# Investigative Studies in Reliability Based Design Approach for Mechanical Systems

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By

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## DECLARATION

I, Saurabh Laximan Raikar, hereby declare that this thesis represents work which has been carried out by me and that it has not been submitted, either in part or full, to any other University or Institution for the award of any research degree.

Place: Farmagudi, Goa
Date : 21/04/2023 Saurabh Laximan Raikar

## CERTIFICATE

I hereby certify that the work was carried out under my supervision and may be placed for evaluation.

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My family, my guide,

and

All Teachers…

# Abstract

Reliability has received utmost importance in today's world as the people are continuously looking for reliable and safe mechanical systems. Mechanical properties like stress and strength require precise designing as these are vital in determining the safety of the component. In real life, we know that properties like stress and strength do not take a fixed value due to the various uncertainties in materials, loading conditions, environmental conditions, etc. Hence, reliability-based design will be suitable in such cases which takes into account the probability of failure if the stress and strength does not take a definite value but follows a certain distribution. It is essential to consider reliability right at the modeling and design stage of mechanical components and systems. In reliability-based design, a large amount of work has been carried out in cases of stress-strength interference models of type P (strength > stress) with various distributions. However, there are some distributions for which the stress-strength models are not developed or do not have a closed form. Also, there is wide scope for improving the stress-strength reliability estimation for various distributions.

In the first part of the thesis, various interference models have been developed when stress and strength follow distributions such as Laplace, exponential, Weibull and gamma. Analysis techniques like Taguchi analysis and response surface analysis have been carried out to study the change in reliability with variation in parameters. Simulation studies have been carried out to validate the proposed models.

The second part of the thesis deals with the estimation of reliability for stress-strength interference. Stress and strength have been considered to be following Weibull distribution as it fits a large number of data and has been used in many applications in mechanical systems. A recent metaheuristic technique, Jaya algorithm has been used in estimation and has been proved to give results with high accuracy and faster compilation time. Simulation studies have been conducted to show the variation of estimated reliability with variation in distribution parameters. Analysis also has been carried out to evaluate the effectiveness of Jaya algorithm in estimation of reliability. The methodology has been applied to real-life data in order to show its implementation.

In the third part of the thesis, the principle of strength degradation has been considered in estimation of reliability as we know that the strength of the material does not remain same but, deteriorates over time. For this part, the strength is considered to be normally distributed as it is widely used in degradation studies. Similar methodology can be applied for other distributions like Weibull, gamma, etc.

**Keywords:** Reliability-based design, Stress-strength interference, Laplace distribution, Exponential distribution, Weibull distribution, Gamma distribution, Normal distribution, Maximum likelihood estimation, Least squares estimation, Weighted least squares estimation, Strength degradation, Jaya optimization algorithm

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# List of Abbreviations and Symbols

## Abbreviations

| | |
|---|---|
| DOE | : Design of experiments |
| pdf | : Probability density function |
| cdf | : Cumulative distribution function |
| MLE | : Maximum likelihood estimation |
| MMLE | : Modified maximum likelihood estimation |
| LSE | : Least squares estimation |
| WLSE | : Weighted least squares estimation |
| MCS | : Monte Carlo simulation |
| SSE | : Sum of squared error |
| MSE | : Mean squared error |
| RMSE | : Root mean squared error |
| SA | : Simulated annealing |
| DE | : Differential evolution |
| HNSA | : Hybrid neighborhood search and simulated annealing |
| PSO | : Particle swarm optimization |

## Symbols

| | |
|---|---|
| $\mu_n$ | : Mean of normal distribution |
| $\delta$ | : Standard deviation of normal distribution |
| $\lambda$ | : Rate parameter of exponential distribution |
| $\theta$ | : Location parameter of Laplace distribution |
| $\phi$ | : Scale parameter of Laplace distribution |
| $\mu$ | : Location parameter of Weibull distribution |
| $\sigma$ | : Scale parameter of Weibull distribution |
| $p$ | : Shape parameter of Weibull distribution |

| | |
|---|---|
| k | : Shape parameter of gamma distribution |
| β | : Scale parameter of gamma distribution |
| R | : Reliability |
| L | : Maximum likelihood function |
| b | : Parameter number in Jaya algorithm |
| a | : Iteration number in Jaya algorithm |
| c | : Population number in Jaya algorithm |
| $U_{b,c,a}$ | : Value of parameter b for iteration a and population number c |
| S | : Strength of material or component |
| l | : Load acting on material or component |
| ˆ | : Estimated value |
| Γ | : Gamma function |
| f | : Function value |
| $t_c$ | : Compilation time |
| R-sq | : R-squared |

