesimal generator of the admitted Lie group in this case is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} \\ &= c_5\left[\frac{1}{\beta(t)+\rho(t)\alpha(t)}\frac{\partial}{\partial t} + \frac{x\alpha(t)}{\beta(t)+\rho(t)\alpha(t)}\frac{\partial}{\partial x}\right] + c_1 x\frac{\partial}{\partial x} + C(t)\frac{\partial}{\partial x}.\end{aligned} \tag{4.50}$$

□

Finally, let us turn to the case when $\rho(t) = 0$. In this case equation (4.34) becomes a first order delay differential equation

$$x'(t) = \alpha(t)x(t) + \beta(t)x(t-r), \tag{4.51}$$

Following the analysis given above equation (4.42) becomes

$$[B(t) - B(t-r)]\beta(t) = \omega(t)\beta'(t) + \omega'(t)\beta(t). \tag{4.52}$$

We then establish the following result

Corollary 4.6.1. *The first order delay differential equation given by equation (4.51), for which $\alpha(t) \neq \alpha(t-r)$, admits a three dimensional group generated by*

$$\zeta_1^* = e^{\displaystyle\int \mu(t) - \frac{\beta'(t)}{\beta(t)}dt}\frac{\partial}{\partial t} + x\alpha(t)e^{\displaystyle\int \mu(t) - \frac{\beta'(t)}{\beta(t)}dt}\frac{\partial}{\partial x}, \quad \zeta_2^* = x\frac{\partial}{\partial x}, \quad \zeta_3^* = D(t)\frac{\partial}{\partial x},$$

where $\mu(t) = \alpha(t) - \alpha(t-r)$ and $D(t)$ solves equation (4.51).

Proof. With $\rho(t) = 0$, equation (4.41) becomes $C'(t) = \alpha(t)C(t) + \beta(t)C(t-r)$.
Using equation (4.40) in equation (4.52), with the notation $\mu(t) = \alpha(t) - \alpha(t-r)$, we get,

$$[\beta(t) + \alpha'(t-r)]\omega'(t) + [\beta'(t) - \mu(t)\beta(t)]\omega(t) = 0,$$

which can be solved to give

$$\omega(t) = c_6 e^{\displaystyle\int \mu(t) - \frac{\beta'(t)}{\beta(t)}dt}, \tag{4.53}$$

where c_6 is an arbitrary constant.
Substituting equation (4.53) into equation (4.39), we get,

$$B(t) = c_6\alpha(t)e^{\displaystyle\int \mu(t) - \frac{\beta'(t)}{\beta(t)}dt} + c_1.$$

Consequently,

$$\Upsilon(t,x) = \left[c_6\alpha(t)e^{\displaystyle\int \mu(t) - \frac{\beta'(t)}{\beta(t)}dt} + c_1\right]x + D(t).$$

Thus, the most general infinitesimal generator of the Lie group is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x}\\ &= c_2\left[e^{\displaystyle\int \mu(t) - \frac{\beta'(t)}{\beta(t)}dt}\frac{\partial}{\partial t} + x\alpha(t)e^{\displaystyle\int \mu(t) - \frac{\beta'(t)}{\beta(t)}dt}\frac{\partial}{\partial x}\right] + c_1x\frac{\partial}{\partial x} + D(t)\frac{\partial}{\partial x}.\end{aligned} \tag{4.54}$$

If $c_6 = 0$, then $\omega(t) = 0$. Consequently, $B(t) = c_1$ and $\Upsilon(t,x) = c_1x + D(t)$. The most general infinitesimal generator of the Lie group in this case is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x}\\ &= c_1x\frac{\partial}{\partial x} + D(t)\frac{\partial}{\partial x}.\end{aligned} \tag{4.55}$$

□

To conclude, we examine what happens when $\alpha(t) = \alpha(t-r)$. We establish the following result for this case:

Corollary 4.6.2. *The delay differential equation given by equation (4.51), satisfying $\alpha(t) = \alpha(t-r)$ admits either*

1. *A two dimensional group generated by*

$$\zeta_1^* = x\frac{\partial}{\partial x}, \quad \zeta_2^* = D(t)\frac{\partial}{\partial x},$$

Or,

2. *A three dimensional group generated by*

$$\zeta_1^* = \frac{1}{\beta(t)}\frac{\partial}{\partial t} + \frac{x\alpha(t)}{\beta(t)}\frac{\partial}{\partial x}, \quad \zeta_2^* = x\frac{\partial}{\partial x}, \quad \zeta_3^* = D(t)\frac{\partial}{\partial x}.$$

depending on $\omega(t)$.

Proof. Suppose $\alpha(t) = \alpha(t-r)$, then from equation (4.40), using the fact that

$\omega(t) = \omega(t-r)$ we get $B(t) = B(t-r)$, and hence from equation (4.52) we get

$$\omega(t)\beta'(t) + \omega'(t)\beta(t) = 0. \tag{4.56}$$

which can be easily solved to give $\omega(t) = \dfrac{c_7}{\beta(t)}$, where c_7 is an arbitrary constant. Then from equation (4.39), we get, $B(t) = \dfrac{c_7\alpha(t)}{\beta(t)} + c_1$ and hence
$\Upsilon(t,x) = \left[\dfrac{c_7\alpha(t)}{\beta(t)} + c_1\right] x + D(t)$.
Thus the infinitesimal generator of the admitted Lie group is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x}\\ &= c_7\left[\frac{1}{\beta(t)}\frac{\partial}{\partial t} + x\frac{\alpha(t)}{\beta(t)}\frac{\partial}{\partial x}\right] + c_1 x\frac{\partial}{\partial x} + D(t)\frac{\partial}{\partial x},\end{aligned} \tag{4.57}$$

provided $c_7 \neq 0$. If $c_7 = 0$, then $\omega(t) = 0$. Consequently, $B(t) = c_1$ and $\Upsilon(t,x) = c_1 x + D(t)$. The most general infinitesimal generator of the Lie group in this case is given by equation (4.55). $\square$

4.7 An Example

Consider the first order neutral differential equation given by $x'(t) + x'(t-\pi) = 0$. The solution of this differential equation is $x(t) = \sin t$.
For this neutral differential equation we have seen that, $\omega(t,x) = 0$ and $\Upsilon(t,x) = c_1 x + \sin t$.
Solving the system,
$\dfrac{d\bar{t}}{d\delta} = \omega(\bar{t},\bar{x}) = 0$, $\dfrac{d\bar{x}}{d\delta} = \Upsilon(\bar{t},\bar{x}) = c_1\bar{x} + \sin\bar{t}$, subject to the conditions, $\bar{t} = t$ and $\bar{x} = x$, when $\delta = 0$, we get the above neutral differential equation invariant under the Lie group $\bar{t} = t, \quad \bar{x} = \dfrac{1}{c_1}\left[e^{c_1\delta}(c_1 x + \sin t) - \sin t\right]$.
The generators of the Lie group (or vector fields of the symmetry algebra) corresponding to this neutral differential equation are given by,
$\zeta_1^* = x\dfrac{\partial}{\partial x}$ and $\zeta_2^* = \sin t\dfrac{\partial}{\partial x}$.

4.8 Symmetries of a Nonlinear Neutral Differential Equation

In this section we consider a nonlinear case obtaining symmetries of

$$x'(t) = x(t)x(t-r) + h(t) + x'(t-r), \tag{4.58}$$

where $h(t)$ is a positive real valued differentiable function.
Applying the operator defined by equation (4.30) to the delay term $g(t) = t - r$, we get $\omega(t,x) = \omega^r$.
Applying the operator defined by equation (4.30) to equation (4.58) we get

$$\Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2 = x\Upsilon^r + x(t-r)\Upsilon + \omega h'(t) + \Upsilon^r_{t-r} + (\Upsilon^r_{x(t-r)} - \omega^r_{t-r})x'(t-r) - \omega^r_{x(t-r)}x'(t-r)^2. \quad (4.59)$$

Splitting equation (4.59) with respect to x'^2 we get $\omega_x = 0$ which can be solved to give $\omega = \omega(t)$.
With this and equation (4.58), equation (4.59) becomes

$$\Upsilon_t + (\Upsilon_x - \omega_t)[xx(t-r) + h(t) + x'(t-r)] = x\Upsilon^r + x(t-r)\Upsilon + \omega h'(t) + \Upsilon^r_{t-r} + (\Upsilon^r_{x(t-r)} - \omega^r_{t-r})x'(t-r). \quad (4.60)$$

Differentiating equation (4.60) with respect to $x(t-r)$ twice and splitting the resulting equation with respect to x, we get $\Upsilon^r_{x(t-r)x(t-r)} = 0$ which can be solved to give $\Upsilon(t,x) = A(t)x + B(t)$.
Substituting this value of $\Upsilon(t,x)$ in equation (4.60) we get

$$A'(t)x+B'(t)+(A(t)-\omega'(t))[xx(t-r)+h(t)+x'(t-r)] = x[A(t-r)x(t-r)+B(t-r)] +x(t-r)[A(t)x+B(t)]+\omega(t)h'(t)+A'(t-r)x(t-r)+B'(t-r)+(A(t-r)-\omega'(t))x'(t-r) = 0. \quad (4.61)$$

Splitting equation (4.61) with respect to x, we get,

$$A'(t) = B(t-r). \quad (4.62)$$

Splitting equation (4.61) with respect to the constant term,

$$B'(t) + h(t)[A(t) - \omega'(t)] = \omega(t)h'(t) + B'(t-r). \quad (4.63)$$

Splitting equation (4.61) with respect to $xx(t-r)$, we get,

$$A(t-r) = -\omega'(t). \quad (4.64)$$

Splitting equation (4.61) with respect to $x'(t-r)$, we get,

$$A(t) = A(t-r). \quad (4.65)$$

Splitting equation (4.61) with respect to $x(t-r)$, we get,

$$A'(t-r) = -B(t). \tag{4.66}$$

Using equation (4.65), equation (4.64) becomes

$$\omega'(t) = -A(t). \tag{4.67}$$

Using equation (4.65), in equation (4.66), we get

$$B(t) = B(t-r). \tag{4.68}$$

Using equation (4.67) and (4.69), in equation (4.63), we solving the resulting equation get

$$\omega(t) = \frac{c_1}{\sqrt{h(t)}}, \tag{4.69}$$

where c_2 is an arbitrary constant.
Hence from equation (4.67) we get $A(t) = c_2 \dfrac{h'(t)}{h^{3/2}(t)}$, and from equation (4.66) we get $B(t) = \dfrac{h'(t)}{h^{5/2}(t)} c_3 - \dfrac{h''(t)}{h^{3/2}(t)} c_4$, where c_2, c_3 and c_4 are arbitrary constants.
Consequently, $\Upsilon(t,x) = \left[c_2 \dfrac{h'(t)}{h^{3/2}(t)} \right] x + c_3 \dfrac{h'(t)}{h^{5/2}(t)} - c_4 \dfrac{h''(t)}{h^{3/2}(t)}$.
Hence the general form of the infinitesimal generator of the admitted Lie group is

$$\begin{aligned} \zeta^* &= \omega \frac{\partial}{\partial t} + \Upsilon \frac{\partial}{\partial x} \\ &= \frac{c_1}{\sqrt{h(t)}} \frac{\partial}{\partial t} + \left[c_2 \frac{h'(t)}{h^{3/2}(t)} \right] x + c_3 \frac{h'(t)}{h^{5/2}(t)} - c_4 \frac{h''(t)}{h^{3/2}(t)} \frac{\partial}{\partial x}. \end{aligned} \tag{4.70}$$

Thus we observe that the Lie group is four dimensional generated by

$$\zeta_1^* = \frac{1}{\sqrt{h(t)}} \frac{\partial}{\partial t}, \quad \zeta_2^* = x \frac{h'(t)}{h^{3/2}(t)} \frac{\partial}{\partial x},$$

$$\zeta_3^* = \frac{h'(t)}{h^{5/2}(t)} \frac{\partial}{\partial x}, \quad \zeta_4^* = \frac{h''(t)}{h^{3/2}(t)} \frac{\partial}{\partial x}.$$

4.9 Summary

We have obtained the symmetries of the first order non-homogeneous neutral differential equation with a general delay. We can make a group classification of the first order neutral differential equation into the following cases. In all cases we see that the first order neutral differential equation admits linear symmetries. The three cases are presented below:

1. If $\rho(t) \neq 0, \beta(t) \neq 0$, and if $x_1(t)$ is a general solution of the associated homogeneous

neutral differential equation, then the non-homogeneous neutral differential equation admits a symmetry algebra of infinite dimension, due to the linear superposition principle, given by the vector field $x_1(t)\frac{\partial}{\partial x}$.

2. If $\rho(t) = 0, \beta(t) \neq 0$, and if $x_2(t)$ is a general solution of the associated homogeneous delay differential equation, then the non-homogeneous delay differential equation admits a symmetry algebra of infinite dimension, due to the linear superposition principle, given by the vector fields $x_2(t)\frac{\partial}{\partial x}$ and $(x - x_3(t))\frac{\partial}{\partial x}$, where $x_3(t)$ is the solution of the non-homogeneous delay differential equation, $x'(t) = \alpha(t)x(t) + \beta(t)x(g(t)) - B_0\gamma(t)$. Further, if the delay differential equation is homogeneous, then it admits a symmetry algebra, again of infinite dimension, given by vector fields $x_2(t)\frac{\partial}{\partial x}$ and $x\frac{\partial}{\partial x}$.

3. If $\rho(t) = 0, \beta(t) = 0$, then the ordinary differential equation admits a symmetry algebra of infinite dimension, due to the linear superposition principle, given by vector fields, $x\frac{\partial}{\partial x}$ and $\exp(\int^t \alpha(s)ds)\frac{\partial}{\partial x}$.

For the first order linear neutral differential equation with variable coefficients and most standard time delay, we have established a Lie type Invariance condition using Taylor's theorem for a function of several variables. We have also illustrated the group methods for a nonlinear first order neutral differential equations. Our results can be summarized as

(i) The general form of the infinitesimal generator of the admitted Lie group for the first order neutral differential equation (4.34), for which $\alpha(t) \neq \alpha(t - r)$, and $\alpha'(t - r) \neq -\dfrac{\beta(t)}{\rho(t)}$, is given by equation (4.46).

(ii) The general form of the infinitesimal generator of the admitted Lie group for the neutral differential equation (4.34), satisfying $\alpha(t) = \alpha(t - r)$ is given by equation (4.50).

(iii) The general form of the infinitesimal generator of the admitted Lie group for the nonlinear neutral differential equation (4.58), is given by equation (4.70).

(iv) The general form of the infinitesimal generator of the admitted Lie group for the first order delay differential equation (4.51), for which $\alpha(t) \neq \alpha(t - r)$, is given by equation (4.54).

(v) The general form of the infinitesimal generator of the admitted Lie group for the first order delay differential equation (4.51), for which $\alpha(t) = \alpha(t - r)$, is given by equation (4.57).

Chapter 5

Group Methods for Second Order Delay Differential Equations

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5.1 Introduction

In this chapter, we study the second order delay differential equation

$$x''(t) = f(t, x(t), x'(t), x(t-r), x'(t-r)), \tag{5.1}$$

where f and x are real valued functions defined on the domain $I \times D^4$, where I is an interval in $\mathbb{R}$, D is an open set in $\mathbb{R}$ and $r > 0$. The notation $x'(t-r)$ means $\dfrac{dx}{dt}(t-r)$. We assume that, $\dfrac{\partial f}{\partial x(t-r)} \neq 0$ and $\dfrac{\partial f}{\partial x'(t-r)} \neq 0$. We specify the delay point $t-r$, in order to completely determine the problem. We shall see different cases of the equation under study and obtain the equivalent symmetries for each of these cases.

In papers [59, 60], we find the application of symmetry analysis to delay differential equations. This research defines and uses an operator equivalent to the canonical Lie-Bäcklund operator. This operator is then used for obtaining symmetries. A research paper by [51] obtains equivalent symmetries of a second order delay differential equation by following an approach different from ours. It should be noted that in [51] too, an operator equivalent to the canonical Lie-Bäcklund operator is defined. The splitting equations obtained by [51] are with respect to terms with double delay as well. Our approach dosen't result in any terms with double delay. Systems of second order linear ordinary differential equations with constant coefficients are thoroughly provided their group classification in [36]. In chapter 2, an admitted Lie group for first order delay differential equations with constant coefficients is defined, and the corresponding generators of the Lie group for this equation are obtained. The approach in chapter 2 consists of using Lie Backlund operators to obtain the determining equations.

Given any equation, the problem of finding all equations, which are equivalent to that given equation, is called an *equivalence problem.* If the given equation is linear, then the equivalence problem is called a *linearization problem.* Consider a linear second order ordinary differential equation

$$x'' + \alpha(t)x' + \gamma(t)x = h(t). \tag{5.2}$$

By an *equivalent Lie group* we mean a Lie group of transformations of the dependent and independent variables, and their coefficients which preserve the differential structure. This group allows simplifying the coefficients of the equations. Sophus Lie showed that any linear second order ordinary differential equation (5.2) is equivalent to the equation

$$x'' = 0. \tag{5.3}$$

In chapter 1, we have seen that the ordinary differential equation given by equation (5.3), admits the eight-dimensional Lie algebra spanned by the generators

$$\zeta_1^* = \frac{\partial}{\partial t}, \quad \zeta_2^* = \frac{\partial}{\partial x}, \quad \zeta_3^* = t\frac{\partial}{\partial t}, \quad \zeta_4^* = x\frac{\partial}{\partial t}, \quad \zeta_5^* = x\frac{\partial}{\partial x},$$
$$\zeta_6^* = t\frac{\partial}{\partial x}, \quad \zeta_7^* = tx\frac{\partial}{\partial t} + x^2\frac{\partial}{\partial x}, \quad \zeta_8^* = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x}.$$

If one tries to find an admitted Lie group for equation (5.2), then the system of determining equations consists of four second-order ordinary differential equations. This system, in general, cannot be solved. In this chapter, we do group classification of

$$x''(t) + \alpha(t)x'(t) + \beta(t)x'(t-r) + \gamma(t)x(t) + \rho(t)x(t-r) = h(t). \tag{5.4}$$

5.2 Lie Type Invariance Condition for Second Order Delay Differential Equations

Formally, a second order delay differential equation is defined as follows:

Definition 5.2.1. (Second Order Delay Differential Equation)
Let J be an interval in $\mathbb{R}$, and let D be an open set in $\mathbb{R}$. Sometimes J will be $[t_0, \beta)$, and sometimes it will be (α, β), where $\alpha \leq t_0 \leq \beta$. Let $f : J \times D^4 \to \mathbb{R}$. Conveniently, a second order delay differential equations is expressed as

$$x''(t) = f(t, x(t), x(t-r), x'(t), x'(t-r)), \tag{5.5}$$

where x and f are real valued functions.
We consider equation (5.5) for $t_0 \leq t \leq \beta$ together with the initial function

$$x(t) = \theta(t), \text{for} \quad \gamma \leq t \leq t_0. \tag{5.6}$$

where θ is a given initial function mapping $[\gamma, t_0] \to D$, and γ is some real number less than t_0.

Definition 5.2.2. (Solution of a Second Order Delay Differential Equation)
By a solution of the delay differential equations (5.5) and (5.6) we mean:
A differentiable function $x : [\gamma, \beta_1) \to D$, for some $\beta_1 \in (t_0, \beta]$, such that,

1. $x(t) = \theta(t)$ for $\gamma \leq t \leq t_0$,
2. $x(t)$ reduces equation (5.5) to an identity on $t_0 \leq t \leq \beta_1$.

We understand $x'(t_0)$ to mean the right-hand derivative.

In this section, we shall obtain a Lie type invariance condition for second order delay differential equations. In order to determine this delay differential equation completely, we need to specify the delay term, where the delayed function is specified, otherwise the problem is not fully determined.

We establish the following Lie type invariance condition using Taylor's theorem which is a novel approach for obtaining symmetries of second order delay differential equations.

Theorem 5.2.1. *Let a function F be defined on $I \times D \times I - r \times D^3$, where D is an open set in $\mathbb{R}$, I is an open interval in $\mathbb{R}$ and $I - r = \{y - r : y \in I\}$. The Lie type invariance condition for*

$$\frac{d^2x}{dt^2} = F(t, x(t), t-r, x(t-r), x'(t), x'(t-r)), \tag{5.7}$$

is given by

$$\begin{aligned}\omega F_t + \varUpsilon F_x + \omega^r F_{t-r} + \varUpsilon^r F_{x(t-r)} + \varUpsilon_{[t]} F_{x'(t)} + \varUpsilon^r_{[t]} F_{x'(t-r)} = \\ \varUpsilon_{tt} + (2\varUpsilon_{tx} - \omega_{tt})x' + (\varUpsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\varUpsilon_x - 2\omega_t)x'' - 3\omega_x x' x'',\end{aligned}$$

where,

$$\varUpsilon_{[t]} = D_t(\varUpsilon) - x' D_t(\omega) = \varUpsilon_t + (\varUpsilon_x - \omega_t)x' - \omega_x x'^2,$$

$$\varUpsilon_{[tt]} = D_t(\varUpsilon_{[t]}) - x'' D_t(\omega), \quad \textit{where} \quad D_t = \frac{\partial}{\partial t} + x'\frac{\partial}{\partial x} + x''\frac{\partial}{\partial x'} + \cdots,$$

and, $\omega^r = \omega(t-r, x(t-r))$, $\varUpsilon^r = \varUpsilon(t-r, x(t-r))$.

Proof. Let the delay differential equation be invariant under the Lie group
$\bar{t} = t + \delta\omega(t, x) + O(\delta^2)$,
$\bar{x} = x + \delta\varUpsilon(t, x) + O(\delta^2)$.
We then naturally define,
$\overline{t-r} = t - r + \delta\omega(t-r, x(t-r)) + O(\delta^2)$ and
$\overline{x(t-r)} = x(t-r) + \delta\varUpsilon(t-r, x(t-r)) + O(\delta^2)$.
Then,

$$\begin{aligned}\frac{d\bar{x}}{d\bar{t}} &= \frac{\frac{d\bar{x}}{dt}}{\frac{d\bar{t}}{dt}} \\ &= \left[\frac{dx}{dt} + (\varUpsilon_t + \varUpsilon_x x')\delta + O(\delta^2)\right][1 - (\omega_t + \omega_x x')\delta + O(\delta^2)] \\ &= \frac{dx}{dt} + [\varUpsilon_t + (\varUpsilon_x - \omega_t)x' - \omega_x x'^2]\delta + O(\delta^2).\end{aligned}$$

With the notation,

$D_t = \dfrac{\partial}{\partial t} + x' \dfrac{\partial}{\partial x},$
we can write,

$$\begin{aligned}\frac{d\bar{x}}{d\bar{t}} &= \frac{dx}{dt} + (D_t(\Upsilon) - x' D_t(\omega))\delta + O(\delta^2) \\ &= \frac{dx}{dt} + \Upsilon_{[t]}\delta + O(\delta^2).\end{aligned}$$

where $\Upsilon_{[t]} = D_t(\Upsilon) - x' D_t(\omega) = \Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2$.
Considering the second-order extended infinitesimals, we can write

$$\begin{aligned}\frac{d^2\bar{x}}{d\bar{t}^2} &= \frac{d}{d\bar{t}}\left(\frac{d\bar{x}}{d\bar{t}}\right) \\ &= \frac{\dfrac{d}{dt}\left[\dfrac{dx}{dt} + [D_t(\Upsilon) - D_t(\omega)x']\delta + O(\delta^2)\right]}{1 + \delta D_t(\omega) + O(\delta^2)} \\ &= \left(\frac{d^2x}{dt^2} + D_t(\Upsilon_{[t]})\delta + O(\delta^2)\right)(1 - \delta D_t(\omega) + O(\delta^2)) \\ &= \frac{d^2x}{dt^2} + (D_t(\Upsilon_{[t]}) - D_t(\omega)x'')\delta + O(\delta^2).\end{aligned}$$

So,
$\Upsilon_{[tt]} = D_t(\Upsilon_{[t]}) - x'' D_t(\omega)$.
As $\Upsilon_{[t]}$ contains t, x and x', we need to extend the definition of D_t, so,
$D_t = \dfrac{\partial}{\partial t} + x' \dfrac{\partial}{\partial x} + x'' \dfrac{\partial}{\partial x'} + \cdots$
Expanding $\Upsilon_{[tt]}$, gives,

$$\Upsilon_{[tt]} = \Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)x'' - 3\omega_x x' x''.$$

With the notations,
$\omega^r = \omega(t-r, x(t-r)), \Upsilon^r = \Upsilon(t-r, x(t-r))$, it follows that,

$$\begin{aligned}\overline{x}'(t-r) &= \frac{d\bar{x}}{d\bar{t}}(\overline{t-r}) \\ &= x'(t-r) + [(\Upsilon^r)_{t-r} + ((\Upsilon^r)_{x(t-r)} \\ &\quad - (\omega^r)_{t-r})x'(t-r) - (x'(t-r))^2(\omega^r)_{x(t-r)}]\delta + O(\delta^2).\end{aligned}$$

Let $\Upsilon^r_{[t]} = (\Upsilon^r)_{t-r} + ((\Upsilon^r)_{x(t-r)} - (\omega^r)_{t-r})x'(t-r) - (x'(t-r))^2(\omega^r)_{x(t-r)}$.
For invariance, $\dfrac{d^2\bar{x}}{d\bar{t}^2} = F(\bar{t}, \overline{x(t)}, \overline{t-r}, \overline{(x(t-r))}, \dfrac{d\bar{x}}{d\bar{t}}, \dfrac{d\bar{x}}{d\bar{t}}(\overline{t-r}))$.

This gives,

$$\begin{aligned}\frac{d^2x}{dt^2} + \Upsilon_{[tt]}\delta + O(\delta^2) = F(t + \delta\omega + O(\delta^2), x + \delta\Upsilon + O(\delta^2),\\ t - r + \delta\omega^r + O(\delta^2), x(t-r) + \delta\Upsilon^r + O(\delta^2),\\ \frac{dx}{dt} + \delta\Upsilon_{[t]} + O(\delta^2), \frac{dx}{dt}(t-r) + \Upsilon^r_{[t]}\delta + O(\delta^2))\\ = F(t, x, t-r, x(t-r), x'(t), x'(t-r)) +\\ (\omega F_t + \Upsilon F_x + \omega^r F_{t-r} + \Upsilon^r F_{x(t-r)} + \Upsilon_{[t]} F_{x'(t)}\\ + \Upsilon^r_{[t]} F_{x'(t-r)})\delta + O(\delta^2).\end{aligned}$$

Comparing the coefficient of δ, we get

$$\begin{aligned}\omega F_t + \Upsilon F_x + \omega^r F_{t-r} + \Upsilon^r F_{x(t-r)} + \Upsilon_{[t]} F_{x'(t)} + \Upsilon^r_{[t]} F_{x'(t-r)} =\\ \Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)x'' - 3\omega_x x' x''.\end{aligned} \quad (5.8)$$

The above obtained equation (5.8) is a Lie type invariance condition for a second order delay differential equation. □

We can define a prolonged operator for the second order delay differential equation as: $\zeta = \omega\frac{\partial}{\partial t} + \omega^r \frac{\partial}{\partial(t-r)} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r \frac{\partial}{\partial x(t-r)}$.

We then, naturally define the extended operator, for a second order delay differential equation as:

$$\zeta^{(1)} = \omega\frac{\partial}{\partial t} + \omega^r\frac{\partial}{\partial(t-r)} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x(t-r)} + \Upsilon_{[t]}\frac{\partial}{\partial x'} + \Upsilon^r_{[t]}\frac{\partial}{\partial x'(t-r)} + \Upsilon_{[tt]}\frac{\partial}{\partial x''}. \quad (5.9)$$

Defining, $\Delta = x''(t) - F(t, x(t), t-r, x(t-r), x'(t), x'(t-r)) = 0$, we get,

$$\zeta^{(1)}\Delta = \Upsilon_{[tt]} - \omega F_t - \Upsilon F_x - \omega^r F_{t-r} - \Upsilon^r F_{x(t-r)} - \Upsilon_{[t]} F_{x'(t)} - \Upsilon^r_{[t]} F_{x'(t-r)}. \quad (5.10)$$

Comparing equation (5.10) and equation (5.8), we get,
$\Upsilon_{[tt]} = \Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)x'' - 3\omega_x x' x''$.
On substituting $x'' = F$ into $\zeta^{(1)}\Delta = 0$, we get an invariance condition for the second order delay differential equation which is $\zeta^{(1)}\Delta \mid_{\Delta=0} = 0$, from which we shall obtain the determining equations.

5.3 Symmetries of A Non-homogeneous Second Order Delay Differential Equation

Consider the delay differential equation with variable coefficients which are twice differentiable:

$$x''(t) + \alpha(t)x'(t) + \beta(t)x'(t-r) + \gamma(t)x(t) + \rho(t)x(t-r) = h(t). \tag{5.11}$$

Proposition 5.3.1. *If $x_1(t)$ is an arbitrary solution of equation (5.11), then by employing the change of variables $\bar{t} = t, \quad \bar{x} = x - x_1(t)$, the delay differential equation given by equation (5.11), gets transformed a homogeneous delay differential equation, namely,*

$$x''(t) + \alpha(t)x'(t) + \beta(t)x'(t-r) + \gamma(t)x(t) + \rho(t)x(t-r) = 0. \tag{5.12}$$

Proof. The propsition easily follows by substituting $t = \bar{t}$ and $x(t) = \bar{x} + x_1(\bar{t})$ in (5.11), and by noting that
$x_1''(t) + \alpha(t)x_1'(t) + \beta(t)x_1'(t-r) + \gamma(t)x_1(t) + \rho(t)x_1(t-r) = h(t)$. □

Proposition 5.3.2. *By employing a suitable change, the delay differential equation*

$$x''(t) + \alpha_1(t)x'(t) + \beta_1(t)x'(t-r) + \gamma_1(t)x(t) + \rho_1(t)x(t-r) = 0, \tag{5.13}$$

with $\alpha_1(t), \beta_1(t), \gamma_1(t)$ and $\rho_1(t)$ twice differentiable functions with variable coefficients can be reduced to a one in which the first order ordinary derivative term is missing.

Proof. By employing a change, $x = u(t)s(t)$, where $u(t) \neq 0$ is some twice differentiable function in t and with $s(t)$ satisfying $s(t) = exp(-\int^{t} \frac{\alpha_1(\xi)d\xi}{2}) + s_0$, where s_0 is an arbitrary constant, equation (5.13), can be reduced to
$u''(t) + \beta_2(t)u'(t-r) + \gamma_2(t)u(t) + \rho_2(t)u(t-r) = 0$, where $\beta_2(t) = \beta_1(t)\frac{s(t-r)}{s(t)}$, $\gamma_2(t) = \frac{s''(t) + \alpha_1(t)s'(t) + \gamma_1(t)s(t)}{s(t)}$, and $\rho_2(t) = \frac{\beta_1(t)s'(t-r) + \rho_1(t)s(t-r)}{s(t)}$. □

This is similar to what is done to second-order ordinary differential equations to remove the coefficient of the first derivative term. This change does not alter the group classification of (5.11).
We shall consider equivalent symmetries of

$$x''(t) + \beta(t)x'(t-r) + \gamma(t)x(t) + \rho(t)x(t-r) = 0. \tag{5.14}$$

Let us specify the delay point,

$$t^r = g(t) = t - r. \tag{5.15}$$

Applying operator $\zeta^{(1)}$ defined by equation (5.9) to equation (5.15), we get,

$$\omega^r = \omega. \tag{5.16}$$

Applying operator $\zeta^{(1)}$ defined by equation (5.9) to equation (5.14), we get,

$$\begin{aligned} &\Upsilon_{tt}+(2\Upsilon_{tx}-\omega_{tt})x'+(\Upsilon_{xx}-2\omega_{tx})x'^2-\omega_{xx}x'^3+(\Upsilon_x-2\omega_t)(-\beta(t)x'(t-r)-\gamma(t)x-\rho x(t-r)) \\ &-3\omega_x x'(-\beta(t)x'(t-r)-\gamma(t)x-\rho x(t-r)) = -[\omega(\beta'(t)x'(t-r)+\gamma'(t)x+\rho'(t)x(t-r)) \\ &\quad +\gamma(t)\Upsilon+\rho(t)\Upsilon^r+\beta(t)(\Upsilon^r_{t-r}+(\Upsilon^r_{x(t-r)}-\omega^r_{t-r})x'(t-r)-\omega^r_{x(t-r)}(x'(t-r)^2))]. \end{aligned} \tag{5.17}$$

Differentiate equation (5.17) with respect to $x'(t-r)$ twice, we get,
$\omega^r_{x(t-r)} = 0$, which we can easily solve to get,

$$\omega(t,x) = A(t). \tag{5.18}$$

Differentiate equation (5.17) with respect to $x(t-r)$ twice, we get,
$\rho(t)\Upsilon^r_{x(t-r)x(t-r)} + \beta(t)\Upsilon^r_{(t-r)(x(t-r))(x(t-r))}$
$+\beta(t)(\Upsilon^r_{x(t-r)x(t-r)x(t-r)} - \omega^r_{(t-r)(x(t-r))(x(t-r))})x'(t-r) = 0.$
Splitting the equation with respect to $x'(t-r)$, and using the fact that $\beta(t) \neq 0$ we get,
$\Upsilon_{xxx} = \omega_{txx} = 0,$
which is solved to give,

$$\Upsilon(t,x) = \frac{1}{2}B(t)x^2 + C(t)x + D(t). \tag{5.19}$$

Substituting equations (5.18) and (5.19) into the determining equation (5.17), we get,

$$\begin{aligned} &\frac{1}{2}B''(t)x^2 + C''(t)x + D''(t) + (2(B'(t)x + C'(t)) - A''(t))x' + B(t)(x'(t-r))^2 \\ &\quad + (B(t)x + C(t) - 2A'(t))(-\beta(t)x'(t-r) - \gamma(t)x - \rho(t)x(t-r)) \\ &= -[A(t)(\beta'(t)x'(t-r) + \gamma'(t)x + \rho'(t)x(t-r)) + \gamma(t)(\frac{1}{2}B(t)x^2 + C(t)x + D(t)) \\ &\quad + \rho(t)(\frac{1}{2}B(t-r)x^2(t-r) + C(t-r)x(t-r) + D(t-r)) \\ &\quad + \beta(t)[(\frac{1}{2}B'(t-r)x^2(t-r) + C'(t-r)x(t-r) + D'(t-r)) \\ &\quad + (B(t-r)x(t-r) + C(t-r) - A'(t-r))x'(t-r)]]. \end{aligned} \tag{5.20}$$

Splitting equation (5.20) with respect to x^2, we get,

$$B''(t) = B(t)\gamma(t). \tag{5.21}$$

Splitting equation (5.20) with respect to x, we get,

$$C''(t) = -\gamma'(t)A(t) - 2\gamma(t)A'(t) - 2B'(t). \tag{5.22}$$

Splitting equation (5.20) with respect to $x'(t-r)$, we get,

$$A''(t) = 2C'(t). \tag{5.23}$$

Splitting equation (5.20) with respect to $(x'(t-r))^2$, we get,

$$B(t) = 0. \tag{5.24}$$

As a consequence of $B(t) = 0$, equation (5.19), reduces to

$$\Upsilon(t, x) = C(t)x + D(t). \tag{5.25}$$

and equation (5.22) reduces to,
$C''(t) = -\gamma'(t)A(t) - 2\gamma(t)A'(t)$.

Using $B(t) = 0$, equation (5.20), simplifies to

$$\begin{aligned} &C''(t)x+D''(t)+(2C'(t)-A''(t))x'+(C(t)-2A'(t))(-\beta(t)x'(t-r)-\gamma(t)x-\rho(t)x(t-r)) \\ &= -[A(t)(\beta'(t)x'(t-r)+\gamma'(t)x+\rho'(t)x(t-r))+\gamma(t)(C(t)x+D(t))+\rho(t)(C(t-r)x(t-r) \\ &+ D(t-r)) + \beta(t)[(C'(t-r)x(t-r) + D'(t-r)) + (C(t-r) - A'(t-r))x'(t-r)]]. \end{aligned} \tag{5.26}$$

Splitting equation (5.26) with respect to $x(t-r)$, we get,

$$\rho(t)[C(t) - C(t-r)] = \rho'(t)A(t) + \beta(t)C'(t-r) + 2A'(t)\rho(t). \tag{5.27}$$

Splitting equation (5.26) with respect to $x'(t-r)$, we get,

$$\beta(t)[C(t) - C(t-r)] = A(t)\beta'(t) + \beta(t)(2A'(t) - A'(t-r)). \tag{5.28}$$

Since $\omega = \omega^r$, equation (5.28) becomes

$$\beta(t)[C(t) - C(t-r)] = A(t)\beta'(t) + \beta(t)A'(t). \tag{5.29}$$

Splitting equation (5.26) with respect to constant term, we get,

$$D''(t) = -\beta(t)D'(t-r) - \gamma(t)D(t) - \rho(t)D(t-r). \tag{5.30}$$

That is, $D(t)$ satisfies the homogeneous second order delay differential equation (5.14)
So far we have obtained from equations (5.16), (5.22), (5.23), (5.25), (5.27), (5.28) and (5.30)

$$\omega = \omega^r, \quad \Upsilon = C(t)x + D(t). \tag{5.31}$$

$$\omega_{tt} = 2C'(t), \quad C''(t) = -\gamma'(t)\omega - 2\gamma(t)\omega_t. \tag{5.32}$$

$$D''(t) = -\beta(t)D'(t-r) - \gamma(t)D(t) - \rho(t)D(t-r). \tag{5.33}$$

$$\beta(t)[C(t) - C(t-r)] = \omega\beta'(t) + \beta(t)\omega'(t). \tag{5.34}$$

$$\rho(t)[C(t) - C(t-r)] = \rho'(t)\omega(t) + \beta(t)C'(t-r) + 2\omega'(t)\rho(t). \tag{5.35}$$

Integrating equation (5.32), we get, $C(t) = \dfrac{\omega_t}{2} + c_1$, where c_1 is a constant.
Since $\omega = \omega^r$, we have, $C(t) = C(t-r)$.
Hence, equation (5.34) gives,

$$\beta(t)\omega(t) = c_2, \tag{5.36}$$

where c_2 is an arbitrary constant.
Equation (5.35) can be written as

$$\begin{aligned} \rho'(t)\omega + 2\omega_t\rho(t) &= -\beta(t)C'(t-r) \\ &= -\frac{\beta(t)\omega_{tt}^r}{2} \\ &= -\frac{\beta(t)}{2}\omega_{tt}. \end{aligned} \tag{5.37}$$

We now make a complete group classification of equation (5.14), by proving the following results:

Theorem 5.3.1. *The second order delay differential equation given by equation (5.14), for which $\beta \neq 0, \rho \neq 0$ admits a three dimensional group generated by*

$$\zeta_1^* = x\frac{\partial}{\partial x}, \quad \zeta_2^* = \frac{1}{\beta(t)}\frac{\partial}{\partial t} + \frac{x}{2}\left(\frac{1}{\beta(t)}\right)'\frac{\partial}{\partial x}, \quad \zeta_3^* = D(t)\frac{\partial}{\partial x}.$$

Proof. Substituting $C(t) = \dfrac{\omega_t}{2} + c_1$, in equation (5.32), we get,
$\omega_{ttt} = -(2\gamma'(t)\omega + 4\gamma(t)\omega_t$ or $\omega\omega_{ttt} = -(2\gamma'(t)\omega^2 + 4\gamma(t)\omega\omega_t$.
Integrating this, we get,

$$\omega\omega_{tt} - \frac{\omega_t^2}{2} + 2\gamma(t)\omega^2 = c_3, \tag{5.38}$$

where c_3 is a constant.
If $c_2 \neq 0$, then from equation (5.36),

$$\omega = \frac{c_2}{\beta(t)}. \tag{5.39}$$

From equation (5.31),

$$\Upsilon(t,x) = x\left(\frac{c_2}{2}\left(\frac{1}{\beta(t)}\right)' + c_1\right) + D(t). \tag{5.40}$$

From equation (5.37), we get, $\rho'(t) - 2\frac{\beta'(t)}{\beta(t)}\rho(t) = \frac{1}{2}\left(\beta''(t) - 2\frac{(\beta'(t))^2}{\beta(t)}\right)$.

This is a linear differential equation yielding solution $\rho(t) = c_4\beta^2(t) + \frac{\beta'(t)}{2}$, where c_4 is an arbitrary constant.

From equation (5.38),

$\gamma(t) = \frac{1}{2}\left[c_5\beta^2(t) - \frac{3}{2}\left(\frac{\beta'(t)}{\beta(t)}\right)^2 + \frac{\beta''(t)}{\beta(t)}\right]$, where $c_5 = \frac{c_3}{c_2^2}$.

Since, $\omega = \omega^r$, $\beta(t) = \beta(t-r)$,

In this case we get coefficients of the infinitesimal transformation as

$$\omega = \frac{c_2}{\beta(t)}, \quad \Upsilon = x\left(\frac{1}{2}\left(\frac{c_2}{\beta(t)}\right)' + c_1\right) + D(t). \tag{5.41}$$

The infinitesimal generator in this case is

$$\zeta^* = c_1 x\frac{\partial}{\partial x} + c_2\left(\frac{1}{\beta(t)}\frac{\partial}{\partial t} + \frac{x}{2}\left(\frac{1}{\beta(t)}\right)'\frac{\partial}{\partial x}\right) + D(t)\frac{\partial}{\partial x}, \tag{5.42}$$

where $D(t)$ is an arbitrary solution of equation (5.14).

If $c_2 = 0$, then

$$\omega = 0, \quad \Upsilon = c_1 x + D(t). \tag{5.43}$$

The infinitesimal generator is given by

$$\zeta^* = (c_1 x + D(t))\frac{\partial}{\partial x}. \tag{5.44}$$

□

Theorem 5.3.2. *The second order delay differential equation given by equation (5.14), for which $\beta \neq 0$ and $\rho = 0$ admits a three dimensional group generated by*

$$\zeta_1^* = \frac{\partial}{\partial t}, \quad \zeta_2^* = x\frac{\partial}{\partial x}, \quad \zeta_3^* = D(t)\frac{\partial}{\partial x}.$$

Proof. We see that from equation (5.35)

$C(t-r) = c_6$, an arbitrary constant.

From equation (5.36),

$\omega = \frac{c_2}{\beta(t)}$.

From equation (5.37),

$\omega = c_7 t + c_8$, both c_7 and c_8 being arbitrary constants.