# Chapter 1

# Introduction

## 1.1 General

Every customer expects that a product they buy should ensure reliability and safety; else, no customer would buy a product that is not reliable and is unsafe. A traditional design for safety includes considering a term 'factor of safety' having a value greater than unity which is the ratio of strength of the body to stress acting on the body. It considers the design to be safe if the strength is greater than stress. But, it is well known that these parameters in the real world are subjected to various uncertainties due to changes in temperatures, pressure, humidity, etc. [1]. The traditional method does not consider these uncertainties and thus, has a greater probability of failure and is unreliable [2]. A good engineering design should prevent accidents, failures, damages and injuries [3]. Reliability-based design ensures safety and quality while at the same time avoiding under-design or over-design of the component. There is a lot of application of component reliability of the form $P(X > Y)$, i.e., strength (X) being greater than stress (Y). Thus, accurate calculation and prediction of reliability is critical in practical applications [4]. In functional testing, components can fail due to many reasons, and material properties can follow different distributions [5]. Hence, reliability calculations for such distributions become extremely important to avoid real-life problems and losses.

## 1.2 Reliability Based Design

The design of complex systems and components used in the modern technological world is crucial because a minor failure can have serious effects. Thus, proper design is vital in order to avoid failure. The conventional design considers the factor of safety which is the ratio of strength to stress. The value of the factor of safety is decided based on the experience. We know that the system properties like stress and strength are random variables. Thus, reliability must be considered at the design stage to optimize safety, cost, weight, etc. The reliability-based design considers the stress and strength as random variables and the uncertainties present in them [6]. The reliability-based design approach originated in the aerospace industry and has now been applied in consumer goods as well because of its effectiveness. The reliability-based design approach is shown in Figure 1.1 below:

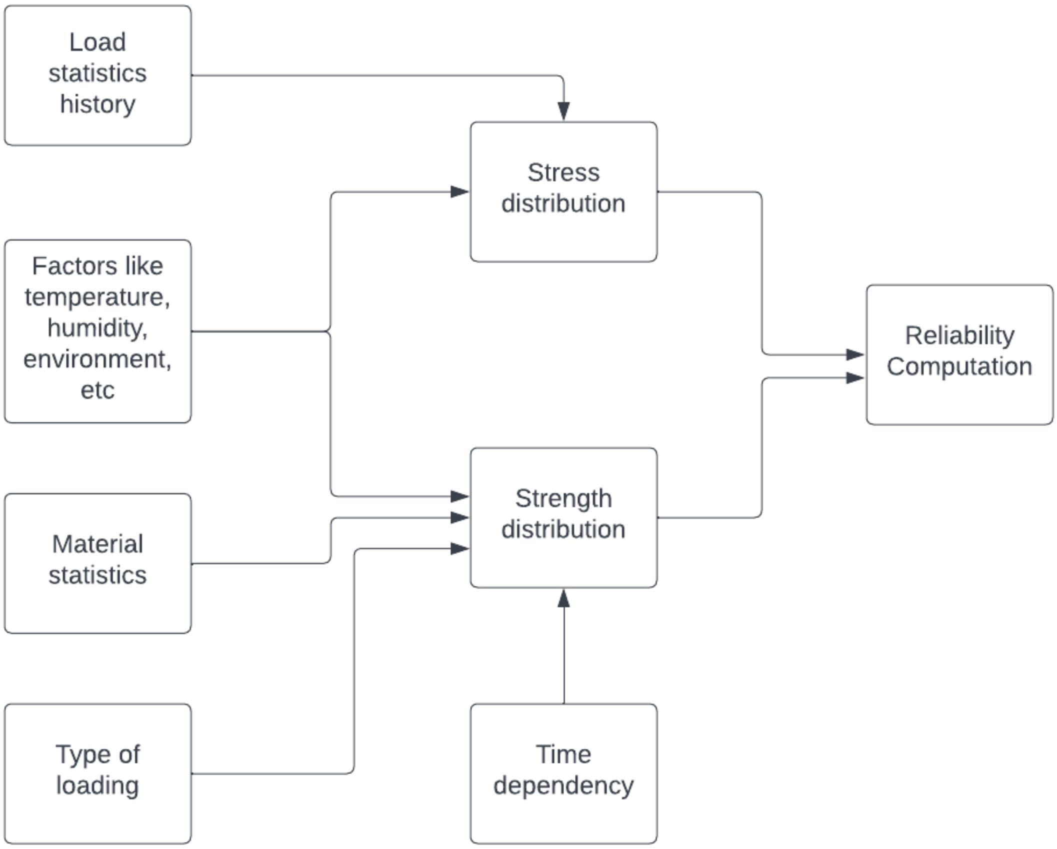


Figure 1.1 Reliability based design approach

The figure shows that stress will follow a particular distribution depending on the load statistics history. The stress uncertainty will also be influenced by external factors like temperature, humidity, environment, etc. These external factors will influence uncertainties in strength as well. The strength distribution will also depend on the material statistics, material properties over time, and the type of loading like static loading, fatigue loading, etc. The reliability of the design will thus depend on the distributions followed by stress and strength. Considering this concept in the design of mechanical systems is crucial, thereby improving safety.

## 1.3 Probability Distributions

In real life, the properties like stress and strength which the mechanical components undergo do not have a definite value but vary in a range. Generally, these properties follow a particular distribution. A probability distribution is a function that gives probable values that a random variable can take within a given range. Two common terms with probability distributions are probability density function (pdf) and cumulative distribution function (cdf). The pdf gives the probability of the random variable taking a value within a particular range. The cdf provides the probability of a random variable taking values lesser than a certain considered value in the distribution. There are many distributions that are used in reliability studies for mechanical systems. Some of the common distributions are considered in this chapter.

### 1.3.1. Normal distribution

A normal distribution is identified by a bell-shaped curve with its mean at the centre and the spread whose probability density decreases as we move away from the mean. The distribution is described by two parameters i.e. mean and standard deviation. The mean is the central tendency located at the peak of bell-shaped curve. The standard deviation is a measure of variability and describes the spread of the distribution. The pdf of normal distribution is given as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_n}{\delta}\right)^2} \tag{1.1}$$

The cdf of normal distribution is

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{t-\mu_n}{\delta}\right)^2} dt \tag{1.2}$$

where x is a random variable following normal distribution, $\mu_n$ is the mean and $\delta$ is the standard deviation of the normal distribution. A normal distribution is symmetrical about the mean with most values at the centre and skewed at the sides. The applications of normal distribution can be seen in many fields like engineering, science, natural, social sciences, etc. as many variables in these fields tend to follow this distribution. The plot of distribution with variation in parameters mean and standard deviation is shown in Figure 1.2.

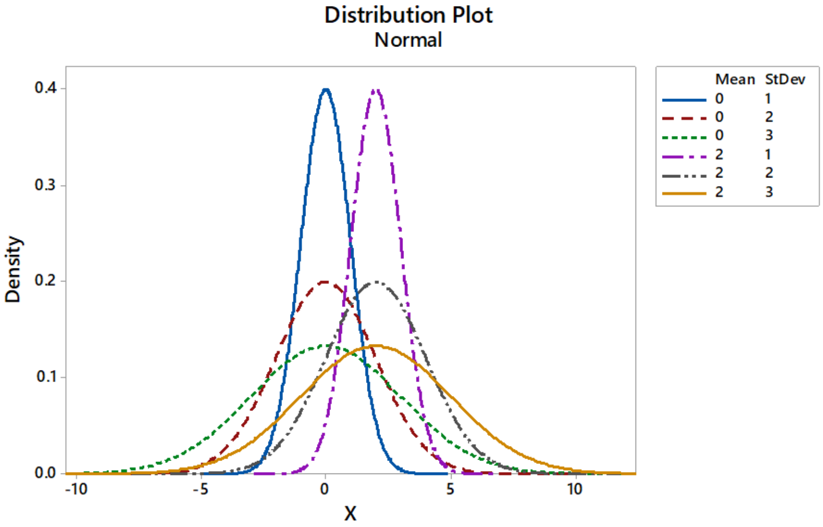


Figure 1.2 Normal distribution plot

### 1.3.2 Lognormal distribution

A lognormal distribution widely used in statistics is a probability distribution function of a random variable whose logarithm is normally distributed. The distribution is used in reliability applications. The pdf of lognormal distribution can be given as

$$f(x) = \frac{1}{x\delta\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(x)-\mu_n}{\delta}\right)^2} \qquad 1.3$$

The cdf of exponential distribution is given as

$$F(x) = \Phi\left(\frac{\ln(x) - \mu_n}{\delta}\right) \qquad 1.4$$

where x is a random variable following lognormal distribution, $\mu_n$ is the mean and δ is the standard deviation of the lognormal distribution. A lognormal plot differs from normal distribution in various ways. The main differences are a lognormal distribution is not a symmetrical distribution like normal. Also, since the logarithm values are positive, the lognormal distribution is right skewed. The mean in lognormal distribution gives the location of the graph whereas δ defines the shape of the distribution plot. The plot of lognormal distribution with variation in parameters mean and standard deviation is shown in Figure 1.3.