From equation (5.38), $\gamma\omega^2 = c_9$, where $c_9 = \dfrac{1}{2}\left[c_3 + \dfrac{c_7^2}{2}\right]$, is an arbitrary constant.
Further, since $\omega = \omega^r$, we get $c_7 = 0$ and $\omega = c_8$.
If $c_8 \neq 0$, then
$\gamma(t) = \dfrac{c_9}{c_8^2}$, $\beta(t) = \dfrac{c_2}{c_8}$.
The infinitesimal generator in this case is given by

$$\zeta^* = c_8\frac{\partial}{\partial t} + (c_6 x + D(t))\frac{\partial}{\partial x}. \tag{5.45}$$

If $c_8 = 0$, then $\omega = 0$ and $\Upsilon = c_6 x + D(t)$.
The infinitesimal generator in this case is given by

$$\zeta^* = (c_6 x + D(t))\frac{\partial}{\partial x}. \tag{5.46}$$

□

Theorem 5.3.3. *The second order delay differential equation given by equation (5.14), for which $\beta = 0, \rho \neq 0$ admits a four dimensional group generated by*

$$\zeta_1^* = \frac{1}{\sqrt{\rho(t)}}\frac{\partial}{\partial t}, \quad \zeta_2^* = \left[\left(-\frac{\rho'(t)}{\rho^{3/2}(t)}\right)x\right]\frac{\partial}{\partial x}, \quad \zeta_3^* = x\frac{\partial}{\partial x}, \quad \zeta_4^* = D(t)\frac{\partial}{\partial x}.$$

Proof. We see that from equation (5.37), we get,
$\omega = \sqrt{\dfrac{c_{10}}{\rho(t)}}$.
Hence,

$$\begin{aligned}\Upsilon &= C(t)x + D(t)\\ &= \left(\frac{\omega_t}{2} + c_1\right)x + D(t)\\ &= \left(-\frac{\sqrt{c_{10}}}{4}\frac{\rho'(t)}{\rho^{3/2}(t)} + c_1\right)x + D(t).\end{aligned}$$

If $c_{10} \neq 0$, then from equation (5.38),
$\gamma(t) = \dfrac{1}{2}\left[\dfrac{c_3}{c_{10}}\rho(t) + \dfrac{\rho''(t)}{2\rho(t)} - \dfrac{5}{8}\left(\dfrac{\rho'(t)}{\rho(t)}\right)^2\right]$.
The infinitesimal generator in this case is given by,

$$\zeta^* = \sqrt{\frac{c_{10}}{\rho(t)}}\frac{\partial}{\partial t} + \left[\left(-\frac{\rho'(t)\sqrt{c_{10}}}{4\rho^{3/2}(t)} + c_1\right)x + D(t)\right]\frac{\partial}{\partial x}. \tag{5.47}$$

If $c_{10} = 0$, then $\omega = 0, \Upsilon = c_1 x + D(t)$.
Hence, the infinitesimal generator in this case is given by,

$$\zeta^* = (c_1 x + D(t))\frac{\partial}{\partial x}. \tag{5.48}$$

□

5.4 An Illustrative Example

We shall apply symmetry analysis and make a group classification of a delay differential equation arising in control systems studied in [9]. Consider, a system whose motion is governed by a second order, linear homogeneous differential equation with positive constant coefficients, given by, $mx''(t) + bx'(t) + kx(t) = 0$. Let b represent the damping coefficient. In [37], the system studied is a ship rolling in the waves and x is the angle of tilt from the normal upright position. As one must be more ingenious in trying to increase b, ballast tanks, partially filled with water, are introduced in each side of the ship. A servomechanism designed to pump water from one tank to the other attempts to counteract the roll of the ship. Hopefully, this introduces another term proportional to $x'(t)$, in the equation, say $qx'(t)$. Thus, we consider,

$$mx''(t) + bx'(t) + qx'(t) + kx(t) = 0. \tag{5.49}$$

If one recognizes that the servomechanism cannot respond instantaneously, then instead of equation (5.49), we must consider,

$$mx''(t) + bx'(t) + qx'(t-r) + kx(t) = 0. \tag{5.50}$$

The control takes time $r > 0$ to respond and thus the control term is proportional to the velocity at earlier instant, $t - r$. It seems possible that such a time lag could result in the force represented by $qx'(t)$ being in the opposite direction to that which is desired. Having explained the model, we can make a group classification of the second order delay differential equation (5.50) representing it. We can rewrite equation (5.50) as:

$$x''(t) + \frac{b}{m}x'(t) + \frac{q}{m}x'(t-r) + \frac{k}{m}x(t) = 0. \tag{5.51}$$

Following the approach given in the previous section, and keeping to the same notations, we see that, $\beta(t) = \dfrac{q}{m}$, a constant and $\rho(t) = 0$. Performing symmetry analysis of equation (5.50), we get, $\omega = c_{11}$, a constant and $\varUpsilon = c_{12}x + E(t)$, where c_{12} is an arbitrary constant and $E(t)$ solves equation (5.50). Hence, the generators of the Lie group (or vector fields of the symmetry algebra) are given by, $\zeta_1^* = \dfrac{\partial}{\partial t}$, $\zeta_2^* = x\dfrac{\partial}{\partial x}$ and $\zeta_3^* = E(t)\dfrac{\partial}{\partial x}$.

Furthermore, solving the system,
$\dfrac{d\bar{t}}{d\delta} = \omega(\bar{t}, \bar{x}) = c_{11}$, $\dfrac{d\bar{x}}{d\delta} = \varUpsilon(\bar{t}, \bar{x}) = c_{12}\bar{x} + E(\bar{t})$, subject to the conditions, $\bar{t} = t$ and $\bar{x} = x$, when $\delta = 0$, we see that the delay differential equation given by (5.50) is invariant

under the Lie group
$\bar{t} = t + c_{11}\delta,\ \bar{x} = \frac{1}{c_{12}}\left[e^{c_{12}\delta}\left(c_{12}x + E(t)\right) - E(t + c_{11}\delta)\right].$
It is noteworthy to mention here that this model actually arose during tests of systems for anti rolling stabilization of a ship before World War II which is seen in [38].

5.5 Summary

We have obtained the infinitesimal generators of equation (5.14), and based on the various cases we can classify the second-order delay differential equation as

1. The delay differential equation (5.14) with $\beta \neq 0, \rho \neq 0$, admits the infinitesimal generator given by equation (5.42).
2. The delay differential equation (5.14) with $\beta \neq 0, \rho = 0$, admits the infinitesimal generator given by equation (5.45).
3. The delay differential equation (5.14) with $\beta = 0, \rho \neq 0$, admits the infinitesimal generator given by equation (5.47).

The results can be summarized as a Table 5.1 below:

Table 5.1: Group Classification of the Second Order Delay Differential Equation

Type of Second Order Delay Differential Equation	Generators
$x''(t)+\beta(t)x'(t-r)+\gamma(t)x(t)+\rho(t)x(t-r)=0$, $\rho(t) = c_4\beta^2(t) + \frac{\beta'(t)}{2}$, $\gamma(t) = \frac{1}{2}\left[c_5\beta^2(t) - \frac{3}{2}\left(\frac{\beta'(t)}{\beta(t)}\right)^2 + \frac{\beta''(t)}{\beta(t)}\right]$	$\zeta_1^* = x\frac{\partial}{\partial x}$, $\zeta_2^* = \frac{1}{\beta(t)}\frac{\partial}{\partial t} + \frac{x}{2}\left(\frac{1}{\beta(t)}\right)'\frac{\partial}{\partial x}$, $\zeta_3^* = D(t)\frac{\partial}{\partial x}$
$x''(t)+\beta(t)x'(t-r)+\gamma(t)x(t)=0$, $\gamma(t) = \frac{c_9}{c_8^2}$	$\zeta_1^* = \frac{\partial}{\partial t}$, $\zeta_2^* = x\frac{\partial}{\partial x}$, $\zeta_3^* = D(t)\frac{\partial}{\partial x}$.
$x''(t)+\gamma(t)x(t)+\rho(t)x(t-r)=0$, $\gamma(t) = \frac{1}{2}\left[\frac{c_3}{c_{10}}\rho(t) + \frac{1}{2}\frac{\rho''(t)}{\rho(t)} - \frac{5}{8}\left(\frac{\rho'(t)}{\rho(t)}\right)^2\right]$	$\zeta_1^* = \frac{1}{\sqrt{\rho(t)}}\frac{\partial}{\partial t}$, $\zeta_2^* = \left[\left(-\frac{\rho'(t)}{\rho^{3/2}(t)}\right)x\right]\frac{\partial}{\partial x}$, $\zeta_3^* = x\frac{\partial}{\partial x}$, $\zeta_4^* = D(t)\frac{\partial}{\partial x}$.

Chapter 6

Group Classification of Second Order Neutral Differential Equations

The contents of this chapter are uploaded as a

Pre-Print on arXiv
with identifier 1912.13228.

6.1 Introduction

In this chapter, we obtain a Lie type invariance condition and make a complete group classification of the second order neutral differential equation

$$x''(t) = f(t, x(t), x'(t), x(t-r), x'(t-r), x''(t-r)), \tag{6.1}$$

where f is defined on $I \times D^5$, where I is an open interval in $\mathbb{R}$, D is an open set in $\mathbb{R}$, $r > 0$ is the delay, $x'(t-r)$ and $x''(t-r)$ mean $\frac{dx}{dt}(t-r)$ and $\frac{d^2x}{dt^2}(t-r)$ respectively. We further assume, $\frac{\partial f}{\partial x(t-r)} \neq 0$, $\frac{\partial f}{\partial x'(t-r)} \neq 0$ and $\frac{\partial f}{\partial x''(t-r)} \neq 0$. We shall first find a group under which this neutral differential equation is invariant. We call this the admitted Lie group by which we mean that one solution curve is carried to another solution curve of the same equation. We then use this group to obtain the desired symmetries. Such group classification of differential equations aid in modeling problems in the fields of mathematics, physics, engineering and mechanics.

In [7], the Lie symmetries of systems of second order linear ordinary differential equations with constant coefficients over both real and complex fields are exhaustively described. The research also proposes an algebraic approach to obtain bounds for the dimensions of the maximal Lie invariance algebras possessed by such systems. Further, such systems are thoroughly provided their group classification in [36, 39], with extensions to linear systems of second order ordinary differential equations with more than two equations. Higher order symmetries for ordinary differential equations are studied in [23]. Another research paper suggests a group method to study functional differential equations based on a search of symmetries of underdetermined differential equations by methods of classical and modern group analysis, using the principle of factorization. The method therein, encompasses the use of a basis of invariants consisting of universal and differential invariants [34]. In chapter 4, we have obtained an invariance condition and used it to make a group classification of first order neutral differential equations with variable coefficients and the most general time delay.

In this chapter, we use Taylor's theorem to obtain a Lie type invariance condition for

$$x''(t) + \alpha(t)x'(t) + \beta(t)x'(t-r) + \gamma(t)x(t) + \rho(t)x(t-r) + \kappa(t)x''(t-r) = h(t), \tag{6.2}$$

where $\alpha(t), \beta(t), \gamma(t), \rho(t), \kappa(t)$ and $h(t)$ are continuously differentiable functions.
Using this, we suitably define an operator, its prolongation and extension and use it to obtain our determining equations. These equations are then split with respect to the independent variables to obtain an over-determined system of partial differential equations, which are then solved to obtain the most general generator of the Lie group and the corresponding equivalent symmetries. It may be noted that while performing

the symmetry analysis of this second order neutral differential equation, we have come across nonlinear ordinary differential equations. It is seen that in most cases, we do not get an explicit solution due to the arbitrariness of the variable coefficients. As such, we do not get explicit infinitesimal generators. By then choosing particular values of the variable coefficients or restricting our differential equation by choosing certain values of the obtained constants (which does not alter the symmetries obtained), we illustrate the infinitesimal generators of the admitted group, which are explicitly obtained, for these special cases. We then obtain the group classification of this second order neutral differential equation and as a special case obtain a group classification of the corresponding second order delay differential equation. The complete classification is presented as tables at the end. It is noteworthy to point out here that there is no existing literature on the group classification of neutral differential equations.

6.2 Lie Type Invariance Condition for Second Order Neutral Differential Equations

Formally, a second order neutral differential equation is defined as follows:

Definition 6.2.1. (Second Order Neutral Differential Equation)
Let J be an interval in $\mathbb{R}$, and let D be an open set in $\mathbb{R}$. Sometimes J will be $[t_0, \beta)$, and sometimes it will be (α, β), where $\alpha \leq t_0 \leq \beta$. Let f map $J \times D^5 \to \mathbb{R}$. Conveniently, a second order neutral differential equation is expressed as,

$$x''(t) = f(t, x(t), x(t-r), x'(t), x'(t-r), x''(t-r)), \tag{6.3}$$

where x and f are real valued functions.
We consider equation (6.3) for $t_0 \leq t \leq \beta$ together with the initial function

$$x(t) = \theta(t), \text{for} \quad \gamma \leq t \leq t_0. \tag{6.4}$$

where $\gamma \in \mathbb{R}$ such that $\gamma < t_0$, and θ is a given initial function mapping $[\gamma, t_0] \to D$.

Definition 6.2.2. (Solution of a Second Order Neutral Differential Equation)
By a solution of the neutral differential equations (6.3) and (6.4), we mean a differentiable function $x : [\gamma, \beta_1) \to D$, for some $\beta_1 \in (t_0, \beta]$ such that,

1. $x(t) = \theta(t)$ for $\gamma \leq t \leq t_0$,
2. $x(t)$ reduces equation (6.3) to an identity on $t_0 \leq t \leq \beta_1$.

We understand $x'(t_0)$ and $x''(t_0)$ to mean the right-hand derivative.

In this section, we extend the results to second order neutral differential equations given by equation (6.2). In order to determine this neutral differential equation completely, we need to specify the delay term, where the delayed function is specified, otherwise the problem is not fully determined.

We establish the following Lie type invariance condition for second order neutral differential equations:

Theorem 6.2.1. *Consider the second order neutral differential equation*

$$\frac{d^2x}{dt^2} = F(t, x, t-r, x(t-r), x'(t), x'(t-r), x''(t-r)), \tag{6.5}$$

where F be defined on a 7-dimensional space $I \times D \times I - r \times D^4$, $D \subset \mathbb{R}$, I is any interval in $\mathbb{R}$, and $I - r = \{y - r : y \in I\}$. Then with the notations, $\omega^r = \omega(t-r, x(t-r))$, $\Upsilon^r = \Upsilon(t-r, x(t-r))$, the Lie type invariance condition is given by

$$\omega F_t + \Upsilon F_x + \omega^r F_{t-r} + \Upsilon^r F_{x(t-r)} + \Upsilon_{[t]} F_{x'(t)} + \Upsilon^r_{[t]} F_{x'(t-r)} + \Upsilon^r_{[tt]} F_{x''(t-r)} =$$
$$\Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)x'' - 3\omega_x x' x''.$$

where,

$$\Upsilon_{[t]} = D_t(\Upsilon) - x' D_t(\omega),$$

$$\Upsilon_{[tt]} = D_t(\Upsilon_{[t]}) - x'' D_t(\omega), \quad \textit{where} \quad D_t = \frac{\partial}{\partial t} + x' \frac{\partial}{\partial x} + x'' \frac{\partial}{\partial x'} + \cdots,$$

$$\Upsilon^r_{[t]} = (\Upsilon^r)_{t-r} + ((\Upsilon^r)_{x(t-r)} - (\omega^r)_{t-r})x'(t-r) - (x'(t-r))^2(\omega^r)_{x(t-r)},$$

$\Upsilon^r_{[tt]} = (\Upsilon^r_{(t-r)(t-r)} + (2\Upsilon^r_{(t-r)x(t-r)} - \omega^r_{(t-r)(t-r)})x'(t-r) + (\Upsilon^r_{x(t-r)x(t-r)} - 2\omega^r_{(t-r)x(t-r)})x'(t-r)^2 - \omega^r_{x(t-r)x(t-r)}x'(t-r)^3 + (\Upsilon^r_{x(t-r)} - 2\omega^r_{t-r})x''(t-r) - 3\omega^r_{x(t-r)}x'(t-r)x''(t-r)).$

Proof. Let the neutral differential equation be invariant under the Lie group
$\bar{t} = t + \delta\omega(t, x) + O(\delta^2)$,
$\bar{x} = x + \delta\Upsilon(t, x) + O(\delta^2)$.
We then naturally define,
$\overline{t-r} = t - r + \delta\omega(t-r, x(t-r)) + O(\delta^2)$ and
$\overline{x(t-r)} = x(t-r) + \delta\Upsilon(t-r, x(t-r)) + O(\delta^2)$.

Then,

$$\begin{aligned}
\frac{d\bar{x}}{d\bar{t}} &= \frac{\frac{d\bar{x}}{dt}}{\frac{d\bar{t}}{dt}} \\
&= \left[\frac{dx}{dt} + (\Upsilon_t + \Upsilon_x x')\delta + O(\delta^2)\right][1 - (\omega_t + \omega_x x')\delta + O(\delta^2)] \\
&= \frac{dx}{dt} + [\Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2]\delta + O(\delta^2).
\end{aligned}$$

With the notation,
$D_t = \frac{\partial}{\partial t} + x'\frac{\partial}{\partial x}$,
we can write,

$$\begin{aligned}
\frac{d\bar{x}}{d\bar{t}} &= \frac{dx}{dt} + (D_t(\Upsilon) - x'D_t(\omega))\delta + O(\delta^2) \\
&= \frac{dx}{dt} + \Upsilon_{[t]}\delta + O(\delta^2).
\end{aligned}$$

where $\Upsilon_{[t]} = D_t(\Upsilon) - x'D_t(\omega) = \Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2$.
Considering the second-order extended infinitesimals, we can write

$$\begin{aligned}
\frac{d^2\bar{x}}{d\bar{t}^2} &= \frac{d}{d\bar{t}}\left(\frac{d\bar{x}}{d\bar{t}}\right) \\
&= \frac{\frac{d}{dt}\left[\frac{dx}{dt} + [D_t(\Upsilon) - x'D_t(\omega)]\delta + O(\delta^2)\right]}{1 + \delta D_t(\omega) + O(\delta^2)} \\
&= \left(\frac{d^2x}{dt^2} + D_t(\Upsilon_{[t]})\delta + O(\delta^2)\right)(1 - \delta D_t(\omega) + O(\delta^2)) \\
&= \frac{d^2x}{dt^2} + (D_t(\Upsilon_{[t]}) - D_t(\omega)x'')\delta + O(\delta^2).
\end{aligned}$$

So,
$\Upsilon_{[tt]} = D_t(\Upsilon_{[t]}) - x''D_t(\omega)$.
As $\Upsilon_{[t]}$ contains t, x and x', we need to extend the definition of D_t, so,
$D_t = \frac{\partial}{\partial t} + x'\frac{\partial}{\partial x} + x''\frac{\partial}{\partial x'} + \cdots$
Expanding $\Upsilon_{[tt]}$, gives,

$$\Upsilon_{[tt]} = \Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)x'' - 3\omega_x x'x''.$$

With the notations,
$\omega^r = \omega(t - r, x(t - r)), \Upsilon^r = \Upsilon(t - r, x(t - r))$,

it follows that,

$$
\begin{aligned}
\overline{x}'(t-r) &= \frac{d\bar{x}}{d\bar{t}}(\overline{t-r}) \\
&= x'(t-r) + [(\Upsilon^r)_{t-r} + ((\Upsilon^r)_{x(t-r)} \\
&- (\omega^r)_{t-r})x'(t-r) - (x'(t-r))^2(\omega^r)_{x(t-r)}]\delta + O(\delta^2),
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{x}''(t-r) &= \frac{d^2\bar{x}}{d\bar{t}^2}(\overline{t-r}) \\
&= x''(t-r) + [\Upsilon^r_{(t-r)(t-r)} + (2\Upsilon^r_{(t-r)x(t-r)} - \omega^r_{(t-r)(t-r)})x'(t-r) \\
&+ (\Upsilon^r_{x(t-r)x(t-r)} - 2\omega^r_{(t-r)x(t-r)})x'(t-r)^2 - \omega^r_{x(t-r)x(t-r)}x'(t-r)^3 \\
&+ (\Upsilon^r_{x(t-r)} - 2\omega^r_{t-r})x''(t-r) - 3\omega^r_{x(t-r)}x'(t-r)x''(t-r)]\delta + O(\delta^2).
\end{aligned}
$$

Let $\Upsilon^r_{[t]} = (\Upsilon^r)_{t-r} + ((\Upsilon^r)_{x(t-r)} - (\omega^r)_{t-r})x'(t-r) - (x'(t-r))^2(\omega^r)_{x(t-r)}$ and $\Upsilon^r_{[tt]} = (\Upsilon^r_{(t-r)(t-r)} + (2\Upsilon^r_{(t-r)x(t-r)} - \omega^r_{(t-r)(t-r)})x'(t-r) + (\Upsilon^r_{x(t-r)x(t-r)} - 2\omega^r_{(t-r)x(t-r)})x'(t-r)^2 - \omega^r_{x(t-r)x(t-r)}x'(t-r)^3 + (\Upsilon^r_{x(t-r)} - 2\omega^r_{t-r})x''(t-r) - 3\omega^r_{x(t-r)}x'(t-r)x''(t-r))$.
For invariance,
$\frac{d^2\bar{x}}{d\bar{t}^2} = F(\bar{t}, \bar{x}, \overline{t-r}, \overline{x(t-r)}, \frac{d\bar{x}}{d\bar{t}}, \frac{d\bar{x}}{d\bar{t}}(\overline{t-r}), \frac{d^2\bar{x}}{d\bar{t}^2}(\overline{t-r}))$.
This gives,

$$
\begin{aligned}
\frac{d^2x}{dt^2} + \Upsilon_{[tt]}\delta + O(\delta^2) &= F(t + \delta\omega + O(\delta^2), x + \delta\Upsilon + O(\delta^2), \\
&\quad t - r + \delta\omega^r + O(\delta^2), x(t-r) + \delta\Upsilon^r + O(\delta^2), \\
&\quad \frac{dx}{dt} + \delta\Upsilon_{[t]} + O(\delta^2), \frac{dx}{dt}(t-r) + \Upsilon^r_{[t]}\delta + O(\delta^2), \\
&\quad \frac{d^2x}{dt^2}(t-r) + \Upsilon^r_{[tt]}\delta + O(\delta^2)) \\
&= F(t, x, t-r, x(t-r), x'(t), x'(t-r), x''(t-r)) + \\
&\quad (\omega F_t + \Upsilon F_x + \omega^r F_{t-r} + \Upsilon^r F_{x(t-r)} + \Upsilon_{[t]}F_{x'(t)} \\
&\quad + \Upsilon^r_{[t]}F_{x'(t-r)} + \Upsilon^r_{[tt]}F_{x''(t-r)})\delta + O(\delta^2).
\end{aligned}
$$

Comparing the coefficient of δ, we get

$$
\begin{aligned}
&\omega F_t + \Upsilon F_x + \omega^r F_{t-r} + \Upsilon^r F_{x(t-r)} + \Upsilon_{[t]}F_{x'(t)} + \Upsilon^r_{[t]}F_{x'(t-r)} + \Upsilon^r_{[tt]}F_{x''(t-r)} = \\
&\quad \Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)x'' - 3\omega_x x'x''. \quad (6.6)
\end{aligned}
$$

The above obtained equation (6.6) is a Lie type invariance condition. □

We can define a prolonged operator for the second order neutral differential equation as:
$\zeta = \omega\frac{\partial}{\partial t} + \omega^r\frac{\partial}{\partial(t-r)} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x(t-r)}$.

We then, naturally define the extended operator, for equation (6.2) as:

$$\zeta^{(1)} = \omega\frac{\partial}{\partial t} + \omega^r\frac{\partial}{\partial(t-r)} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x(t-r)} + \Upsilon_{[t]}\frac{\partial}{\partial x'} + \Upsilon^r_{[t]}\frac{\partial}{\partial x'(t-r)} + \Upsilon_{[tt]}\frac{\partial}{\partial x''} + \Upsilon^r_{[tt]}\frac{\partial}{\partial x''(t-r)}.$$

Defining, $\Delta = x''(t) - F(t, x(t), t-r, x(t-r), x'(t), x'(t-r), x''(t-r)) = 0$, we get,

$$\zeta^{(1)}\Delta = \Upsilon_{[tt]} - \omega F_t - \Upsilon F_x - \omega^r F_{t-r} - \Upsilon^r F_{x(t-r)} - \Upsilon_{[t]}F_{x'(t)} - \Upsilon^r_{[t]}F_{x'(t-r)} - \Upsilon^r_{[tt]}F_{x''(t-r)}. \quad (6.7)$$

Comparing equation (6.7) and equation (6.6), we get,
$\Upsilon_{[tt]} = \Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)x'' - 3\omega_x x'x''$.
On substituting $x'' = F$ into $\zeta^{(1)}\Delta = 0$, we get an invariance condition for the second order neutral differential equation which is $\zeta^{(1)}\Delta \mid_{\Delta=0} = 0$, from which we shall obtain the determining equations.

6.3 Symmetries of A Non-homogeneous Second Order Neutral Differential Equation

Consider the neutral differential equation with continuously differentiable variable coefficients given by

$$x''(t) + a(t)x'(t) + b(t)x'(t-r) + c(t)x(t) + d(t)x(t-r) + k(t)x''(t-r) = h(t). \quad (6.8)$$

Proposition 6.3.1. *If $x_1(t)$ is an arbitrary solution of equation (6.8), then by employing the change of variables $\bar{t} = t, \quad \bar{x} = x - x_1(t)$, the neutral differential equation given by equation (6.8), gets transformed a homogeneous neutral differential equation, namely,*

$$x''(t) + a(t)x'(t) + b(t)x'(t-r) + c(t)x(t) + d(t)x(t-r) + k(t)x''(t-r) = 0. \quad (6.9)$$

Proof. The proposition easily follows by substituting $t = \bar{t}$ and $x(t) = \bar{x} + x_1(\bar{t})$ in (6.8), by noting that
$x_1''(t) + a(t)x_1'(t) + b(t)x_1'(t-r) + c(t)x_1(t) + d(t)x_1(t-r) + k(t)x_1''(t-r) = h(t)$. □

Proposition 6.3.2. *By employing a suitable change, the neutral differential equation*

$$x''(t) + a_1(t)x'(t) + b_1(t)x'(t-r) + c_1(t)x(t) + d_1(t)x(t-r) + k_1(t)x''(t-r) = 0, \quad (6.10)$$

with $a_1(t), b_1(t), c_1(t), d_1(t)$ and $k_1(t)$ twice differentiable functions with variable coefficients can be reduced to a one in which the first order ordinary derivative term is missing.

Proof. By employing a change, $x = u(t)s(t)$, where $u(t) \neq 0$ is some twice differentiable function in t and with $s(t)$ satisfying $s(t) = exp(-\int^{t} \frac{a_1(\xi)d\xi}{2}) + s_0$, where s_0 is an arbitrary constant, equation (6.10), can be reduced to
$u''(t) + b_2(t)u'(t-r) + c_2(t)u(t) + d_2(t)u(t-r) + k_2(t)u''(t-r) = 0$, where
$b_2(t) = \frac{b_1(t)s(t-r) + 2k(t)s'(t-r)}{s(t)}$, $c_2(t) = \frac{s''(t) + a_1(t)s'(t) + c_1(t)s(t)}{s(t)}$,
$d_2(t) = \frac{b_1(t)s'(t-r) + d_1(t)s(t-r) + k_1(t)s''(t-r)}{s(t)}$ and $k_2(t) = \frac{k_1(t)}{u(t)}$. □

This is similar to what is done to second-order ordinary differential equations to remove the coefficient of the first derivative term. This change does not alter the group classification of (6.8).
We shall consider equivalent symmetries of

$$x''(t) + b(t)x'(t-r) + c(t)x(t) + d(t)x(t-r) + k(t)x''(t-r) = 0. \tag{6.11}$$

Let us specify the delay point,

$$t^r = g(t) = t - r. \tag{6.12}$$

Applying operator $\zeta^{(1)}$ defined by equation (6.2) to equation (6.12), we get,

$$\omega^r = \omega. \tag{6.13}$$

Applying operator $\zeta^{(1)}$ defined by equation (6.2) to equation (6.11), we get,

$$\begin{aligned}
&\Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)\Big(-b(t)x'(t-r) - c(t)x \\
&- d(t)x(t-r) - k(t)x''(t-r)\Big) - 3\omega_x x'(-b(t)x'(t-r) - c(t)x - d(t)x(t-r) - k(t)x''(t-r)) \\
&= -\Big[\omega(b'(t)x'(t-r) + c'(t)x(t) + d'(t)x(t-r) + k'(t)x''(t-r)) + c(t)\Upsilon + d(t)\Upsilon^r + b(t)(\Upsilon^r_{t-r} \\
&+ (\Upsilon^r_{x(t-r)} - \omega^r_{t-r})x'(t-r) - \omega^r_{x(t-r)}x'^2(t-r)) + k(t)\Big(\Upsilon^r_{(t-r)(t-r)} + (2\Upsilon^r_{(t-r)x(t-r)} \\
&- \omega^r_{(t-r)(t-r)})x'(t-r) + (\Upsilon^r_{x(t-r)x(t-r)} - 2\omega^r_{(t-r)x(t-r)})x'^2(t-r) - \omega^r_{x(t-r)x(t-r)}x'^3(t-r) \\
&+ (\Upsilon^r_{x(t-r)} - 2\omega^r_{t-r})x''(t-r) - 3\omega^r_{x(t-r)}x'(t-r)x''(t-r)\Big)\Big].
\end{aligned} \tag{6.14}$$

Splitting equation (6.14) with respect to $x'^3(t-r)$, we get,
$k(t)\omega^r_{x(t-r)x(t-r)} = 0$, which we can easily solve to get,

$$\omega(t, x) = \alpha(t)x + \beta(t). \tag{6.15}$$

Differentiating equation (6.14) with respect to $x''(t-r)$, we get,
$k(t)(2\omega_t - \Upsilon_x) + 3k(t)\omega_x x' = 3k(t)\omega^r_{x(t-r)}x'(t-r) - (\omega k'(t) + k(t)(\Upsilon^r_{x(t-r)} - 2\omega^r_{t-r}))$
Splitting this equation with respect to $x'(t-r)$, and using the fact that $k(t) \neq 0$ we get,
$\omega_x = 0$.
This with equation (6.15) gives,

$$\omega(t,x) = \beta(t). \tag{6.16}$$

Splitting equation (6.14) with x'^2, we get,
$\Upsilon_{xx} = 0$, which solves to give,

$$\Upsilon(t,x) = \gamma(t)x + \rho(t). \tag{6.17}$$

Substituting equations (6.16) and (6.17) into the determining equation (6.14), we get,

$$\begin{aligned}
&\gamma''(t)x + \rho''(t) + (2\gamma'(t) - \beta''(t))x' + (\gamma(t) - 2\beta'(t))(-b(t)x'(t-r) - c(t)x - d(t)x(t-r)\\
&-k(t)x''(t-r)) = -\Big[\beta(t)(b'(t)x'(t-r) + c'(t)x + d'(t)x(t-r) + k'(t)x''(t-r)) + c(t)(\gamma(t)x\\
&+\rho(t)) + d(t)(\gamma(t-r)x(t-r) + \rho(t-r)) + b(t)(\gamma'(t-r)x(t-r) + \rho'(t-r) + (\gamma(t-r) - \beta'(t-r))\\
&x'(t-r)) + k(t)\Big(\gamma''(t-r)x(t-r) + \rho''(t-r) + (2\gamma'(t-r) - \beta''(t-r))x'(t-r) + (\gamma(t-r)\\
&- 2\beta'(t-r))x''(t-r)\Big)\Big].
\end{aligned} \tag{6.18}$$

From (6.13), we have,

$$\beta(t) = \beta(t-r). \tag{6.19}$$

Splitting (6.18) with respect to $x(t)$, we get,

$$\gamma''(t) + 2\beta'(t)c(t) + \beta(t)c'(t) = 0. \tag{6.20}$$

Splitting (6.18) with respect to $x'(t)$, we get,

$$\gamma(t) = \frac{1}{2}[\beta'(t) + c_1]. \tag{6.21}$$

Using (6.19), we get,

$$\gamma(t) = \gamma(t-r). \tag{6.22}$$

Splitting (6.18) with respect to the constant terms, we get,

$$\rho''(t) + b(t)\rho'(t-r) + c(t)\rho(t) + d(t)\rho(t-r) + k(t)\rho''(t-r) = 0. \tag{6.23}$$

That is, $\rho(t)$ satisfies the homogeneous neutral differential equation of second order given by (6.11).

Splitting (6.18) with respect to $x''(t-r)$, and using (6.19) and (6.22), we get,

$$\beta(t)k'(t) = 0. \tag{6.24}$$

Theorem 6.3.1. *The neutral differential equation given by equation (6.11) for which $k(t) \neq$ constant, admits a two dimensional group generated by*

$$\zeta_1^* = x\frac{\partial}{\partial x}, \quad \zeta_2^* = \rho(t)\frac{\partial}{\partial x}.$$

Proof. Equation (6.24), having to be true for an arbitrary $\beta(t)$ and $k(t)$ implies that for a non-constant $k(t)$, we must have, $\beta(t) = 0$, and consequently,
$\omega(t,x) = 0$ and $\Upsilon(t,x) = \frac{c_1}{2}x + \rho(t)$.
The infinitesimal generator of the Lie group is given by,

$$\zeta^* = \frac{c_1}{2}x\frac{\partial}{\partial x} + \rho(t)\frac{\partial}{\partial x}, \tag{6.25}$$

where c_1 is an arbitrary constant and $\rho(t)$ satisfies (6.11). □

Having obtained the infinitesimal generator for the case when $k(t)$ is non-constant, we now perform symmetry analysis and a complete group classification of the second order neutral differential equation given by (6.8), for which,

$$k(t) = c_2, \tag{6.26}$$

where c_2 is an arbitrary constant.
Splitting (6.18) with respect to $x(t-r)$, and using (6.22), we get,

$$k(t)\beta'''(t) + 2\beta(t)d'(t) + 4\beta'(t)d(t) + 2b(t)\gamma'(t) = 0. \tag{6.27}$$

Splitting (6.18) with respect to $x'(t-r)$, and using (6.19) and (6.22), we get,

$$b(t)\beta'(t) + \beta(t)b'(t) = 0. \tag{6.28}$$

Equation (6.28) can be easily integrated to give,

$$b(t)\beta(t) = c_3, \tag{6.29}$$

where c_3 is an arbitrary constant.
Using (6.16), we can rewrite equations (6.17), (6.20), (6.21), (6.27) and (6.29) as,

$$\Upsilon(t,x) = \left[\frac{1}{2}(\omega_t + c_1)\right]x + \rho(t), \tag{6.30}$$

$$\omega_{ttt} + 4c(t)\omega_t + 2c'(t)\omega = 0, \tag{6.31}$$

$$\gamma(t) = \frac{1}{2}(\omega_t + c_1), \tag{6.32}$$

$$c_2\omega_{ttt} + 2d'(t)\omega(t) + 4d(t)\omega_t + b(t)\omega_{tt} = 0, \tag{6.33}$$

$$\omega(t, x) = \frac{c_3}{b(t)}, \tag{6.34}$$

where c_1, c_2 and c_3 are arbitrary constants.
The following theorems make a complete group classification of the second order neutral differential equation:

Theorem 6.3.2. *The neutral differential equation given by equation (6.11) for which $b(t) \neq 0, d(t) \neq 0, k(t) = c_2$ admits a three dimensional group generated by*

$$\zeta_1^* = x\frac{\partial}{\partial x}, \quad \zeta_2^* = \frac{1}{b(t)}\frac{\partial}{\partial t} + \frac{x}{2}\left(\frac{1}{b(t)}\right)'\frac{\partial}{\partial x}, \quad \zeta_3^* = \rho(t)\frac{\partial}{\partial x}.$$

Proof. If $c_3 \neq 0$, from (6.34) we get

$$b(t) = \frac{c_3}{\omega(t, x)}. \tag{6.35}$$

From (6.30), we can write,

$$\Upsilon(t, x) = [\frac{1}{2}(c_2(\frac{1}{b(t)})' + c_1)]x + \rho(t). \tag{6.36}$$

Using (6.35) in (6.33), we get,

$$c_2\omega\omega_{ttt} + 2\omega^2 d'(t) + 4\omega\omega_t d(t) + c_3\omega_{tt} = 0. \tag{6.37}$$

Equation (6.37) can be easily integrated to give,

$$c_2\omega\omega_{tt} - \frac{c_0}{2}\omega_t^2 + 2\omega^2 d(t) + c_3\omega_t = c_4, \tag{6.38}$$

where c_4 is an arbitrary constant.
Using (6.34), we can solve (6.38) for $d(t)$ to get,
$d(t) = \frac{1}{2}\left[c_5 b^2(t) + b'(t) + c_2\left(\frac{b''(t)}{b(t)} - 2\left(\frac{b'(t)}{b(t)}\right)^2 + \frac{b'(t)}{b^2(t)}\right)\right]$, where $c_5 = c_2 c_3^2$.
Since $\omega = \omega^r$, we get, $b(t) = b(t - r)$.
Using (6.34) in (6.31), we get,

$$c'(t) - 2\frac{b'(t)}{b(t)}c(t) = -\frac{1}{2}b(t)\left[6\frac{b'(t)b''(t)}{b^3(t)} - \frac{b'''(t)}{b^2(t)} - 6\frac{b'^3(t)}{b^4(t)}\right]. \tag{6.39}$$

Equation (6.39) is a first order linear ordinary differential equation which can be solved to give,

$$c(t) = \frac{1}{2}\left[\frac{b''(t)}{b(t)} - \frac{3}{2}\left(\frac{b'(t)}{b(t)}\right)^2 + \frac{c_6}{2}b^2(t)\right], \tag{6.40}$$

where c_6 is an arbitrary constant.

In this case, we have obtained the coefficients of the infinitesimal transformation as,

$$\omega = \frac{c_3}{b(t)}, \quad \Upsilon = \frac{x}{2}\left[c_3\left(\frac{1}{b(t)}\right)' + c_1\right] + \rho(t). \tag{6.41}$$

The infinitesimal generator in this case is given by,

$$\zeta^* = \frac{c_1}{2}x\frac{\partial}{\partial x} + c_3\left(\frac{1}{b(t)}\frac{\partial}{\partial t} + \frac{x}{2}\left(\frac{1}{b(t)}\right)'\frac{\partial}{\partial x}\right) + \rho(t)\frac{\partial}{\partial x}, \tag{6.42}$$

where $\rho(t)$ is an arbitrary solution of equation (6.11).

If $c_3 = 0$ then,

$$\omega(t,x) = 0, \quad \Upsilon(t,x) = \frac{c_1}{2}x + \rho(t). \tag{6.43}$$

The infinitesimal generator is given by,

$$\zeta^* = \left(\frac{c_1}{2}x + \rho(t)\right)\frac{\partial}{\partial x}. \tag{6.44}$$

□

Theorem 6.3.3. *The neutral differential equation given by equation (6.11) for which $b(t) \neq 0, d(t) = 0, k(t) = c_2$, admits the infinitesimal generator given by*

$$\zeta^* = \Phi_1(t)\frac{\partial}{\partial t} + \Psi_1(t,x)\frac{\partial}{\partial x},$$

where $\Phi_1(t)$ solves $\int^{\omega(t)} \frac{c_2}{E\tan A}\mathrm{d}\theta - t - c_9 = 0$, for $\omega(t)$ where A is a root (or zero) of $\left[B\ln\left(\frac{B^2(1+\tan^2 y)}{c_2\theta}\right) + D + 2c_3 y\right]$ for y, with $B = \sqrt{2c_7c_2^2 - c_3^2}, \quad D = c_8B$, $E = c_3 + B$, and $\Psi_1(t,x) = \frac{1}{2}[(\Phi_1(t))_t + c_1]x + \rho(t)$.

Proof. If $c_3 \neq 0$, then substituting (6.35) into (6.33), we get,

$$c_2\omega\omega_{ttt} + c_3\omega_{tt} = 0. \tag{6.45}$$

This is a nonlinear third order differential equation, the solution of which is given by,

$$\int^{\omega(t)} \frac{c_2}{E\tan A}d\theta - t - c_9 = 0, \tag{6.46}$$

where A is a root (or zero) of $\left[B\ln\left(\dfrac{B^2(1+\tan^2 y)}{c_2\theta}\right)+D+2c_3y\right]$ for y, with
$B=\sqrt{2c_7c_2^2-c_3^2}$,
$D=c_8B$, and
$E=c_3+B$.
It is to be noted that the expression in (6.46) may be complex valued and we are finding the zeroes for y. In this solution, c_7, c_8 and c_9 are arbitrary constants. To obtain the corresponding infinitesimal generator, we have to solve (6.46) for $\omega(t)$.
The infinitesimal generator in this case is given by

$$\zeta^* = \Phi_1(t)\frac{\partial}{\partial t}+\Psi_1(t,x)\frac{\partial}{\partial x}, \tag{6.47}$$

where $\Phi_1(t)$ solves (6.46) for $\omega(t)$ and $\Psi_1(t,x)=\dfrac{1}{2}[(\Phi_1(t))_t+c_1]x+\rho(t)$. □

Remark 6.3.1. As the above is not easy to solve in general, choosing $B=0$ that is $k(t)=\dfrac{c_3}{\sqrt{2c_7}}$, we see that, $\omega(t,x)=\dfrac{c_3}{c_2}(t+c_{10})$, solves (6.46).
But the condition $\omega=\omega^r$ gives $c_3=0$.
Consequently, $\omega(t,x)=0 \quad \Upsilon(t,x)=\dfrac{1}{2}c_1x+\rho(t)$, and the infinitesimal generator is given by

$$\zeta^* = \frac{1}{2}x\frac{\partial}{\partial x}+\rho(t)\frac{\partial}{\partial x}. \tag{6.48}$$

By considering a very special case in which $c_2=1=c_3$, we obtain from (6.33),

$$\omega\omega_{ttt}+\omega_{tt}=0. \tag{6.49}$$

Equation (6.49) yields a solution for which some infinitesimal generators can be explicitly found. It's solution is given by

$$\int^{\omega(t)}\frac{d\theta}{1+c_{11}\tan G}-t-c_{13}=0, \tag{6.50}$$

where G is a root (or zero) of $\left[\ln\left(\dfrac{c_{11}^2}{\cos^2 y}\right)c_{11}-c_{11}\ln\theta+c_{11}c_{12}+2y\right]$ for y.
In (6.50), c_{11}, c_{12} and c_{13} are arbitrary constants.
The infinitesimal generator in this case is

$$\zeta^* = \Phi_2(t)\frac{\partial}{\partial t}+\Psi_2(t,x)\frac{\partial}{\partial x}, \tag{6.51}$$

where $\Phi_2(t)$ solves (6.50) for $\omega(t)$ and $\Psi_2(t,x)=\dfrac{1}{2}[(\Phi_2(t))_t+c_1]x+\rho(t)$.

Corollary 6.3.1. *The neutral differential equation given by equation (6.11) for which*

$b(t) \neq 0, d(t) = 0, k(t) = 1 = c_3, c_{11} = 0,$ *admits the three dimensional group given by*

$$\zeta_1^* = \frac{\partial}{\partial t}, \quad \zeta_2^* = \frac{x}{2}\frac{\partial}{\partial x}, \quad \zeta_3^* = \rho(t)\frac{\partial}{\partial x}.$$

Proof. It can be easily seen that the generators corresponding to $c_{11} = 0$ can be explicitly obtained. In this case $\omega(t,x) = c_{14}t + c_{15}$ is a solution of (6.49), where c_{14} and c_{15} are arbitrary constants.
The condition $\omega = \omega^r$ implies $c_{14} = 0$.
Hence, $\omega(t,x) = c_{15}, \quad \Upsilon(t,x) = \dfrac{c_1}{2}x + \rho(t).$
If $c_{15} \neq 0$,
The infinitesimal generator is given by

$$\zeta^* = c_{15}\frac{\partial}{\partial t} + \frac{1}{2}c_1 x\frac{\partial}{\partial x} + \rho(t)\frac{\partial}{\partial x}. \tag{6.52}$$

Using (6.40), $c(t) = \dfrac{1}{4}\dfrac{c_6 c_3^2}{c_{15}^2}$.
If $c_{15} = 0$, then the infinitesimal generator is given by (6.48). Finally, if $c_3 = 0$, then the infinitesimal generator is given by (6.44). □

Theorem 6.3.4. *The neutral differential equation given by equation (6.11) for which* $b(t) = 0, d(t) \neq 0, k(t) = c_2$ *admits the infinitesimal generator given by*

$$\zeta^* = \Phi_3(t)\frac{\partial}{\partial t} + \Psi_3(t,x)\frac{\partial}{\partial x},$$

where $\Phi_3(t)$ *solves* $c_2\omega\omega_{tt} - c_2\dfrac{\omega_t^2}{2} + 2\omega^2(t)d(t) = c_{16}$, *for* $\omega(t)$, *and*
$\Psi_3(t,x) = \dfrac{1}{2}[(\Phi_3(t))_t + c_1]x + \rho(t).$

Proof. Then from equation (6.33),

$$c_2\omega_{ttt} + 2d'(t)\omega(t) + 4d(t)\omega_t = 0. \tag{6.53}$$

Equation (6.53) can be integrated once to obtain,

$$c_2\omega\omega_{tt} - c_2\frac{\omega_t^2}{2} + 2\omega^2(t)d(t) = c_{16}, \tag{6.54}$$

where c_{16} is an arbitrary constant.
Equation (6.54) is extremely difficult to solve for an arbitrary $d(t)$.
If $\omega(t) = \Phi_3(t)$ solves equation (6.54), then, The infinitesimal generator in this case is given by

$$\zeta^* = \Phi_3(t)\frac{\partial}{\partial t} + \Psi_3(t,x)\frac{\partial}{\partial x}, \tag{6.55}$$

where $\Phi_3(t)$ solves (6.54) for $\omega(t)$ and $\Psi_3(t,x) = \dfrac{1}{2}\left[(\Phi_2(t))_t + c_1\right]x + \rho(t)$. □

Remark 6.3.2. As can be seen the infinitesimal generator given by (6.55) cannot be explicitly solved due to the arbitrariness of $d(t)$. However, by choosing a few explicit values of $d(t)$, we obtain the corresponding different infinitesimal generator.

Corollary 6.3.2. *The neutral differential equation given by equation (6.11) for which $b(t) = 0, d(t) = 1, k(t) = c_2$ admits the five dimensional Lie group generated by*

$$\zeta_1^* = \frac{\partial}{\partial t}, \quad \zeta_2^* = \frac{x}{2}\frac{\partial}{\partial x}, \quad \zeta_3^* = \sin\left(\frac{2t}{\sqrt{k(t)}}\right)\frac{\partial}{\partial t} + \frac{x}{\sqrt{k(t)}}\cos\left(\frac{2t}{\sqrt{k(t)}}\right)\frac{\partial}{\partial x},$$

$$\zeta_4^* = \cos\left(\frac{2t}{\sqrt{k(t)}}\right)\frac{\partial}{\partial t} - \frac{x}{\sqrt{k(t)}}\sin\left(\frac{2t}{\sqrt{k(t)}}\right)\frac{\partial}{\partial x}, \quad \zeta_5^* = \rho(t)\frac{\partial}{\partial x}.$$

Proof. Taking $d(t) = 1$, equation (6.54) becomes $c_2\omega\omega_{tt} - c_2\dfrac{\omega_t^2}{2} + 2\omega^2(t) = c_{16}$, which can be solved to give

$$\omega\,(t) = \frac{\sqrt{c_{17}}}{2} + c_{18}\,\sin\left(\frac{2t}{\sqrt{c_2}}\right) + c_{19}\,\cos\left(\frac{2t}{\sqrt{c_2}}\right), \tag{6.56}$$

where $c_{17} = 4c_{18}^2 + 4c_{19}^2 + 2c_{16}$, c_{18}, c_{19} are arbitrary constants. Using (6.56), equation (6.30) gives,

$$\Upsilon(t,x) = \frac{1}{2}\left[2\frac{c_{18}}{\sqrt{c_2}}\cos\left(\frac{2t}{\sqrt{c_2}}\right) - 2\frac{c_{19}}{\sqrt{c_2}}\sin\left(\frac{2t}{\sqrt{c_2}}\right) + c_1\right]x + \rho(t).$$

Using (6.31), we see that,

$$\begin{aligned} c(t) = \Big(&2c_{18}^2\cos\Big(\frac{4t}{\sqrt{c_2}}\Big) - 2c_{19}^2\cos\Big(\frac{4t}{\sqrt{c_2}}\Big) - 4c_{18}c_{19}\sin\Big(\frac{4t}{\sqrt{c_2}}\Big) - 4\cos\Big(\frac{4t}{\sqrt{c_2}}\Big)\sqrt{c_{17}}c_{19} \\ &- 4\sin\Big(\frac{4t}{\sqrt{c_2}}\Big)\sqrt{c_{17}}c_{18} - c_{20}c_2\Big) \Big/ \Big(2\cos\Big(\frac{4t}{\sqrt{c_2}}\Big)c_2c_{18}^2 - 2\cos\Big(\frac{4t}{\sqrt{c_2}}\Big)c_2c_{19}^2 - 4\sin\Big(\frac{4t}{\sqrt{c_2}}\Big) \\ &c_2c_{18}c_{19} - 4\cos\Big(\frac{4t}{\sqrt{c_2}}\Big)c_2\sqrt{c_{17}}c_{19} - 4\sin\Big(\frac{4t}{\sqrt{c_2}}\Big)c_2\sqrt{c_{17}}c_{18} - 6c_2c_{18}^2 - 6c_2c_{19}^2 - 2c_2c_{16}\Big). \end{aligned}$$

We note that, from equation (6.26), $k(t) = c_2$.
The infinitesimal generator in this case is explicitly given by,

$$\begin{aligned} \zeta^* = c_{21}\frac{\partial}{\partial t} + \frac{c_1}{2}x\frac{\partial}{\partial x} + c_{18}\left(\sin\left(\frac{2t}{\sqrt{c_2}}\right)\frac{\partial}{\partial t} + \frac{x}{\sqrt{c_2}}\cos\left(\frac{2t}{\sqrt{c_2}}\right)\frac{\partial}{\partial x}\right) \\ + c_{19}\left(\cos\left(\frac{2t}{\sqrt{c_2}}\right)\frac{\partial}{\partial t} - \frac{x}{\sqrt{c_2}}\sin\left(\frac{2t}{\sqrt{c_2}}\right)\frac{\partial}{\partial x}\right) + \rho(t)\frac{\partial}{\partial x}. \end{aligned} \tag{6.57}$$

where c_{18}, c_{19}, c_{20} and $c_{21} = \dfrac{\sqrt{c_{17}}}{2}$ are arbitrary constants. $\square$

Corollary 6.3.3. *The neutral differential equation given by equation (6.11) for which $b(t) = 0, d(t) = e^t, k(t) = c_2$ admits the five dimensional Lie group generated by*

$$\zeta_1^* = \Big(J_0\big(2\lambda\big)\Big)^2 \frac{\partial}{\partial t} + \frac{xe^t}{\sqrt{k(t)e^t}} J_0\big(2\lambda\big) J_1\big(2\lambda\big) \frac{\partial}{\partial x},$$

$$\zeta_2^* = \Big(Y_0\big(2\lambda\big)\Big)^2 \frac{\partial}{\partial t} - \frac{xe^t}{\sqrt{k(t)e^t}} Y_0\big(2\lambda\big) Y_1\big(2\lambda\big) \frac{\partial}{\partial x},$$

$$\zeta_3^* = J_0\big(2\lambda\big) Y_0\big(2\lambda\big) \frac{\partial}{\partial t} - \frac{xe^t}{\sqrt{k(t)e^t}} \Big(J_1\big(2\lambda\big) Y_0\big(2\lambda\big) + J_0\big(2\lambda\big) Y_1\big(2\lambda\big)\Big) \frac{\partial}{\partial x},$$

$$\zeta_4^* = \frac{x}{2}\frac{\partial}{\partial x}, \quad \zeta_5^* = \rho(t) \frac{\partial}{\partial x},$$

where $\lambda = \dfrac{\sqrt{k(t)e^t}}{k(t)}$.

Proof. Taking $d(t) = e^t$, equation (6.54) becomes $c_2 \omega \omega_{tt} - c_2 \dfrac{\omega_t^2}{2} + 2e^t \omega^2(t) = c_{16}$, which can be solved to give

$$\omega(t) = \frac{1}{4} \frac{c_{23}^2(1 + 2c_{16})}{c_{22}} \left(J_0\left(2\frac{\sqrt{c_2 e^t}}{c_2}\right)\right)^2 + c_{22} \left(Y_0\left(2\frac{\sqrt{c_2 e^t}}{c_2}\right)\right)^2 + c_{23} J_0\left(2\frac{\sqrt{c_2 e^t}}{c_2}\right) Y_0\left(2\frac{\sqrt{c_2 e^t}}{c_2}\right), \quad (6.58)$$

where c_{22} and c_{23} are arbitrary constants,
From (6.30), we get

$$\Upsilon(t,x) = \frac{1}{2}\Big[\frac{-1}{2} \frac{e^t {c_{23}}^2(1+2c_{16})}{c_{22}\sqrt{c_2 e^t}} J_0\Big(2\,\frac{\sqrt{c_2 e^t}}{c_2}\Big) J_1\Big(2\,\frac{\sqrt{c_2 e^t}}{c_2}\Big) - 2\,\frac{c_{22}\,e^t}{\sqrt{c_2 e^t}} Y_0\Big(2\,\frac{\sqrt{c_2 e^t}}{c_2}\Big) Y_1\Big(2\,\frac{\sqrt{c_2 e^t}}{c_2}\Big) - \frac{c_{23}\,e^t}{\sqrt{c_2 e^t}} J_1\Big(2\,\frac{\sqrt{c_2 e^t}}{c_2}\Big) Y_0\Big(2\,\frac{\sqrt{c_2 e^t}}{c_2}\Big) - \frac{c_{23}\,e^t}{\sqrt{c_2 e^t}} J_0\Big(2\,\frac{\sqrt{c_2 e^t}}{c_2}\Big) Y_1\Big(2\,\frac{\sqrt{c_2 e^t}}{c_2}\Big) + c_1\Big] x + \rho(t).$$

Using (6.31), we see that,

$$c(t) = \left[\int \frac{q(t)}{r(t)} e^{-4\int \frac{p_1(t)}{p_2(t)} dt} dt + c_{24}\right] e^{4\int \frac{s_(t)}{s_2(t)} dt},$$

where,

$$p_1(t) = e^t \Big[(1+2c_{16}) c_{23}^2 J_0\left(\frac{2\sqrt{c_2 e^t}}{c_2}\right) J_1\left(\frac{2\sqrt{c_2 e^t}}{c_2}\right) + 2c_{22}c_{23}\Big(J_0\left(\frac{2\sqrt{c_2 e^t}}{c_2}\right) Y_1\left(\frac{2\sqrt{c_2 e^t}}{c_2}\right) + J_1\left(\frac{2\sqrt{c_2 e^t}}{c_2}\right) Y_0\left(\frac{2\sqrt{c_2 e^t}}{c_2}\right)\Big) + 4c_{22}^2 Y_0\left(\frac{2\sqrt{c_2 e^t}}{c_2}\right) Y_1\left(\frac{2\sqrt{c_2 e^t}}{c_2}\right)\Big],$$

$$p_2(t) = \sqrt{c_2e^t}\Big[(1+2c_{16})c_{23}^2\left(J_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)\right)^2 + 4c_{22}c_{23}J_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)Y_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right) + 4c_{22}^2\left(Y_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)\right)^2\Big],$$

$$\begin{aligned} q(t) \;=\; & (1+2c_{16})c_{23}^2e^t\sqrt{c_2e^t}\left(J_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)\right)^2 \\ & +4c_{22}e^t\sqrt{ke^t}\Big(J_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)Y_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)c_{23} + \left(Y_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)\right)^2 c_{22}\Big) \\ & -4(1+2c_{16})e^{2t}c_{23}^2J_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)J_1\left(\frac{2\sqrt{c_2e^t}}{c_2}\right) \\ & -8c_{22}c_{23}e^{2t}\left(J_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)Y_1\left(\frac{2\sqrt{c_2e^t}}{c_2}\right) + J_1\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)Y_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)\right) \\ & -16e^{2t}c_{22}^2Y_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)Y_1\left(\frac{2\sqrt{c_2e^t}}{c_2}\right), \end{aligned}$$

$$r(t) = c_2\sqrt{c_2e^t}\Big[(1+2c_{16})c_{23}^2\left(J_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)\right)^2 + 4c_{22}\Big(J_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)Y_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)c_{23} + \left(Y_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)\right)^2 c_{22}\Big)\Big],$$

$$s_1(t) = c_2^2e^{3t}\Big[(1+2c_{16})c_{23}^2J_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)J_1\left(\frac{2\sqrt{c_2e^t}}{c_2}\right) + 2c_{22}c_{23}\Big(J_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right) Y_1\left(\frac{2\sqrt{c_2e^t}}{c_2}\right) + J_1\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)Y_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)\Big) + 4c_{22}^2Y_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)Y_1\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)\Big],$$

and

$$s_2(t) = (c_2e^t)^{5/2}\Big[(1+2c_{16})c_{23}^2\left(J_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)\right)^2 + 4c_{22}Y_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)\Big(J_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)c_{23} + Y_0\left(\frac{2\sqrt{c_2e^t}}{c_2}\right)c_{22}\Big)\Big],$$

where c_{24} is an arbitrary constant.

The infinitesimal generator is given by

$$\zeta^* = \frac{c_{23}^2(1+2c_{16})}{4c_{22}}\Big[\Big(J_0\Big(2\frac{\sqrt{c_2e^t}}{c_2}\Big)\Big)^2\frac{\partial}{\partial t} - \frac{xe^t}{\sqrt{c_2e^t}}J_0\Big(2\frac{\sqrt{c_2e^t}}{c_2}\Big)J_1\Big(2\frac{\sqrt{c_2e^t}}{c_2}\Big)\frac{\partial}{\partial x}\Big]$$
$$+ c_{22}\Big[\Big(Y_0\Big(2\frac{\sqrt{c_2e^t}}{c_2}\Big)\Big)^2\frac{\partial}{\partial t} - \frac{xe^t}{\sqrt{c_2e^t}}Y_0\Big(2\frac{\sqrt{c_2e^t}}{c_2}\Big)Y_1\Big(2\frac{\sqrt{c_2e^t}}{c_2}\Big)\frac{\partial}{\partial x}\Big]$$
$$+ c_{23}\Big[J_0\Big(2\frac{\sqrt{c_2e^t}}{c_2}\Big)Y_0\Big(2\frac{\sqrt{c_2e^t}}{c_2}\Big)\frac{\partial}{\partial t} - \frac{xe^t}{\sqrt{c_2e^t}}\Big(J_1\Big(2\frac{\sqrt{c_2e^t}}{c_2}\Big)Y_0\Big(2\frac{\sqrt{c_2e^t}}{c_2}\Big)$$
$$+ J_0\Big(2\frac{\sqrt{c_2e^t}}{c_2}\Big)Y_1\Big(2\frac{\sqrt{c_2e^t}}{c_2}\Big)\Big)\frac{\partial}{\partial x}\Big] + c_1\frac{x}{2}\frac{\partial}{\partial x} + \rho(t)\frac{\partial}{\partial x}. \quad (6.59)$$

□

Corollary 6.3.4. *The neutral differential equation given by equation (6.11) for which $b(t) = 0, d(t) = \sin t, k(t) = c_2$, admits the five dimensional Lie group generated by*
$\zeta_1^* = \Big(MathieuC\Big(0, -\frac{2}{k(t)}, \frac{-\pi}{4} + \frac{t}{2}\Big)\Big)^2\frac{\partial}{\partial t}$
$+ \frac{x}{2}MathieuC\Big(0, -\frac{2}{k(t)}, \frac{-\pi}{4} + \frac{t}{2}\Big)MathieuCPrime\Big(0, -\frac{2}{k(t)}, \frac{-\pi}{4} + \frac{t}{2}\Big)\frac{\partial}{\partial x},$
$\zeta_2^* = \Big(MathieuS\Big(0, -\frac{2}{k(t)}, \frac{-\pi}{4} + \frac{t}{2}\Big)\Big)^2\frac{\partial}{\partial t}$
$+ \frac{x}{2}MathieuS\Big(0, -\frac{2}{k(t)}, \frac{-\pi}{4} + \frac{t}{2}\Big)MathieuSPrime\Big(0, -\frac{2}{k(t)}, \frac{-\pi}{4} + \frac{t}{2}\Big)\frac{\partial}{\partial x},$
$\zeta_3^* = MathieuC\Big(0, -\frac{2}{k(t)}, \frac{-\pi}{4} + \frac{t}{2}\Big)MathieuS\Big(0, -\frac{2}{k(t)}, \frac{-\pi}{4} + \frac{t}{2}\Big)\frac{\partial}{\partial t}$
$+ \frac{x}{4}\Big(MathieuCPrime\Big(0, -\frac{2}{k(t)}, \frac{-\pi}{4} + \frac{t}{2}\Big)MathieuS\Big(0, -\frac{2}{k(t)}, \frac{-\pi}{4} + \frac{t}{2}\Big)$
$+ MathieuC\Big(0, -\frac{2}{k(t)}, \frac{-\pi}{4} + \frac{t}{2}\Big)MathieuSPrime\Big(0, -\frac{2}{k(t)}, \frac{-\pi}{4} + \frac{t}{2}\Big)\Big)\frac{\partial}{\partial x},$
$\zeta_4^* = \frac{x}{2}\frac{\partial}{\partial x}, \quad \zeta_5^* = \rho(t)\frac{\partial}{\partial x}.$

Proof. Taking $d(t) = \sin t$, equation (6.54) becomes $c_2\omega\omega_{tt} - c_2\frac{\omega_t^2}{2} + 2\omega^2(t)\sin t = c_{16}$, which can be solved to give

$$\omega\,(t) = \frac{1}{4}\,\frac{{c_{26}}^2}{c_{25}}(1+8c_{16})\left(MathieuC\left(0, -\frac{2}{k(t)}, \frac{-\pi}{4} + \frac{t}{2}\right)\right)^2$$
$$+ c_{25}\left(MathieuS\left(0, -\frac{2}{k(t)}, \frac{-\pi}{4} + \frac{t}{2}\right)\right)^2 + c_{26}\,MathieuC\left(0, -\frac{2}{k(t)}, \frac{-\pi}{4} + \frac{t}{2}\right)$$
$$MathieuS\Big(0, -\frac{2}{k(t)}, \frac{-\pi}{4} + \frac{t}{2}\Big), \quad (6.60)$$

where c_{25}, c_{26} are arbitrary constants. Using (6.60), equation (6.30) gives,

$$\Upsilon(t,x)=\frac{1}{2}\Big[\frac{1}{4}\frac{c_{26}^2}{c_{25}}(1+8c_{16})MathieuC\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)MathieuCPrime\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)+c_{25}\,MathieuS\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)MathieuSPrime\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)+\frac{1}{2}c_{26}\,MathieuCPrime\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)MathieuS\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)+\frac{1}{2}c_{26}\,MathieuC\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)MathieuSPrime\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)+c_1\Big]x+\rho(t).$$

Using (6.31), we see that,

$$c(t)=e^{-2\int r_1(t)\mathrm{d}t}\int\frac{q_1(t)}{q_2(t)}e^{2\int r_1(t)\mathrm{d}t}\mathrm{d}t+c_{27}e^{-2\int r_1(t)\mathrm{d}t},$$

where,

$$\begin{aligned}r_1(t)=\Big(&c_{26}(1+8c_{16})^2MathieuC\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)MathieuCPrime\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)\\&+2c_{25}c_{26}\Big(MathieuCPrime\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)MathieuS\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)\\&+MathieuC\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)MathieuSPrime\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)\Big)\\&+4c_{25}^2MathieuS\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)MathieuSPrime\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)\Big)\Big/\Big(c_{26}(1+8c_{16})^2\\&\Big(MathieuC\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)\Big)^2+4c_{25}c_{26}MathieuC\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)\\&MathieuS\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)+4c_{25}^2\Big(MathieuS\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)\Big)^2\Big),\end{aligned}$$

$$\begin{aligned}q_1(t)=\;&2c_{26}(1+8c_{16})^2MathieuC\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)MathieuCPrime\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)\\&\sin t+4c_{25}c_{26}\Big(MathieuCPrime\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)MathieuS\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)\\&+MathieuC\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)MathieuSPrime\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)\Big)\sin t\\&+8c_{25}^2MathieuS\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)MathieuSPrime\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)\sin t\\&+c_{26}(1+8c_{16})^2\Big(MathieuC\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)\Big)^2\cos t+4c_{25}c_{26}MathieuC\Big(0,-\frac{2}{k(t)},\\&\frac{-\pi}{4}+\frac{t}{2}\Big)MathieuS\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)\cos t+4c_{25}^2\Big(MathieuS\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)\Big)^2\cos t,\end{aligned}$$

and,

$$q_2(t) = k(t)\Big(c_{26}(1+8c_{16})^2\Big(MathieuC\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)\Big)^2 + 4c_{25}c_{26}$$
$$MathieuC\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)MathieuS\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)$$
$$+4c_{25}^2\Big(MathieuS\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)\Big)^2\Big),$$

where c_{27} is an arbitrary constant.
The infinitesimal generator in this case is explicitly given by,

$$\zeta^* = \frac{1}{4}\frac{c_{26}^2}{c_{25}}(1+8c_{16})\Big[\Big(MathieuC\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)\Big)^2\frac{\partial}{\partial t}$$
$$+\frac{x}{2}MathieuC\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)MathieuCPrime\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)\frac{\partial}{\partial x}\Big]$$
$$+c_{25}\Big[\Big(MathieuS\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)\Big)^2\frac{\partial}{\partial t}+\frac{x}{2}MathieuS\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)$$
$$MathieuSPrime\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)\frac{\partial}{\partial x}\Big]+c_{26}\Big[MathieuC\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)$$
$$MathieuS\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)\frac{\partial}{\partial t}+\frac{x}{4}\Big(MathieuCPrime\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)$$
$$MathieuS\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)+MathieuC\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)$$
$$MathieuSPrime\Big(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\Big)\Big)\frac{\partial}{\partial x}\Big]+c_1\frac{x}{2}\frac{\partial}{\partial x}+\rho(t)\frac{\partial}{\partial x}. \quad (6.61)$$

□

Corollary 6.3.5. *The neutral differential equation given by equation (6.11) for which* $b(t)=0, d(t)=t^m$ *where* m *is any constant,* $k(t)=c_2$ *admits the five dimensional Lie group generated by*

$$\zeta_1^* = t(J_\nu(\mu))^2\frac{\partial}{\partial t}+x\Big(\frac{1}{2}(J_\nu(\mu))^2+\frac{2}{m+2}J_\nu(\mu)\Big(-J_{\nu+1}(\mu)+\frac{J_\nu(\mu)}{2\tau}\Big)\tau(m/2+1)\Big)\frac{\partial}{\partial x},$$
$$\zeta_2^* = t(Y_\nu(\mu))^2\frac{\partial}{\partial t}+x\Big(\frac{1}{2}(Y_\nu(\mu))^2+\frac{2}{m+2}Y_\nu(\mu)\Big(-Y_{\nu+1}(\mu)+\frac{Y_\nu(\mu)}{2\tau}\Big)\tau(m/2+1)\Big)\frac{\partial}{\partial x},$$
$$\zeta_3^* = tJ_\nu(\mu)Y_\nu(\mu)\frac{\partial}{\partial t}+x\Big(\frac{1}{2}J_\nu(\mu)Y_\nu(\mu)+\frac{1}{m+2}\Big(\Big(-J_{\nu+1}(\mu)+\frac{J_\nu(\mu)}{2\tau}Y_\nu(\mu)$$
$$+J_\nu(\mu)\Big(-Y_{\nu+1}(\mu)+\frac{Y_\nu(\mu)}{2\tau}\Big)\Big)\tau(m/2+1)\Big)\Big)\frac{\partial}{\partial x},$$
$$\zeta_4^* = \frac{x}{2}\frac{\partial}{\partial x},\quad \zeta_5^* = \rho(t)\frac{\partial}{\partial x},$$

where $\nu=(m+2)^{-1}$, $\tau=\sqrt{(k(t))^{-1}}t^{m/2+1}$, $\mu=2\tau\nu$.

Proof. Taking $d(t)=t^m$, where m is any constant, equation (6.54) becomes

$c_2\omega\omega_{tt} - c_2\dfrac{\omega_t^2}{2} + 2\omega^2(t)t^m = c_{16}$, which can be solved to give

$$\begin{aligned}\omega\,(t,x) = \frac{1}{4}\,\frac{{c_{29}}^2 t}{c_{28}}(1+2c_{16})\Big(J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)\Big)^2 \\ + c_{28}\,t\Big(Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{{c_2}^{-1}}t^{m/2+1}}{m+2}\Big)\Big)^2 \\ + c_{29}\,tJ_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{{c_2}^{-1}}t^{m/2+1}}{m+2}\Big)Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{{c_2}^{-1}}t^{m/2+1}}{m+2}\Big),\end{aligned} \tag{6.62}$$

where c_{28} and c_{29} are arbitrary constants,
From (6.30), we get

$$\begin{aligned}\Upsilon(t,x) = \frac{1}{2}\Big[\frac{1}{4}\,\frac{{c_{29}}^2}{c_{28}}(1+2c_{16})\Big(J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)\Big)^2 + \frac{1+2c_{16}}{c_{28}(m+2)} \\ \Big(c_{29}^2 J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)\Big(-J_{(m+2)^{-1}+1}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big) + \frac{J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)}{2\sqrt{c_2^{-1}}t^{m/2+1}}\Big) \\ \sqrt{c_2^{-1}}t^{m/2+1}\,(m/2+1)\Big) + c_{28}\Big(Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{{c_2}^{-1}}t^{m/2+1}}{m+2}\Big)\Big)^2 \\ + \frac{1}{m+2}\Big(4c_{28}Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{{c_2}^{-1}}t^{m/2+1}}{m+2}\Big)\Big(-Y_{(m+2)^{-1}+1}\Big(2\,\frac{\sqrt{{c_2}^{-1}}t^{m/2+1}}{m+2}\Big) \\ + \frac{Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)}{2\sqrt{c_2^{-1}}t^{m/2+1}}\Big)\sqrt{c_2^{-1}}t^{m/2+1}\,(m/2+1)\Big) + c_{29}J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big) \\ Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big) + \frac{1}{m+2}\Big(2c_{29}\Big(-J_{(m+2)^{-1}+1}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big) \\ + \frac{J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)}{2\sqrt{c_2^{-1}}t^{m/2+1}}\Big)\sqrt{c_2^{-1}}t^{m/2+1}\,(m/2+1)\,Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)\Big) \\ + \frac{1}{m+2}\Big(2c_{29}J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)\Big(-Y_{(m+2)^{-1}+1}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big) \\ + \frac{Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)}{2\sqrt{c_2^{-1}}t^{m/2+1}}\Big)\sqrt{c_2^{-1}}t^{m/2+1}\,(m/2+1)\Big) + c_1\Big]x + \rho(t).\end{aligned}$$

Using (6.31), we see that,

$$c(t) = \left[\int -\frac{l(t)}{c_2 j(t)} e^{-4\int \frac{e(t)}{j(t)}\mathrm{d}t}\,\mathrm{d}t + c_{30}\right] e^{4\int \frac{y(t)}{j(t)}}\,\mathrm{d}t,$$

where,

$$
\begin{aligned}
e(t) = {} & \sqrt{c_2^{-1}}(1+2c_{16})c_{29}^2 J_{(3+m)(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big) J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big) \\
& t^{m/2+1} + 2\sqrt{c_2^{-1}}c_{28}c_{29}\Big(J_{(3+m)(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big) \\
& + J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)Y_{(3+m)(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)\Big)t^{m/2+1} \\
& + 4\sqrt{c_2^{-1}}c_{28}^2 Y_{(3+m)(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)t^{m/2+1} \\
& - (1+2c_{16})c_{29}^2\Big(J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)\Big)^2 - 4c_{28}Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big) \\
& \Big(J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)c_{29} + Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)c_{28}\Big),
\end{aligned}
$$

$$
\begin{aligned}
j(t) = {} & t\Big[(1+2c_{16})c_{29}^2\Big(J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)\Big)^2 + 4c_{28}Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big) \\
& \Big(J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)c_{29} + Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)c_{28}\Big)\Big],
\end{aligned}
$$

$$l(t)=4\sqrt{c_2^{-1}}(1{+}2c_{16})c_{29}^2J_{(3+m)(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)t^{1.5m+1}$$
$$-2mc_{29}^2c_{16}\Big(J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)\Big)^2t^m{+}8\sqrt{c_2^{-1}}c_{28}c_{29}\Big(J_{(3+m)(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)$$
$$Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)+J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)Y_{(3+m)(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)\Big)$$
$$t^{3m/2+1}+16\sqrt{c_2^{-1}}c_{28}^2Y_{(3+m)(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)t^{3m/2+1}$$
$$-8c_{29}^2c_{16}\Big(J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)\Big)^2t^m-mc_{29}^2\Big(J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)\Big)^2t^m$$
$$-4mc_{28}c_{29}J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)t^m$$
$$-4mc_{28}^2\Big(Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)\Big)^2t^m-4c_{29}^2\Big(J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)\Big)^2t^m$$
$$-16c_{28}Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)\Big(J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)c_{29}$$
$$+Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)c_{28}\Big)t^m,$$

$$y(t)=\sqrt{c_2^{-1}}c_{29}^2(1+2c_{16})J_{(3+m)(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)$$
$$t^{m/2+1}+2\sqrt{c_2^{-1}}c_{28}c_{29}\Big(J_{(3+m)(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)$$
$$+J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)Y_{(3+m)(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)\Big)t^{m/2+1}+4\sqrt{c_2^{-1}}c_{28}^2$$
$$Y_{(3+m)(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)t^{m/2+1}-(1+2c_{16})c_{29}^2$$
$$\Big(J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)\Big)^2-4c_{28}Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)$$
$$\Big(J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)c_{29}+Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)c_{28}\Big),$$

where c_{30} is an arbitrary constant.

The infinitesimal generator is given by

$$\zeta^* = \frac{1}{4}\frac{c_{29}^2}{c_{28}}(1+2c_{16})\Big[t\Big(J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)\Big)^2\frac{\partial}{\partial t}+\Big(\frac{x}{2}\Big(J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)\Big)^2$$
$$+\frac{2x}{m+2}J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)\Big(-J_{(m+2)^{-1}+1}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)$$
$$+\frac{J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)}{2\sqrt{c_2^{-1}}t^{m/2+1}}\Big)\sqrt{c_2^{-1}}t^{m/2+1}\,(m/2+1)\Big)\frac{\partial}{\partial x}\Big]+c_{28}\Big[t\Big(Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2{}^{-1}}t^{m/2+1}}{m+2}\Big)\Big)^2$$
$$\frac{\partial}{\partial t}+\Big(\frac{x}{2}\Big(Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2{}^{-1}}t^{m/2+1}}{m+2}\Big)\Big)^2+\frac{2x}{m+2}Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2{}^{-1}}t^{m/2+1}}{m+2}\Big)$$
$$\Big(-Y_{(m+2)^{-1}+1}\Big(2\,\frac{\sqrt{c_2{}^{-1}}t^{m/2+1}}{m+2}\Big)+\frac{Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)}{2\sqrt{c_2^{-1}}t^{m/2+1}}\Big)\sqrt{c_2^{-1}}t^{m/2+1}\,(m/2+1)\Big)\frac{\partial}{\partial x}\Big]$$
$$+c_{29}\Big[tJ_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2{}^{-1}}t^{m/2+1}}{m+2}\Big)Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2{}^{-1}}t^{m/2+1}}{m+2}\Big)\frac{\partial}{\partial t}$$
$$+\Big(\frac{x}{2}J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2{}^{-1}}t^{m/2+1}}{m+2}\Big)Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2{}^{-1}}t^{m/2+1}}{m+2}\Big)$$
$$+\frac{x}{m+2}\Big(-J_{(m+2)^{-1}+1}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)+\frac{J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)}{2\sqrt{c_2^{-1}}t^{m/2+1}}\Big)\sqrt{c_2^{-1}}t^{m/2+1}\,(m/2+1)$$
$$Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)+\frac{x}{m+2}J_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)\Big(-Y_{(m+2)^{-1}+1}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)$$
$$+\frac{Y_{(m+2)^{-1}}\Big(2\,\frac{\sqrt{c_2^{-1}}t^{m/2+1}}{m+2}\Big)}{2\sqrt{c_2^{-1}}t^{m/2+1}}\Big)\sqrt{c_2^{-1}}t^{m/2+1}\,(m/2+1)\Big)\frac{\partial}{\partial x}\Big]+c_1\frac{x}{2}\frac{\partial}{\partial x}+\rho(t)\frac{\partial}{\partial x}. \quad (6.63)$$

□

Remark 6.3.3. In all our cases above, we gave assumed $k(t) \neq 0$, that is $c_2 \neq 0$. However, if $c_2 = 0$, then equation (6.8) reduces to a second order delay differential equation. As special cases of our group classification, we study the cases for which $c_2 = 0$.

Theorem 6.3.5. *The delay differential equation given by equation (6.11) for which* $b(t) \neq 0, d(t) \neq 0, k(t) = 0$ *admits a three dimensional group generated by*

$$\zeta_1^* = x\frac{\partial}{\partial x}, \quad \zeta_2^* = \frac{1}{b(t)}\frac{\partial}{\partial t} + \frac{x}{2}\left(\frac{1}{b(t)}\right)'\frac{\partial}{\partial x}, \quad \zeta_3^* = \rho(t)\frac{\partial}{\partial x}.$$

Proof. Equation (6.31) reduces to ,
$\omega_{ttt} = -(2c'(t)\omega + 4c(t)\omega_t$ or $\omega\omega_{ttt} = -(2c'(t)\omega^2 + 4c(t)\omega\omega_t$.

Integrating this, we get,

$$\omega\omega_{tt} - \frac{\omega_t^2}{2} + 2c(t)\omega^2 = c_{31}, \tag{6.64}$$

where c_{31} is an arbitrary constant.
If $c_3 \neq 0$, then from equation (6.34),

$$\omega = \frac{c_3}{b(t)}. \tag{6.65}$$

From equation (6.30),

$$\Upsilon(t,x) = x\frac{1}{2}\left(c_3\left(\frac{1}{\beta(t)}\right)' + c_1\right) + \rho(t). \tag{6.66}$$

From equation (6.33), we get,
$d'(t) - 2\frac{b'(t)}{b(t)}d(t) = \frac{1}{2}\left(b''(t) - 2\frac{(b'(t))^2}{b(t)}\right).$
This is a linear differential equation yielding solution
$d(t) = c_{32}b^2(t) + \frac{b'(t)}{2},$
where c_{32} is an arbitrary constant.
From equation (6.64),
$c(t) = \frac{1}{2}\left[c_{33}b^2(t) - \frac{3}{2}\left(\frac{b'(t)}{b(t)}\right)^2 + \frac{b''(t)}{b(t)}\right]$, where $c_{33} = \frac{c_{31}}{c_3^2}$ is an arbitrary constant.
Since, $\omega = \omega^r$, we get, $b(t) = b(t-r)$,
In this case we get coefficients of the infinitesimal transformation as

$$\omega(t,x) = \frac{c_3}{b(t)}, \quad \Upsilon(t,x) = x\frac{1}{2}\left(c_3\left(\frac{1}{b(t)}\right)' + c_1\right) + \rho(t). \tag{6.67}$$

The infinitesimal generator in this case is

$$\zeta^* = \frac{c_1}{2}x\frac{\partial}{\partial x} + c_3\left(\frac{1}{b(t)}\frac{\partial}{\partial t} + \frac{x}{2}\left(\frac{1}{b(t)}\right)'\frac{\partial}{\partial x}\right) + \rho(t)\frac{\partial}{\partial x}, \tag{6.68}$$

where $\rho(t)$ is an arbitrary solution of equation (6.11).
If $c_3 = 0$, then the coefficients of the infinitesimal transformation are given by (6.43) and the infinitesimal generator is given by (6.44). □

Theorem 6.3.6. *The delay differential equation given by equation (6.11) for which* $b(t) \neq 0, d(t) = 0, k(t) = 0$ *admits a three dimensional group generated by*

$$\zeta_1^* = \frac{\partial}{\partial t}, \quad \zeta_2^* = x\frac{\partial}{\partial x}, \quad \zeta_3^* = \rho(t)\frac{\partial}{\partial x}.$$

Proof. From equation (6.33),
$b(t)\omega_{tt} = 0$, which can be solved to give, $\omega(t,x) = c_{34}t + c_{35}$, where c_{34} and c_{35} are arbitrary constants.

From equation (6.64), $c(t)\omega^2(t,x) = c_{36}$, where $c_{36} = \dfrac{c_{31}}{2} + \dfrac{c_{34}^2}{4}$, is an arbitrary constant. Further, as $\omega = \omega^r$, we get $c_{34} = 0$ and hence, $\omega(t,x) = c_{35}$.
If $c_{35} \neq 0$, then
$c(t) = \dfrac{c_{36}}{c_{35}^2}$, $b(t) = \dfrac{c_3}{c_{35}}$.
The infinitesimal generator in this case is given by

$$\zeta^* = c_{35}\frac{\partial}{\partial t} + (\frac{c_1}{2}x + \rho(t))\frac{\partial}{\partial x}. \tag{6.69}$$

If $c_{35} = 0$, then $\omega(t,x) = 0$ and $\Upsilon(t,x) = \dfrac{c_1}{2}x + \rho(t)$.
The infinitesimal generator in this case is given by(6.44). □

Theorem 6.3.7. *The delay differential equation given by equation (6.11) for which* $b(t) = 0, d(t) \neq 0, k(t) = 0$ *admits a four dimensional group generated by*

$$\zeta_1^* = \frac{1}{\sqrt{d(t)}}\frac{\partial}{\partial t}, \quad \zeta_2^* = \left[\left(-\frac{d'(t)}{d^{3/2}(t)}\right)x\right]\frac{\partial}{\partial x}, \quad \zeta_3^* = \frac{x}{2}\frac{\partial}{\partial x}, \quad \zeta_4^* = \rho(t)\frac{\partial}{\partial x}.$$

Proof. From equation (6.33), we get,
$\omega(t,x) = \sqrt{\dfrac{c_{37}}{d(t)}}$, where c_{37} is an arbitrary constant.
Then from equation (6.30),

$$\begin{aligned}\Upsilon(t,x) &= \left[\frac{1}{2}\left(\left(\sqrt{\frac{c_{37}}{d(t)}}\right)' + c_1\right)\right]x + \rho(t)\\ &= \left(-\frac{\sqrt{c_{37}}}{4}\frac{d'(t)}{d^{3/2}(t)} + \frac{c_1}{2}\right)x + \rho(t).\end{aligned}$$

If $c_{37} \neq 0$, then from equation (6.64),
$c(t) = \dfrac{1}{2}\left[\dfrac{c_{31}}{c_{37}}d(t) + \dfrac{d''(t)}{2d(t)} - \dfrac{5}{8}\left(\dfrac{d'(t)}{d(t)}\right)^2\right]$.
The infinitesimal generator in this case is given by,

$$\zeta^* = \sqrt{\frac{c_{37}}{d(t)}}\frac{\partial}{\partial t} + \left[\left(-\frac{d'(t)\sqrt{c_{37}}}{4d^{3/2}(t)} + \frac{c_1}{2}\right)x + \rho(t)\right]\frac{\partial}{\partial x}. \tag{6.70}$$

If $c_{37} = 0$, then $\omega(t,x) = 0, \Upsilon(t,x) = \dfrac{c_1}{2}x + \rho(t)$.
Hence, the infinitesimal generator in this case is given by (6.44). □

6.4 Some Illustrative Examples

Example 6.4.1. *Consider the second order neutral differential equation given by* $x''(t) + x''(t-\pi) = 0$. *The solution of this differential equation is* $x(t) = \sin t$.

Following the procedure given in the previous section, we can show that,
$\omega(t,x) = c_{38}$, *a constant, and* $\Upsilon(t,x) = \frac{c_1}{2}x + \sin t$.
Solving the system,
$\frac{d\bar{t}}{d\delta} = \omega(\bar{t},\bar{x}) = c_{38}$, $\frac{d\bar{x}}{d\delta} = \Upsilon(\bar{t},\bar{x}) = \frac{c_1}{2}\bar{x} + \sin\bar{t}$, *subject to the conditions,* $\bar{t} = t$ *and* $\bar{x} = x$, *when* $\delta = 0$, *we get the above neutral differential equation invariant under the Lie group*
$\bar{t} = t + c_{38}\delta$, $\bar{x} = \frac{2}{c_1}\left[e^{c_1\delta/2}\left(\frac{c_1}{2}x + \sin t\right) - \sin(t + c_{38}\delta)\right]$.
The generators of the Lie group (or vector fields of the symmetry algebra) corresponding to this neutral differential equation are given by,
$\zeta_1^* = \frac{\partial}{\partial t}, \zeta_2^* = x\frac{\partial}{\partial x}$ *and* $\zeta_3^* = \sin t\frac{\partial}{\partial x}$.

Example 6.4.2. *Consider the Cauchy problem,*
$x'(t) = \int_{-r}^{0} x(s)\mathrm{d}s$.
This is equivalent to the second order delay differential equation given by,
$x''(t) - x(t) + x(t-r) = 0$.
Following the procedure in the previous section, from Theorem 6.3.7, we get,
$\omega(t,x) = c_{39}$, *where* $c_{39} = \sqrt{c_{37}}$, *is a constant and* $\Upsilon(t,x) = \frac{c_1}{2}x + \tilde{x}(t)$.
Solving the system,
$\frac{d\bar{t}}{d\delta} = \omega(\bar{t},\bar{x}) = c_{39}$, $\frac{d\bar{x}}{d\delta} = \Upsilon(\bar{t},\bar{x}) = \frac{c_1}{2}\bar{x} + \tilde{x}(\bar{t})$, *subject to the conditions,* $\bar{t} = t$ *and* $\bar{x} = x$, *when* $\delta = 0$, *we get the above neutral differential equation invariant under the Lie group*
$\bar{t} = t + c_{39}\delta$, $\bar{x} = \frac{2}{c_1}\left[e^{c_1\delta/2}\left(\frac{c_1}{2}x + \tilde{x}(t)\right) - \tilde{x}(t + c_{39}\delta)\right]$.
The generators of the Lie group (or vector fields of the symmetry algebra) corresponding to this delay differential equation are given by,
$\zeta_1^* = \frac{\partial}{\partial t}, \zeta_2^* = x\frac{\partial}{\partial x}$ *and* $\zeta_3^* = \tilde{x}(t)\frac{\partial}{\partial x}$.

6.5 Summary

We have obtained the infinitesimal generators of equation (6.11) and based on the various cases we can classify the linear second-order neutral differential equation as follows:

1. Equation (6.11) with $b(t) \neq 0, d(t) \neq 0, k(t) =$ a non constant function, admits the infinitesimal generator given by equation (6.25).

2. Equation (6.11) with $b(t) \neq 0, d(t) \neq 0, k(t) =$ a non-zero constant, admits the infinitesimal generator given by equation (6.42).

3. Equation (6.11) with $b(t) \neq 0, d(t) = 0, k(t) =$ a non-zero constant, admits the infinitesimal generator given by equation (6.47).

4. Equation (6.11) with $b(t) \neq 0, d(t) = 0, k(t) = 1$, admits the infinitesimal generator given by equation (6.52).

5. Equation (6.11) with $b(t) = 0, d(t) \neq 0, k(t) =$ a non-zero constant, admits the infinitesimal generator given by equation (6.55).