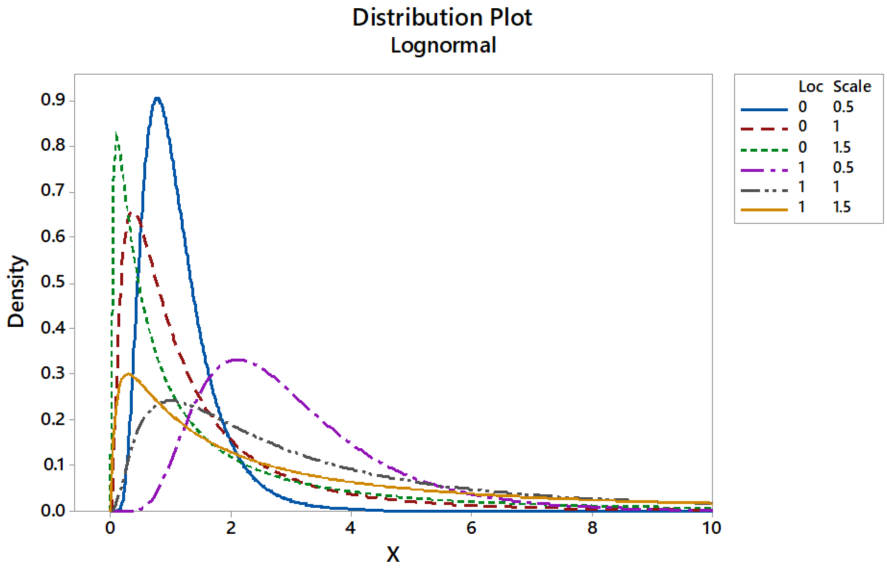


Figure 1.3 Lognormal distribution plot

### 1.3.3 Exponential distribution

The exponential distribution gives time between events in a Poisson process. It is also used in reliability studies to model the failure rate of components. A single parameter exponential distribution is defined by rate parameter or scale parameter (reciprocal of rate parameter). The pdf of exponential distribution is given as

$$f(x) = \lambda e^{-\lambda x} \qquad x \geq 0$$
$$f(x) = 0 \qquad x < 0 \tag{1.5}$$

The cdf of exponential distribution is given as

$$F(x) = 1 - e^{-\lambda x} \qquad x \geq 0$$
$$F(x) = 0 \qquad x < 0 \tag{1.6}$$

where x is a random variable following exponential distribution and λ is the rate parameter. The probability distribution plot of exponential distribution with variation in rate parameter is shown in Figure 1.4.