6. Equation (6.11) with $b(t) = 0, d(t) = e^t, k(t) =$ a non-zero constant, admits the infinitesimal generator given by equation (6.59).

7. Equation (6.11) with $b(t) = 0, d(t) = \sin t, k(t) =$ a non-zero constant, admits the infinitesimal generator given by equation (6.61).

8. Equation (6.11) with $b(t) = 0, d(t) = t^m, k(t) =$ a non-zero constant, admits the infinitesimal generator given by equation (6.63).

9. Equation (6.11) with $b(t) = 0, d(t) = 1, k(t) =$ a non-zero constant, admits the infinitesimal generator given by equation (6.57).

The neutral differential equation (6.11) with $k(t) = 0$ becomes a delay differential equation, the results for which are summarized below:

10. With $k(t) = 0$, equation (6.11) together with $b(t) \neq 0, d(t) \neq 0$, admits the infinitesimal generator given by equation (6.68).

11. With $k(t) = 0$, equation (6.11) together with $b(t) \neq 0, d(t) = 0$, admits the infinitesimal generator given by equation (6.69).

12. With $k(t) = 0$, equation (6.11) together with $b(t) = 0, d(t) \neq 0$, admits the infinitesimal generator given by equation (6.70).

The results can be summarized in the following tables:

Table 6.1: Group Classification of the Second Order Neutral Differential Equation

Type of Second order Neutral Differential Equation	Generators
$x''(t)+b(t)x'(t-r)+c(t)x(t)+d(t)x(t-r)+k(t)x''(t-r)=0$, $k(t)\neq constant$	$\zeta_1^*=x\frac{\partial}{\partial x}$, $\zeta_2^*=\rho(t)\frac{\partial}{\partial x}$
$x''(t)+b(t)x'(t-r)+c(t)x(t)+d(t)x(t-r)+k(t)x''(t-r)=0$, $k(t)=c_2$, $d(t)=\frac{1}{2}\left[c_5b^2(t)+b'(t)+c_2\left(\frac{b''(t)}{b(t)}-2\left(\frac{b'(t)}{b(t)}\right)^2+\frac{b'(t)}{b^2(t)}\right)\right]$, $c(t)=\frac{1}{2}\left[\frac{b''(t)}{b(t)}-\frac{3}{2}\left(\frac{b'(t)}{b(t)}\right)^2+\frac{c_6}{2}b^2(t)\right]$	$\zeta_1^*=x\frac{\partial}{\partial x}$, $\zeta_2^*=\frac{1}{b(t)}\frac{\partial}{\partial t}+\frac{x}{2}\left(\frac{1}{b(t)}\right)'\frac{\partial}{\partial x}$, $\zeta_3^*=\rho(t)\frac{\partial}{\partial x}$
$x''(t)+b(t)x'(t-r)+c(t)x(t)+k(t)x''(t-r)=0$, $c(t)=\frac{1}{2}\left[\frac{b''(t)}{b(t)}-\frac{3}{2}\left(\frac{b'(t)}{b(t)}\right)^2+\frac{c_6}{2}b^2(t)\right]$, $k(t)=\frac{c_3}{\sqrt{2c_7}}$	$\zeta_1^*=\frac{x}{2}\frac{\partial}{\partial x}$, $\zeta_2^*=\rho(t)\frac{\partial}{\partial x}$.
$x''(t)+b(t)x'(t-r)+c(t)x(t)+k(t)x''(t-r)=0$, $k(t)=1$ $c(t)=\frac{1}{4}\frac{c_6c_3^2}{c_{15}^2}$, $c_3=1$	$\zeta_1^*=\frac{\partial}{\partial t}$, $\zeta_2^*=\frac{x}{2}\frac{\partial}{\partial x}$, $\zeta_3^*=\rho(t)\frac{\partial}{\partial x}$.
$x''(t)+c(t)x(t)+d(t)x(t-r)+k(t)x''(t-r)=0$, $d(t)=1$, $c(t)=\Big(2c_{18}^2\cos\big(\frac{4t}{\sqrt{c_2}}\big)-2c_{19}^2\cos\big(\frac{4t}{\sqrt{c_2}}\big)-4c_{18}c_{19}\sin\big(\frac{4t}{\sqrt{c_2}}\big)$ $-4\cos\big(\frac{4t}{\sqrt{c_2}}\big)\sqrt{c_{17}}c_{19}-4\sin\big(\frac{4t}{\sqrt{c_2}}\big)\sqrt{c_{17}}c_{18}-c_{20}c_2\Big)\Big/$ $\Big(2\cos\big(\frac{4t}{\sqrt{c_2}}\big)c_2c_{18}^2-2\cos\big(\frac{4t}{\sqrt{c_2}}\big)c_2c_{19}^2-4\sin\big(\frac{4t}{\sqrt{c_2}}\big)c_2c_{18}c_{19}$ $-4\cos\big(\frac{4t}{\sqrt{c_2}}\big)c_2\sqrt{c_{17}}c_{19}-4\sin\big(\frac{4t}{\sqrt{c_2}}\big)c_2\sqrt{c_{17}}c_{18}-6c_2c_{18}^2$ $-6c_2c_{19}^2-2c_2c_{16}\Big)$	$\zeta_1^*=\frac{\partial}{\partial t}$, $\zeta_2^*=\frac{x}{2}\frac{\partial}{\partial x}$ $\zeta_3^*=\sin\left(\frac{2t}{\sqrt{k(t)}}\right)\frac{\partial}{\partial t}$ $+\frac{x}{\sqrt{k(t)}}\cos\left(\frac{2t}{\sqrt{k(t)}}\right)\frac{\partial}{\partial x}$, $\zeta_4^*=\cos\left(\frac{2t}{\sqrt{k(t)}}\right)\frac{\partial}{\partial t}$ $-\frac{x}{\sqrt{k(t)}}\sin\left(\frac{2t}{\sqrt{k(t)}}\right)\frac{\partial}{\partial x}$, $\zeta_5^*=\rho(t)\frac{\partial}{\partial x}$.

Table 6.2: Group Classification of the Second Order Neutral Differential Equation

Type of Second order Neutral Differential Equation	Generators
$x''(t) + c(t)x(t) + d(t)x(t-r) + k(t)x''(t-r) = 0,$ $d(t) = e^t,$ $$c(t) = \left[\int \frac{q(t)}{r(t)} e^{-4\int \frac{p_1(t)}{p_2(t)} dt} dt + c_{24} \right] e^{4\int \frac{s_(t)}{s_2(t)} dt}$$ $$p_1(t) = e^t \Big[(1+2c_{16})c_{23}^2 J_0(\lambda)J_1(\lambda) + 2c_{22}c_{23}\Big(J_0(\lambda)Y_1(\lambda) + J_1(\lambda)Y_0(\lambda)\Big) + 4c_{22}^2 Y_0(\lambda)Y_1(\lambda)\Big]$$ $$p_2(t) = \sqrt{c_2 e^t}\Big[(1+2c_{16})c_{23}^2 \left(J_0(\lambda)\right)^2 + 4c_{22}c_{23}J_0(\lambda)Y_0(\lambda) + 4c_{22}^2 \left(Y_0(\lambda)\right)^2\Big]$$ $$q(t) = (1+2c_{16})c_{23}^2 e^t \sqrt{c_2 e^t} \left(J_0(\lambda)\right)^2 + 4c_{22}e^t\sqrt{ke^t}\Big(J_0(\lambda)Y_0(\lambda)c_{23} + \left(Y_0(\lambda)\right)^2 c_{22}\Big) - 4(1+2c_{16})e^{2t}c_{23}^2 J_0(\lambda)J_1(\lambda) - 8c_{22}c_{23} e^{2t}\left(J_0(\lambda)Y_1(\lambda) + J_1(\lambda)Y_0(\lambda)\right) - 16e^{2t}c_{22}^2 Y_0(\lambda)Y_1(\lambda)$$ $$r(t) = c_2\sqrt{c_2 e^t}\Big[(1+2c_{16})c_{23}^2 \left(J_0(\lambda)\right)^2 + 4c_{22}\Big(J_0(\lambda)Y_0(\lambda)c_{23} + \left(Y_0(\lambda)\right)^2 c_{22}\Big)\Big].$$ $$s_1(t) = c_2^2 e^{3t}\Big[(1+2c_{16})c_{23}^2 J_0(\lambda)J_1(\lambda) + 2c_{22}c_{23}\Big(J_0(\lambda)Y_1(\lambda) + J_1(\lambda)Y_0(\lambda)\Big) + 4c_{22}^2 Y_0(\lambda)Y_1(\lambda)\Big]$$ $$s_2(t) = (c_2 e^t)^{5/2}\Big[(1+2c_{16})c_{23}^2 \left(J_0(\lambda)\right)^2 + 4c_{22}Y_0(\lambda)\Big(J_0(\lambda)c_{23} + Y_0(\lambda)c_{22}\Big)\Big]$$	With $\lambda = \dfrac{\sqrt{k(t)e^t}}{k(t)}$, $$\zeta_1^* = \Big(J_0\big(2\lambda\big)\Big)^2 \frac{\partial}{\partial t} + \frac{xe^t}{\sqrt{k(t)e^t}} J_0\big(2\lambda\big)J_1\big(2\lambda\big)\frac{\partial}{\partial x}$$ $$\zeta_2^* = \Big(Y_0\big(2\lambda\big)\Big)^2 \frac{\partial}{\partial t} - \frac{xe^t}{\sqrt{k(t)e^t}} Y_0\big(2\lambda\big)Y_1\big(2\lambda\big)\frac{\partial}{\partial x}$$ $$\zeta_3^* = J_0\big(2\lambda\big)Y_0\big(2\lambda\big)\frac{\partial}{\partial t} - \frac{xe^t}{\sqrt{k(t)e^t}}\Big(J_1\big(2\lambda\big)Y_0\big(2\lambda\big) + J_0\big(2\lambda\big)Y_1\big(2\lambda\big)\Big)\frac{\partial}{\partial x}$$ $\zeta_4^* = \dfrac{x}{2}\dfrac{\partial}{\partial x},$ $\zeta_5^* = \rho(t)\dfrac{\partial}{\partial x}.$

Table 6.3: Group Classification of the Second Order Neutral Differential Equation

Type of Second order Neutral Differential Equation	Generators
$x''(t)+c(t)x(t)+d(t)x(t-r)+k(t)x''(t-r)=0$, $d(t)=\sin t$, $c(t)=e^{-2\int r_1(t)\mathrm{d}t}\int\frac{q_1(t)}{q_2(t)}e^{2\int r_1(t)\mathrm{d}t}\mathrm{d}t+c_{27}e^{-2\int r_1(t)\mathrm{d}t}.$ where, $r_1(t)=\Big(c_{26}(1+8c_{16})^2\mathit{MathieuC}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\mathit{MathieuCPrime}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)+2c_{25}c_{26}\Big(\mathit{MathieuCPrime}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\mathit{MathieuS}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)+\mathit{MathieuC}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\mathit{MathieuSPrime}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\Big)+4c_{25}^2\mathit{MathieuS}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\mathit{MathieuSPrime}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\Big)\Big/\Big(c_{26}(1+8c_{16})^2\Big(\mathit{MathieuC}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\Big)^2+4c_{25}c_{26}\mathit{MathieuC}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\mathit{MathieuS}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)+4c_{25}^2\Big(\mathit{MathieuS}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\Big)^2\Big),$ $q_1(t)=2c_{26}(1+8c_{16})^2\mathit{MathieuC}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\mathit{MathieuCPrime}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\sin t+4c_{25}c_{26}\Big(\mathit{MathieuCPrime}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\mathit{MathieuS}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)+\mathit{MathieuC}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\mathit{MathieuSPrime}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\Big)\sin t+8c_{25}^2\mathit{MathieuS}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\mathit{MathieuSPrime}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\sin t+c_{26}(1+8c_{16})^2\Big(\mathit{MathieuC}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\Big)^2\cos t+4c_{25}c_{26}\mathit{MathieuC}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\mathit{MathieuS}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\cos t+4c_{25}^2\Big(\mathit{MathieuS}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\Big)^2\cos t,$ and, $q_2(t)=k(t)\Big(c_{26}(1+8c_{16})^2\Big(\mathit{MathieuC}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\Big)^2+4c_{25}c_{26}\mathit{MathieuC}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\mathit{MathieuS}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)+4c_{25}^2\Big(\mathit{MathieuS}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\Big)^2\Big).$	$\zeta_1^*=\Big(\mathit{MathieuC}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\Big)^2\frac{\partial}{\partial t}+\frac{x}{2}\mathit{MathieuC}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\mathit{MathieuCPrime}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\frac{\partial}{\partial x}$ $\zeta_2^*=\Big(\mathit{MathieuS}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\Big)^2\frac{\partial}{\partial t}+\frac{x}{2}\mathit{MathieuS}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\mathit{MathieuSPrime}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\frac{\partial}{\partial x}$ $\zeta_3^*=\mathit{MathieuC}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\mathit{MathieuS}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\frac{\partial}{\partial t}+\frac{x}{4}\Big(\mathit{MathieuCPrime}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\mathit{MathieuS}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)+\mathit{MathieuC}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\mathit{MathieuSPrime}\left(0,-\frac{2}{k(t)},\frac{-\pi}{4}+\frac{t}{2}\right)\Big)\frac{\partial}{\partial x}$ $\zeta_4^*=\frac{x}{2}\frac{\partial}{\partial x},$ $\zeta_5^*=\rho(t)\frac{\partial}{\partial x}.$

Table 6.4: Group Classification of the Second Order Neutral Differential Equation

Type of Second order Neutral Differential Equation	Generators
$x''(t) + c(t)x(t) + d(t)x(t-r) + k(t)x''(t-r) = 0$, $d(t) = t^m$, where m is a constant, $c(t) = \left[\int -\frac{l(t)}{c_2 j(t)} e^{-4\int \frac{e(t)}{j(t)} dt} dt + c_{30}\right] e^{4\int \frac{y(t)}{j(t)} dt}.$ With $\theta = \left(2\frac{\sqrt{(k(t))^{-1}}t^{m/2+1}}{m+2}\right)$, $e(t) = \sqrt{(k(t))^{-1}}(1+2c_{16})c_{29}^2 J_{(3+m)(m+2)^{-1}}(\theta)J_{(m+2)^{-1}}(\theta)t^{m/2+1} + 2\sqrt{(k(t))^{-1}}c_{28}c_{29}\Big(J_{(3+m)(m+2)^{-1}}(\theta)Y_{(m+2)^{-1}}(\theta) + J_{(m+2)^{-1}}(\theta)Y_{(3+m)(m+2)^{-1}}(\theta)\Big)t^{m/2+1} + 4\sqrt{(k(t))^{-1}}c_{28}^2 Y_{(3+m)(m+2)^{-1}}(\theta)Y_{(m+2)^{-1}}(\theta)t^{m/2+1} - (1+2c_{16})c_{29}^2\Big(J_{(m+2)^{-1}}(\theta)\Big)^2 - 4c_{28}Y_{(m+2)^{-1}}(\theta)\Big(J_{(m+2)^{-1}}(\theta)c_{29} + Y_{(m+2)^{-1}}(\theta)c_{28}\Big),$ $j(t) = t\Big[(1+2c_{16})c_{29}^2\Big(J_{(m+2)^{-1}}(\theta)\Big)^2 + 4c_{28}Y_{(m+2)^{-1}}(\theta)\Big(J_{(m+2)^{-1}}(\theta)c_{29} + Y_{(m+2)^{-1}}(\theta)c_{28}\Big)\Big],$ $l(t) = 4\sqrt{(k(t))^{-1}}(1+2c_{16})c_{29}^2 J_{(3+m)(m+2)^{-1}}(\theta)J_{(m+2)^{-1}}(\theta)t^{3m/2+1} - 2mc_{29}^2c_{16}\Big(J_{(m+2)^{-1}}(\theta)\Big)^2 t^m + 8\sqrt{(k(t))^{-1}}c_{28}c_{29}\Big(J_{(3+m)(m+2)^{-1}}(\theta) Y_{(m+2)^{-1}}(\theta) + J_{(m+2)^{-1}}(\theta)Y_{(3+m)(m+2)^{-1}}(\theta)\Big)t^{3m/2+1} + 16\sqrt{(k(t))^{-1}}c_{28}^2 Y_{(3+m)(m+2)^{-1}}(\theta)Y_{(m+2)^{-1}}(\theta)t^{3m/2+1} - 8c_{29}^2c_{16}\Big(J_{(m+2)^{-1}}(\theta)\Big)^2 t^m - mc_{29}^2\Big(J_{(m+2)^{-1}}(\theta)\Big)^2 t^m - 4mc_{28}c_{29}J_{(m+2)^{-1}}(\theta)Y_{(m+2)^{-1}}(\theta)t^m - 4mc_{28}^2\Big(Y_{(m+2)^{-1}}(\theta)\Big)^2 t^m - 4c_{29}^2\Big(J_{(m+2)^{-1}}(\theta)\Big)^2 t^m - 16c_{28}Y_{(m+2)^{-1}}(\theta)\Big(J_{(m+2)^{-1}}(\theta)c_{29} + Y_{(m+2)^{-1}}(\theta)c_{28}\Big)t^m,$ $y(t) = \sqrt{(k(t))^{-1}}c_{29}^2(1+2c_{16})J_{(3+m)(m+2)^{-1}}(\theta)J_{(m+2)^{-1}}(\theta)t^{m/2+1} + 2\sqrt{(k(t))^{-1}}c_{28}c_{29}\Big(J_{(3+m)(m+2)^{-1}}(\theta)Y_{(m+2)^{-1}}(\theta) + J_{(m+2)^{-1}}(\theta) Y_{(3+m)(m+2)^{-1}}(\theta)\Big)t^{m/2+1} + 4\sqrt{(k(t))^{-1}}c_{28}^2 Y_{(3+m)(m+2)^{-1}}(\theta) Y_{(m+2)^{-1}}(\theta)t^{m/2+1} - (1+2c_{16})c_{29}^2\Big(J_{(m+2)^{-1}}(\theta)\Big)^2 - 4c_{28}Y_{(m+2)^{-1}}(\theta)\Big(J_{(m+2)^{-1}}(\theta)c_{29} + Y_{(m+2)^{-1}}(\theta)c_{28}\Big).$	With $\nu = (m+2)^{-1}$, $\tau = \sqrt{(k(t))^{-1}}t^{m/2+1}, \mu = 2\tau\nu$, $\zeta_1^* = t(J_\nu(\mu))^2\frac{\partial}{\partial t} + x\Big(\frac{1}{2}(J_\nu(\mu))^2 + \frac{2}{m+2}J_\nu(\mu)\Big(-J_{\nu+1}(\mu) + \frac{J_\nu(\mu)}{2\tau}\Big)\tau(m/2+1)\Big)\frac{\partial}{\partial x}$ $\zeta_2^* = t(Y_\nu(\mu))^2\frac{\partial}{\partial t} + x\Big(\frac{1}{2}(Y_\nu(\mu))^2 + \frac{2}{m+2}Y_\nu(\mu)\Big(-Y_{\nu+1}(\mu) + \frac{Y_\nu(\mu)}{2\tau}\Big)\tau(m/2+1)\Big)\frac{\partial}{\partial x}$ $\zeta_3^* = tJ_\nu(\mu)Y_\nu(\mu)\frac{\partial}{\partial t} + x\Big(\frac{1}{2}J_\nu(\mu)Y_\nu(\mu) + \frac{1}{m+2}\Big(\Big(-J_{\nu+1}(\mu) + \frac{J_\nu(\mu)}{2\tau}\Big)Y_\nu(\mu) + J_\nu(\mu)\Big(-Y_{\nu+1}(\mu) + \frac{Y_\nu(\mu)}{2\tau}\Big)\Big)\tau(m/2+1)\Big)\Big)\frac{\partial}{\partial x}$ $\zeta_4^* = \frac{x}{2}\frac{\partial}{\partial x},$ $\zeta_5^* = \rho(t)\frac{\partial}{\partial x}.$

Table 6.5: Group Classification of the Second Order Delay Differential Equation

Type of Second order Delay Differential Equation	Generators
$x''(t)+b(t)x'(t-r)+c(t)x(t)+d(t)x(t-r)=0$, $d(t)=c_{32}b^2(t)+\frac{b'(t)}{2}$, $c(t)=\frac{1}{2}\left[c_{33}b^2(t)-\frac{3}{2}\left(\frac{b'(t)}{b(t)}\right)^2+\frac{b''(t)}{b(t)}\right]$	$\zeta_1^*=x\frac{\partial}{\partial x}$, $\zeta_2^*=\frac{1}{b(t)}\frac{\partial}{\partial t}+\frac{x}{2}\left(\frac{1}{b(t)}\right)'\frac{\partial}{\partial x}$, $\zeta_3^*=\rho(t)\frac{\partial}{\partial x}$
$x''(t)+b(t)x'(t-r)+c(t)x(t)=0$, $c(t)=\frac{c_{36}}{c_{35}^2}$	$\zeta_1^*=\frac{\partial}{\partial t}$, $\zeta_2^*=x\frac{\partial}{\partial x}$, $\zeta_3^*=\rho(t)\frac{\partial}{\partial x}$.
$x''(t)+c(t)x(t)+d(t)x(t-r)=0$, $c(t)=\frac{1}{2}\left[\frac{c_{31}}{c_{37}}d(t)+\frac{d''(t)}{2d(t)}-\frac{5}{8}\left(\frac{d'(t)}{d(t)}\right)^2\right]$	$\zeta_1^*=\frac{1}{\sqrt{d(t)}}\frac{\partial}{\partial t}$, $\zeta_2^*=\left[\left(-\frac{d'(t)}{d^{3/2}(t)}\right)x\right]\frac{\partial}{\partial x}$, $\zeta_3^*=\frac{x}{2}\frac{\partial}{\partial x}$, $\zeta_4^*=\rho(t)\frac{\partial}{\partial x}$.

Chapter 7

Classification of First Order Functional Differential Equations With Constant Coefficients to Solvable Lie Algebras

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7.1 Introduction

In this chapter, we make a complete classification of first order functional (delay and neutral) differential equations with constant coefficients to solvable Lie algebras. We provide a basis for the Lie algebra given by the first order linear and nonlinear functional differential equations. In all the chapters seen so far, the only drawback is that the inverse of the obtained classification cannot be found.

It may be noted that Lie group–Lie algebra correspondence allows one to study Lie groups which are geometric objects, in terms of Lie algebras, which are linear objects. Any Lie group gives rise to a Lie algebra, which is its tangent space at the identity. Conversely, by Lie's third theorem, every finite dimensional real Lie algebra is the Lie algebra of some simply connected Lie group.

We shall be studying the functional differential equation

$$\Phi(t, x(t), x(t-r), x'(t), x'(t-r)) = 0, \tag{7.1}$$

where Φ is a real valued function defined on $I \times D^4$ where D is an open set in $\mathbb{R}$, I is an open interval in $\mathbb{R}$ and $r > 0$ is the delay. We use the notations $x'(t-r)$ to mean $\dfrac{dx}{dt}(t-r)$ and the notation x^r to denote $x(t-r)$. For first order functional differential equations we assume that, $\dfrac{\partial \Phi}{\partial x(t-r)} \neq 0$ or $\dfrac{\partial \Phi}{\partial x'(t-r)} \neq 0$ depending on whether the differential equation is a delay or neutral type. We shall find a Lie group under which these functional differential equations are invariant. We call this the admitted Lie group by which we mean that one solution curve is carried to another solution curve of the same equation. In this chapter we obtain a Lie type invariance condition by setting up a procedure slightly different from the way we set up in the previous chapters.

The rest of this chapter is organised as follows: The next section extends the results for ordinary differential equations to functional differential equations by obtaining a Lie type invariance condition using Taylor's theorem for a function of several variables. In the sections to follow, each section will consist of two subsections — one for linear and the other for nonlinear functional differential equations with constant coefficients. Each section will independently be concerned with (i) First order delay differential equations (ii) First order neutral differential equations. We conclude with representation of our results, which are the basis for the Lie algebras, in a tabular form.

7.2 Lie Type Invariance Condition for First Order Functional Differential Equations

In this section, we extend the results for ordinary differential equations to functional differential equations. We prove a Lie type invariance condition using Taylor's theorem for a function of several variables. A careful look at the proof will see that it is slightly different from the proof given in chapter 3 and 4.

Theorem 7.2.1. *Let a function F be defined on $I \times D^3$, where D is an open set in $\mathbb{R}$, and I is an open interval in $\mathbb{R}$. The Lie type invariance condition for*

$$\frac{dx}{dt} = F(t, x(t), x(t-r), x'(t-r)), \tag{7.2}$$

is given by

$$\omega F_t + \Upsilon F_x + \Upsilon^r F_{x(t-r)} + \Upsilon^r_{[t]} F_{x'(t-r)} = \Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2,$$

where

$$\Upsilon_{[t]} = D_t(\Upsilon) - x' D_t(\omega) = \Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2,$$

$$\Upsilon^r_{[t]} = (\Upsilon_t)^r + ((\Upsilon_x)^r - (\omega_t)^r)x'(t-r) - (x'(t-r))^2(\omega_x)^r,$$

where $D_t = \frac{\partial}{\partial t} + x'\frac{\partial}{\partial x}$,
and $\omega^r = \omega(t-r, x(t-r))$, $\Upsilon^r = \Upsilon(t-r, x(t-r))$.

Proof. Let the neutral differential equation be invariant under the Lie group

$$\bar{t} = t + \delta\omega(t,x) + O(\delta^2), \quad \bar{x} = x + \delta\Upsilon(t,x) + O(\delta^2).$$

We then naturally define $\overline{t-r} = t - r + \delta\omega(t-r, x(t-r)) + O(\delta^2)$ and
$\overline{x(t-r)} = x(t-r) + \delta\Upsilon(t-r, x(t-r)) + O(\delta^2)$.

With the notations, $\omega^r = \omega(t-r, x(t-r))$, and $\Upsilon^r = \Upsilon(t-r, x(t-r))$, it follows that,

$$\begin{aligned} \overline{x'(t-r)} &= \tfrac{d\bar{x}}{dt}(\overline{t-r}) \\ &= x'(t-r) + (\Upsilon_t)^r + ((\Upsilon_x)^r - (\omega_t)^r)x'(t-r) \\ &- (x'(t-r))^2(\omega_x)^r)\delta + O(\delta^2). \end{aligned} \tag{7.3}$$

For invariance, $\frac{d\bar{x}}{dt} = F(\bar{t}, \bar{x}, \overline{(x(t-r))}, \overline{x'(t-r)})$.
This gives,

$$\begin{aligned}
& \tfrac{dx}{dt} + [\Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2]\delta + O(\delta^2) \\
= \;& F(t + \delta\omega + O(\delta^2),\ x + \delta\Upsilon + O(\delta^2),\ x(t-r) + \delta\Upsilon^r + O(\delta^2), \\
& x'(t-r) + ((\Upsilon_t)^r + ((\Upsilon_x)^r - (\omega_t)^r)x'(t-r) - (x'(t-r))^2(\omega_x)^r)\delta + O(\delta^2)) \\
= \;& F(t, x, x(t-r), x'(t-r)) + (\omega F_t + \Upsilon F_x + \Upsilon^r F_{x(t-r)} \\
& + \Upsilon^r_{[t]} F_{x'(t-r)})\delta + O(\delta^2),
\end{aligned} \tag{7.4}$$

where $\Upsilon^r_{[t]} = (\Upsilon_t)^r + ((\Upsilon_x)^r - (\omega_t)^r)x'(t-r) - (x'(t-r))^2(\omega_x)^r$.
Comparing the coefficient of δ, we get

$$\omega F_t + \Upsilon F_x + \Upsilon^r F_{x(t-r)} + \Upsilon^r_{[t]} F_{x'(t-r)} = \Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2. \tag{7.5}$$

The above obtained equation (7.5) is a Lie type invariance condition. □

We can define a prolonged operator (the general infinitesimal generator associated with the Lie algebra) for neutral differential equations as:

$$\zeta = \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x(t-r)}.$$

With the notation $D_t = \frac{\partial}{\partial t} + x'\frac{\partial}{\partial x}$, we can write,

$$\begin{aligned}
\tfrac{d\bar{x}}{dt} &= \tfrac{dx}{dt} + (D_t(\Upsilon) - x' D_t(\omega))\delta + O(\delta^2). \\
&= \tfrac{dx}{dt} + \Upsilon_{[t]}\delta + O(\delta^2),
\end{aligned} \tag{7.6}$$

where $\Upsilon_{[t]} = D_t(\Upsilon) - x' D_t(\omega)$. We then define the extended operator as:

$$\zeta^{(1)} = \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x(t-r)} + \Upsilon_{[t]}\frac{\partial}{\partial x'} + \Upsilon^r_{[t]}\frac{\partial}{\partial x'(t-r)}. \tag{7.7}$$

Defining $\Delta = x'(t) - F(t, x(t), x(t-r), x'(t-r)) = 0$, we get

$$\zeta^{(1)}\Delta = \Upsilon_{[t]} - \omega F_t + \Upsilon F_x + \Upsilon^r F_{x(t-r)} + \Upsilon^r_{[t]} F_{x'(t-r)}. \tag{7.8}$$

Comparing equations (7.5) and (7.8), we get
$\Upsilon_{[t]} = \Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2$.
On substituting $x' = F$ into $\zeta^{(1)}\Delta = 0$, we get an invariance condition for the neutral differential equation which is $\zeta^{(1)}\Delta \mid_{\Delta=0} = 0$, from which we shall obtain the determining equations.

We point out here that equations (7.6)-(7.8) is an easy way of working with higher order differential equations as compared to equations (7.3)-(7.5) which is simpler to use for lower order differential equations.

Remark 7.2.1. If the term $x'(t-r)$ is absent, then the corresponding first order neutral differential equation reduces to a first order delay differential equation.

We conclude this section by proving a very elementary result which is used in our subsequent sections:

Proposition 7.2.1. *If the linear functional differential equation is given by*

$$x'(t) + ax'(t-r) + bx(t) + cx(t-r) = d(t), \tag{7.9}$$

then by employing a change of variables namely $\bar{t} = t, \bar{x} = x - \tilde{x}$*, where* $\tilde{x}$ *is a solution of the functional differential equation, we can convert the given non-homogeneous linear functional differential equation to a homogeneous one, namely*
$x'(t) + ax'(t-r) + bx(t) + cx(t-r) = 0.$

Proof. The proposition easily follows by substituting $t = \bar{t}$ and $x(t) = \bar{x} + \tilde{x}(\bar{t})$ in (7.9), by noting that $\tilde{x}'(t) + a\tilde{x}'(t-r) + b\tilde{x}(t) + c\tilde{x}(t-r) = h(t)$. □

7.3 Classification of First Order Delay Differential Equations to Solvable Lie Algebras

7.3.1 The Linear Case

We shall make a classification of the first order delay differential equation with constant coefficients,

$$x'(t) + \alpha x(t) + \beta x(t-r) = 0. \tag{7.10}$$

The extension and prolongation operator for equation (7.10) is given by,

$$\zeta^{(1)} = \omega \frac{\partial}{\partial t} + \omega^r \frac{\partial}{\partial(t-r)} + \Upsilon \frac{\partial}{\partial x} + \Upsilon^r \frac{\partial}{\partial x(t-r)} + \Upsilon_{[t]} \frac{\partial}{\partial x'}. \tag{7.11}$$

Applying the operator defined by equation (7.11), to the delay equation $g(t) = t - r$, we get

$$\omega(t, x) = \omega(t-r, x(t-r)). \tag{7.12}$$

Applying the operator defined by equation (7.11), to equation (7.10), we get,

$$\Upsilon_t + (\Upsilon_x - \omega_t)(-\alpha x - \beta x^r) - \omega_x(\alpha^2 x^2 - 2\alpha\beta x x^r + \beta^2 x^{r2}) + \alpha\Upsilon + \beta\Upsilon^r = 0. \tag{7.13}$$

Splitting equation (7.13) with respect to the constant term we get,

$$\Upsilon_t + \alpha\Upsilon + \beta\Upsilon^r = 0. \tag{7.14}$$

Splitting equation (7.13) with respect to x we get,

$$-\alpha(\Upsilon_x - \omega_t) = 0. \tag{7.15}$$

Splitting equation (7.13) with respect to x^2, x^{r2} or xx^r, we get,

$$\omega_x = 0. \tag{7.16}$$

Splitting equation (7.13) with respect to x^r we get,

$$-\beta(\Upsilon_x - \omega_t) = 0. \tag{7.17}$$

We solve the above equations by studying all possible cases and make a complete classification of (7.10) to solvable Lie algebras by proving the following theorems, with the notation $u = x^r$.

Theorem 7.3.1. *The first order delay differential equation (7.10) for which*

1. $\alpha \neq -\beta$, *admits the two dimensional Lie algebra generated by*

$$S_1 = \frac{\partial}{\partial t}, \quad S_2 = x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right),$$

with the infinite dimensional Lie sub-algebra given by

$$S_3^i = -\left(\frac{\omega_t}{\alpha+\beta}\right)\frac{\partial}{\partial t} + \Big[\theta - (\alpha+\beta)\omega x\Big]\frac{\partial}{\partial x} - \left[\frac{\alpha}{\beta}\theta + \frac{1}{\beta}\theta_t + (\alpha+\beta)\omega x\right]\frac{\partial}{\partial u}.$$

2. $\alpha = -\beta$, *admits the two dimensional Lie algebra generated by*

$$S_1 = t\frac{\partial}{\partial t} + x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right), \quad S_2 = \frac{\partial}{\partial t},$$

with the infinite dimensional Lie sub-algebra given by

$$S_3^i = \theta\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right) - \frac{1}{\beta}\theta_t\frac{\partial}{\partial u}.$$

Proof. (1) Let α, β be arbitrary non-zero constants, $\alpha \neq -\beta$. Then, from equation (7.16), we get $\omega = \omega(t)$.

From equation (7.15), we get,

$\Upsilon = \omega_t x + \theta(t), \quad \Upsilon^r = \omega_t x + \psi(t-r).$

From equation (7.14), we get, $\omega_t = c_1 - (\alpha+\beta)\omega, \quad \psi = -\frac{\alpha}{\beta}\theta - \frac{1}{\beta}\theta_t,$

$$\omega = c_2 - \frac{\omega_t}{\alpha+\beta}, \tag{7.18}$$

where c_1 is an arbitrary constant and $c_2 = \frac{c_1}{\alpha+\beta}$. Hence,

$$\Upsilon = [c_1 - (\alpha+\beta)\omega]x + \theta, \tag{7.19}$$

and,

$$\Upsilon^r = [c_1 - (\alpha+\beta)\omega]x + \psi. \tag{7.20}$$

The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= \left(c_2 - \frac{\omega_t}{\alpha+\beta}\right)\frac{\partial}{\partial t} + ([c_1 - (\alpha+\beta)\omega]x + \psi)\frac{\partial}{\partial x}\\ &\quad + \left([c_1 - (\alpha+\beta)\omega]x - \left(\frac{\alpha}{\beta}\theta - \frac{1}{\beta}\theta_t\right)\right)\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by $S_1 = \frac{\partial}{\partial t}$, $S_2 = x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right)$ with

$$S_3 = -\left(\frac{\omega_t}{\alpha+\beta}\right)\frac{\partial}{\partial t} + [\theta - (\alpha+\beta)\omega x]\frac{\partial}{\partial x} - \left[(\alpha+\beta)\omega x + \frac{\alpha}{\beta}\theta + \frac{1}{\beta}\theta_t\right]\frac{\partial}{\partial u}.$$

as the infinite dimensional Lie sub-algebra.
The commutator table is given by,

	S_1	S_2
S_1	0	0
S_2	0	0

.

Then $L = \{S_1, S_2\}$ is a solvable Lie algebra.

(2) Let α, β be arbitrary non-zero constants, $\alpha = -\beta$.
Then equation (7.14), becomes $\Upsilon_t + \alpha(\Upsilon - \Upsilon^r) = 0$, which can be solved to give

$$\omega = c_3 t + c_4, \tag{7.21}$$

$$\Upsilon = c_3 x + \theta, \tag{7.22}$$

$$\Upsilon^r = c_3 x + \psi, \tag{7.23}$$

where c_3, c_4 are arbitrary constants and $\psi = \dfrac{\theta_t}{\alpha} + \alpha$.
The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r} \\ &= (c_3 t + c_4)\frac{\partial}{\partial t} + (c_3 x + \theta)\frac{\partial}{\partial x} + \left(c_3 x + \frac{\theta_t}{\alpha} + \alpha\right)\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by $S_1 = t\dfrac{\partial}{\partial t} + x\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial u}\right)$, $S_2 = \dfrac{\partial}{\partial t}$ with $S_3 = \theta\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial u}\right) - \dfrac{1}{\beta}\theta_t\dfrac{\partial}{\partial u}$ as the infinite dimensional Lie sub-algebra.

The commutator table is given by,

	S_1	S_2
S_1	0	$-S_2$
S_2	S_2	0

.

Then $L = \{S_1, S_2\}$ is a solvable Lie algebra. □

Corollary 7.3.1. *The first order delay differential equation given by equation (7.10) for which $\alpha = 0$, β is an arbitrary non-zero constant, admits the same generators as the Theorem 7.3.1 (part (1)) above, except that the infinite dimensional Lie sub algebra is given by* $S_3 = -\left(\dfrac{\omega_t}{\beta}\right)\dfrac{\partial}{\partial t} + (\theta - \beta x\omega)\dfrac{\partial}{\partial x} - \left[\dfrac{\theta_t}{\beta} + \beta x\omega\right]\dfrac{\partial}{\partial u}$.

7.3.2 A Nonlinear Case

We make a classification of

$$x'(t) = k\left[1 - \frac{x(t-r)}{P}\right]x(t), \tag{7.24}$$

a nonlinear delay differential equation extensively studied by [28, 30] in modeling population growth problems.
Applying the operator defined by equation (7.11), to the delay equation $g(t) = t - r$, we get equation (7.12).
Applying the operator defined by equation (7.11), to equation (7.24), we get,

$$\Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2 = k\Upsilon - \frac{k}{P}[\Upsilon_x^r + x\Upsilon^r]. \tag{7.25}$$

Splitting equation (7.25) with respect to constant term, x' and x'^2 respectively, we get,

$$\Upsilon_t = k\Upsilon - \frac{k}{P}x^r\Upsilon - \frac{k}{P}x\Upsilon^r, \tag{7.26}$$

$$\Upsilon_x - \omega_t = 0, \tag{7.27}$$

$$\omega_x = 0. \tag{7.28}$$

These equations can be solved to give,

$$\omega = c_1 \tag{7.29}$$

$$\Upsilon = \theta, \quad \Upsilon^r = \psi, \tag{7.30}$$

where c_1 is an arbitrary constant and $\theta_t = \psi\theta - \frac{k}{P}\theta x^r - \frac{k}{P}x\psi$.
The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= c_1\frac{\partial}{\partial t} + \theta\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by $S_1 = \frac{\partial}{\partial t}$ with $S_2 = \theta\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial u}$ as the infinite dimensional Lie sub-algebra.

7.4 Classification of First Order Neutral Differential Equations to Solvable Lie Algebras

7.4.1 The Linear Case

We shall make a classification of the first order neutral differential equation with constant coefficients,

$$x'(t) + \alpha x(t) + \beta x(t-r) + \gamma x'(t-r) = 0. \tag{7.31}$$

The extension and prolongation operator for equation (7.31) is given by,

$$\zeta^{(1)} = \omega\frac{\partial}{\partial t} + \omega^r\frac{\partial}{\partial(t-r)} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x(t-r)} + \Upsilon_{[t]}\frac{\partial}{\partial x'} + \Upsilon^r_{[t]}\frac{\partial}{\partial x^{r\prime}}. \tag{7.32}$$

Applying the operator defined by equation (7.32), to the delay equation $g(t) = t - r$, we get equation (7.12).
Applying the operator defined by equation (7.32), to equation (7.31), we get,

$$\begin{aligned}\Upsilon_t + (\Upsilon_x - \omega_t)(-\alpha x - \beta x^r - \gamma x^{r\prime}) - \omega_x(\alpha^2x^2 + 2\alpha\beta x x^r + 2\beta\gamma x^r x^{r\prime} + 2\alpha\gamma x x^{r\prime} + \beta^2 x^{r2}\\ + \gamma^2 x^{r\prime 2}) + \alpha\Upsilon + \beta\Upsilon^r + \gamma[\Upsilon^r_t + (\Upsilon^r_x - \omega^r_t)x^{r\prime} - \omega^r_x x^{r\prime 2}] = 0.\end{aligned} \tag{7.33}$$

Splitting equation (7.33) with respect to the constant term we get,

$$\Upsilon_t + \alpha\Upsilon + \beta\Upsilon^r + \gamma\Upsilon_t^r = 0. \tag{7.34}$$

Splitting equation (7.33) with respect to x we get,

$$-\alpha(\Upsilon_x - \omega_t) = 0. \tag{7.35}$$

Splitting equation (7.33) with respect to $x^2, x^{r2}, x^r x^{r\prime}, xx^{r\prime}$ or xx^r, we get,

$$\omega_x = 0. \tag{7.36}$$

Splitting equation (7.33) with respect to x^r we get,

$$-\beta(\Upsilon_x - \omega_t) = 0. \tag{7.37}$$

Splitting equation (7.33) with respect to $x^{r'2}$ we get,

$$\gamma\omega_x - \omega_x^r = 0. \tag{7.38}$$

Splitting equation (7.33) with respect to $x^{r'}$ we get,

$$\Upsilon_x - \omega_t = \Upsilon_x^r - \omega_t^r. \tag{7.39}$$

We solve the above equations by studying all possible cases and make a complete classification of (7.31) to solvable Lie algebras by proving the following theorems, with the notation $u = x^r$.

Theorem 7.4.1. *The first order neutral differential equation (7.31) for which*

1. $\alpha \neq -\beta$, *admits the two dimensional Lie algebra generated by*

$$S_1 = \frac{\partial}{\partial t}, \quad S_2 = x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right),$$

with the infinite dimensional Lie sub-algebra given by

$$S_3^i = -\left(\frac{1+\gamma}{\alpha+\beta}\right)\omega_t\frac{\partial}{\partial t} + \left[\theta - \left(\frac{\alpha+\beta}{1+\gamma}\right)\omega x\right]\frac{\partial}{\partial x} + \left[\psi - \left(\frac{\alpha+\beta}{1+\gamma}\right)\omega x\right]\frac{\partial}{\partial u}.$$

2. $\alpha = -\beta$, *admits the two dimensional Lie algebra generated by*

$$S_1 = t\frac{\partial}{\partial t} + x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right), \quad S_2 = \frac{\partial}{\partial t},$$

with the infinite dimensional Lie sub-algebra given by $S_3^i = \theta\dfrac{\partial}{\partial x} + \psi\dfrac{\partial}{\partial u}$.

3. $\alpha = -\beta$, $\gamma = -1$, *admits the one dimensional Lie algebra generated by* $S_1 = \dfrac{\partial}{\partial t}$ *with the infinite dimensional Lie sub-algebra given by* $S_2^i = \theta\dfrac{\partial}{\partial x} + \psi\dfrac{\partial}{\partial u}$.

Proof. (1) Let α, β, γ be arbitrary non-zero constants, $\alpha \neq -\beta$, $\gamma \neq -1$. Then, from equation (7.36), we get $\omega = \omega(t)$.
From equations (7.35), (7.37) and (7.39), we get,
$\Upsilon = \omega_t x + \theta(t), \quad \Upsilon^r = \omega_t x + \psi(t-r)$.
From equation (7.34), we get, $\omega_t = c_3 - \dfrac{(\alpha+\beta)}{1+\gamma}\omega, \quad \theta_t + \alpha\theta + \beta\psi + \gamma\psi_t = 0,$

$$\omega = c_2 - \frac{1+\gamma}{\alpha+\beta}\omega_t, \tag{7.40}$$

where c_1 is an arbitrary constant, $c_2 = \dfrac{c_1}{\alpha+\beta}$ and $c_3 = \dfrac{c_1}{1+\gamma}$. Hence,

$$\Upsilon = \left[c_3 - \frac{(\alpha+\beta)}{1+\gamma}\omega\right]x + \theta, \tag{7.41}$$

and,

$$\Upsilon^r = \left[c_3 - \frac{(\alpha+\beta)}{1+\gamma}\omega\right]x + \psi. \tag{7.42}$$

The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= \left(c_2 - \frac{1+\gamma}{\alpha+\beta}\omega_t\right)\frac{\partial}{\partial t} + \left(\left[c_3 - \frac{(\alpha+\beta)}{1+\gamma}\omega\right]x + \theta\right)\frac{\partial}{\partial x}\\ &\quad + \left(\left[c_3 - \frac{(\alpha+\beta)}{1+\gamma}\omega\right]x + \psi\right)\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by $S_1 = \dfrac{\partial}{\partial t}$, $S_2 = x\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial u}\right)$ with

$$S_3 = -\left(\frac{1+\gamma}{\alpha+\beta}\right)\omega_t\frac{\partial}{\partial t} + \left[\theta - \left(\frac{\alpha+\beta}{1+\gamma}\right)\omega x\right]\frac{\partial}{\partial x} - \left[\psi - \left(\frac{\alpha+\beta}{1+\gamma}\right)\omega x\right]\frac{\partial}{\partial u}$$

as the infinite dimensional Lie sub-algebra.
The commutator table is given by,

	S_1	S_2
S_1	0	0
S_2	0	0

.

Then $L = \{S_1, S_2\}$ is a solvable Lie algebra.

(2) Let α, β be arbitrary non-zero constants, $\alpha = -\beta$, $\gamma \neq -1$.
Then equation (7.34), becomes $\Upsilon_t + \alpha(\Upsilon - \Upsilon^r) + \gamma\Upsilon_t^r = 0$, which can be solved to give

$$\omega = c_4 t + c_5, \tag{7.43}$$

$$\Upsilon = c_4 x + \theta, \tag{7.44}$$

$$\Upsilon^r = c_4 x + \psi, \tag{7.45}$$

where c_4, c_5 are arbitrary constants and $\theta_t + \alpha(\theta - \psi) + \gamma\psi_t = 0$.
The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= (c_4 t + c_5)\frac{\partial}{\partial t} + (c_4 x + \theta)\frac{\partial}{\partial x} + (c_4 x + \psi)\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by $S_1 = t\frac{\partial}{\partial t} + x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right)$, $S_2 = \frac{\partial}{\partial t}$ with $S_3 = \theta\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial u}$ as the infinite dimensional Lie sub-algebra.

The commutator table is given by,

	S_1	S_2
S_1	0	$-S_2$
S_2	S_2	0

.

Then $L = \{S_1, S_2\}$ is a solvable Lie algebra.

(3) Let $\alpha \neq -\beta$, $\gamma = -1$.
Then equation (7.34), becomes $\Upsilon_t + \alpha\Upsilon + \beta\Upsilon^r - \Upsilon_t^r = 0$, which can be solved to give

$$\omega = c_7, \tag{7.46}$$

$$\Upsilon = \theta(t), \tag{7.47}$$

$$\Upsilon^r = \psi(t - r), \tag{7.48}$$

where c_6 is an arbitrary constant, $c_7 = \dfrac{c_6}{\alpha + \beta}$ and $\theta_t + \alpha\theta + \beta\psi - \psi_t = 0$.
The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= c_7\frac{\partial}{\partial t} + \theta\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by $S_1 = \dfrac{\partial}{\partial t}$ with
$S_2 = \theta\dfrac{\partial}{\partial x} + \psi\dfrac{\partial}{\partial u}$ as the infinite dimensional Lie sub-algebra.

□

Corollary 7.4.1. *Subject to the conditions $\gamma = -1$, in equation (7.31), the same result of Theorem 7.4.1 (part (3)) is obtained if $\alpha \neq -\beta$ but either $\alpha = 0$ or $\beta = 0$.*

Theorem 7.4.2. *The first order neutral differential equation (7.31) for which $\alpha = 0 = \beta, \quad \gamma = 1$ admits the three dimensional Lie algebra generated by*

$$S_1 = t\frac{\partial}{\partial t} + x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right), \quad S_2 = \frac{\partial}{\partial t}, \quad S_3 = \frac{\partial}{\partial u},$$

with the infinite dimensional Lie sub-algebra given by

$$S_4^i = \theta\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial u}\right].$$

Proof. Let $\alpha = 0 = \beta$, $\gamma = 1$.
Then equation (7.34), becomes $\Upsilon_t + \Upsilon_t^r = 0$, which can be solved to give

$$\omega = c_8 t + c_9, \tag{7.49}$$

$$\Upsilon = c_8 x + \theta(t), \tag{7.50}$$

$$\Upsilon^r = c_8 x + \psi(t - r), \tag{7.51}$$

where c_8, c_9, c_{10} are arbitrary constants and $\psi = c_{10} - \theta$.
The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= (c_8 t + c_9)\frac{\partial}{\partial t} + (c_8 x + \theta)\frac{\partial}{\partial x} + (c_8 x + c_{10} - \theta)\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by $S_1 = t\dfrac{\partial}{\partial t} + x\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial u}\right)$, $S_2 = \dfrac{\partial}{\partial t}$,
$S_3 = \dfrac{\partial}{\partial u}$ with $S_4 = \theta\left(\dfrac{\partial}{\partial x} - \dfrac{\partial}{\partial u}\right)$ as the infinite dimensional Lie sub-algebra.

The commutator table is given by

	S_1	S_2	S_3
S_1	0	$-S_2$	0
S_2	S_2	0	0
S_3	0	0	0

.

Then $L = \{S_1, S_2, S_3\}$ is a solvable Lie algebra.

□

Corollary 7.4.2. *For the first order neutral differential equation given by (7.31) with $\alpha = 0 = \beta$, $\gamma \neq 0, -1$, the same generators as in the previous Theorem are obtained, only that the infinite dimensional Lie sub-algebra is given by*

$$S_4 = \theta\left(\frac{\partial}{\partial x} - \frac{1}{\gamma}\frac{\partial}{\partial u}\right).$$

7.4.2 A Nonlinear Case

We make a classification of

$$x'(t) + x(t)x(t-r) + x'(t-r) = v(t), \tag{7.52}$$

This is a nonlinear and nonhomogeneous equation.
Applying the operator defined by equation (7.32), to the delay equation $g(t) = t - r$, we get equation (7.12).
Applying the operator defined by equation (7.32), to equation (7.52), we get,

$$\Upsilon_t + (\Upsilon_x - \omega_t)x' - \omega_x x'^2 + x\Upsilon^r + x^r\Upsilon + \Upsilon_t^r + (\Upsilon_x^r - \omega_t^r)x^{r\prime} - \omega_x^r {x^{r\prime}}^2 = \omega v'. \tag{7.53}$$

Splitting equation (7.53) with respect to constant term, x', x'^2, $x^{r\prime}$ and ${x^{r\prime}}^2$ respectively, we get,

$$\Upsilon_t + x\Upsilon^r + x^r\Upsilon + \Upsilon_t^r = \omega v', \tag{7.54}$$

$$\Upsilon_x - \omega_t = 0, \tag{7.55}$$

$$\omega_x = 0, \tag{7.56}$$

$$\Upsilon_x^r - \omega_t^r = 0, \tag{7.57}$$

$$\omega_x^r = 0. \tag{7.58}$$

These equations can be solved to give,

$$\omega = c_1, \tag{7.59}$$

$$\Upsilon = \theta, \quad \Upsilon^r = \psi, \tag{7.60}$$

where c_1, c_2 are arbitrary constants and $\theta = c_1 v + c_2 - \psi_t$.
The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r} \\ &= c_1\frac{\partial}{\partial t} + (c_1 v + c_2 - \psi_t)\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by $S_1 = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x}$, $S_2 = \frac{\partial}{\partial x}$ with $S_3 = -\psi_t\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial u}$ as the infinite dimensional Lie sub-algebra.

The commutator table is given by

	S_1	S_2
S_1	0	0
S_2	0	0

Then $L = \{S_1, S_2\}$ is a solvable Lie algebra.

7.5 Summary

With the notation L_n^m, where m denotes the dimension of the solvable Lie algebra and S^i to mean the infinite dimensional Lie sub-algebra, the entire classification of first order functional differential equations with constant coefficients to solvable Lie algebras is summarized in Table 7.1 and Table 7.2 below:

Table 7.1: Group Classification of First Order Functional Differential Equations

Type of Functional Differential Equation	Basis for the Lie Algebra	Solvable Lie algebra
$x'(t)+\alpha x(t)+\beta x(t-r)=0$, $\alpha\neq-\beta$	$S_1=\frac{\partial}{\partial t},\quad S_2=x\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right),$ $S_3^i=-\left(\frac{\omega_t}{\alpha+\beta}\right)\frac{\partial}{\partial t}+\left[\theta-(\alpha+\beta)\omega x\right]\frac{\partial}{\partial x}-\left[(\alpha+\beta)\omega x+\frac{\alpha}{\beta}\theta+\frac{1}{\beta}\theta_t\right]\frac{\partial}{\partial u}$	L_1^2
$x'(t)+\alpha(x(t)-x(t-r))=0.$	$S_1=t\frac{\partial}{\partial t}+x\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right),\quad S_2=\frac{\partial}{\partial t},$ $S_3^i=\theta\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right)-\frac{1}{\beta}\theta_t\frac{\partial}{\partial u}$	L_2^2
$x'(t)+\beta x(t-r)=0.$	$S_1=\frac{\partial}{\partial t},\quad S_2=x\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right),$ $S_3^i=-\left(\frac{\omega_t}{\beta}\right)\frac{\partial}{\partial t}+(\theta-\beta x\omega)\frac{\partial}{\partial x}-\beta x\omega\frac{\partial}{\partial u}$	L_3^2
$x'(t)=k\left[1-\frac{x(t-r)}{P}\right]x(t).$	$S_1=\frac{\partial}{\partial t},\quad S_2^i=\theta\frac{\partial}{\partial x}+\psi\frac{\partial}{\partial u}$	L_4^1
$x'(t)+\alpha x(t)+\beta x(t-r)+\gamma x'(t-r)=0$, $\alpha\neq-\beta,\quad\gamma\neq-1$	$S_1=\frac{\partial}{\partial t},\quad S_2=x\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right)$ $S_3^i=-\left(\frac{1+\gamma}{\alpha+\beta}\right)\omega_t\frac{\partial}{\partial t}+\left[\theta-\left(\frac{\alpha+\beta}{1+\gamma}\right)\omega x\right]\frac{\partial}{\partial x}-\left[\psi-\left(\frac{\alpha+\beta}{1+\gamma}\right)\omega x\right]\frac{\partial}{\partial u}$	L_5^2
$x'(t)+\alpha(x(t)-x(t-r))+\gamma x'(t-r)=0$, $\gamma\neq-1.$	$S_1=t\frac{\partial}{\partial t}+x\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right),\quad S_2=\frac{\partial}{\partial t}$ $S_3^i=\theta\frac{\partial}{\partial x}+\psi\frac{\partial}{\partial u}$	L_6^2

Table 7.2: Group Classification of First Order Functional Differential Equations

Type of Functional Differential Equation	Basis for the Lie Algebra	Solvable Lie algebra
$x'(t)+\alpha x(t)+\beta x(t-r)-x'(t-r)=0$, $\alpha \neq -\beta$	$S_1 = \frac{\partial}{\partial t}$ $S_2^i = \theta\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$	L_7^1
$x'(t)+x'(t-r)=0.$	$S_1 = t\frac{\partial}{\partial t} + x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right), \quad S_2 = \frac{\partial}{\partial t}$ $S_3 = \frac{\partial}{\partial u},\ S_4^i = \theta\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial u}\right]$	L_8^3
$x'(t)+\gamma x'(t-r)=0$, $\gamma \neq 0, -1$	$S_1 = t\frac{\partial}{\partial t} + x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right), \quad S_2 = \frac{\partial}{\partial t}$ $S_3 = \frac{\partial}{\partial u},\ S_4^i = \theta\left[\frac{\partial}{\partial x} - \frac{1}{\gamma}\frac{\partial}{\partial u}\right].$	L_9^3
$x'(t)+x(t)x(t-r)+x'(t-r)=v(t).$	$S_1 = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x}, \quad S_2 = \frac{\partial}{\partial x}$ $S_3^i = -\psi_t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$	L_{10}^2

Chapter 8

Classification of Second Order Functional Differential Equations With Constant Coefficients to Solvable Lie Algebras

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8.1 Introduction

In this chapter, we make a complete classification of second order functional (delay and neutral) differential equations with constant coefficients to solvable Lie algebras. We shall use certain facts stated later to simplify our second order functional differential equations. We provide a basis for the Lie algebra given by second order linear and nonlinear functional differential equations by simplifying them and using an approach different from the existing literature. We also make a classification for some second order nonlinear functional differential equations.

It is noteworthy to mention here that by using Lie-Bäcklund operator and invariant manifold theorem, [40] classifies second order delay differential equations to solvable Lie algebras. We see that [40] performs a symmetry analysis without simplification of the linear delay differential equations, the simplification of which will be seen in this chapter. In addition several crucial cases are not considered in [40]. The approach for classification of delay differential equations to solvable Lie algebras is extended to some nonlinear differential equations in [41, 42]. The drawback of the analysis in [51, 59, 60] is that the inverse of the obtained classification cannot be found. We extend the results we have obtained in the previous chapter to obtain symmetries for linear and nonlinear functional differential equations.

We shall be studying the functional differential equation

$$\Phi(t, x(t), x(t-r), x'(t), x'(t-r), x''(t), x''(t-r)) = 0, \tag{8.1}$$

where Φ is defined on $I \times D^6$ where D is an open set in $\mathbb{R}$, I is an interval in $\mathbb{R}$ and $r > 0$ is the delay. We assume that, $\dfrac{\partial \Phi}{\partial x''(t-r)} \neq 0$. We shall find a Lie group under which these functional differential equations are invariant. We call this the admitted Lie group by which we mean that one solution curve is carried to another solution curve of the same equation.

The rest of this chapter is organised as follows: The next section extends the results for ordinary differential equations to functional differential equations by obtaining a Lie type invariance condition using Taylor's theorem for a function of several variables. In the sections to follow, each section will consist of two subsections — one for linear and the other for nonlinear functional differential equations with constant coefficients. Each section will independently be concerned with (i) Second order delay differential equations (ii) Second order neutral differential equations. We conclude with representation of our results, which are the basis for the Lie algebras, in a tabular form.

8.2 Lie Type Invariance Condition for Second Order Functional Differential Equations

In this section, we extend the results for ordinary differential equations to functional differential equations. The notation x^r whenever it appears will denote $x(t-r)$.
We establish the following Lie type invariance condition for second order neutral differential equations. A careful look at the proof will see that it is slightly different from the proof given in chapter 5 and 6.

Theorem 8.2.1. *Consider the second order neutral differential equation*

$$\frac{d^2x}{dt^2} = F(t, x, x(t-r), x'(t), x'(t-r), x''(t-r)), \tag{8.2}$$

where F be defined on a 6-dimensional space $I \times D^5$, D is an open set in $\mathbb{R}$ and I is any interval in $\mathbb{R}$. Then the Lie type invariance condition is given by

$$\omega F_t + \Upsilon F_x + \Upsilon^r F_{x(t-r)} + \Upsilon_{[t]} F_{x'(t)} + \Upsilon^r_{[t]} F_{x'(t-r)} + \Upsilon^r_{[tt]} F_{x''(t-r)} =$$
$$\Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)x'' - 3\omega_x x' x'',$$

where,

$$\Upsilon_{[t]} = D_t(\Upsilon) - x' D_t(\omega),$$

$$\Upsilon_{[tt]} = D_t(\Upsilon_{[t]}) - x'' D_t(\omega), \quad \textit{where} \quad D_t = \frac{\partial}{\partial t} + x'\frac{\partial}{\partial x} + x''\frac{\partial}{\partial x'} + \cdots,$$

$$\Upsilon^r_{[t]} = (\Upsilon_t)^r + ((\Upsilon_x)^r - (\omega_t)^r)x'(t-r) - (x'(t-r))^2(\omega_x)^r,$$

$\Upsilon^r_{[tt]} = (\Upsilon_{tt})^r + (2(\Upsilon_{tx})^r - (\omega_{tt})^r)x'(t-r) + ((\Upsilon_{xx})^r - 2(\omega_{tx})^r)x'(t-r)^2$
$- (\omega_{xx})^r x'(t-r)^3 + ((\Upsilon_x)^r - 2(\omega_t)^r)x''(t-r) - 3(\omega_x)^r x'(t-r)x''(t-r)),$
and $\omega^r = \omega(t-r, x(t-r))$, $\Upsilon^r = \Upsilon(t-r, x(t-r))$.

Proof. Let the neutral differential equation be invariant under the Lie group

$$\bar{t} = t + \delta\omega(t,x) + O(\delta^2), \quad \bar{x} = x + \delta\Upsilon(t,x) + O(\delta^2).$$

We then naturally define $\overline{t-r} = t - r + \delta\omega(t-r, x(t-r)) + O(\delta^2)$ and
$\overline{x(t-r)} = x(t-r) + \delta\Upsilon(t-r, x(t-r)) + O(\delta^2)$.

With the notations, $\omega^r = \omega(t-r, x(t-r))$, and $\Upsilon^r = \Upsilon(t-r, x(t-r))$, it follows that,

$$\begin{aligned} \overline{x'(t-r)} &= \tfrac{d\bar{x}}{d\bar{t}}(\overline{t-r}) \\ &= x'(t-r) + (\Upsilon_t)^r + ((\Upsilon_x)^r - (\omega_t)^r)x'(t-r) \\ &- (x'(t-r))^2(\omega_x)^r)\delta + O(\delta^2). \end{aligned} \tag{8.3}$$

Considering the second-order extended infinitesimals, we can write

$$\begin{aligned} \frac{d^2\bar{x}}{d\bar{t}^2} &= \frac{d}{d\bar{t}}\left(\frac{d\bar{x}}{d\bar{t}}\right) \\ &= \frac{\dfrac{d}{dt}\left[\dfrac{dx}{dt} + [D_t(\Upsilon) - x'D_t(\omega)]\delta + O(\delta^2)\right]}{1 + \delta D_t(\omega) + O(\delta^2)} \\ &= \left(\frac{d^2x}{dt^2} + D_t(\Upsilon_{[t]})\delta + O(\delta^2)\right)(1 - \delta D_t(\omega) + O(\delta^2)) \\ &= \frac{d^2x}{dt^2} + (D_t(\Upsilon_{[t]}) - D_t(\omega)x'')\delta + O(\delta^2). \end{aligned}$$

So, $\Upsilon_{[tt]} = D_t(\Upsilon_{[t]}) - x''D_t(\omega)$.
As $\Upsilon_{[t]}$ contains t, x and x', we need to extend the definition of D_t.
Let $D_t = \dfrac{\partial}{\partial t} + x'\dfrac{\partial}{\partial x} + x''\dfrac{\partial}{\partial x'} + \cdots$.
Expanding $\Upsilon_{[tt]}$, gives,

$$\Upsilon_{[tt]} = \Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)x'' - 3\omega_x x'x''.$$

It follows that,

$$\begin{aligned} \overline{x''(t-r)} &= \frac{d^2\bar{x}}{d\bar{t}^2}(\overline{t-r}) \\ &= x''(t-r) + \big[(\Upsilon_{tt})^r + (2(\Upsilon_{tx})^r - (\omega_{tt})^r)x'(t-r) \\ &+ ((\Upsilon_{xx})^r - 2(\omega_{tx})^r)x'(t-r)^2 - (\omega_{xx})^r x'(t-r)^3 \\ &+ ((\Upsilon_x)^r - 2(\omega_t)^r)x''(t-r) - 3(\omega_x)^r x'(t-r)x''(t-r)\big]\delta + O(\delta^2). \end{aligned}$$

Let $\Upsilon^r_{[t]} = (\Upsilon_t)^r + ((\Upsilon_x)^r - (\omega_t)^r)x'(t-r) - (x'(t-r))^2(\omega_x)^r$ and
$\Upsilon^r_{[tt]} = (\Upsilon_{tt})^r + (2(\Upsilon_{tx})^r - (\omega_{tt})^r)x'(t-r) + ((\Upsilon_{xx})^r - 2(\omega_{tx})^r)x'(t-r)^2$
$- (\omega_{xx})^r x'(t-r)^3 + ((\Upsilon_x)^r - 2(\omega_t)^r)x''(t-r) - 3(\omega_x)^r x'(t-r)x''(t-r))$.
For invariance,
$\dfrac{d^2\bar{x}}{d\bar{t}^2} = F(\bar{t}, \bar{x}, \overline{x(t-r)}, \dfrac{d\bar{x}}{d\bar{t}}, \dfrac{d\bar{x}}{d\bar{t}}(\overline{t-r}), \dfrac{d^2\bar{x}}{d\bar{t}^2}(\overline{t-r}))$.

This gives,

$$\begin{aligned}\frac{d^2x}{dt^2}+\Upsilon_{[tt]}\delta+O(\delta^2) &= F(t+\delta\omega+O(\delta^2), x+\delta\Upsilon+O(\delta^2), x(t-r)+\delta\Upsilon^r+O(\delta^2),\\ &\quad \frac{dx}{dt}+\delta\Upsilon_{[t]}+O(\delta^2), \frac{dx}{dt}(t-r)+\Upsilon^r_{[t]}\delta+O(\delta^2),\\ &\quad \frac{d^2x}{dt^2}(t-r)+\Upsilon^r_{[tt]}\delta+O(\delta^2))\\ &= F(t,x,x(t-r),x'(t),x'(t-r),x''(t-r))+\\ &\quad (\omega F_t+\Upsilon F_x+\Upsilon^r F_{x(t-r)}+\Upsilon_{[t]}F_{x'(t)}+\Upsilon^r_{[t]}F_{x'(t-r)}\\ &+\Upsilon^r_{[tt]}F_{x''(t-r)})\delta+O(\delta^2).\end{aligned}$$

Comparing the coefficient of δ, we get

$$\begin{aligned}&\omega F_t+\Upsilon F_x+\Upsilon^r F_{x(t-r)}+\Upsilon_{[t]}F_{x'(t)}+\Upsilon^r_{[t]}F_{x'(t-r)}+\Upsilon^r_{[tt]}F_{x''(t-r)}=\\ &\qquad \Upsilon_{tt}+(2\Upsilon_{tx}-\omega_{tt})x'+(\Upsilon_{xx}-2\omega_{tx})x'^2-\omega_{xx}x'^3+(\Upsilon_x-2\omega_t)x''-3\omega_x x'x''. \quad (8.4)\end{aligned}$$

The above obtained equation (8.4) is a Lie type invariance condition. □

We can define a prolonged operator (the general infinitesimal generator associated with the Lie algebra) for the second order neutral differential equation as:
$\zeta=\omega\frac{\partial}{\partial t}+\Upsilon\frac{\partial}{\partial x}+\Upsilon^r\frac{\partial}{\partial x(t-r)}$.

We then, naturally define the extended operator, for second order neutral differential equations as:

$$\zeta^{(1)}=\omega\frac{\partial}{\partial t}+\Upsilon\frac{\partial}{\partial x}+\Upsilon^r\frac{\partial}{\partial x(t-r)}+\Upsilon_{[t]}\frac{\partial}{\partial x'}+\Upsilon^r_{[t]}\frac{\partial}{\partial x'(t-r)}+\Upsilon_{[tt]}\frac{\partial}{\partial x''}+\Upsilon^r_{[tt]}\frac{\partial}{\partial x''(t-r)}. \quad (8.5)$$

Defining, $\Delta=x''(t)-F(t,x(t),x(t-r),x'(t),x'(t-r),x''(t-r))=0$, we get,

$$\zeta^{(1)}\Delta=\Upsilon_{[tt]}-\omega F_t-\Upsilon F_x-\Upsilon^r F_{x(t-r)}-\Upsilon_{[t]}F_{x'(t)}-\Upsilon^r_{[t]}F_{x'(t-r)}-\Upsilon^r_{[tt]}F_{x''(t-r)}. \quad (8.6)$$

Comparing equation (8.6) and equation (8.4), we get,
$\Upsilon_{[tt]}=\Upsilon_{tt}+(2\Upsilon_{tx}-\omega_{tt})x'+(\Upsilon_{xx}-2\omega_{tx})x'^2-\omega_{xx}x'^3+(\Upsilon_x-2\omega_t)x''-3\omega_x x'x''$.
On substituting $x''=F$ into $\zeta^{(1)}\Delta=0$, we get an invariance condition for the second order neutral differential equation which is $\zeta^{(1)}\Delta\mid_{\Delta=0}=0$, from which we shall obtain the determining equations.

Remark 8.2.1. If the term $x''(t-r)$ is absent, then the corresponding second order neutral differential equation reduces to a second order delay differential equation.

We conclude this section by proving two very elementary results which we shall be using in our subsequent sections:

Proposition 8.2.1. *If the linear functional differential equation is given by*

$$x''(t) + bx'(t) + cx'(t-r) + dx''(t-r) + ex(t) + jx(t-r) = m(t), \tag{8.7}$$

then by employing a change of variables namely $\bar{t} = t, \bar{x} = x - \tilde{x}$, *where* $\tilde{x}$ *is a solution of equation (8.7), we can convert the given non-homogeneous linear functional differential equation to a homogeneous one, namely*
$x''(t) + bx'(t) + cx'(t-r) + dx''(t-r) + ex(t) + jx(t-r) = 0.$

Proof. The proposition easily follows by substituting $t = \bar{t}$ and $x(t) = \bar{x} + \tilde{x}(\bar{t})$ in (8.7), by noting that $\tilde{x}''(t) + b\tilde{x}'(t) + c\tilde{x}'(t-r) + d\tilde{x}''(t-r) + e\tilde{x}(t) + j\tilde{x}(t-r) = m(t)$. □

The next proposition is particularly useful in simplifying second order functional differential equations.

Proposition 8.2.2. *If the linear functional differential equation is given by*

$$x''(t) + bx'(t) + cx'(t-r) + dx''(t-r) + ex(t) + jx(t-r) = 0, \tag{8.8}$$

then by employing a suitable transformation, we can convert the given non-homogeneous linear functional differential equation to a one in which the first derivative (ordinary derivative) term is missing, that is to the equation

$$x''(t) + cx'(t-r) + dx''(t-r) + ex(t) + jx(t-r) = 0. \tag{8.9}$$

Proof. We have seen a proof of this proposition in chapter 6. □

Remark 8.2.2. It should be noted that this transformation does not affect the symmetries of equation (8.8).

8.3 Classification of Second Order Delay Differential Equations to Solvable Lie Algebras

8.3.1 The Linear Case

We shall make a classification of

$$x''(t) + \alpha x'(t) + \beta x'(t-r) + \gamma x(t) + \rho x(t-r) = 0. \tag{8.10}$$

By using Proposition 8.2.2, we make a classification of

$$x''(t) + \beta x'(t-r) + \gamma x(t) + \rho x(t-r) = 0. \tag{8.11}$$

The extension and prolongation operator for equation (8.11) is given by,

$$\zeta^{(1)} = \omega\frac{\partial}{\partial t} + \omega^r\frac{\partial}{\partial(t-r)} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x(t-r)} + \Upsilon_{[t]}\frac{\partial}{\partial x'} + \Upsilon^r_{[t]}\frac{\partial}{\partial x^{r\prime}} + \Upsilon_{[tt]}\frac{\partial}{\partial x''}. \tag{8.12}$$

Applying the operator defined by equation (8.12), to the delay equation $g(t) = t - r$, we get $\omega(t, x) = \omega(t - r, x(t - r))$.
Applying the operator defined by equation (8.12), to equation (8.11), we get,

$$\begin{aligned} &\Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)(-\beta x'(t-r) - \gamma x - \rho x(t-r)) \\ &- 3\omega_x x'(-\beta x'(t-r) - \gamma x - \rho x(t-r)) + \beta(\Upsilon^r_t + (\Upsilon^r_x - \omega^r_t)x^{r\prime} - \omega^r_x x^{r\prime 2}) + \gamma\Upsilon + \rho\Upsilon^r = 0. \end{aligned} \tag{8.13}$$

Splitting equation (8.13) with respect to the constant term we get,

$$\Upsilon_{tt} + \beta\Upsilon^r_t + \gamma\Upsilon + \rho\Upsilon^r = 0. \tag{8.14}$$

Splitting equation (8.13) with respect to x we get,

$$\gamma(\Upsilon_x - \omega_t) = 0. \tag{8.15}$$

Splitting equation (8.13) with respect to x' we get,

$$2\Upsilon_{tx} = \omega_{tt}. \tag{8.16}$$

Splitting equation (8.13) with respect to x'^2 we get,

$$\Upsilon_{xx} = 2\omega_{tx}. \tag{8.17}$$

Splitting equation (8.13) with respect to x'^3 we get,

$$\omega_{xx} = 0. \tag{8.18}$$

Splitting equation (8.13) with respect to $x'x^{r\prime}, xx'$ or $x'x^r$, we get,

$$\omega_x = 0. \tag{8.19}$$

Splitting equation (8.13) with respect to x^r we get,

$$-\rho(\Upsilon_x - 2\omega_t) = 0. \tag{8.20}$$

Splitting equation (8.13) with respect to ${x^{r}}'^{2}$ we get,

$$-\beta\omega_x^r = 0. \tag{8.21}$$

Splitting equation (8.13) with respect to $x^{r'}$ we get,

$$-\beta(\Upsilon_x - 2\omega_t) + \beta(\Upsilon_x^r - \omega_t^r) = 0. \tag{8.22}$$

From equation (8.19) and (8.21), we get $\omega = \omega(t)$.
From equation (8.17), $\Upsilon = A(t)x + \theta(t)$.
From equation (8.15) or (8.20), we get,

$$\omega_t = \frac{1}{2}A(t). \tag{8.23}$$

From equation (8.22) and using equation (8.23), we get $\Upsilon^r = \dfrac{1}{2}A(t)x + \psi(t-r)$.

The following theorems make a complete classification of the second order delay differential equation to solvable Lie algebras. The notation u is used to denote x^r.

Theorem 8.3.1. *The delay differential equation given by equation (8.11) for which $\beta \neq 0, \gamma \neq -\dfrac{\rho}{2}$ admits a three dimensional group generated by*

$$S_1 = \frac{\partial}{\partial t}, \quad S_2 = x\frac{\partial}{\partial x}, \quad S_3 = x\frac{\partial}{\partial u},$$

with the infinite dimensional Lie sub-algebra given by

$$S_4^i = -\left(\frac{1}{2(\gamma+\frac{\rho}{2})} + \frac{4(\gamma+\frac{\rho}{2})}{\beta}A\right)\frac{\partial}{\partial t} + \left[\theta - x\left(\frac{2}{\beta}A_t + \frac{4(\gamma+\frac{\rho}{2})}{\beta}\omega\right)\right]\frac{\partial}{\partial x}$$
$$+\left[\psi - x\left(\frac{1}{\beta}A_t + \frac{2(\gamma+\frac{\rho}{2})}{\beta}\omega\right)\right]\frac{\partial}{\partial u}.$$

Proof. Let β, γ, ρ be arbitrary non-zero constants, $\gamma \neq -\dfrac{\rho}{2}$. Then from equation (8.14), we get,

$$A_{tt} + \frac{\beta}{2}A_t + \gamma A + \frac{\rho}{2}A = 0, \tag{8.24}$$

and $\theta_{tt} + \beta\psi_t + \gamma\theta + \rho\psi = 0$.
Solving equation (8.24) using equation (8.23), we get,

$$\omega = c_1 - \frac{A_t + \frac{\beta}{2}A}{2(\gamma+\frac{\rho}{2})}, \tag{8.25}$$

where c_1 is an arbitrary constant. From equation (8.25),

$$A(t) = c_2 - \frac{2}{\beta}A_t - \frac{4\omega(\gamma + \frac{\rho}{2})}{\beta}, \tag{8.26}$$

where $c_2 = \dfrac{4c_1(\gamma + \frac{\rho}{2})}{\beta}$.
This yields,

$$\Upsilon = \left(c_2 - \frac{2}{\beta}A_t - \frac{4\omega(\gamma + \frac{\rho}{2})}{\beta}\right)x + \theta, \tag{8.27}$$

and,

$$\Upsilon^r = \left(c_3 - \frac{1}{\beta}A_t - \frac{2\omega(\gamma + \frac{\rho}{2})}{\beta}\right)x + \psi, \tag{8.28}$$

where $c_3 = \dfrac{c_2}{2}$.
The infinitesimal generator is given by

$$\begin{aligned}
\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r} \\
&= \left(c_1 - \frac{A_t + \frac{\beta}{2}A}{2(\gamma + \frac{\rho}{2})}\right)\frac{\partial}{\partial t} + \left[\left(c_2 - \frac{2}{\beta}A_t - \frac{4\omega(\gamma + \frac{\rho}{2})}{\beta}\right)x + \theta\right]\frac{\partial}{\partial x} \\
&\quad + \left[\left(c_3 - \frac{1}{\beta}A_t - \frac{2\omega(\gamma + \frac{\rho}{2})}{\beta}\right)x + \psi\right]\frac{\partial}{\partial x^r}.
\end{aligned}$$

The Lie algebra is spanned by $S_1 = \dfrac{\partial}{\partial t}, \quad S_2 = x\dfrac{\partial}{\partial x}, \quad S_3 = x\dfrac{\partial}{\partial u}$.
With $g = 2\omega_t$, we get

$$\begin{aligned}
S_4 = &-\left(\frac{1}{2(\gamma + \frac{\rho}{2})} + \frac{\beta}{4(\gamma + \frac{\rho}{2})}A\right)\frac{\partial}{\partial t} + \left[\theta - x\left(\frac{2}{\beta}A_t + \frac{4(\gamma + \frac{\rho}{2})}{\beta}\omega\right)\right]\frac{\partial}{\partial x} \\
&+ \left[\psi - x\left(\frac{1}{\beta}A_t + \frac{2(\gamma + \frac{\rho}{2})}{\beta}\omega\right)\right]\frac{\partial}{\partial u}
\end{aligned}$$

is the infinite dimensional Lie sub-algebra.

The commutator table is given by

	S_1	S_2	S_3
S_1	0	0	0
S_2	0	0	S_3
S_3	0	$-S_3$	0

.

Then $L = \{S_1, S_2, S_3\}$ is a solvable Lie algebra. □

Theorem 8.3.2. *The delay differential equation given by equation (8.11) for which $\beta \neq 0, \gamma = -\frac{\rho}{2}$ admits a four dimensional group generated by*

$$S_1 = t\frac{\partial}{\partial t}, \quad S_2 = \frac{\partial}{\partial t}, \quad S_3 = tx\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right], \quad S_4 = x\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right],$$

with the infinite dimensional Lie sub-algebra given by

$$S_5^i = -\frac{A}{\beta}\frac{\partial}{\partial t} + (\theta - \beta x\omega)\frac{\partial}{\partial x} + (\psi - \frac{\beta x\omega}{4})\frac{\partial}{\partial u}.$$

Proof. Let β, γ, ρ be arbitrary non-zero constants, $\gamma = -\frac{\rho}{2}$. Then from equation (8.14), we get,

$$\omega_{ttt} + \frac{\beta}{2}\omega_{tt} = 0, \tag{8.29}$$

and $\theta_{tt} + \beta\psi_t + \frac{\rho}{2}\theta + \rho\psi = 0$.
Solving equation (8.29) we get,

$$\omega = c_6 t + c_7 - \frac{A}{\beta}, \tag{8.30}$$

where c_4, c_5 are arbitrary constants and $c_6 = \frac{2c_4}{\beta}, c_7 = \frac{2c_5}{\beta}$. From equation (8.30),

$$A(t) = c_8 t + c_9 - \beta\omega, \tag{8.31}$$

where $c_8 = \beta c_6,\ c_9 = \beta c_7$. This yields,

$$\Upsilon = (c_8 t + c_9 - \beta\omega)x + \theta(t), \tag{8.32}$$

and,

$$\Upsilon^r = \frac{1}{2}(c_8 t + c_9 - \beta\omega)x + \psi(t - r). \tag{8.33}$$

The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= \left(c_6t + c_7 - \frac{A}{\beta}\right)\frac{\partial}{\partial t} + [(c_8t + c_9 - \beta\omega)x + \theta]\frac{\partial}{\partial x} + [\frac{1}{2}(c_8t + c_9 - \beta\omega)x + \psi]\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by $S_1 = t\frac{\partial}{\partial t}$, $\quad S_2 = \frac{\partial}{\partial t}$,
$S_3 = tx\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right]$, $\quad S_4 = x\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right]$ with
$S_5 = -\frac{A}{\beta}\frac{\partial}{\partial t} + (\theta - \beta x\omega)\frac{\partial}{\partial x} + (\psi - \frac{\beta x\omega}{4})\frac{\partial}{\partial u}$ as the infinite dimensional Lie sub-algebra.

The commutator table is given by

	S_1	S_2	S_3	S_4
S_1	0	$-S_2$	S_3	0
S_2	S_2	0	S_4	0
S_3	$-S_3$	$-S_4$	0	0
S_4	0	0	0	0

.

Then $L = \{S_1, S_2, S_3, S_4\}$ is a solvable Lie algebra. □

Theorem 8.3.3. *The delay differential equation given by equation (8.11) for which $\beta = 1, \gamma = 0 = \rho$ admits a three dimensional group generated by*

$$S_1 = t\frac{\partial}{\partial t} + tx\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right], \quad S_2 = \frac{\partial}{\partial t} + x\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right], \quad S_3 = \frac{\partial}{\partial u},$$

with the infinite dimensional Lie sub-algebra given by

$$S_4^i = -A\frac{\partial}{\partial t} + (\theta - x\omega)\frac{\partial}{\partial x} - \left(\theta_t + \frac{x\omega}{2}\right)\frac{\partial}{\partial u}.$$

Proof. Let $\beta = 1$, $\gamma = 0 = \rho$. Then equation (8.14) becomes $\Upsilon_{tt} + \Upsilon_t^r = 0$, which yields,

$$A_{tt} + \frac{1}{2}A_t = 0, \tag{8.34}$$

and $\psi = -\theta_t + c_{10}$.
Solving equation (8.34) we get,

$$\omega = c_{11}t + c_{12} - A(t), \tag{8.35}$$

where c_{10}, c_{11}, c_{12} are arbitrary constants. From equation (8.35),

$$A(t) = c_{11}t + c_{12} - \omega. \tag{8.36}$$

This yields,

$$\Upsilon = (c_{11}t + c_{12} - \omega)x + \theta(t), \tag{8.37}$$

and,

$$\Upsilon^r = \frac{1}{2}(c_{11}t + c_{12} - \omega)x + c_{10} - \theta_t. \tag{8.38}$$

The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= (c_{11}t + c_{12} - A(t))\frac{\partial}{\partial t} + [(c_{11}t + c_{12} - \omega)x + \theta(t)]\frac{\partial}{\partial x}\\ &+ [\frac{1}{2}(c_{11}t + c_{12} - \omega)x + c_{10} - \theta_t]\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by
$S_1 = t\frac{\partial}{\partial t} + tx\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right]$, $S_2 = \frac{\partial}{\partial t} + x\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right]$, $S_3 = \frac{\partial}{\partial u}$,
with $S_4 = -A\frac{\partial}{\partial t} + (\theta - x\omega)\frac{\partial}{\partial x} - \left(\theta_t + \frac{x\omega}{2}\right)\frac{\partial}{\partial u}$ as the infinite dimensional Lie sub-algebra.

The commutator table is given by

	S_1	S_2	S_3
S_1	0	$-S_2$	0
S_2	S_2	0	0
S_3	0	0	0

.

Then $L = \{S_1, S_2, S_3\}$ is a solvable Lie algebra. □

Theorem 8.3.4. *The delay differential equation given by equation (8.11) for which* $\beta \neq 0, \gamma = 0 = \rho$ *admits a five dimensional group generated by*

$$S_1 = t\frac{\partial}{\partial t}, \quad S_2 = \frac{\partial}{\partial t}, \quad S_3 = tx\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right],$$

$$S_4 = x\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right], \quad S_5 = \frac{\partial}{\partial u},$$

with the infinite dimensional Lie sub-algebra given by

$$S_6^i = -\frac{A}{\beta}\frac{\partial}{\partial t} + (\theta - \beta x\omega)\frac{\partial}{\partial x} + \left(\frac{\theta_t}{\beta} + \beta x\omega\right)\frac{\partial}{\partial u}.$$

Proof. Let $\gamma = 0 = \rho$, β be an arbitrary non zero constant. Then equation (8.14) becomes $\Upsilon_{tt} + \beta\Upsilon_t^r = 0$, which yields,

$$A_{tt} + \frac{\beta}{2}A_t = 0, \tag{8.39}$$

and $\psi = c_{14} - \frac{\theta_t}{\beta}$, where c_{13} is an arbitrary constant and $c_{14} = \frac{c_{13}}{\beta}$.
Solving equation (8.39) we get,

$$\omega = c_{17}t + c_{18} - \frac{\theta}{\beta}, \tag{8.40}$$

where c_{15}, c_{16} are arbitrary constants and $c_{17} = \frac{c_{15}}{\beta}, \quad c_{18} = \frac{c_{16}}{\beta}$. From equation (8.40),

$$A(t) = c_{15}t + c_{16} - \beta\omega. \tag{8.41}$$

This yields,

$$\Upsilon = (c_{15}t + c_{16} - \beta\omega)x + \theta(t), \tag{8.42}$$

and,

$$\Upsilon^r = \frac{1}{2}(c_{15}t + c_{16} - \beta\omega)x + c_{14} - \frac{\theta_t}{\beta}. \tag{8.43}$$

The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= (c_{17}t + c_{18} - \frac{\theta}{\beta})\frac{\partial}{\partial t} + [(c_{15}t + c_{16} - \beta\omega)x + \theta(t)]\frac{\partial}{\partial x}\\ &\quad + \left[\frac{1}{2}(c_{15}t + c_{16} - \beta\omega)x + c_{14} - \frac{\theta_t}{\beta}\right]\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by $S_1 = t\frac{\partial}{\partial t}, \quad S_2 = \frac{\partial}{\partial t}$,
$S_3 = tx\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right], \quad S_4 = x\left[\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u}\right], \quad S_5 = \frac{\partial}{\partial u}$ with
$S_6 = -\frac{A}{\beta}\frac{\partial}{\partial t} + (\theta - \beta x\omega)\frac{\partial}{\partial x} + \left(\frac{\theta_t}{\beta} + \beta x\omega\right)\frac{\partial}{\partial u}$ as the infinite dimensional Lie sub-algebra.

The commutator table is given by

	S_1	S_2	S_3	S_4	S_5
S_1	0	$-S_2$	S_3	0	0
S_2	S_2	0	S_4	0	0
S_3	$-S_3$	$-S_4$	0	0	0
S_4	0	0	0	0	0
S_5	0	0	0	0	0

.

Then $L = \{S_1, S_2, S_3, S_4, S_5\}$ is a solvable Lie algebra. □

8.3.2 A Nonlinear Case

We make a classification of

$$x''(t) + x'(t) + x'(t-r)x(t) = 0. \tag{8.44}$$

Applying the operator defined by equation (8.11), to the delay equation $g(t) = t - r$, we get equation $\omega(t,x) = \omega(t-r, x(t-r))$.
Applying the operator defined by equation (8.11), to equation (8.44), we get,

$$\Upsilon_{tt}+(2\Upsilon_{tx}-\omega_{tt})x'+(\Upsilon_{xx}-2\omega_{tx})x'^2-\omega_{xx}x'^3+(\Upsilon_x-2\omega_t)x''-3\omega_x x'x''+\Upsilon_t+(\Upsilon_x-\omega_t)x' \\ -\omega_x x'^2 + x^{r\prime}\Upsilon + x[\Upsilon_t^r + (\Upsilon_x^r - \omega_t^r)x^{r\prime} - \omega_x^r {x^{r\prime}}^2] = 0. \tag{8.45}$$

Splitting equation (8.45) with respect to constant term, x', x'^2, x'^3, x'', $x'x''$, $x^{r\prime}$ and ${x^{r\prime}}^2$ respectively, we get,

$$\Upsilon_{tt} + \Upsilon_t + x\Upsilon_t^r = 0, \tag{8.46}$$

$$2\Upsilon_{tx} - \omega_{tt} + \Upsilon_x - \omega_t = 0, \tag{8.47}$$

$$\Upsilon_{xx} - 2\omega_{tx} - \omega_x = 0, \tag{8.48}$$

$$\omega_{xx} = 0, \tag{8.49}$$

$$\Upsilon_x - 2\omega_t = 0, \tag{8.50}$$

$$\omega_x = 0, \tag{8.51}$$

$$\Upsilon + x(\Upsilon_x^r - \omega_t^r) = 0, \tag{8.52}$$

$$x\omega_x^r = 0. \tag{8.53}$$

From these equations we get, $\omega = \omega(t)$, $\Upsilon = A(t)x + \theta(t)$, $\Upsilon^r = \frac{1}{2}A(t)x + \psi(t-r)$, where $A(t) = 2\omega_t$.

Substituting the values of Υ, Υ^r in equation (8.46) and solving it, we get,
$A(t) = c_{19},\ \psi = c_{20},\ \theta = c_{21} - \theta_t$, and

$$\omega = c_{22}t + c_{23}, \tag{8.54}$$

$$\Upsilon = c_{19}x + c_{21} - \theta_t, \quad \Upsilon^r = c_{24}x + c_{20}, \tag{8.55}$$

where $c_{19}, c_{20}, c_{21}, c_{22}, c_{23}, c_{24}$ are arbitrary constants.
The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= (c_{22}t + c_{23})\frac{\partial}{\partial t} + (c_{19}x + c_{21} - \theta_t)\frac{\partial}{\partial x} + (c_{24}x + c_{20})\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by
$S_1 = x\dfrac{\partial}{\partial x}, \quad S_2 = \dfrac{\partial}{\partial x}, \quad S_3 = t\dfrac{\partial}{\partial t},$
$S_4 = \dfrac{\partial}{\partial t}, \quad S_5 = x\dfrac{\partial}{\partial u}, \quad S_6 = \dfrac{\partial}{\partial u}$ with $S_7 = -\theta_t\dfrac{\partial}{\partial x}$ as the infinite dimensional Lie sub-algebra.

The commutator table is given by

	S_1	S_2	S_3	S_4	S_5	S_6
S_1	0	$-S_2$	0	0	S_5	0
S_2	S_2	0	0	0	S_6	0
S_3	0	0	0	$-S_4$	0	0
S_4	0	0	S_4	0	0	0
S_5	$-S_5$	$-S_6$	0	0	0	0
S_6	0	0	0	0	0	0

Then $L = \{S_1, S_2, S_3, S_4, S_5, S_6\}$ is a solvable Lie algebra.

Remark 8.3.1. For the non-homogeneous nonlinear second order delay differential equation $x''(t) + x'(t) + x'(t-r)x(t) = h(t)$, we get exactly the same generators as in the homogeneous case, only that $S_7 = (\theta_t - y)\dfrac{\partial}{\partial x}$ is the corresponding infinite dimensional Lie sub-algebra, where $y = c_{22}\int t\, h' dt + c_{23}h$.

8.4 Classification of Second Order Neutral Differential Equations to Solvable Lie Algebras

8.4.1 The Linear Case

We shall make a classification of

$$x''(t) + \alpha x'(t) + \beta x'(t-r) + \gamma x(t) + \rho x(t-r) + \kappa x''(t-r) = 0. \tag{8.56}$$

By using the Theorem 8.2.1, we make a classification of

$$x''(t) + \beta x'(t-r) + \gamma x(t) + \rho x(t-r) + \kappa x''(t-r) = 0. \tag{8.57}$$

The extension and prolongation operator for equation (8.11) is given by,

$$\zeta^{(1)} = \omega\frac{\partial}{\partial t} + \omega^r\frac{\partial}{\partial(t-r)} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x(t-r)} + \Upsilon_{[t]}\frac{\partial}{\partial x'} + \Upsilon^r_{[t]}\frac{\partial}{\partial x^{r\prime}} + \Upsilon_{[tt]}\frac{\partial}{\partial x''} + \Upsilon^r_{[tt]}\frac{\partial}{\partial x^{r\prime\prime}}. \tag{8.58}$$

Applying the operator defined by equation (8.58), to the delay equation $g(t) = t - r$, we get equation $\omega(t,x) = \omega(t-r, x(t-r))$.
Applying the operator defined by equation (8.58), to equation (8.57), we get,

$$\begin{aligned}0 &= \Upsilon_{tt} + (2\Upsilon_{tx} - \omega_{tt})x' + (\Upsilon_{xx} - 2\omega_{tx})x'^2 - \omega_{xx}x'^3 + (\Upsilon_x - 2\omega_t)(-\beta x'(t-r) - \gamma x - \rho x(t-r)\\ &-\kappa x''(t-r)) - 3\omega_x x'(-\beta x'(t-r) - \gamma x - \rho x(t-r) - \kappa x''(t-r)) + \beta\left[\Upsilon^r_t + (\Upsilon^r_x - \omega^r_t)x^{r\prime} - \omega^r_x {x^{r\prime}}^2\right]\\ &+\gamma\Upsilon + \rho\Upsilon^r + \kappa\left[\Upsilon^r_{tt} + (2\Upsilon^r_{tx} - \omega^r_{tt})x^{r\prime} + (\Upsilon^r_{xx} - 2\omega^r_{tx}){x^{r\prime}}^2 - \omega^r_{xx}{x^{r\prime}}^3 + (\Upsilon^r_x - 2\omega^r_t)x^{r\prime\prime} - 3\omega^r_x x^{r\prime}x^{r\prime\prime}\right].\end{aligned} \tag{8.59}$$

Splitting equation (8.59) with respect to the constant term we get,

$$\Upsilon_{tt} + \beta\Upsilon^r_t + \gamma\Upsilon + \rho\Upsilon^r + \kappa\Upsilon^r_{tt} = 0. \tag{8.60}$$

Splitting equation (8.59) with respect to x we get,

$$\gamma(\Upsilon_x - 2\omega_t) = 0. \tag{8.61}$$

Splitting equation (8.59) with respect to x' we get,

$$2\Upsilon_{tx} - \omega_{tt} = 0. \tag{8.62}$$

Splitting equation (8.59) with respect to x'^2 we get,

$$\Upsilon_{xx} - 2\omega_{tx} = 0. \tag{8.63}$$

Splitting equation (8.59) with respect to ${x'}^3$ we get,

$$\omega_{xx} = 0. \tag{8.64}$$

Splitting equation (8.59) with respect to $x'x^{r\prime}, xx', x'x^r$, or $x'x^{r\prime\prime}$ we get,

$$\omega_x = 0. \tag{8.65}$$

Splitting equation (8.59) with respect to x^r we get,

$$-\rho(\Upsilon_x - 2\omega_t) = 0. \tag{8.66}$$

Splitting equation (8.59) with respect to ${x^{r\prime}}^2$ we get,

$$-\beta\omega^r_x + \kappa(\Upsilon^r_{xx} - 2\omega^r_{tx}) = 0. \tag{8.67}$$

Splitting equation (8.59) with respect to $x^{r\prime}$ we get,

$$-\beta(\Upsilon_x - 2\omega_t) + \beta(\Upsilon^r_x - \omega^r_t) + \kappa(2\Upsilon^r_{tx} - \omega^r_{tt}) = 0. \tag{8.68}$$

Splitting equation (8.59) with respect to ${x^{r\prime}}^3$ we get,

$$-\kappa\omega^r_{xx} = 0. \tag{8.69}$$

Splitting equation (8.59) with respect to $x^{r\prime}$ or $x^{r\prime\prime}$ we get,

$$-\kappa\omega^r_x = 0. \tag{8.70}$$

Splitting equation (8.59) with respect to $x^{r\prime\prime}$ we get,

$$-\kappa(\Upsilon_x - 2\omega_t) + \kappa(\Upsilon^r_x - 2\omega^r_t) = 0. \tag{8.71}$$

From equation (8.65), we get $\omega = \omega(t)$.
From equation (8.63), $\Upsilon = A(t)x + \theta(t)$.
From equation (8.61) or (8.66), we get,

$$\omega_t = \frac{1}{2}A(t). \tag{8.72}$$

From equation (8.71) and using equations (8.70) and (8.72), we get $\Upsilon^r = A(t)x + \psi(t-r)$.

The following theorems make a complete classification of the second order neutral differential equation to solvable Lie algebras. The notation u is used to denote x^r:

Theorem 8.4.1. *The neutral differential equation given by equation (8.57) for which*

$\beta \neq 0, \kappa \neq 0, \gamma \neq -\rho$ admits a two dimensional group generated by

$$S_1 = \frac{\partial}{\partial t}, \quad S_2 = x\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right),$$

with the infinite dimensional Lie sub-algebra given by

$$S_3^i = -\left(\frac{1+\kappa}{2(\gamma+\rho)}g_t + \frac{\beta}{2(\gamma+\rho)}A\right)\frac{\partial}{\partial t} + \left[\theta - x\left(\frac{1+\kappa}{\beta}A_t + \frac{2(\gamma+\rho)}{\beta}\omega\right)\right]\frac{\partial}{\partial x} + \left[\psi - x\left(\frac{1+\kappa}{\beta}A_t + \frac{2(\gamma+\rho)}{\beta}\omega\right)\right]\frac{\partial}{\partial u}.$$

Proof. Let $\beta, \gamma, \rho, \kappa$ be arbitrary non-zero constants, $\gamma \neq -\rho$. Then from equation (8.60), we get,

$$(1+\kappa)A_{tt} + \beta A_t + (\gamma+\rho)A = 0, \tag{8.73}$$

and $\theta_{tt} + \beta\psi_t + \gamma\theta + \rho\psi + \kappa\psi_{tt} = 0$.
Solving equation (8.73) by using equation (8.72), we get,

$$\omega = c_2 - \frac{1+\kappa}{2(\gamma+\rho)}A_t - \frac{\beta}{2(\gamma+\rho)}A, \tag{8.74}$$

where c_1 is an arbitrary constant and $c_2 = \dfrac{c_1}{\gamma+\rho}$. From equation (8.74),

$$A(t) = c_3 - \frac{1+\kappa}{\beta}A_t - \frac{2(\gamma+\rho)}{\beta}\omega, \tag{8.75}$$

where $c_3 = \dfrac{2c_2(\gamma+\rho)}{\beta}$.
This yields,

$$\Upsilon = \left(c_3 - \frac{1+\kappa}{\beta}A_t - \frac{2(\gamma+\rho)}{\beta}\omega\right)x + \theta, \tag{8.76}$$

and,

$$\Upsilon^r = \left(c_3 - \frac{1+\kappa}{\beta}A_t - \frac{2(\gamma+\rho)}{\beta}\omega\right)x + \psi. \tag{8.77}$$

The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= \left(c_2 - \frac{1+\kappa}{2(\gamma+\rho)}A_t - \frac{\beta}{2(\gamma+\rho)}A\right)\frac{\partial}{\partial t} + \left[\left(c_3 - \frac{1+\kappa}{\beta}A_t - \frac{2(\gamma+\rho)}{\beta}\omega\right)x + \theta\right]\frac{\partial}{\partial x}\\ &\quad + \left[\left(c_3 - \frac{1+\kappa}{\beta}A_t - \frac{2(\gamma+\rho)}{\beta}\omega\right)x + \psi\right]\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by $S_1 = \dfrac{\partial}{\partial t}, \quad S_2 = x\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial u}\right)$.

With $g = 2\omega_t$, we get

$$S_3 = -\left(\frac{1+\kappa}{2(\gamma+\rho)}A_t + \frac{\beta}{2(\gamma+\rho)}A\right)\frac{\partial}{\partial t} + \left[\theta - x\left(\frac{1+\kappa}{\beta}A_t + \frac{2(\gamma+\rho)}{\beta}\omega\right)\right]\frac{\partial}{\partial x} + \left[\psi - x\left(\frac{1+\kappa}{\beta}A_t + \frac{2(\gamma+\rho)}{\beta}\omega\right)\right]\frac{\partial}{\partial u}$$

is the infinite dimensional Lie sub-algebra.

The commutator table is given by

	S_1	S_2
S_1	0	0
S_2	0	0

.

Then $L = \{S_1, S_2\}$ is a solvable Lie algebra. □

Corollary 8.4.1. *For the neutral differential equation given by equation (8.57), we obtain the same result as in Theorem 8.4.1, if either γ or ρ is* 0.

Theorem 8.4.2. *The neutral differential equation given by equation (8.57) for which $\beta \neq 0, \gamma = -\rho, \kappa \neq -1$, admits a four dimensional group generated by*

$$S_1 = t\frac{\partial}{\partial t}, \quad S_2 = \frac{\partial}{\partial t}, \quad S_3 = tx\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right], \quad S_4 = x\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right],$$

with the infinite dimensional Lie sub-algebra given by

$$S_5^i = -\frac{1+\kappa}{2\beta}A\frac{\partial}{\partial t} + \left[\theta - \frac{2\beta\omega x}{1+\kappa}\right]\frac{\partial}{\partial x} + \left[\psi - \frac{2\beta x\omega}{1+\kappa}\right]\frac{\partial}{\partial u}.$$

Proof. Let $\beta, \gamma, \rho, \kappa$ be arbitrary non-zero constants, $\gamma = -\rho, \kappa \neq -1$. Then from equation (8.60), we get,

$$(1+\kappa)\omega_{ttt} + \beta\omega_{tt} = 0, \tag{8.78}$$

and $\theta_{tt} + \beta\psi_t + \gamma(\theta - \psi) + \kappa\psi_{tt} = 0$.
Solving equation (8.78) we get,

$$\omega = c_6 t + c_7 - \frac{1+\kappa}{2\beta}A, \tag{8.79}$$

where c_4, c_5 are arbitrary constants and $c_6 = \dfrac{c_4}{\beta}, c_7 = \dfrac{c_5}{\beta}$. From equation (8.79),

$$A(t) = c_8 t + c_9 - \frac{2\beta}{A}\omega, \tag{8.80}$$

where $c_8 = \dfrac{2c_1}{1+\kappa}$, $\quad c_9 = \dfrac{2c_2}{1+\kappa}$. This yields,

$$\Upsilon = \left(c_8 t + c_9 - \frac{2\beta}{A}\omega\right) x + \theta(t), \tag{8.81}$$

and,

$$\Upsilon^r = \left(c_8 t + c_9 - \frac{2\beta}{A}\omega\right) x + \psi(t-r). \tag{8.82}$$

The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= \left(c_6 t + c_7 - \frac{1+\kappa}{2\beta}A\right)\frac{\partial}{\partial t} + \left[\left(c_8 t + c_9 - \frac{2\beta}{A}\omega\right)x + \theta\right]\frac{\partial}{\partial x}\\ &\quad + \left[\left(c_8 t + c_9 - \frac{2\beta}{A}\omega\right)x + \psi\right]\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by $S_1 = t\dfrac{\partial}{\partial t}$, $\quad S_2 = \dfrac{\partial}{\partial t}$,
$S_3 = tx\left[\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial u}\right]$, $\quad S_4 = x\left[\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial u}\right]$ with
$S_5 = -\dfrac{1+\kappa}{2\beta}A\dfrac{\partial}{\partial t} + \left[\theta - \dfrac{2\beta\omega x}{1+\kappa}\right]\dfrac{\partial}{\partial x} + \left[\psi - \dfrac{2\beta x\omega}{1+\kappa}\right]\dfrac{\partial}{\partial u}$ as the infinite dimensional Lie sub-algebra.

The commutator table is given by,

	S_1	S_2	S_3	S_4
S_1	0	$-S_2$	S_3	0
S_2	S_2	0	S_4	0
S_3	$-S_3$	$-S_4$	0	0
S_4	0	0	0	0

.

Then $L = \{S_1, S_2, S_3, S_4\}$ is a solvable Lie algebra. □

Theorem 8.4.3. *The neutral differential equation given by equation (8.57) for which $\beta \neq 0, \gamma = -\rho, \kappa = -1$, admits a three dimensional group generated by*

$$S_1 = t\frac{\partial}{\partial t}, \quad S_2 = \frac{\partial}{\partial t}, \quad S_3 = x\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right],$$

with the infinite dimensional Lie sub-algebra given by

$$S_4^i = \theta\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial u}.$$

Proof. Let $\beta, \gamma, \rho, \kappa$ be arbitrary non-zero constants, $\gamma = -\rho, \kappa = -1$. Then equation (8.60) becomes $\Upsilon_{tt} + \beta\Upsilon_t^r + \gamma(\Upsilon - \Upsilon^r) - \Upsilon_{tt}^r = 0$, which yields,

$$A_{tt} = 0, \tag{8.83}$$

and $\theta_{tt} + \beta\psi_t + \gamma(\theta - \psi) - \psi_{tt} = 0$.
Solving equation (8.83) we get,

$$\omega = c_{10}t + c_{11}, \tag{8.84}$$

where c_{10}, c_{11} are arbitrary constants. From equation (8.84),

$$A(t) = c_{12}, \tag{8.85}$$

where $c_{12} = 2c_{10}$.
This yields,

$$\Upsilon = c_{12}x + \theta(t), \tag{8.86}$$

and,

$$\Upsilon^r = c_{12}x + \psi. \tag{8.87}$$

The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= (c_{10}t + c_{11})\frac{\partial}{\partial t} + (c_{12}x + \theta)\frac{\partial}{\partial x} + (c_{12}x + \psi)\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by
$S_1 = t\frac{\partial}{\partial t}$, $S_2 = \frac{\partial}{\partial t}$, $S_3 = x\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right]$ with
$S_4 = \theta\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial u}$ as the infinite dimensional Lie sub-algebra.

The commutator table is given by

	S_1	S_2	S_3
S_1	0	$-S_2$	0
S_2	S_2	0	0
S_3	0	0	0

.

Then $L = \{S_1, S_2, S_3\}$ is a solvable Lie algebra. □

Theorem 8.4.4. *The neutral differential equation given by equation (8.57) for which* $\beta \neq 0, \gamma = 0 = \rho, \kappa \neq 0$, *admits a five dimensional group generated by*

$$S_1 = t\frac{\partial}{\partial t}, \quad S_2 = \frac{\partial}{\partial t}, \quad S_3 = tx\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right],$$

$$S_4 = x\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right], \quad S_5 = \frac{\partial}{\partial u},$$

with the infinite dimensional Lie sub-algebra given by

$$S_6^i = -\frac{1+\kappa}{2\beta}A\frac{\partial}{\partial t} + \left[\theta - \frac{2\beta\omega x}{1+\kappa}\right]\frac{\partial}{\partial x} - \left[\frac{\kappa}{\beta}\psi_t + \frac{1}{\beta} + \frac{2\beta\omega}{1+\kappa}\right]\frac{\partial}{\partial u}.$$

Proof. Let β, κ be arbitrary non-zero constants, $\gamma = 0 = \rho$.
Then equation (8.60) becomes $\Upsilon_{tt} + \beta\Upsilon_t^r + \kappa\Upsilon_{tt}^r = 0$, which yields,

$$(1+\kappa)A_{tt} + \beta A_t = 0, \tag{8.88}$$

and $\psi = c_{13} - \frac{\kappa}{\beta}\psi_t - \frac{1}{\beta}\theta_t$, where c_{13} is an arbitrary constant.
Solving equation (8.88) we get,

$$\omega = c_{16}t + c_{17} - \frac{1+\kappa}{2\beta}A, \tag{8.89}$$

where c_{14}, c_{15} are arbitrary constants and $c_{16} = \frac{c_{14}}{\beta}, \quad c_{17} = \frac{c_{15}}{\beta}$. From equation (8.89),

$$A(t) = c_{18}t + c_{19} - \frac{2\beta}{1+\kappa}\omega, \tag{8.90}$$

where $c_{18} = \frac{2c_{16}\beta}{1+\kappa}$ and $c_{19} = \frac{2c_{17}\beta}{1+\kappa}$. This yields,

$$\Upsilon = \left(c_{18}t + c_{19} - \frac{2\beta}{1+\kappa}\omega\right)x + \theta, \tag{8.91}$$

and,

$$\Upsilon^r = \left(c_{18}t + c_{19} - \frac{2\beta}{1+\kappa}\omega\right)x + c_{13} - \frac{\kappa}{\beta}\psi_t - \frac{1}{\beta}\theta_t. \tag{8.92}$$

The infinitesimal generator is given by

$$\begin{aligned}
\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r} \\
&= \left(c_{16}t + c_{17} - \frac{1+\kappa}{2\beta}A\right)\frac{\partial}{\partial t} + \left[\left(c_{18}t + c_{19} - \frac{2\beta}{1+\kappa}\omega\right)x + \theta\right]\frac{\partial}{\partial x} \\
&\quad + \left[\left(c_{18}t + c_{19} - \frac{2\beta}{1+\kappa}\omega\right)x + c_{13} - \frac{\kappa}{\beta}\psi_t - \frac{1}{\beta}\theta_t\right]\frac{\partial}{\partial x^r}.
\end{aligned}$$

The Lie algebra is spanned by $S_1 = t\dfrac{\partial}{\partial t}$, $\quad S_2 = \dfrac{\partial}{\partial t}$,
$S_3 = tx\left[\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial u}\right]$, $\quad S_4 = x\left[\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial u}\right]$, $\quad S_5 = \dfrac{\partial}{\partial u}$ with

$$S_6 = -\frac{1+\kappa}{2\beta}A\frac{\partial}{\partial t} + \left[\theta - \frac{2\beta\omega x}{1+\kappa}\right]\frac{\partial}{\partial x} - \left[\frac{\kappa}{\beta}\psi_t + \frac{1}{\beta} + \frac{2\beta\omega}{1+\kappa}\right]\frac{\partial}{\partial u}$$

as the infinite dimensional Lie sub-algebra.

The commutator table is given by

	S_1	S_2	S_3	S_4	S_5
S_1	0	$-S_2$	S_3	0	0
S_2	S_2	0	S_4	0	0
S_3	$-S_3$	$-S_4$	0	0	0
S_4	0	0	0	0	0
S_5	0	0	0	0	0

.

Then $L = \{S_1, S_2, S_3, S_4, S_5\}$ is a solvable Lie algebra. □

Theorem 8.4.5. *The neutral differential equation given by equation (8.57) for which $\beta = 1, \kappa = 1$, admits a three dimensional group generated by*

$$S_1 = t\frac{\partial}{\partial t} + tx\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right], \quad S_2 = \frac{\partial}{\partial t} + x\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right], \quad S_3 = \frac{\partial}{\partial u},$$

with the infinite dimensional Lie sub-algebra given by

$$S_4^i = -A\frac{\partial}{\partial t} + [\theta - \omega x]\frac{\partial}{\partial x} - [\omega x + (\theta_t + \psi_t)]\frac{\partial}{\partial u}.$$

Proof. Let $\beta = 1 = \kappa$, $\gamma = -\rho, \kappa \neq -1$. Then equation (8.60) becomes $\Upsilon_{tt} + \Upsilon_t^r + \Upsilon_{tt}^r = 0$, which yields,

$$2A_{tt} + A_t = 0, \tag{8.93}$$

and $\psi = c_{20} - (\theta_t + \psi_t)$, where c_{20} is an arbitrary constant.
Solving equation (8.93) we get,

$$\omega = c_{21}t + c_{22} - A, \tag{8.94}$$

where c_{21}, c_{22} are arbitrary constants. From equation (8.94),

$$A(t) = c_{21}t + c_{22} - \omega. \tag{8.95}$$

This yields,

$$\Upsilon = (c_{21}t + c_{22} - \omega)x + \theta, \tag{8.96}$$

and,

$$\Upsilon^r = (c_{21}t + c_{22} - \omega)x + c_{20} - (\theta_t + \psi_t). \tag{8.97}$$

The infinitesimal generator is given by

$$\begin{aligned}\zeta^* &= \omega\frac{\partial}{\partial t} + \Upsilon\frac{\partial}{\partial x} + \Upsilon^r\frac{\partial}{\partial x^r}\\ &= (c_{21}t + c_{22} - A)\frac{\partial}{\partial t} + [(c_{21}t + c_{22} - \omega)x + \theta]\frac{\partial}{\partial x}\\ &\quad + [(c_{21}t + c_{22} - \omega)x + c_{20} - (\theta_t + \psi_t)]\frac{\partial}{\partial x^r}.\end{aligned}$$

The Lie algebra is spanned by $S_1 = t\frac{\partial}{\partial t} + tx\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right]$,
$S_2 = \frac{\partial}{\partial t} + x\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right]$, $\quad S_3 = \frac{\partial}{\partial u}$ with
$S_4 = -A\frac{\partial}{\partial t} + [\theta - \omega x]\frac{\partial}{\partial x} - [\omega x + (\theta_t + \psi_t)]\frac{\partial}{\partial u}$ as the infinite dimensional Lie sub-algebra.

The commutator table is given by

	S_1	S_2	S_3
S_1	0	$-S_2$	0
S_2	S_2	0	0
S_3	0	0	0

.

Then $L = \{S_1, S_2, S_3\}$ is a solvable Lie algebra. □

8.4.2 A Nonlinear Case

We make a classification of

$$x''(t) + x''(t - r) + x'(t - r) + x'(t)x(t) = 0. \tag{8.98}$$

Applying the operator defined by equation (8.57), to the delay equation $g(t) = t - r$, we get equation $\omega(t, x) = \omega(t - r, x(t - r))$.

Applying the operator defined by equation (8.57), to equation (8.98), we get,

$$\Upsilon_{tt}+(2\Upsilon_{tx}-\omega_{tt})x'+(\Upsilon_{xx}-2\omega_{tx})x'^2-\omega_{xxx}x'^3+(\Upsilon_x-2\omega_t)x''-3\omega_x x'x''+\Upsilon^r_{tt}+(2\Upsilon^r_{tx}-\omega^r_{tt})x^{r\prime}$$
$$+(\Upsilon^r_{xx}-2\omega^r_{tx})x^{r\prime 2}-\omega^r_{xx}x^{r\prime 3}+(\Upsilon^r_x-2\omega^r_t)x^{r\prime\prime}-3\omega^r_x x^{r\prime}x^{r\prime\prime}+\Upsilon^r_t+(\Upsilon^r_x-\omega^r_t)x^{r\prime}-\omega^r_x x^{r\prime 2}$$
$$+x'\Upsilon+x[\Upsilon_t+(\Upsilon_x-\omega_t)x'-\omega_x x'^2]=0. \quad (8.99)$$

Splitting equation (8.99) with respect to constant term, x', x'^2, x'^3, x'', $x'x''$, $x^{r\prime}$, $x^{r\prime 2}$, $x^{r\prime 3}$, $x^{r\prime\prime}$ and $x^{r\prime}x^{r\prime\prime}$ respectively, we get,

$$\Upsilon_{tt}+\Upsilon^r_{tt}+\Upsilon^r_t+x\Upsilon_t=0, \quad (8.100)$$

$$2\Upsilon_{tx}-\omega_{tt}+\Upsilon=0, \quad (8.101)$$

$$\Upsilon_{xx}-2\omega_{tx}=0, \quad (8.102)$$

$$\omega_{xx}=0, \quad (8.103)$$

$$\Upsilon_x-2\omega_t=0, \quad (8.104)$$

$$\omega_x=0, \quad (8.105)$$

$$2\Upsilon^r_{tx}-\omega^r_{tt}+\Upsilon^r_x-\omega^r_t=0, \quad (8.106)$$

$$\Upsilon^r_{xx}-2\omega^r_{tx}-\omega^r_x=0, \quad (8.107)$$

$$\omega^r_{xx}=0, \quad (8.108)$$

$$\Upsilon^r_x-2\omega^r_t=0, \quad (8.109)$$

$$\omega^r_x=0. \quad (8.110)$$

From these equations we get, $\omega=\omega(t)$, $\quad \Upsilon=A(t)x+\theta(t)$, $\quad \Upsilon^r=\frac{1}{2}A(t)x+\psi(t-r)$, where $A(t)=2\omega_t$.

Substituting the values of Υ,Υ^r in equation (8.100) and solving it, we get,
$A(t)=c_{23}$, $\quad \theta=c_{24}$, $\quad \psi=c_{25}-\psi_t$, and,

$$\omega=c_{26}t+c_{27}, \quad (8.111)$$

$$\Upsilon=c_{23}x+c_{24}, \quad \Upsilon^r=c_{23}x+c_{25}-\psi_t, \quad (8.112)$$

where $c_{23},c_{24},c_{25},c_{27}$ are arbitrary constants and $c_{26}=\frac{c_{23}}{2}$.

The infinitesimal generator is given by

$$\zeta^*=\omega\frac{\partial}{\partial t}+\Upsilon\frac{\partial}{\partial x}+\Upsilon^r\frac{\partial}{\partial x^r}$$
$$=(c_{26}t+c_{27})\frac{\partial}{\partial t}+(c_{23}x+c_{24})\frac{\partial}{\partial x}+(c_{23}x+c_{25}-\psi_t)\frac{\partial}{\partial x^r}.$$

The Lie algebra is spanned by

$$S_1 = t\frac{\partial}{\partial t}, \quad S_2 = \frac{\partial}{\partial t}, \quad S_3 = x\Big[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\Big],$$

$$S_4 = \frac{\partial}{\partial x}, \quad S_5 = \frac{\partial}{\partial u}$$ with $S_6 = -\psi_t \frac{\partial}{\partial u}$ as the infinite dimensional Lie sub-algebra.

The commutator table is given by

	S_1	S_2	S_3	S_4	S_5
S_1	0	$-S_2$	0	0	0
S_2	S_2	0	0	0	0
S_3	0	0	0	$-S_4 - S_5$	0
S_4	0	0	$S_4 + S_5$	0	0
S_5	0	0	0	0	0

Then $L = \{S_1, S_2, S_3, S_4, S_5\}$ is a solvable Lie algebra.

Remark 8.4.1. For the non-homogeneous nonlinear second order neutral differential equation $x''(t) + x''(t-r) + x'(t-r) + x'(t)x(t) = h(t)$, we get exactly the same generators as in the homogeneous case, only that $S_6 = (y - \psi_t)\frac{\partial}{\partial x^r}$ is the corresponding infinite dimensional Lie sub-algebra, where $y = c_{26} \int t\, h' dt + c_{27} h$.

8.5 Summary

With the notation L_n^m, where m denotes the dimension of the solvable Lie algebra and S^i to mean the infinite dimensional Lie sub-algebra, the entire classification of second order functional differential equations with constant coefficients to solvable Lie algebras is summarized in Table 8.1 and Table 8.2 below:

Table 8.1: Group Classification of Second Order Functional Differential Equations

Type of Functional Differential Equation	Basis for the Lie Algebra	Solvable Lie algebra
$x''(t)+\beta x'(t-r)+\gamma x(t)+\rho x(t-r)=0$, $\gamma \neq \frac{\rho}{2}$.	$S_1=\frac{\partial}{\partial t}, \quad S_2=x\frac{\partial}{\partial x}, \quad S_3 x\frac{\partial}{\partial u},$ $S_4^i=-\left(\frac{1}{2(\gamma+\frac{\rho}{2})}+\frac{4(\gamma+\frac{\rho}{2})}{\beta}A\right)\frac{\partial}{\partial t} + \left[\theta - x\left(\frac{2}{\beta}A_t+\frac{4(\gamma+\frac{\rho}{2})}{\beta}\omega\right)\right]\frac{\partial}{\partial x} + \left[\psi - x\left(\frac{1}{\beta}A_t+\frac{2(\gamma+\frac{\rho}{2})}{\beta}\omega\right)\right]\frac{\partial}{\partial u}.$	L_1^3
$x''(t)+\beta x'(t-r)+\frac{\rho}{2}x(t)+\rho x(t-r)=0$.	$S_1=t\frac{\partial}{\partial t}, \quad S_2=\frac{\partial}{\partial t},$ $S_3=tx\left[\frac{\partial}{\partial x}+\frac{1}{2}\frac{\partial}{\partial u}\right],$ $S_4=x\left[\frac{\partial}{\partial x}+\frac{1}{2}\frac{\partial}{\partial u}\right],$ $S_5^i=-\frac{A}{\beta}\frac{\partial}{\partial t}+(\theta-\beta x\omega)\frac{\partial}{\partial x}+(\psi-\frac{\beta x\omega}{4})\frac{\partial}{\partial u}.$	L_2^4
$x''(t)+x'(t-r)=0$.	$S_1=t\frac{\partial}{\partial t}+tx\left[\frac{\partial}{\partial x}+\frac{1}{2}\frac{\partial}{\partial u}\right],$ $S_2=\frac{\partial}{\partial t}+x\left[\frac{\partial}{\partial x}+\frac{1}{2}\frac{\partial}{\partial u}\right], \quad S_3=\frac{\partial}{\partial u},$ $S_4^i=-A\frac{\partial}{\partial t}+(\theta-x\omega)\frac{\partial}{\partial x}-\left(\theta_t+\frac{x\omega}{2}\right)\frac{\partial}{\partial u}.$	L_3^3
$x''(t)+\beta x'(t-r)=0$.	$S_1=t\frac{\partial}{\partial t}, \; S_2=\frac{\partial}{\partial t},$ $S_3=tx\left[\frac{\partial}{\partial x}+\frac{1}{2}\frac{\partial}{\partial u}\right],$ $S_4=x\left[\frac{\partial}{\partial x}+\frac{1}{2}\frac{\partial}{\partial u}\right], \quad S_5=\frac{\partial}{\partial u},$ $S_6^i=-\frac{A}{\beta}\frac{\partial}{\partial t}+(\theta-\beta x\omega)\frac{\partial}{\partial x}+\left(\frac{\theta_t}{\beta}+\beta x\omega\right)\frac{\partial}{\partial u}.$	L_4^5
$x''(t)+x'(t)+x'(t-r)x(t)=v(t)$.	$S_1=x\frac{\partial}{\partial x}, \quad S_2=\frac{\partial}{\partial x},$ $S_3=t\frac{\partial}{\partial t}, \quad S_4=\frac{\partial}{\partial t},$ $S_5=x\frac{\partial}{\partial u}, \quad S_6=\frac{\partial}{\partial u},$ $S_7^i=(\theta_t-y)\frac{\partial}{\partial x}.$	L_5^6

Table 8.2: Group Classification of Second Order Functional Differential Equations

Type of Functional Differential Equation	Basis for the Lie Algebra	Solvable Lie algebra
$x''(t)+\beta x'(t-r)+\gamma x(t)+\rho x(t-r)+\kappa x''(t-r)=0$, $\gamma\neq-\rho$.	$S_1=\frac{\partial}{\partial t},\quad S_2=x\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right),$ $S_3^i=-\left(\frac{1+\kappa}{2(\gamma+\rho)}g_t+\frac{\beta}{2(\gamma+\rho)}A\right)\frac{\partial}{\partial t}+\left[\theta-x\left(\frac{1+\kappa}{\beta}A_t+\frac{2(\gamma+\rho)}{\beta}\omega\right)\right]\frac{\partial}{\partial x}+\left[\psi-x\left(\frac{1+\kappa}{\beta}A_t+\frac{2(\gamma+\rho)}{\beta}\omega\right)\right]\frac{\partial}{\partial u}.$	L_6^2
$x''(t)+\beta x'(t-r)+\gamma(x(t)-x(t-r))+\kappa x''(t-r)=0$, $\kappa\neq-1$.	$S_1=t\frac{\partial}{\partial t},\quad S_2=\frac{\partial}{\partial t},$ $S_3=tx\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right],\ S_4=x\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right],$ $S_5^i=-\frac{1+\kappa}{2\beta}A\frac{\partial}{\partial t}+\left[\theta-\frac{2\beta\omega x}{1+\kappa}\right]\frac{\partial}{\partial x}+\left[\psi-\frac{2\beta x\omega}{1+\kappa}\right]\frac{\partial}{\partial u}.$	L_7^4
$x''(t)+\beta x'(t-r)+\gamma(x(t)-x(t-r))-x''(t-r)=0$.	$S_1=t\frac{\partial}{\partial t},\ S_2=\frac{\partial}{\partial t},$ $S_3=x\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right],$ $S_4^i=\theta\frac{\partial}{\partial x}+\psi\frac{\partial}{\partial u}.$	L_8^3
$x''(t)+\beta x'(t-r)+\kappa x''(t-r)=0$.	$S_1=t\frac{\partial}{\partial t},\ S_2=\frac{\partial}{\partial t},$ $S_3=tx\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right],$ $S_4=x\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right],\quad S_5=\frac{\partial}{\partial u},$ $S_6^i=-\frac{1+\kappa}{2\beta}A\frac{\partial}{\partial t}+\left[\theta-\frac{2\beta\omega x}{1+\kappa}\right]\frac{\partial}{\partial x}-\left[\frac{\kappa}{\beta}\psi_t+\frac{1}{\beta}+\frac{2\beta\omega}{1+\kappa}\right]\frac{\partial}{\partial u}.$	L_9^5
$x''(t)+x'(t-r)+x''(t-r)=0$.	$S_1=t\frac{\partial}{\partial t}+tx\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right],$ $S_2=\frac{\partial}{\partial t}+x\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right],\quad S_3=\frac{\partial}{\partial u},$ $S_4^i=-A\frac{\partial}{\partial t}+[\theta-\omega x]\frac{\partial}{\partial x}-[\omega x+(\theta_t+\psi_t)]\frac{\partial}{\partial u}.$	L_{10}^3
$x''(t)+x''(t-r)+x'(t-r)+x'(t)x(t)=v(t)$.	$S_1=t\frac{\partial}{\partial t},\quad S_2=\frac{\partial}{\partial t},\quad S_3=\frac{\partial}{\partial x},$ $S_4=x\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right],\quad S_5=\frac{\partial}{\partial u},$ $S_6^i=(y-\psi_t)\frac{\partial}{\partial u}.$	L_{11}^5

Chapter 9

Group Analysis of the Inviscid Burgers' Equation With Delay

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9.1 Introduction

The Burger equation also known as Bateman-Burger equation is a fundamental partial differential equation most commonly occuring in fluid mechanics, nonlinear acoustics, gas dynamics and traffic flow. For a given field $u(t, x)$ and diffusion coefficient ν, the general form of the Burgers' equation is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}.$$

When the diffusion term is absent, the Burgers' equation becomes the Inviscid Burger equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0,$$

which is a prototype for conservation laws that can develop discontinuities called *shock waves*.
Literature on the Burger equation with its applications can be found in [6]. Further periodic wave shock solutions for the Burgers' equation can be found in [4]. Also in [52] we can find numerical solutions of the non-homogeneous and nonlinear one dimensional Burgers' equation. As differential equations with delay cannot easily be solved, our biggest motivation was to apply symmetry methods in finding a representation of analytic solutions to the Burgers' equation with delay. These solutions obtained will be of utmost importance to researchers in fluid mechanics. The group methods indeed help us in studying the properties of solutions of this partial differential equation with delay. The analytic solutions obtained can further be studied to understand their qualitative properties which will help in better modelling of the physical problem at hand.

Not many research papers on symmetry analysis of partial differential equations with delay are available. Symmetry ideas and techniques for solutions of partial differential equations are presented in [62]. In [16] symmetry analysis is used to reduce the number of independent variables of time fractional partial differential equations and obtain some exact solutions. Approximate symmetries of a class of nonlinear reaction-diffusion equations are comprehensively analyzed in [44] and also one-dimensional sub algebras are constructed. Lie symmetry theory is extended to the class of space-time fractional differential equations with a delay in [46] and the admitted symmetries of the time fractional Poisson equation with constant delay have been found. Group analysis to the reaction-diffusion delay differential equation has been applied in [35] and the complete group classification of the reaction-diffusion equation with delay is made. The research interest in group analysis of differential equations is so much that admitted Lie groups for stochastic differential equations have been defined in [56] and stochastic differential equations with multi-Brownian motion have been studied. New exact explicit solutions of (2+1)-dimensional dispersive long wave equations using similarity transformation method are obtained in [55]. The method therein, reduces the dimension of the partial differential equations by one. Further, very recently in [10] Lie group theoretic method is used to carry out the similarity reduction and solitary wave solutions of (2+1)-dimensional Date–Jimbo–Kashiwara–Miwa equation. The infinitesimal generators for the governing equation have been obtained therein. These papers add to the motivation for applying symmetry analysis to differential equations with delay as many systems are accurately modelled by delay differential equations and finding analytic solutions to them become of paramount importance. The existing computational technique to obtain symmetries of delay differential equations are based on the Lie-Bäcklund operator and result in magnification in the delay term when obtaining the determining equations. No such magnification appears while using our technique obtained from Taylor's theorem for a function of several variables.

In this chapter, we perform group analysis of the Inviscid Burgers' type equation with delay, which is of the form,

$$\frac{\partial u}{\partial t}(t,x) + u(t,x)\frac{\partial u}{\partial x}(t,x) = G(u(t-r,x)), \tag{9.1}$$

where u is a real valued function defined on $I \times D$, and where I is an open interval in $\mathbb{R}$ and D is an open set in $\mathbb{R}$.
Equation (9.1) under study is a nonlinear first-order partial differential equation with delay and an arbitrary differentiable functional G.
We have used Taylor's theorem for a function of several variables to obtain a Lie type invariance condition for first-order partial differential equations with delay. We have studied the Inviscid Burgers' type equation with delay and an arbitrary differentiable functional and have obtained its symmetries and made a group classification. Further, we have found the kernel and extensions of the kernel to classify (9.1) with respect to its symmetries for an arbitrary and the special case for its functional G. We have obtained a representation of solutions from the invariants and used these representations to reduce the equations to ordinary functional differential equations.

9.2 Lie Type Invariance Condition for First Order Partial Differential Equations With Delay

Let $u = u(t,x)$. Then we consider transformations of the form,

$$\bar{t} = f_1(t,x,u;\delta), \quad \bar{x} = f_2(t,x,u;\delta), \quad \bar{u} = f_3(t,x,u;\delta),$$

where f_1, f_2, f_3 are smooth functions of t, x, u having a convergent Taylor series in δ. Defining,

$$T(t,x,u) = \frac{\partial f_1}{\partial \delta}\bigg|_{\delta=0}, \quad X(t,x,u) = \frac{\partial f_2}{\partial \delta}\bigg|_{\delta=0}, \quad U(t,x,u) = \frac{\partial f_3}{\partial \delta}\bigg|_{\delta=0},$$

we can write the transformations as,

$$\begin{cases} \bar{t} = t + \delta T(t,x,u) + O(\delta^2), \\ \bar{x} = x + \delta X(t,x,u) + O(\delta^2), \\ \bar{u} = u + \delta U(t,x,u) + O(\delta^2). \end{cases}$$

We establish the following Lie type invariance condition for first-order partial differential equations with delay using Taylor's theorem for a function of several variables:

Theorem 9.2.1. *Consider the first-order partial differential equation with delay*

$$F\left(t, t-r, x, u, u(t-r,x), \frac{\partial u}{\partial t}(t,x), \frac{\partial u}{\partial x}(t,x), \frac{\partial u}{\partial x}(t-r,x)\right) = 0, \tag{9.2}$$

where F is defined on a 8-dimensional space $I \times I - r \times D^6$, where D is an open set in $\mathbb{R}$, I is any interval in $\mathbb{R}$, and $I - r = \{y - r : y \in I\}$. Then the Lie type invariance condition is given by

$$TF_t + T^r F_{t^r} + XF_x + UF_u + U^r F_{u^r} + U_{[t]} F_{u_t} + U_{[x]} F_{u_x} + U^r_{[x]} F_{u^r_x} = 0,$$

where $T^r = T(t-r, x, u(t-r,x)), U^r = U(t-r, x, u(t-r,x))$ and the total differential operators given by,

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots,$$

and,

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + \cdots.$$

The extended infinitesimals are given by,

$$U_{[t]} = D_t(U) - u_t D_t(T) - u_x D_t(X),$$

$$U_{[x]} = D_x(U) - u_t D_x(T) - u_x D_x(X),$$

$$U^r_{[x]} = U^r_x + u^r_x U^r_{u^r} - u^r_{t^r}(T^r_x + u^r_x T^r_{u^r}) - u^r_x (X^r_x + u^r_x X^r_{u^r}).$$

Proof. We seek the invariance of equation (9.2) under Lie group of infinitesimal transformations given by,

$$\bar{t} = t + \delta T(t, x, u) + O(\delta^2),$$

$$\bar{x} = x + \delta X(t, x, u) + O(\delta^2),$$

$$\bar{u} = u + \delta U(t, x, u) + O(\delta^2).$$

Then, it naturally follows that,

$$\overline{t-r} = t - r + \delta T(t-r, x, u(t-r, x)) + O(\delta^2),$$

$$\bar{u}(\overline{t-r}, \bar{x}) = u(t-r, x) + \delta U(t-r, x, u(t-r, x)) + O(\delta^2).$$

Let $T(t-r, x, u(t-r, x)) = T^r, \quad X(t-r, x, u(t-r, x)) = X^r, \quad U(t-r, x, u(t-r, x)) = U^r$. As the partial differential equation given by equation (9.2) contains first-order derivatives $\dfrac{\partial u}{\partial t}(t,x) = u_t(t,x)$ and $\dfrac{\partial u}{\partial x}(t,x) = u_x(t,x)$, it is necessary to obtain extended transformations for these.

In analogy with ordinary differential equations, we define the extended transformations $\bar{u}_{\bar{t}}$ and $\bar{u}_{\bar{x}}$ as,

$$\bar{u}_{\bar{t}} = u_t + \delta U_{[t]} + O(\delta^2),$$

$$\bar{u}_{\bar{x}} = u_x + \delta U_{[x]} + O(\delta^2).$$

We introduce the total differential operators D_t and D_x where,

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots,$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + \cdots.$$

We can write equation (9.2) as $F(t, t-r, x, u, u^r, u_t, u_x, u^r_x) = 0$ where $u^r = u(t-r, x)$.

We shall now construct the extended transformations for u_t and u_x as follows,

$$\begin{aligned}
\bar{u}_{\bar{t}} &= \frac{\partial(\bar{u},\bar{x})}{\partial(\bar{t},\bar{x})} = \frac{\partial(\bar{u},\bar{x})}{\partial(t,x)} \Bigg/ \frac{\partial(\bar{t},\bar{x})}{\partial(t,x)} \\
&= \begin{vmatrix} \bar{u}_t & \bar{u}_x \\ \bar{x}_t & \bar{x}_x \end{vmatrix} \Bigg/ \begin{vmatrix} \bar{t}_t & \bar{t}_x \\ \bar{x}_t & \bar{x}_x \end{vmatrix} \\
&= \frac{\begin{vmatrix} u_t + \delta D_t(U) + O(\delta^2) & u_x + \delta D_x(U) + O(\delta^2) \\ \delta D_t(X) + O(\delta^2) & 1 + \delta D_x(X) + O(\delta^2) \end{vmatrix}}{\begin{vmatrix} 1 + \delta D_t(T) + O(\delta^2) & \delta D_x(T) + O(\delta^2) \\ \delta D_t(X) + O(\delta^2) & 1 + \delta D_x(X) + O(\delta^2) \end{vmatrix}} \\
&= \frac{u_t + \delta[D_t(U) + u_t D_x(X) - u_x D_t(X)] + O(\delta^2)}{1 + \delta[D_t(T) + D_x(X)] + O(\delta^2)} \\
&= u_t + \delta[D_t(U) - u_t D_t(T) - u_x D_t(X)] + O(\delta^2).
\end{aligned}$$

Thus we define,

$$U_{[t]} = D_t(U) - u_t D_t(T) - u_x D_t(X),$$

$$U_{[x]} = D_x(U) - u_t D_x(T) - u_x D_x(X).$$

Then,

$$\bar{u}(\overline{t-r},\bar{x}) = u_x(t-r,x) + \delta U_{[x]}(t-r,x,u(t-r,x)) + O(\delta^2).$$

Now,

$$U_{[t]} = U_t + u_t U_u - u_t(T_t + u_t T_u) - u_x(X_t + u_t X_u), \tag{9.3}$$

and,

$$U_{[x]} = U_x + u_x U_u - u_t(T_x + u_x T_u) - u_x(X_x + u_x X_u). \tag{9.4}$$

$$\begin{aligned}
U^r_{[x]} &= U_x(t-r,x,u(t-r,x)) \\
&\quad + u_x(t-r,x)U_{u(t-r,x)}(t-r,x,u(t-r,x,u(t-r,x))) \\
&\quad - u_{t-r}(t-r,x)(T_x(t-r,x,u(t-r,x)) \\
&\quad + u_x(t-r,x)T_{u(t-r,x)}(t-r,x,u(t-r,x))) \\
&\quad - u_x(t-r,x)(X_x(t-r,x,u(t-r,x)) \\
&\quad + u_x X_{u(t-r,x)}(t-r,x,u(t-r,x))).
\end{aligned}$$

Then,

$$U^r_{[x]} = U^r_x + u^r_x U^r_{u^r} - u^r_{t^r}(T^r_x + u^r_x T^r_{u^r}) - u^r_x(X^r_x + u^r_x X^r_{u^r}).$$

Then, it follows that,

$$\bar{u}_{\bar{x}}(\overline{t-r}, \bar{x}) = u_x(t-r, x) + \delta U^r_{[x]} + O(\delta^2).$$

For invariance, we need to have,

$$\begin{aligned}
0 &= F(\bar{t}, \overline{t-r}, \bar{x}, \bar{u}, \bar{u}^r, \bar{u}_{\bar{t}}, \bar{u}_{\bar{x}}, \bar{u}^r_{\bar{x}}) \\
&= F(t + \delta T + O(\delta^2), t - r + \delta T^r + O(\delta^2), x + \delta X + O(\delta^2), \\
&\quad u + \delta U + O(\delta^2), u^r + \delta U^r + O(\delta^2), u_t + \delta U_{[t]} + O(\delta^2), \\
&\quad u_x + \delta U_{[x]} + O(\delta^2), u^r_x + \delta U^r_{[x]} + O(\delta^2)) \\
&= F(t, t-r, x, u, u^r, u_t, u_x, u^r_x) + \delta(TF_t + T^r F_{t^r} + XF_x + UF_u + U^r F_{u^r} \\
&\quad + U_{[t]} F_{u_t} + U_{[x]} F_{u_x} + U^r_{[x]} F_{u^r_x}) + O(\delta^2).
\end{aligned}$$

Equating the coefficient of δ, we get,

$$TF_t + T^r F_{t^r} + XF_x + UF_u + U^r F_{u^r} + U_{[t]} F_{u_t} + U_{[x]} F_{u_x} + U^r_{[x]} F_{u^r_x} = 0.$$

□

The infinitesimal generator of the admitted group for the equation given by (9.2) is,

$$\zeta^* = T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + U\frac{\partial}{\partial u}.$$

The system of characteristics for this is

$$\frac{dt}{T} = \frac{dx}{X} = \frac{du}{U}. \tag{9.5}$$

The first extension is given by,

$$\zeta^{(1)} = T\frac{\partial}{\partial t} + T^r\frac{\partial}{\partial t^r} + X\frac{\partial}{\partial x} + U\frac{\partial}{\partial u} + U^r\frac{\partial}{\partial u^r} + U_{[t]}\frac{\partial}{\partial u_t} + U_{[x]}\frac{\partial}{\partial u_x} + U^r_{[x]}\frac{\partial}{\partial u^r_x}. \tag{9.6}$$

The Lie type invariance condition is given by $\zeta^{(1)}\Delta \mid_{\Delta=0} = 0$, where
$\Delta = F(t, t-r, x, u, u^r, u_t, u_x, u^r_x) = 0$.

9.3 Symmetries of the Inviscid Burgers' Equation With Delay

In this section, we shall obtain symmetries of the Invisid Burgers' equation with delay given by equation (9.1). We can rewrite equation (9.1) as $u_t + uu_x = G(u^r)$. Here, $\Delta = u_t + uu_x - G(u^r) = 0$.
We establish the following result:

Theorem 9.3.1. *The time-delayed Inviscid Burgers' equation given by equation (9.1) admits the three dimensional Lie group generated by*

$$\zeta_1^* = \frac{\partial}{\partial t}, \quad \zeta_2^* = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \quad \zeta_3^* = \frac{\partial}{\partial x}.$$

Proof. Applying the first extension $\zeta^{(1)}$ to the delay condition $h(t) = t - r$, we get, $T(t,x,u) = T^r$.
Applying the Lie type invariance condition given by equation (9.6) to equation (9.1), we get,

$$u_x U - G'(u^r)U^r + U_{[t]} + uU_{[x]} = 0.$$

Using equations (9.3) and (9.4), we get the determining equation,

$$\begin{aligned} u_x U - G'(u^r)U^r + U_t + u_t U_u - u_t T_t - u_t^2 T_u - u_x X_t - u_x u_t X_u \\ + u(U_x + u_x U_u - u_t T_x - u_x u_t T_u - u_x X_x - u_x^2 X_u) = 0. \end{aligned} \quad (9.7)$$

Since u_x, uu_x, uu_t and u_t are independent variables, we can split (9.7) with respect to these variables.
Splitting equation (9.7) with respect to u_x^2 we get $X_u = 0$ which can be solved to give,

$$X(t,x,u) = A(t,x). \quad (9.8)$$

Splitting equation (9.7) with respect to u_t^2 we get $T_u = 0$ which can be solved to give,

$$T(t,x,u) = B(t,x). \quad (9.9)$$

Splitting equation (9.7) with respect to uu_x and solving we get $U_u = A_x(t,x)$ which can be solved to give,

$$U(t,x,u) = A_x(t,x)u + C(t,x). \quad (9.10)$$

Substituting equations (9.8), (9.9) and (9.10) in equation (9.7), we get,

$$\begin{aligned} &2A_x(t,x)G(u^r) - A_x(t^r,x)u^r G'(u^r) + u^2 A_{xx}(t,x) + u_x C(t,x) + uC_x(t,x) \\ &+ uA_{xt}(t,x) - u_x A_t(t,x) + C_t(t,x) - u_t B_t(t,x) - uu_t B_x(t,x) - G'(u^r)C(t^r,x) = 0. \end{aligned} \quad (9.11)$$

Equating the coefficient of u^2 to 0, we get $A_{xx}(t,x) = 0$, which can be solved to give,

$$A(t,x) = a_1 tx + b_1(t). \quad (9.12)$$

Equating the coefficient of u to 0, we get,

$$C_x(t,x) + A_{xt}(t,x) - u_t B_x(t,x) = 0. \quad (9.13)$$

Splitting equation (9.13) with respect to u_t, we get, $B_x(t,x) = 0$, which can be solved to give,

$$B = B(t). \tag{9.14}$$

Using equations (9.12) and (9.14), equation (9.13) gives,

$$C(t,x) = -a_1'(t)x + a_3(t). \tag{9.15}$$

Splitting equation (9.11) with respect to u_t, we get,

$$B(t,x) = c_1, \tag{9.16}$$

where c_1 is an arbitrary constant.
Splitting equation (9.11) with respect to u_x, we get,

$$C(t,x) = A_t(t,x). \tag{9.17}$$

Using equations (9.12) and (9.15), equation (9.17) can be solved to give, $a_1(t) = c_2$, where c_2 is a constant, and $b_1'(t) = a_3(t)$.
Hence equation (9.15) gives,

$$C(t,x) = a_3(t). \tag{9.18}$$

So far we have obtained,

$$A(t,x) = a_1 x + b_1(t), \quad B(t,x) = c_1, \quad C(t,x) = a_3(t).$$

With these values of A, B, C equation (9.11), simplifies to

$$c_2[u^r G'(u^r) - 2G(u^r)] = 0, \tag{9.19}$$

and,

$$a_3'(t) = G'(u^r)a_3(t^r). \tag{9.20}$$

Since equation (9.20) is true for any functional G, we conclude that $a_3(t) = 0$. Consequently, $b_1(t) = c_3$, where c_3 is an arbitrary constant.
Hence we get the coefficients of the infinitesimal transformation as

$$T(t,x,u) = c_1, \quad X(t,x,u) = c_2 x + c_3, \quad U(t,x,u) = c_2 u.$$

Thus, the infinitesimal generator of the admitted Lie group is given by

$$\zeta^* = T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + U\frac{\partial}{\partial u} \tag{9.21}$$

$$= c_1\frac{\partial}{\partial t} + (c_2 x + c_3)\frac{\partial}{\partial x} + c_2 u\frac{\partial}{\partial u}. \tag{9.22}$$

□

9.4 Kernel of the Admitted Generators

Definition 9.4.1. A *kernel* of admitted generators is the set of symmetries, which are admitted for any functional G appearing in the equation.

This implies that for the Inviscid Burgers' equation with delay (9.1) the coefficients of $G'u^r$ and G vanish. Therefore, $c_2 = 0$. Thus,

$$T(t,x,u) = c_1, \quad X(t,x,u) = c_3, \quad U(t,x,u) = 0.$$

Thus, the obtained infinitesimal generator (which is admitted for any functional G) is given by,

$$\zeta^* = c_1\frac{\partial}{\partial t} + c_3\frac{\partial}{\partial x}. \tag{9.23}$$

We have just obtained the following result:

Theorem 9.4.1. *The kernel of the admitted Lie group for the time-delayed Inviscid Burgers' equation (9.1) is two dimensional with generators*

$$\zeta_1^* = \frac{\partial}{\partial t}, \quad \zeta_2^* = \frac{\partial}{\partial x}.$$

To obtain the symmetry derived by this infinitesimal generator, we need to solve,

$$\frac{d\bar{t}}{d\delta} = T(\bar{t},\bar{x},\bar{u}) = c_1, \quad \text{subject to } \bar{t} = t, \quad \text{when} \quad \delta = 0,$$
$$\frac{d\bar{x}}{d\delta} = X(\bar{t},\bar{x},\bar{u}) = c_3, \quad \text{subject to } \bar{x} = x, \quad \text{when} \quad \delta = 0,$$
$$\frac{d\bar{u}}{d\delta} = U(\bar{t},\bar{x},\bar{u}) = 0, \quad \text{subject to } \bar{u} = u, \quad \text{when} \quad \delta = 0.$$

Solving this, we get,

$$\bar{t} = t + \delta c_1, \quad \bar{x} = x + \delta c_3, \quad \bar{u} = u. \tag{9.24}$$

9.5 Extensions of the Kernel

Definition 9.5.1. Extensions are symmetries for the particular functional G only.

In the case of the Inviscid Burgers' equation with delay (9.1), $\exists\, G(u^r)$ satisfying equation (9.19).

Here the extensions of the kernel given by (9.23) will be considered.

Since $T(t,x,u) = c_1, \quad X(t,x,u) = c_3, \quad U(t,x,u) = 0$, are considered in the case of the kernel, the functions for this case are,

$$T(t,x,u) = 0, \quad X(t,x,u) = c_3 x, \quad U(t,x,u) = c_3 u.$$

For the non-trivial case $c_2 \neq 0$, we seek a solution of equation (9.19), which is

$$u^r G'(u^r) = 2G(u^r).$$

This is a separable equation, which can be solved to give,

$$G(u^r) = (u^r)^2 c_4,$$

where c_4 is an arbitrary non-zero constant.
Then the extension of the kernel given by equation (9.23) is

$$\zeta^* = c_2 \left[x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} \right]. \tag{9.25}$$

We have just proved the following result:

Theorem 9.5.1. *The admitted Lie group for the extensions of the kernel of the time-delayed Inviscid Burgers' equation (9.1) is one dimensional with generator*

$$\zeta_1^* = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$$

To obtain the symmetry derived from equation (9.25) we solve,

$\dfrac{d\bar{t}}{d\delta} = T(\bar{t}, \bar{x}, \bar{u}) = 0,$ subject to $\bar{t} = t,$ when $\delta = 0,$

$\dfrac{d\bar{x}}{d\delta} = X(\bar{t}, \bar{x}, \bar{u}) = c_2\bar{x},$ subject to $\bar{x} = x,$ when $\delta = 0,$

$\dfrac{d\bar{u}}{d\delta} = U(\bar{t}, \bar{x}, \bar{u}) = c_2\bar{u},$ subject to $\bar{u} = u,$ when $\delta = 0.$

Solving this, we get the symmetries derived from equation (9.25) which are given by,

$$\bar{t} = t, \quad \bar{x} = xe^{c_2\delta}, \quad \bar{u} = ue^{c_2\delta}. \tag{9.26}$$

9.6 Representations of Analytical Solutions for the Time-Delayed Inviscid Burgers' Equation

In this section, we obtain the representation of invariant solutions for equation (9.1) which is given in our following result:

Theorem 9.6.1. *The representation of invariant solutions for the time-delayed Inviscid Burgers' equation given by equation (9.1), for which*

1. $G(u^r)$ *is arbitrary is* $u(t,x) = \phi_1(c_3 t - c_1 x)$.
2. $G(u^r) = c_4(u^r)^2$ *and for which*

    (a) $c_1 = 0$ *is* $u(t,x) = (x + c_5)\phi_2(t)$.

(b) $c_1 \neq 0$ *is* $u(t,x) = e^{c_6 t}\phi_3((x+c_5)e^{-c_6 t})$,

where ϕ_1, ϕ_2, ϕ_3 *are arbitrary functions.*

Proof. We give a proof as follows:

9.6.1 Representations of Solutions for $\mathbf{u_t(t,x)+u(t,x)u_x(t,x)=G(u(t-r,x))}$ for an Arbitrary Functional G.

The system of characteristics for equation (9.23) is given by

$$\frac{dt}{c_1} = \frac{dx}{c_3} = \frac{du}{0}. \tag{9.27}$$

Solving equation (9.27) we get, $u =$ constant and $c_3 t - c_1 x =$ constant.
Hence the invariants are u and $c_3 t - c_1 x$.
For constructing a representation of solutions, the relation between these two invariants is

$$u(t,x) = \phi_1(c_3 t - c_1 x), \tag{9.28}$$

where ϕ_1 is an arbitrary function.
We call u in equation (9.28) as a *representation of solutions* of equation (9.1) for the infinitesimal generator given by equation (9.23).
At the outset we immediately observe that, if ϕ_1 is bounded, then all solutions $u(t,x)$ are also bounded.
We now discuss the solution for some choices of the function ϕ_1:
Case i Let $\phi_1(.) = e^{(.)}$.
Then, $u(t,x) = e^{c_3 t - c_1 x}$.
We observe that if $c_3 < 0$, then all solutions are positive, decrease with time and eventually tend to zero, as t tends to infinity.
However, if $c_3 > 0$, then all solutions increase exponentially with time and eventually tend to infinity, as t tends to infinity.
Case ii Let $\phi_1(.) = (.)^m$, for any $m \in \mathbb{N}$.
In this case, all solutions tend to infinity as $t \to \infty$.
The graphical representations of the analytical solutions for some choices of ϕ_1 are shown in Figure 9.4. Here t varies over $[-10, 10]$ and x varies over $[-5, 5]$.

9.6.2 Representations of solutions of $\mathbf{u_t(t,x)+u(t,x)u_x(t,x)=G(u(t-r,x))}$ for $G = c_4(u^r)^2$.

The infinitesimal generator for the equation

$$u_t(t,x) + u(t,x)u_x(t,x) = c_4 u^2(t-r,x), \tag{9.29}$$

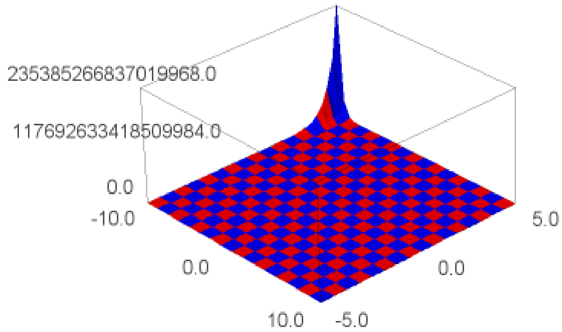

Figure 9.1: $\phi_1(.) = e^{(.)}, c_3 = -3, c_1 = -2.$

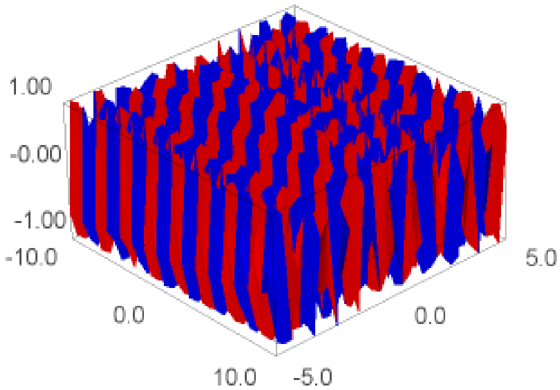

Figure 9.2: $\phi_1(.) = \sin(.), c_3 = 5, c_1 = -7.$

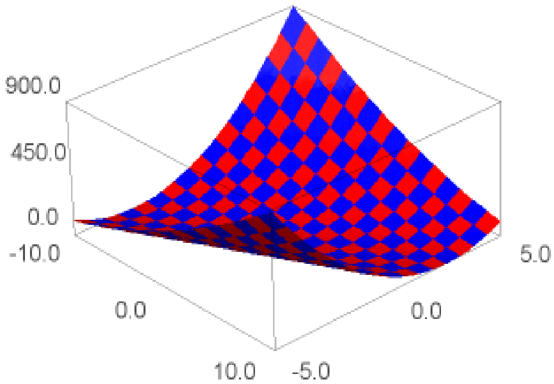

Figure 9.3: $\phi_1(.) = (.)^2, c_3 = 1, c_1 = 4.$

Figure 9.4: Graphical representation of solutions for some choices of ϕ_1.

is a linear combination of the kernel given by equation (9.23) and extension of kernel given by equation (9.25). Thus,

$$\zeta^* = c_1 \frac{\partial}{\partial t} + (c_2 x + c_3) \frac{\partial}{\partial x} + c_2 u \frac{\partial}{\partial u}. \tag{9.30}$$

The system of characteristics for the infinitesimal generator given by equation (9.30) is

$$\frac{dt}{c_1} = \frac{dx}{c_2 x + c_3} = \frac{du}{c_2 u}. \tag{9.31}$$

<u>Case 1:</u> Let $c_1 = 0$.
Then solving the equation relating the first and third term of equation (9.31) we get, $t =$ constant.
Solving the equation relating the second and third term of equation (9.31) we get, $\dfrac{u}{x + c_3/c_2} =$ constant.
In this case, the invariants are t and $\dfrac{u}{x + c_3/c_2}$.
Since c_2 and c_3 are arbitrary constants and $c_2 \neq 0$, we shall denote $c_5 = \dfrac{c_3}{c_2}$.
Hence a representation of solutions of equation (9.29) is

$$\frac{u}{x + c_5} = \phi_2(t).$$

That is a representation of solutions for equation (9.29) in this case, is given by

$$u(t, x) = (x + c_5)\phi_2(t), \tag{9.32}$$

where ϕ_2 is an arbitrary function and c_5 is an arbitrary constant.
We immediately observe that if ϕ_2 is a bounded function, then all solutions $u(t, x)$ are bounded if and only if, x belongs to a bounded set.
We now discuss the solution by taking certain choices for the function ϕ_2:
<u>Case i:</u> Let $\phi_2(t) = e^{\alpha t}, \alpha \in \mathbb{R}$.
In this case, all solutions $u(t, x)$ tend to 0 as $t \to \infty$, provided $\alpha < 0$.
On the other hand, all solutions $u(t, x)$ tend to ∞ as $t \to \infty$, if $\alpha > 0$.
<u>Case ii:</u> Let $\phi_2(t) = t^m, m \in \mathbb{N}$.
In this case, all solutions $u(t, x)$ tend to ∞ as $t \to \infty$.
The graphical representations of the analytical solutions for some choices of ϕ_2 are shown in Figure 9.8. Here t varies over $[-20, 20]$ and x varies over $[-10, 10]$.

<u>Case 2:</u> Let $c_1 \neq 0$.
Then solving the equation relating the first and second term of equation (9.31) we get, $(x + c_5)e^{-c_2 t/c_1} =$ constant.
Solving the equation relating the first and third term of equation (9.31) we get, $ue^{-c_2 t/c_1} =$ constant.

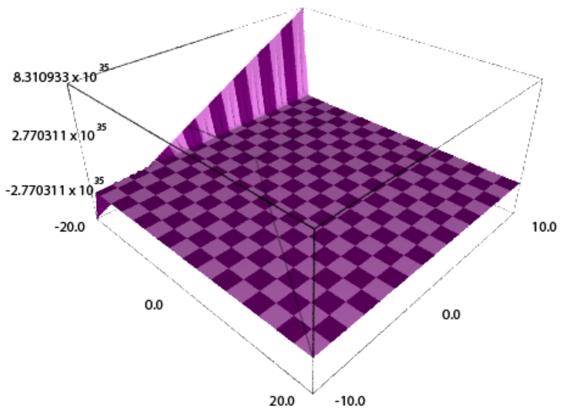

Figure 9.5: $\phi_2(t) = e^{\alpha t}, \alpha = -4, c_5 = 5.$

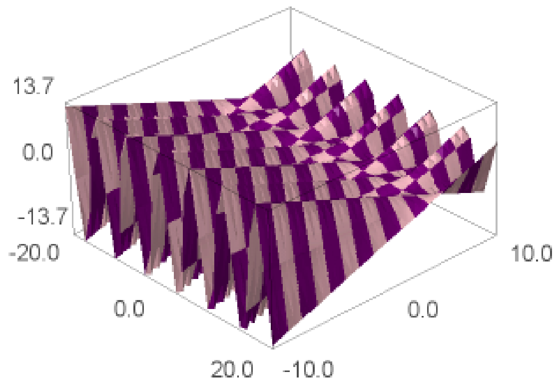

Figure 9.6: $\phi_2(t) = \sin(t), c_5 = -4.$

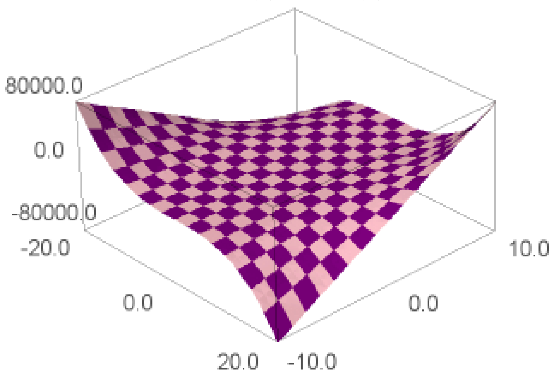

Figure 9.7: $\phi_2(t) = t^3, c_5 = 0.$

Figure 9.8: Graphical representation of solutions for some choices of ϕ_2.

In this case the invariants are $(x + c_5)e^{-c_2t/c_1}$ and ue^{-c_2t/c_1}.
A representation of solutions of equation (9.29) in this case is

$$u = e^{c_2t/c_1}\phi_3\left((x + c_5)e^{-c_2t/c_1}\right),$$

where ϕ_3 is an arbitrary function. Letting $c_6 = \dfrac{c_2}{c_1}$, we get a representation of solutions for equation (9.29) as

$$u(t, x) = e^{c_6t}\phi_3\left((x + c_5)e^{-c_6t}\right). \tag{9.33}$$

We discuss the behavior of the solutions with passing time for some choices of ϕ_3 :
Case i: We first note that if ϕ_3 is bounded and $c_6 < 0$, then all solutions tend to zero, with increase in time, that is, as t tends to infinity.
Case ii: If ϕ_3 is bounded and $c_6 > 0$, then all solutions increase with increase in time, become unbounded and approach infinity as t tends to infinity.
Case iii: If ϕ_3 is an identity function, i.e $\phi_3(.) = (.)$, then $u(t, x) = x + c_5$. In this case, we see that the solution is independent of time, and represents the "steady state" solution.
Case iv: Let $\phi_3(.) = e^{(.)}$.
Then, $u(t, x) = e^{c_6t}e^{(x+c_5)e^{-c_6t}}$.
We deduce from here that, if $c_6 > 0$, then all solutions are positive, become unbounded and tend to infinity as time increases to infinity.
However, if $c_6 < 0$, then all solutions exponentially decrease to 0 as $t \to \infty$, provided $x + c_5$ is non-positive.
The graphical representations of the analytical solutions for some choices of ϕ_3 are shown in Figure 9.12. In Figure 9.9 and Figure 9.10, t varies over $[-30, 30]$ and x varies over $[-15, 15]$ while in Figure 9.11, t varies over $[-0.3, 0.3]$ and x varies over $[-0.1, 0.1]$. □

9.7 Reduced Equations

Representations of solutions that we have obtained simplify the Inviscid Burgers' equation with delay (9.1). They reduce the number of independent variables appearing in the equation. The representation when substituted in equation (9.1) reduces it to an ordinary functional differential equation called a *reduced equation*.

Case 1: Let $u = \phi_1(c_3t - c_1x)$.
Then by substituting this, equation (9.1) becomes

$$c_3\phi_1'(c_3t - c_1x) - c_1\phi_1'(c_3t - c_1x)\phi_1(c_3t - c_1x) = G(\phi_1(c_3(t - r) - c_1x)).$$

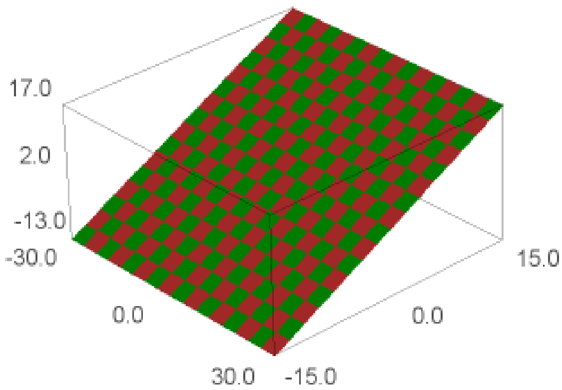

Figure 9.9: $\phi_3(.) = (.), c_5 = 2$.

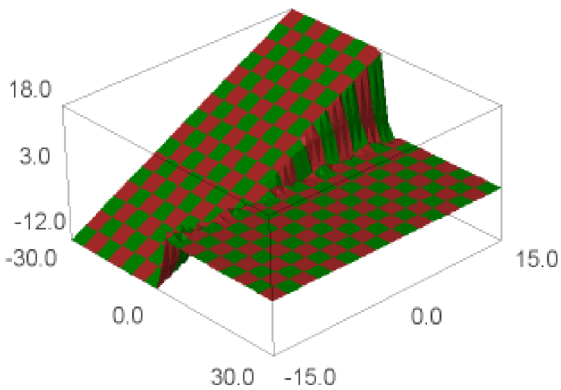

Figure 9.10: $\phi_3(.) = \sin(.), c_5 = 3, c_6 = -1$.

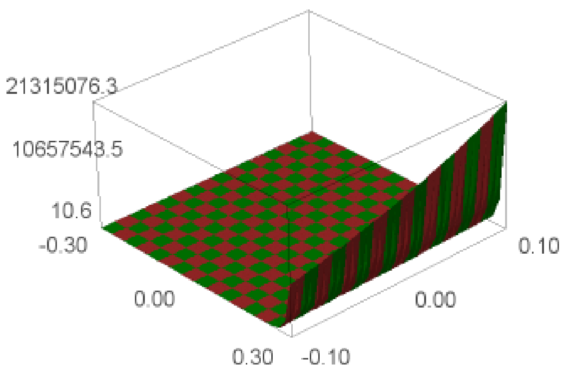

Figure 9.11: $\phi_3(.) = e^{(.)}, c_5 = 4, c_6 = -5$.

Figure 9.12: Graphical representation of solutions for some choices of ϕ_3.

Putting $\xi = c_3t - c_1x$ and simplifying the above equation we get,

$$\phi_1'(\xi) = \frac{G(\phi_1(\xi - c_3r))}{c_3 - c_1\phi_1(\xi)}. \tag{9.34}$$

Case 2: Let $u = (x + c_5)\phi_2(t)$.
Then by substituting this, equation (9.29) becomes

$$(x + c_5)\phi_2'(t) + (x + c_5)[\phi_2(t)]^2 = c_4(x + c_5)^2[\phi_2(t - r)]^2,$$

which can be solved to give

$$\phi_2'(t) = c_4(x + c_5)[\phi_2(t - r)]^2 - [\phi_2(t)]^2. \tag{9.35}$$

Case 3: Let $u = e^{c_6t}\phi_3\left((x + c_5)e^{-c_6t}\right)$.
Then by substituting this and putting $\eta = (x + c_5)e^{-c_6t}$, equation (9.29) becomes

$$-c_6\eta e^{c_6t}\phi_3'(\eta) + c_6e^{c_6t}\phi_3(\eta) + e^{c_6t}\phi_3(\eta)\phi_3'(\eta) = c_4e^{c_6(t-r)}\phi_3\left((x + c_5)e^{-c_6(t-r)}\right),$$

which gives,

$$-c_6\phi_3'(\eta) + c_6\phi_3(\eta) + \phi_3(\eta)\phi_3'(\eta) = c_4e^{-c_6r}\phi_3(\eta e^{c_6r}),$$

which on simplification yields,

$$\phi_3'(\eta) = \frac{c_4e^{-c_6r}\phi_3(\eta e^{c_6r}) - c_6\phi_3(\eta)}{\phi_3(\eta) - \eta c_6}. \tag{9.36}$$

Remark 9.7.1. It should be noted that equations (9.34), (9.35) and (9.36) are nonlinear ordinary functional differential equations.

9.8 Summary

The following are our results of symmetry analysis of inviscid Burgers' equation with delay (9.1):

1. The inviscid Burgers' equation with delay (9.1) admits the three dimensional Lie algebra with generators

$$\zeta_1^* = \frac{\partial}{\partial t}, \quad \zeta_2^* = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \quad \zeta_3^* = \frac{\partial}{\partial x}.$$

2. The kernel of the admitted Lie group is two dimensional with generators

$$\zeta_1^* = \frac{\partial}{\partial t}, \quad \zeta_2^* = \frac{\partial}{\partial x}.$$

3. The extensions of the kernel of the admitted Lie group is one dimensional with generator
$$\zeta_1^* = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$$

4. The symmetry (admitted Lie group) for the inviscid Burgers' equation with delay (9.1) for an arbitrary function G is given by (9.24).

5. The symmetry (admitted Lie group) for the inviscid Burgers' equation with delay (9.1) for $G = c_4u^2(t - r, x)$ is given by (9.26).

6. A representation of solutions for the inviscid Burgers' equation with delay (9.1) for an arbitrary function G is given by (9.28), while a representation of solutions for the inviscid Burgers' equation with delay (9.1) for $G = c_4u^2(t - r, x)$ is given by (9.32) and (9.33).

7. The further analysis of these solutions discussed will aid researchers studying Burgers' equation with delay to accurately model and predict the behavior of the system with the passage of time.

8. The reduced equations of the inviscid Burgers' equation with delay (9.1) are found and given by equations (9.34), (9.35) and (9.36).

Chapter 10

Lie Group Analysis of the One-Dimensional Wave Equation With Delay

The contents of this chapter are published in

Applied Mathematics and Computation.
(A UGC Care Listed Journal indexed in SCI, SCOPUS and Web of Science).

10.1 Introduction

Existing research on symmetry analysis of partial differential equations with delay include the complete group classification of the reaction-diffusion equation with delay. More literature on symmetry analysis of first order partial differential equations with delay, using Lie-Bäcklund operators can be found in [58]. Wave equations with delay are discussed in [29, 63].

In this chapter, we perform group analysis of the one-dimensional wave equation with delay, which is of the form,

$$\frac{\partial^2 u}{\partial t^2}(t,x) - c^2 \frac{\partial^2 u}{\partial x^2}(t,x) = G(u(t-r,x)). \tag{10.1}$$

Here u is a real valued function defined on $I \times D$, I is an open interval in $\mathbb{R}$ and D is an open set in $\mathbb{R}$. Equation (10.1) is a nonlinear second order partial differential equation with delay term in the arbitrary differentiable nonzero functional G, with c as the constant speed.

Wave equations find applications in modeling the air column of a clarinet or organ pipe, modeling tension via springs, motion of a vibrating string, study of damping, elastic waves in a rod, acoustic model for seismic waves, sound waves in liquids and gases, etc. We have used Taylor's theorem for a function of several variables to obtain a Lie type invariance condition for second order partial differential equations with delay. The procedure of getting and splitting the determining equations is different from any literature on symmetry analysis of partial differential equations with delay. We have studied the wave equation with delay and an arbitrary functional and have obtained its symmetries and made a group classification. Further we have found the kernel and extensions of the kernel to classify (10.1) with respect to its symmetries for an arbitrary and the special case for its functional G.

10.2 Lie Type Invariance Condition For Second Order Partial Differential Equations With Delay

Let $u = u(t,x)$. Then we consider transformations of the form,

$$\bar{t} = f_1(t,x,u;\delta), \quad \bar{x} = f_2(t,x,u;\delta), \quad \bar{u} = f_3(t,x,u;\delta),$$

where f_1, f_2, f_3 are smooth functions of t, x, u having a convergent Taylor series in δ. Defining,

$$T(t,x,u) = \left.\frac{\partial f_1}{\partial \delta}\right|_{\delta=0}, \quad X(t,x,u) = \left.\frac{\partial f_2}{\partial \delta}\right|_{\delta=0}, \quad U(t,x,u) = \left.\frac{\partial f_3}{\partial \delta}\right|_{\delta=0},$$

we can write the transformations as,

$$\begin{cases} \bar{t} = t + \delta T(t,x,u) + O(\delta^2), \\ \bar{x} = x + \delta X(t,x,u) + O(\delta^2), \\ \bar{u} = u + \delta U(t,x,u) + O(\delta^2). \end{cases}$$

We establish the following Lie type invariance condition for second order partial differential equations with delay using Taylor's theorem for a function of several variables:

Theorem 10.2.1. *Consider the second order partial differential equation with delay*

$$F\Big(t,x,u,t-r,u(t-r,x),\frac{\partial u}{\partial t}(t,x),\frac{\partial u}{\partial x}(t,x),\frac{\partial u}{\partial x}(t-r,x), \frac{\partial^2 u}{\partial t^2}(t,x),\frac{\partial^2 u}{\partial x^2}(t,x),\frac{\partial^2 u}{\partial x^2}(t-r,x),\frac{\partial^2 u}{\partial t\partial x}(t,x)\Big)=0, \quad (10.2)$$

where F is defined on a 12-dimensional space $I\times D^2\times I-r\times D^8$, D is an open set in $\mathbb{R}$, I is any interval in $\mathbb{R}$, and $I-r=\{y-r:y\in I\}$. Then, the Lie type invariance condition is given by

$$TF_t + T^rF_{t^r} + XF_x + UF_u + U^rF_{u^r} + U_{[t]}F_{u_t} + U_{[x]}F_{u_x} + U^r_{[x]}F_{u^r_x} + U_{[tt]}F_{u_{tt}} + U_{[xx]}F_{u_{xx}} + U^r_{[xx]}F_{u^r_{xx}} + U_{[tx]}F_{u_{tx}} = 0, \quad (10.3)$$

where, $T^r = T(t-r,x,u(t-r,x)), U^r = U(t-r,x,u(t-r,x))$, and the total differential operators given by,

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + \cdots,$$

and,

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_t} + u_{xx}\frac{\partial}{\partial u_x} + \cdots$$

the extended infinitesimals are given by,

$$U_{[t]} = D_t(U) - u_tD_t(T) - u_xD_t(X),$$

$$U_{[x]} = D_x(U) - u_tD_x(T) - u_xD_x(X),$$

$$U^r_{[x]} = U^r_x + u^r_xU^r_{u^r} - u^r_{tr}(T^r_x + u^r_xT^r_{u^r}) - u^r_x(X^r_x + u^r_xX^r_{u^r}),$$

$$U_{[tt]} = D_t(U_{[t]}) - u_{tt}D_t(T) - u_{tx}D_t(X),$$

$$U_{[xx]} = D_x(U_{[x]}) - u_{tx}D_x(T) - u_{xx}D_x(X),$$

$$U^r_{[xx]} = U^r_{xx} - u^r_{tr}T^r_{xx} + u^r_x(2U^r_{xu^r} - X^r_{xx}) - 2u^r_{tr}u^r_xT^r_{xu^r} + (u^r_x)^2(U^r_{u^ru^r} - 2X^r_{xu^r})$$
$$- u^r_{tr}(u^r_x)^2T^r_{u^ru^r} - (u^r_x)^3X^r_{u^ru^r} - 2u^r_{tr x}T^r_x + u^r_{xx}(U^r_{u^r} - 2X^r_x) - 2u^r_xu^r_{tr x}T^r_{u^r}$$
$$- u^r_{tr}u^r_{xx}T^r_{u^r} - 3u^r_xu^r_{xx}X^r_{u^r},$$

$$U_{[tx]} = D_t(U_{[x]}) - u_{tx}D_t(T) - u_{xx}D_t(X) = D_x(U_{[t]}) - u_{tt}D_x(T) - u_{tx}D_x(X).$$

Proof. Extending transformations to partial differential equations, we seek invariance of equation (10.2) under Lie group of infinitesimal transformations given by,

$$\bar{t} = t + \delta T(t, x, u) + O(\delta^2),$$

$$\bar{x} = x + \delta X(t, x, u) + O(\delta^2),$$

$$\bar{u} = u + \delta U(t, x, u) + O(\delta^2).$$

It naturally follows that,

$$\overline{t-r} = t - r + \delta T(t - r, x, u(t - r, x)) + O(\delta^2),$$

$$\bar{u}(\overline{t-r}, \bar{x}) = u(t - r, x) + \delta U(t - r, x, u(t - r, x)) + O(\delta^2).$$

Let, $T(t-r, x, u(t-r, x)) = T^r$, $X(t-r, x, u(t-r, x)) = X^r$, $U(t-r, x, u(t-r, x)) = U^r$.

As the partial differential equation given by equation (10.2) contains first order derivatives $\frac{\partial u}{\partial t}(t, x) = u_t(t, x)$ and $\frac{\partial u}{\partial x}(t, x) = u_x(t, x)$, it is necessary to obtain extended transformations for these.

In analogy with existing work for partial differential equations without delay, we define the extended transformations $\bar{u}_{\bar{t}}$ and $\bar{u}_{\bar{x}}$ as,

$$\bar{u}_{\bar{t}} = u_t + \delta U_{[t]} + O(\delta^2),$$

$$\bar{u}_{\bar{x}} = u_x + \delta U_{[x]} + O(\delta^2).$$

Equation (10.2) can be written as $F(t, t - r, x, u, u^r, u_t, u_x, u^r_x, u_{tt}, u_{xx}, u^r_{xx}, u_{tx}) = 0$, where $u^r = u(t - r, x)$.

We shall now construct the extended transformations for u_t and u_x as follows

$$
\begin{aligned}
\bar{u}_{\bar{t}} = \frac{\partial(\bar{u},\bar{x})}{\partial(\bar{t},\bar{x})} &= \frac{\partial(\bar{u},\bar{x})}{\partial(t,x)} \Big/ \frac{\partial(\bar{t},\bar{x})}{\partial(t,x)} \\
&= \begin{vmatrix} \bar{u}_t & \bar{u}_x \\ \bar{x}_t & \bar{x}_x \end{vmatrix} \Big/ \begin{vmatrix} \bar{t}_t & \bar{t}_x \\ \bar{x}_t & \bar{x}_x \end{vmatrix} \\
&= \frac{\begin{vmatrix} u_t + \delta D_t(U) + O(\delta^2) & u_x + \delta D_x(U) + O(\delta^2) \\ \delta D_t(X) + O(\delta^2) & 1 + \delta D_x(X) + O(\delta^2) \end{vmatrix}}{\begin{vmatrix} 1 + \delta D_t(T) + O(\delta^2) & \delta D_x(T) + O(\delta^2) \\ \delta D_t(X) + O(\delta^2) & 1 + \delta D_x(X) + O(\delta^2) \end{vmatrix}} \\
&= \frac{u_t + \delta[D_t(U) + u_t D_x(X) - u_x D_t(X)] + O(\delta^2)}{1 + \delta[D_t(T) + D_x(X)] + O(\delta^2)} \\
&= u_t + \delta[D_t(U) - u_t D_t(T) - u_x D_t(X)] + O(\delta^2).
\end{aligned}
$$

Thus we define,

$$
\begin{aligned}
U_{[t]} &= D_t(U) - u_t D_t(T) - u_x D_t(X), \\
U_{[x]} &= D_x(U) - u_t D_x(T) - u_x D_x(X).
\end{aligned}
$$

Then,

$$\bar{u}(\overline{t-r},\overline{x}) = u_x(t-r,x) + \delta U_{[x]}(t-r,x,u(t-r,x)) + O(\delta^2).$$

Now,

$$U_{[t]} = U_t + u_t U_u - u_t(T_t + u_t T_u) - u_x(X_t + u_t X_u), \tag{10.4}$$

and,

$$U_{[x]} = U_x + u_x U_u - u_t(T_x + u_x T_u) - u_x(X_x + u_x X_u). \tag{10.5}$$

Let $t^r = t - r, \quad u^r = u(t-r,x)$. It can then be seen that,

$$U^r_{[x]} = U^r_x + u^r_x U^r_{u^r} - u^r_{t^r}(T^r_x + u^r_x T^r_{u^r}) - u^r_x(X^r_x + u^r_x X^r_{u^r}). \tag{10.6}$$

It then follows that,

$$\bar{u}_{\overline{x}}(\overline{t-r},\overline{x}) = u_x(t-r,x) + \delta U^r_{[x]} + O(\delta^2).$$

In line with the procedure described above, we can obtain the second order extended transformations. We obtain the extended transformation to u_{tt}. Similar analysis can be used to obtain the extended transformations for u_{xx}, u^r_{xx}, u_{tx}.

$$\begin{aligned}
\bar{u}_{\bar{t}\bar{t}} &= \frac{\partial(\bar{u}_{\bar{t}}, \bar{x})}{\partial(\bar{t}, \bar{x})} = \frac{\partial(\bar{u}_{\bar{t}}, \bar{x})}{\partial(t, x)} \Bigg/ \frac{\partial(\bar{t}, \bar{x})}{\partial(t, x)} \\
&= \begin{vmatrix} (\bar{u}_{\bar{t}})_t & (\bar{u}_{\bar{t}})_x \\ \bar{x}_t & \bar{x}_x \end{vmatrix} \Bigg/ \begin{vmatrix} \bar{t}_t & \bar{t}_x \\ \bar{x}_t & \bar{x}_x \end{vmatrix} \\
&= \frac{\begin{vmatrix} u_{tt} + \delta D_t(U_{[t]}) + O(\delta^2) & u_{tx} + \delta D_x(U_{[t]}) + O(\delta^2) \\ \delta D_t(X) + O(\delta^2) & 1 + \delta D_x(X) + O(\delta^2) \end{vmatrix}}{\begin{vmatrix} 1 + \delta D_t(T) + O(\delta^2) & \delta D_x(T) + O(\delta^2) \\ \delta D_t(X) + O(\delta^2) & 1 + \delta D_x(X) + O(\delta^2) \end{vmatrix}} \\
&= \frac{u_{tt} + \delta[u_{tt} D_x(X) + D_t(U_{[t]}) - u_{tx} D_t(X)] + O(\delta^2)}{1 + \delta[D_t(T) + D_x(X)] + O(\delta^2)} \\
&= u_{tt} + \delta[D_t(U_{[t]}) - u_{tt} D_t(T) - u_{tx} D_t(X)] + O(\delta^2).
\end{aligned}$$

Thus we have obtained,

$$\bar{u}_{\bar{t}\bar{t}} = u_{tt} + \delta(D_t(U_{[t]}) - u_{tt} D_t(T) - u_{tx} D_t(X)) + O(\delta^2),$$

$$\bar{u}_{\bar{x}\bar{x}} = u_{xx} + \delta(D_x(U_{[x]}) - u_{tx} D_x(T) - u_{xx} D_x(X)) + O(\delta^2),$$

$$\begin{aligned}
\bar{u}_{\bar{t}\bar{x}} &= u_{tx} + \delta(D_t(U_{[x]}) - u_{tx} D_t(T) - u_{xx} D_t(X)) + O(\delta^2), \\
&= u_{tx} + \delta(D_x(U_{[t]}) - u_{tt} D_x(T) - u_{tx} D_x(X)) + O(\delta^2).
\end{aligned}$$

Using the expressions for D_t and D_x, we see that,

$$\begin{aligned}
U_{[tt]} &= U_{tt} + u_t(2U_{tu} - T_{tt}) - u_x X_{tt} - 2u_t u_x X_{tu} + u_t^2(U_{uu} - 2T_{tu}) - u_x u_t^2 X_{uu} \\
&\quad - u_t^3 T_{uu} - 2u_{tx} X_t + u_{tt}(U_u - 2T_t) - 2u_t u_{tx} X_u - u_x u_{tt} X_u - 3u_t u_{tt} T_u, \qquad (10.7)
\end{aligned}$$

$$\begin{aligned}
U_{[xx]} &= U_{xx} - u_t T_{xx} + u_x(2U_{xu} - X_{xx}) - 2u_t u_x T_{xu} + u_x^2(U_{uu} - 2X_{xu}) - u_t u_x^2 T_{uu} \\
&\quad - u_x^3 X_{uu} - 2u_{tx} T_x + u_{xx}(U_u - 2X_x) - 2u_x u_{tx} T_u - u_t u_{xx} T_u - 3u_x u_{xx} X_u, \\
&\qquad (10.8)
\end{aligned}$$

$$\begin{aligned}
U^r_{[xx]} &= U^r_{xx} - u^r_{tr} T^r_{xx} + u^r_x(2U^r_{xu^r} - X^r_{xx}) - 2u^r_{tr} u^r_x T^r_{xu^r} + (u^r_x)^2(U^r_{u^r u^r} - 2X^r_{xu^r}) \\
&\quad - u^r_{tr}(u^r_x)^2 T^r_{u^r u^r} - (u^r_x)^3 X^r_{u^r u^r} - 2u^r_{tr x} T^r_x + u^r_{xx}(U^r_{u^r} - 2X^r_x) - 2u^r_x u^r_{tr x} T^r_{u^r} \\
&\quad - u^r_{tr} u^r_{xx} T^r_{u^r} - 3u^r_x u^r_{xx} X^r_{u^r}, \qquad (10.9)
\end{aligned}$$

For invariance, we need to have,

$$\begin{aligned}
0 &= F(\bar{t}, \overline{t-r}, \bar{x}, \bar{u}, \bar{u}^r, \bar{u}_{\bar{t}}, \bar{u}_{\bar{x}}, \bar{u}^r_{\bar{x}}, \bar{u}_{\overline{tt}}, \bar{u}_{\overline{xx}}, \bar{u}^r_{\overline{xx}}, \bar{u}_{\overline{tx}}) \\
&= F(t + \delta T + O(\delta^2), t - r + \delta T^r + O(\delta^2), x + \delta X + O(\delta^2), \\
&\quad u + \delta U + O(\delta^2), u^r + \delta U^r + O(\delta^2), u_t + \delta U_{[t]} + O(\delta^2), \\
&\quad u_x + \delta U_{[x]} + O(\delta^2), u^r_x + \delta U^r_{[x]} + O(\delta^2), \\
&\quad u_{tt} + \delta U_{[tt]} + O(\delta^2), u_{xx} + \delta U_{[xx]} + O(\delta^2), u^r_{xx} + \delta U^r_{[xx]} + O(\delta^2), u_{tx} + \delta U_{[tx]} + O(\delta^2)) \\
&= F(t, t-r, x, u, u^r, u_t, u_x, u^r_x, u_{tt}, u_{xx}, u^r_{xx}, u_{tx}) + \delta\Big(TF_t + T^r F_{t^r} + XF_x + UF_u \\
&\quad + U^r F_{u^r} + U_{[t]}F_{u_t} + U_{[x]}F_{u_x} + U^r_{[x]}F_{u^r_x} + U_{[tt]}F_{u_{tt}} + U_{[xx]}F_{u_{xx}} + U^r_{[xx]}F_{u^r_{xx}} + U_{[tx]}F_{u_{tx}}\Big) \\
&\quad + O(\delta^2)
\end{aligned}$$

Equating the coefficient of δ, we get equation (10.3) which proves the theorem. □

The infinitesimal generator of the admitted group for the equation given by (10.2) is,

$$\zeta^* = T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + U\frac{\partial}{\partial u}.$$

The extension is given by,

$$\begin{aligned}
\zeta^{(1)} &= TF_t + T^r F_{t^r} + XF_x + UF_u + U^r F_{u^r} + U_{[t]}F_{u_t} + U_{[x]}F_{u_x} \\
&\qquad + U^r_{[x]}F_{u^r_x} + U_{[tt]}F_{u_{tt}} + U_{[xx]}F_{u_{xx}} + U^r_{[xx]}F_{u^r_{xx}} + U_{[tx]}F_{u_{tx}} = 0.
\end{aligned}$$

The Lie type invariance condition is given by $\zeta^{(1)}\Delta\ |_{\Delta=0} = 0$, where
$\Delta = F(t, t^r, x, u, u^r, u_t, u_x, u^r_x, u_{tt}, u_{xx}, u^r_{xx}, u_{tx}) = 0$.

10.3 Symmetry Analysis of the Wave Equation with Delay

We shall be making a complete group classification of

$$u_{tt}(t, x) - c^2 u_{xx}(t, x) = G(u(t - r, x)) \tag{10.10}$$

Applying the operator defined by equation (10.3) on the delay term $g(t) = t - r$, we get $T = T^r$.
The Lie type invariance condition for the time-delayed wave equation (10.10) gives

$$U_{[tt]} - c^2 U_{[xx]} = U^r G'(u^r). \tag{10.11}$$

Using equations (10.7) and (10.8), we get,

$$\begin{aligned} &U_{tt}+u_t(2U_{tu}-T_{tt})-u_xX_{tt}-2u_tu_xX_{tu}+u_t^2(U_{uu}-2T_{tu})-u_xu_t^2X_{uu}-u_t^3T_{uu}-2u_{tx}X_t \\ &+u_{tt}(U_u-2T_t)-2u_tu_{tx}X_u-u_xu_{tt}X_u-3u_tu_{tt}T_u-c^2\Big(U_{xx}-u_tT_{xx}+u_x(2U_{xu}-X_{xx}) \\ &-2u_tu_xT_{xu}+u_x^2(U_{uu}-2X_{tu})-u_tu_x^2T_{uu}-u_x^3X_{uu}-2u_{tx}T_x+u_{xx}(U_u-2X_x)-2u_xu_{tx}T_u \\ &-u_tu_{xx}T_u-3u_xu_{xx}X_u\Big)=U^rG'(u^r). \qquad (10.12) \end{aligned}$$

Substitute $u_{tt}=c^2u_{xx}+G(u^r)$ in equation (10.12), we get,

$$\begin{aligned} &U_{tt}+u_t(2U_{tu}-T_{tt})-u_xX_{tt}-2u_tu_xX_{tu}+u_t^2(U_{uu}-2T_{tu})-u_xu_t^2X_{uu}-u_t^3T_{uu}-2u_{tx}X_t \\ &+c^2u_{xx}U_u-2c^2u_{xx}T_t+G(u^r)U_u-2G(u^r)T_t-2u_tu_{tx}X_u-c^2u_xu_{xx}X_u-u_xG(u^r)X_u \\ &-3c^2u_tu_{xx}T_u-3u_tG(u^r)T_u-c^2\Big(U_{xx}-u_tT_{xx}+u_x(2U_{xu}-X_{xx})-2u_tu_xT_{xu} \\ &+u_x^2(U_{uu}-2X_{xu})-u_tu_x^2T_{uu}-u_x^3X_{uu}-2u_{tx}T_x+u_{xx}(U_u-2X_x)-2u_xu_{tx}T_u \\ &-u_tu_{xx}T_u-3u_xu_{xx}X_u\Big)=U^rG'(u^r). \qquad (10.13) \end{aligned}$$

Splitting equation (10.13) with respect to u_t^3, we get,

$$T_{uu}=0,$$

which can be solved to give,

$$T(t,x,u)=A(t,x)u+B(t,x). \qquad (10.14)$$

Splitting equation (10.13) with respect to u_x^3, we get,

$$c^2X_{uu}=0,$$

which can be solved to give,

$$X(t,x,u)=C(t,x)u+D(t,x). \qquad (10.15)$$

Splitting equation (10.13) with respect to u_t we get,

$$2U_{tu}-T_{tt}-3G(u^r)T_u+c^2T_{xx}=0. \qquad (10.16)$$

Splitting equation (10.13) with respect to u_x we get,

$$-X_{tt}-G(u^r)X_u-2c^2U_{xu}+c^2X_{xx}=0. \qquad (10.17)$$

Splitting equation (10.13) with respect to u_{xx} we get,

$$2c^2(X_x - T_t) = 0. \tag{10.18}$$

Splitting equation (10.13) with respect to u_{xt} we get,

$$2(c^2T_x - X_t) = 0. \tag{10.19}$$

Splitting equation (10.13) with respect to u_x^2 we get,

$$c^2(2X_{xu} - U_{uu}) = 0. \tag{10.20}$$

Splitting equation (10.13) with respect to u_t^2 we get,

$$U_{uu} - 2T_{tu} = 0. \tag{10.21}$$

Splitting equation (10.13) with respect to $u_t^2u_x$ we get,

$$-X_{uu} = 0. \tag{10.22}$$

Splitting equation (10.13) with respect to $u_tu_x^2$ we get,

$$c^2T_{uu} = 0. \tag{10.23}$$

Splitting equation (10.13) with respect to u_tu_x we get,

$$-2X_{tu} + 2c^2T_{xu} = 0. \tag{10.24}$$

Splitting equation (10.13) with respect to u_tu_{xt} we get,

$$-2X_u = 0. \tag{10.25}$$

Splitting equation (10.13) with respect to u_xu_{xt} we get,

$$2c^2T_u = 0. \tag{10.26}$$

Splitting equation (10.13) with respect to u_tu_{xx} we get,

$$-2c^2T_u = 0. \tag{10.27}$$

Splitting equation (10.13) with respect to u_xu_{xx} we get,

$$2c^2X_u = 0. \tag{10.28}$$

Splitting equation (10.13) with respect to the constant term we get,

$$U_{tt} + G(u^r)U_u - 2G(u^r)T_t - c^2U_{xx} - U^rG'(u^r) = 0. \tag{10.29}$$

From equations (10.14) and (10.26), we get, $A(t, x) = 0$.
From equations (10.15) and (10.28), we get, $C(t, x) = 0$.
Hence,

$$T(t, x, u) = B(t, x), \quad X(t, x, u) = D(t, x). \tag{10.30}$$

From equations (10.20) and (10.30), we get,

$$U(t, x, u) = E(t, x)u + F(t, x). \tag{10.31}$$

From equations (10.18) and (10.19), we get,

$$c^2B_x = D_t, \quad D_x = B_t. \tag{10.32}$$

As $T = T^r$, from equation (10.32) we see that $X = X^r$.
From equation (10.16) we get,

$$2E_t - B_{tt} + c^2B_{xx} = 0. \tag{10.33}$$

From equation (10.17) we get,

$$-D_{tt} - 2c^2E_x + c^2D_{xx} = 0. \tag{10.34}$$

Using equation (10.32), equations (10.33) and (10.34) give $E(t, x) = H$, a constant. Hence, equation (10.31) gives

$$U(t, x, u) = Hu + F(t, x). \tag{10.35}$$

It follows that, $U^r = Hu^r + F^r$, where $F^r = F(t - r, x)$.
Solving the system of first order partial differential equations given by equation (10.32), we get,

$$T(t, x, u) = \frac{1}{c}f(-ct-x) + \frac{1}{c}g(-ct+x), \quad X(t, x, u) = f(-ct-x) - g(-ct+x). \tag{10.36}$$

Since $T = T^r$ and $X = X^r$, it follows that f and g are periodic, that is, $f(t) = f(t - r)$ and $g(t) = g(t - r)$.
Using equations (10.35) and (10.36), equation (10.29) gives,

$$G'(u^r)[Hu^r + F^r] - [2(f' + g') + H]G(u^r) + [c^2F_{xx} - F_{tt}] = 0. \tag{10.37}$$

The infinitesimal generator of the admitted Lie group for the wave equation with delay given by equation (10.10) is

$$\zeta^* = \left[\frac{1}{c}f(-ct-x) + \frac{1}{c}g(-ct+x)\right]\frac{\partial}{\partial t} + [f(-ct-x) - g(-ct+x)]\frac{\partial}{\partial x} + [Hu + F(t,x)]\frac{\partial}{\partial u}. \quad (10.38)$$

10.4 Kernel of the Admitted Lie Group

Definition 10.4.1. A *kernel* of admitted generators are the set of symmetries, which are admitted for any functional G appearing in the equation.

We prove our following result:

Theorem 10.4.1. *The symmetry of the wave equation given by (10.1) admit a two-dimensional Lie group generated by*

$$\zeta_1^* = \frac{\partial}{\partial t}, \quad \zeta_2^* = \frac{\partial}{\partial x}.$$

Further the representation of the invariant solution is found to be $u(t,x) = g_1(c_5 t - c_4 x)$, *where* c_4, c_5 *are arbitrary constants and* g_1 *is an arbitrary function.*

Proof. We assume that equation (10.37) is valid for an arbitrary functional G. Without loss of generality we consider the particular case,

$$G(u^r) = a_0 + a_1 u^r + a_2 (u^r)^2 + a_3 (u^r)^3,$$

where $a_i, \quad 0 \leq i \leq 3$ are arbitrary constants.
Substituting this value of $G(u^r)$ in equation (10.37) we get

$$2a_3[H - (f' + g')](u^r)^3 + [3a_3 F^r + Ha_2 - 2a_2(f' + g')](u^r)^2 + 2[a_2 F^r - a_1(f' + g')]u^r + [a_1 F^r - Ha_0 - 2a_0(f' + g') + c^2 F_{xx} - F_{tt}] = 0. \quad (10.39)$$

Since u^r is arbitrary, equating the various powers of u^r to zero, we get,

$$f' + g' = H, \quad a_3 F^r + H = 0, \quad a_2 F^r - a_1 K = 0, \quad a_1 F^r - 3a_0 H + c^2 F_{xx} - F_{tt} = 0. \quad (10.40)$$

From the second equation in (10.40), we see that F^r must be a constant. Further since a_3 is arbitrary $H = F = 0$.
Therefore, $U(t,x,u) = 0$ and $g'(-ct+x) = -f'(-ct-x)$. Hence,

$$g(-ct+x) = c_1(-ct+x) + c_2, \quad f(-ct-x) = -c_1(-ct-x) + c_3, \quad (10.41)$$

where c_1, c_2, c_3 are arbitrary constants. But since $g(t) = g(t-r)$ and $f(t) = f(t-r)$, we must have $c_1 = 0$. Thus,

$$g(-ct + x) = c_2, \quad f(-ct - x) = c_3. \tag{10.42}$$

Consequently, using equations (10.36) and (10.42) we get,

$$T(t, x, u) = c_4, \quad X(t, x, u) = c_5, \quad U(t, x, u) = 0, \tag{10.43}$$

where $c_4 = \dfrac{1}{c}(c_2 + c_3)$ and $c_5 = c_3 - c_2$. The general form of the infinitesimal generator is

$$\zeta^* = c_4 \frac{\partial}{\partial t} + c_5 \frac{\partial}{\partial x}. \tag{10.44}$$

Hence the Kernel of the admitted Lie group is defined by the infinitesimal generators,

$$\zeta_1^* = \frac{\partial}{\partial t}, \quad \zeta_2^* = \frac{\partial}{\partial x}.$$

A solution which is invariant with respect to this generator has to satisfy the equation $\zeta^* u(t, x) = 0$. Hence solving the corresponding system, namely

$$\frac{dt}{T(t, x, u)} = \frac{dx}{X(t, x, u)} = \frac{du}{U(t, x, u)}, \tag{10.45}$$

that is, by solving,

$$\frac{dt}{c_4} = \frac{dx}{c_5} = \frac{du}{0},$$

we get the representation of an invariant solution namely $u(t, x) = g_1(c_5 t - c_4 x)$. This solution is a travelling wave which reduces equation (10.1) to the second order ordinary functional differential equation

$$g_1''(\theta) = \frac{G(g_1(\theta - c_5 r))}{c_5^2 - c^2 c_4^2},$$

where $\theta = c_5 t - c_4 x$ and g_1 is an arbitrary function. □

10.5 Extensions of the Kernel

Definition 10.5.1. Extensions are symmetries for the particular functional G only.

We establish the following result:

Theorem 10.5.1. *The wave equation with delay $u_{tt} + c^2 u_{xx} = G(u^r)$ for which*

1. *$G(u^r) = K_1 e^{-K_2 u^r}$ admits the infinitesimal generator given by*

$$\zeta^* = \frac{1}{c}(f+g)\frac{\partial}{\partial t} + (f-g)\frac{\partial}{\partial x} - \frac{2}{K_2}(f'+g')\frac{\partial}{\partial u}.$$

Further a representation of the invariant solution is given by

$$u(t,x) = \frac{2}{K_2}\ln|f(-ct-x)+g(-ct+x)| + g_2\Big(c\int(f(-ct-x)-g(-ct+x))dt - \int(f(-ct-x)-g(-ct+x))dx\Big),$$

where K_2 is a non-zero constant and g_2 is an arbitrary function.

2. *$G(u^r) = K_{11}u^r + K_{12}$ admits the infinitesimal generator given by*

$$\zeta^* = c_8\frac{\partial}{\partial t} + c_9\frac{\partial}{\partial x} + [Hu + F(t,x)]\frac{\partial}{\partial u},$$

where H is an arbitrary constant and F solves $K_{11}F^r - HK_{12} = F_{tt} - c^2F_{xx}$, with K_{11} as a non-zero constant. Further a representation of the invariant solution is given by $u = \frac{1}{H}\left[e^{H(t-c_8g_3(c_9t-c_8x))/c_8} - F(t,x)\right]$, where c_8, c_9 are arbitrary constants and g_3 is an arbitrary function.

Proof. Differentiating equation (10.37) with respect to u^r we get

$$Hu^rG''(u^r) + F^rG''(u^r) - 2(f'+g')G'(u^r) = 0,$$

which can be written as

$$H\alpha + F^r\beta - 2(f'+g')\gamma = 0. \tag{10.46}$$

That is,

$$< H, F^r, -2(f'+g') > . < \alpha, \beta, \gamma >= 0, \tag{10.47}$$

where $\alpha = u^rG''$, $\beta = G''$, $\gamma = G'$. Analysis of equation (10.46) is similar to the analysis given for gas dynamics in [49].

Let us consider the vector space $\mathbb{V} = span\{< \alpha, \beta, \gamma >\}$.

10.5.1 dim($\mathbb{V}$) = 3

Then from equation (10.46), using the fact that α, β, γ are linearly independent, we get

$$H = 0, \quad F^r = 0, \quad f' + g' = 0,$$

which implies that,

$$f(t,x) = c_1(-ct - x) + c_2, \quad g(t,x) = -c_1(-ct + x) + c_3.$$

Since $f(t) = f(t-r)$ and $g(t) = g(t-r)$, we get $c_1 = 0$.
Consequently using equation (10.36), we see that $T(t,x,u)$ and $X(t,x,u)$ are constants. Also $U(t,x,u) = 0$ and hence in this case there is no extension of the Kernel.

10.5.2 dim($\mathbb{V}$) = 2

if $\dim(\mathbb{V}) = 2$, then by the Gram-Schmidt orthogonalisation process there exists a non-zero constant $< \xi, \eta, \varsigma >$ which is orthogonal to $\mathbb{V}$.
That is, $< \xi, \eta, \varsigma > . < \alpha, \beta, \gamma >= 0$, which implies

$$\xi u^r G'' + \eta G'' + \varsigma G' = 0.$$

Letting $z = G'$, we get

$$(\xi u^r + \eta) z' + \varsigma z = 0. \tag{10.48}$$

We discuss the following cases:

Case 1: Let $\xi = 0$. Then necessarily $\eta \neq 0$, and

$$\eta z' = -\varsigma z, \tag{10.49}$$

which can be solved to give,

$$z = G'(u^r) = K_1 e^{-K_2 u^r}, \tag{10.50}$$

where K_1 is a non-zero constant and $K_2 = \dfrac{\varsigma}{\eta}$.
Since the integration for z depends on K_2, we consider the following two cases:

Case 2: If $K_2 = 0$, then from equation (10.49), we get $G(u^r) = K_3 u^r + K_4$, where K_3 and K_4 are arbitrary constants.
This implies that $\dim(\mathbb{V}) = 0$ which contradicts the fact that $\dim(\mathbb{V}) = 2$.

Case 3: If $K_2 \neq 0$, then from equation (10.49), we get $z = G'(u^r) = K_1 e^{-K_2 u^r}$.
Integrating this equation with respect to u^r, we get,

$$G(u^r) = -\frac{K_1}{K_2} e^{-K_2 u^r} + K_5, \tag{10.51}$$

where K_5 is an arbitrary constant.

Substituting equation (10.51) in equation (10.37) we get,

$$(Hu^r+F^r)(K_1e^{-K_2u^r})-[2(f'+g')+H]\left(-\frac{K_1}{K_2}e^{-K_2u^r}+K_5\right)+c^2F_{xx}-F_{tt}=0. \quad (10.52)$$

Splitting equation (10.52) with respect to $u^re^{-K_2u^r}$ and using the fact that $K_1 \neq 0$, we get $H=0$.
Splitting equation (10.52) with respect to $e^{-K_2u^r}$ and using the fact that $H=0$, we get $F^r = -\frac{2}{K_2}[g'(-ct+x)-f'(-ct-x)]$.
Splitting equation (10.52) with respect to the constant term we get
$c^2F_{xx}-F_{tt}-2K_5(g'(-ct+x)+f'(-ct-x))=0$.
Thus from these equations we get

$$K_5(g'(-ct+x)+f'(-ct-x))=0.$$

We shall study two cases according to K_5.

Case 4: If $K_5 \neq 0$, then this lead to f and g being constants, which does not extend the Kernel of the admitted Lie group.

Case 5: If $K_5=0$, then,

$$T(t,x,u)=\frac{1}{c}[g(-ct+x)+f(-ct-x)], \quad X(t,x,u)=f(-ct-x)-g(-ct+x)$$

and

$$U(t,x,u)=Hu+F^r=F^r=-\frac{2}{K_2}[g'(-ct+x)+f'(-ct-x)].$$

Thus the admitted infinitesimal generator in this case is

$$\zeta^*=\frac{1}{c}(g+f)\frac{\partial}{\partial t}+(f-g)\frac{\partial}{\partial x}-\frac{2}{K_2}(f'+g')\frac{\partial}{\partial u}. \quad (10.53)$$

An invariant solution of equation $u_{tt}-c^2u_{xx}=K_1e^{-K_2u^r}$ is obtained by solving the corresponding system, namely equation (10.45) that is, by solving,

$$\frac{cdt}{f(-ct-x)+g(-ct+x)}=\frac{dx}{f(-ct-x)-g(-ct+x)}=-\frac{K_2}{2}\frac{du}{f'(-ct-x)+g'(-ct+x)},$$

we get the representation of an invariant solution namely

$$u(t,x)=\frac{2}{K_2}\ln|f(-ct-x)+g(-ct+x)|+g_2\Big(c\int(f(-ct-x) - g(-ct+x))dt-\int(f(-ct-x)-g(-ct+x))dx\Big), \quad (10.54)$$

where g_2 is an arbitrary function.

Case 6: If $\xi \neq 0$, then from equation (10.48) we get $(\xi u^r + \eta)z' = -\varsigma z$, which ca be solved to give

$$z = G'(u^r) = K_6(\xi u^r + \eta)^{-\varsigma/\xi}, \tag{10.55}$$

where K_6 is a non-zero constant.
Since further integration depends on $-\dfrac{\varsigma}{\xi}$, we study two cases:

Case 7: If $\dfrac{\varsigma}{\xi} \neq 1$, then integrating equation (10.55) we get

$$G(u^r) = K_6\frac{(\xi u^r + \eta)^{(\xi-\varsigma)/\xi}}{\xi - \varsigma} + K_7, \tag{10.56}$$

where K_7 is an arbitrary constant.
Differentiating equation (10.56) with respect to u^r we get

$$G'(u^r) = K_6(\xi u^r + \eta)^{-\varsigma/\xi}. \tag{10.57}$$

Substituting equations (10.56) and (10.57) in equation (10.37), we get

$$K_6[Hu^r + F^r](\xi u^r + \eta)^{-\varsigma/\xi} - [2(f' + g') + H]\left[\frac{K_6(\xi u^r + \eta)^{(\xi-\varsigma)/\xi}}{\xi - \varsigma} + K_7\right] + c^2 F_{xx} - F_{tt} = 0. \tag{10.58}$$

Differentiating equation (10.58) with respect to u^r we get

$$\frac{\varsigma H u^r + \varsigma F^r}{\xi u^r + \eta} + 2(f' + g') = 0. \tag{10.59}$$

Differentiating equation (10.59) with respect to u^r we get

$$\varsigma(F^r\xi - \eta H) = 0. \tag{10.60}$$

We consider two cases depending on ς.

Case 8: If $\varsigma = 0$, then from equation (10.55), we get $G'(u^r) = K_6$, which yields $G(u^r) = K_6 u^r + K_8$, where K_8 is an arbitrary constant.
This implies that $\dim(\mathbb{V}) = 0$ and contradicts the assumption that $\dim(\mathbb{V}) = 2$.

Case 9: If $\varsigma \neq 0$, then from equation (10.59), we get

$$\varsigma H u^r + \varsigma F^r = -2(f' + g')(\xi u^r + \eta).$$

Splitting this equation with respect to u^r we get

$$f' + g' = -\frac{\varsigma H}{2\xi} \tag{10.61}$$

This implies that $H = 0$. Further as $f(t) = f(t - r)$ and $g(t) = g(t - r)$, this case does not give extensions of the Kernel.

Case 10: If $\xi = \varsigma$, then from equation (10.61) we get

$$f' + g' = -\frac{H}{2},$$

which again implies that $H = 0$ and hence does not give an extension of the Kernel.

10.5.3 $\textbf{dim}(\mathbb{V}) = 1$

This implies the existence of a non-zero constant vector $< \xi, \eta, \varsigma >$ such that

$$< \alpha, \beta, \gamma >= f(u^r) < \xi, \eta, \varsigma >,$$

where f is any arbitrary non-constant function.
Without loss of generality assume that $\varsigma = 1$.
From equation (10.47) we see that

$$< u^r G'', G'', G' >= f(u^r) < \xi, \eta, \varsigma >,$$

from which it follows that $\xi = \eta u^r$ which gives $\eta = 0$ and consequently $\xi = 0$.
Hence G' has to be a constant say K_9, which implies that $G(u^r) = K_9 u^r + K_{10}$, with K_{10} as an arbitrary constant, contradicting the dimension of $\mathbb{V} = 1$.

Remark 10.5.1. If G' is a constant or G is linear with respect to u^r then $\dim(\mathbb{V}) = 0$.

10.5.4 $\textbf{dim}(\mathbb{V}) = 0$

This implies that $< \alpha, \beta, \gamma >$ is a constant vector say $< \xi, \eta, \varsigma >$.
That is $< u^r G'', G'', G' >=< \xi, \eta, \varsigma >$, which gives $\xi = \eta = 0$. Without loss of generality assume that $\varsigma \neq 0$.
Then $G(u^r) = \varsigma u^r + K_{12} = K_{11} u^r + K_{12}$, where $K_{11} = \varsigma$ and K_{12} are arbitrary constants.
Substituting this value of $G(u^r)$ in equation (10.37) we get

$$K_{11}(Hu^r + F^r) - [2(f' + g') + H](K_{11} u^r + K_{12}) + c^2 F_{xx} - F_{tt} = 0. \tag{10.62}$$

Splitting equation (10.62) with respect to u^r we get $f' + g' = 0$.
As $f(t) = f(t - r)$ and $g(t) = g(t - r)$, we get f and g are constants say

$f(t,x) = c_6,\ g(t,x) = c_7.$
Consequently $T(t,x,u) = c_8, \quad X(t,x,u) = c_9$, where $c_8 = \dfrac{1}{c}(c_6 + c_7), c_9 = c_7 - c_6$.
Splitting equation (10.62) with respect to the constant term we get

$$K_{11}F^r - HK_{12} = F_{tt} - c^2 F_{xx}. \tag{10.63}$$

Hence the infinitesimal generator is given by

$$\zeta^* = c_8\frac{\partial}{\partial t} + c_9\frac{\partial}{\partial x} + [Hu + F]\frac{\partial}{\partial u}, \tag{10.64}$$

where F is an arbitrary solution of equation (10.63). An invariant solution of equation $u_{tt} - c^2 u_{xx} = K_{11}u^r + K_{12}$ is obtained by solving the corresponding system, namely equation (10.45) that is, by solving,

$$\frac{dt}{c_8} = \frac{dx}{c_9} = \frac{du}{Hu + F},$$

we get the representation of an invariant solution namely

$$u(t,x) = \frac{1}{H}\left[e^{H(t - c_8 g_3(c_9 t - c_8 x))/c_8} - F(t,x)\right]. \tag{10.65}$$

□

This solution reduces equation (10.1) to the second order ordinary functional differential equation

$$g_3''(\psi) = \frac{c_8^2 e^{-\tau}\left[G\left(\frac{1}{H}(e^{\tau^r} - F^r)\right) + \frac{1}{H}(F_{tt} - c^2 F_{xx})\right] + H\left[c^2 c_8^4 [g_3'(\psi)]^2 - (1 - c_8 c_9 g'(\psi))^2\right]}{c_8^2(c^2 c_8^2 - c_9^2)},$$

where $\psi = c_9 t - c_8 x, \quad \tau(t,x) = \dfrac{H}{c_8}(t - c_8 g_3(c_9 t - c_8 x)), \quad \tau(t-r,x) = \tau^r$ and g_3 is an arbitrary function.

10.6 Summary

This chapter deals with the symmetry analysis of the one-dimensional wave equation with delay and constant speed given by

$$\frac{\partial^2 u}{\partial t^2}(t,x) - c^2\frac{\partial^2 u}{\partial x^2}(t,x) = G(u(t-r,x)).$$

The complete group classification is given in Table 10.1 below.
Further this classification leads to the complete set of invariant solutions given Table 10.2 below.

Table 10.1: Group classification of $u_{tt} - c^2 u_{xx} = G(u^r)$.

Functional $G(u^r)$	Generator
$G(u^r)$ is arbitrary	$\zeta^* = c_4 \frac{\partial}{\partial t} + c_5 \frac{\partial}{\partial x}.$
$G(u^r) = K_{11} u^r + K_{12}$	$\zeta^* = c_8 \frac{\partial}{\partial t} + c_9 \frac{\partial}{\partial x} + [Hu + F(t,x)] \frac{\partial}{\partial u}.$
$G(u^r) = K_1 e^{-K_2 u^r}$	$\zeta^* = \frac{1}{c}(f+g)\frac{\partial}{\partial t} + (f-g)\frac{\partial}{\partial x} - \frac{2}{K_2}(f'+g')\frac{\partial}{\partial u}.$

Table 10.2: Representation of invariant solutions of $u_{tt} - c^2 u_{xx} = G(u^r)$.

Functional $G(u^r)$	Representation of the invariant solution of $u_{tt} - c^2 u_{xx} = G(u^r)$
$G(u^r)$ is arbitrary	$u = g_1(c_t - c_4 x).$
$G(u^r) = K_{11} u^r + K_{12}$	$u = \frac{1}{H}\left[e^{H(t - c_8 g_3(c_9 t - c_8 x))/c_8} - F(t,x)\right].$
$G(u^r) = K_1 e^{-K_2 u^r}$	$u = \frac{2}{K_2} \ln\lvert f+g\rvert + g_2(c\int(f-g)dt - \int(f+g)dx).$

Future Scope

Symmetry analysis can be applied to

1. Classify ordinary functional differential equations with variable coefficients to solvable Lie algebras.
2. Integro-differential equations with delay specifically arising in population dynamics of species and prey-predator models.
3. Systems of delay differential equations.
4. Linear, semi-linear and quasilinear partial differential equations with delay.
5. Classify partial differential equations with delay and with constant and variable coefficients to solvable Lie algebras.
6. Neutral partial differential equations.

to obtain Lie type invariance conditions and study their group classification/invariant solutions.

In addition special partial differential equations with delay like the Laplace equation, Potential Burgers' equation, Chaplygin's equation, Korteweg de Vries equation, Boussinesq quation, etc. can be studied using Lie group analysis and provided with their group classification and invariant solutions depending on the dimensions of the underlying Vector Space. Research in this area can also be extended to systems of partial differential equations with delay.

Finally symmetry analysis can be applied to discrete systems and a study of difference equations can be done.

Presentations and Publications

LIST OF PAPERS PRESENTED IN CONFERENCES

1. Presented a paper titled “Finding Solutions of the One Dimensional Wave Equation Using Group Methods, ” in the International Conference on Algebra and Discrete Mathematics, organized by Madurai Kamraj University, Tamil Nadu, between $24^{th} - 26^{th}$ June, 2020.

2. Presented a paper titled “Group Analysis of the One Dimensional Wave Equation With Delay, ” in the 85^{th} Annunal Conference of the Indian Mathematical Society, held at IIT Kharagpur, West Bengal, between $22^{nd} - 25^{th}$ November, 2019.

3. Presented a paper titled “Lie Group Analysis of the Time-Delayed Inviscid Burgers’ Equation, ” in the International Conference on Applied Mathematics and Computational Sciences, held at DIT University, Dehradun, Uttarakhand, between $17^{th} - 19^{th}$ October, 2019.

4. Presented a paper titled “Classification of Some Functional Differential Equations With Constant Coefficients to Solvable Lie Algebras, ” in the International Conference on Differential Equations and Control Problems: Modeling, Analysis and Computations, held at IIT Mandi, Himachal Pradesh, between $17^{th} - 19^{th}$ June, 2019.

5. Presented a paper titled “Symmetry Analysis for First Order Delay Differential Equations With Constant Coefficients, ” in the National Conference on Mathematical and Computational Sciences, held at Alagappa University, Karaikudi, Tamil Nadu, between $23^{rd} - 24^{th}$ October, 2018.

LIST OF PAPERS ACCEPTED/PUBLISHED IN JOURNALS

1. Jervin Zen Lobo and Y. S. Valaulikar, Classification of Second Order Functional Differential Equations with Constant Cofficients to Solvable Lie Algebras, accepted for publication in *Journal of Mathematical Extension*. (UGC Care list, Web of Science).

2. Jervin Zen Lobo and Y. S. Valaulikar, Lie Group Analysis of the Time-Delayed Inviscid Burgers’ Equation, published in the *Journal of the Indian Mathematical Society*, **88**(1-2), 2021, pp. 105-124. (UGC Care list, SCOPUS).

3. Jervin Zen Lobo and Y. S. Valaulikar, Classification of First Order Functional Dif-

ferential Equations with Constant Coefficients to Solvable Lie Algebras, published in *Applications and Applied Mathematics: An International Journal*, **15**(2), 2020, pp. 985-1003. (UGC Care list, Web of Science).

4. Jervin Zen Lobo and Y. S. Valaulikar, Group Analysis of the One Dimensional Wave Equation With Delay, published by Elsevier in *Applied Mathematics and Computation*, **378**, Article ID 125193, 2020, 12 pages. (UGC Care list, Science Citation Index, SCOPUS, Web of Science).

5. Jervin Zen Lobo and Y. S. Valaulikar, Group Methods for Second Order Delay Differential Equations, accepted in 2020, for publication in *TWMS Journal of Applied and Engineering Mathematics.* (UGC Care list, SCOPUS, Web of Science).

6. Jervin Zen Lobo and Y. S. Valaulikar, On Symmetry Analysis in Finding Solutions of the One Dimensional Wave Equation, published in *Proceedings of International Conference on Applied Mathematics and Computational Sciences*, 2020, pp. 133-141, DOI: https://doi.org/10.21467/proceedings.100.13. (An AIJR Proceedings with ISSN: 2582-3922).

7. Jervin Zen Lobo and Y. S. Valaulikar, Lie Symmetries of First Order Neutral Differential Equations, published in *Journal of Applied Mathematics and Comutational Mechanics*, **18**(1), 2019, pp. 29-40. (UGC Care list, Web of Science).

8. Jervin Zen Lobo and Y. S. Valaulikar, On Defining an Admitted Lie Group for First Order Delay Differential Equations with Constant Coefficients, published in *Journal of Applied Science and Computations*, **5**(11), 2018, pp. 1301-1307. (UGC listed till June 2019).

LIST OF PAPERS COMMUNICATED TO JOURNALS

1. Jervin Zen Lobo and Y. S. Valaulikar, On Symmetry Analysis of First Order Delay Differential Equations.

2. Jervin Zen Lobo and Y. S. Valaulikar, Group Methods for First Order Neutral Differential Equations.

3. Jervin Zen Lobo and Y. S. Valaulikar, Group Classification of Second Order Neutral Differential Equations. (*Uploaded on arXiv as a pre-print with identifier 1912.13228*).

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