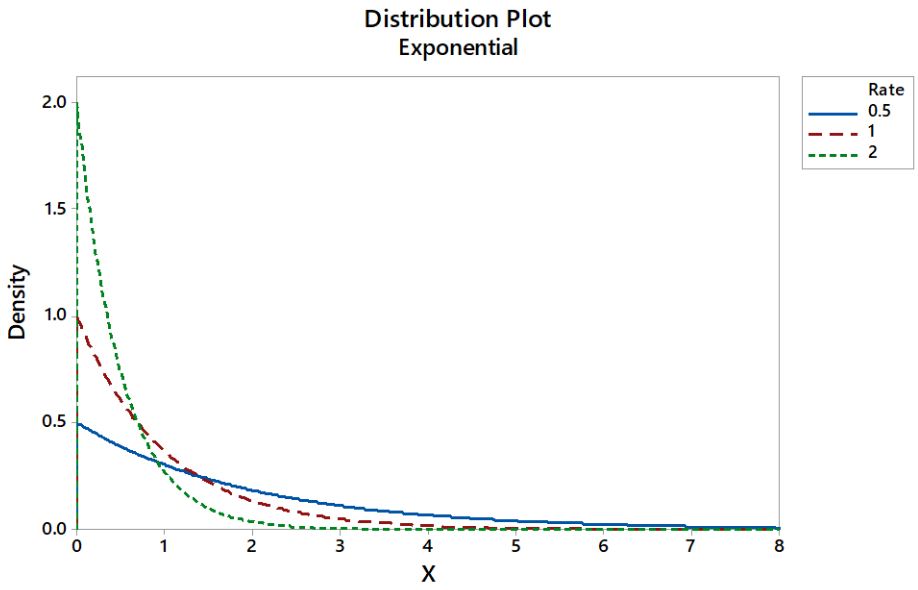


Figure 1.4 Exponential distribution plot

### 1.3.4 Laplace distribution

Laplace distribution which is sometimes also called as double-sided exponential distribution since it looks like a symmetrical image of exponential distribution joined together at one end. The pdf of Laplace distribution is give by

$$f(x) = \frac{1}{2\phi} e^{-\frac{|x-\theta|}{\phi}} \qquad 1.7$$

$$f(x) = \frac{1}{2\phi} e^{-\frac{(x-\theta)}{\phi}} \quad x \geq \theta \qquad 1.8$$

$$f(x) = \frac{1}{2\phi} e^{-\frac{(\theta-x)}{\phi}} \quad x < \theta \qquad 1.9$$

The cdf of Laplace distribution is

$$F(x) = 1 - \frac{1}{2} e^{-\frac{(x-\theta)}{\phi}} \quad x \geq \theta \qquad 1.10$$

$$F(x) = \frac{1}{2} e^{\frac{(x-\theta)}{\phi}} \quad x < \theta \qquad 1.11$$

where x is a random variable following Laplace distribution, θ is the location parameter and ϕ is the scale parameter. The probability distribution plot for Laplace distribution with varying parameters is shown in Figure 1.5.

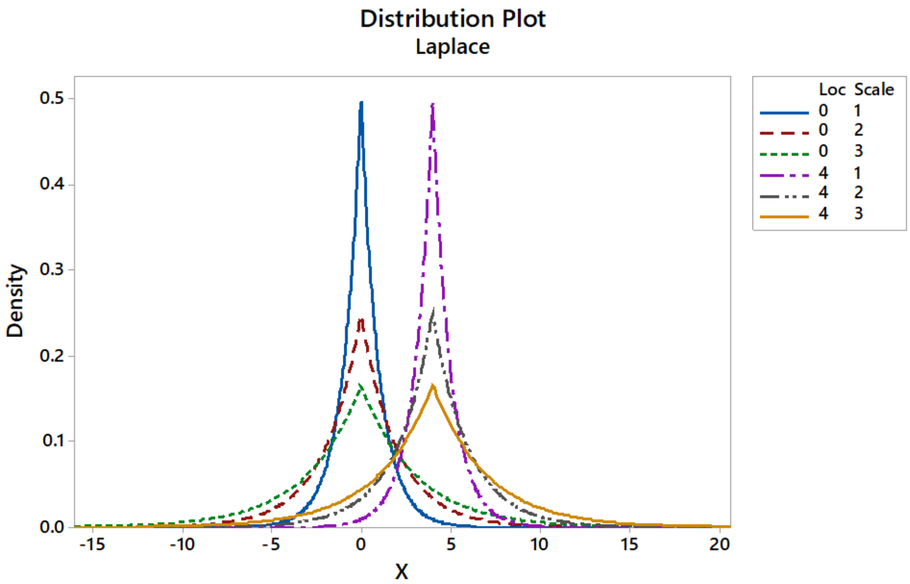


Figure 1.5 Laplace distribution plot

### 1.3.4 Weibull distribution

Three-parameter Weibull distribution has been extensively used in reliability and lifetime studies as it is known to fit a wide range of data and is immensely flexible [7–9]. The pdf of three parameter Weibull distribution is given by

$$f(x;\mu,\sigma,p)=\frac{p}{\sigma^p}(x-\mu)^{p-1}\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^p\right\},x>\mu,\sigma>0,p>0 \qquad 1.12$$

and cdf is given by

$$F(x;\mu,\sigma,p)=1-\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^p\right\} \qquad 1.13$$

where, x is a random variable following Weibull distribution, μ is the location parameter, $\sigma$ is the scale parameter and $p$ is the shape parameter. Random variable X has a pdf denoted by X ~ W(μ, $\sigma$, $p$).

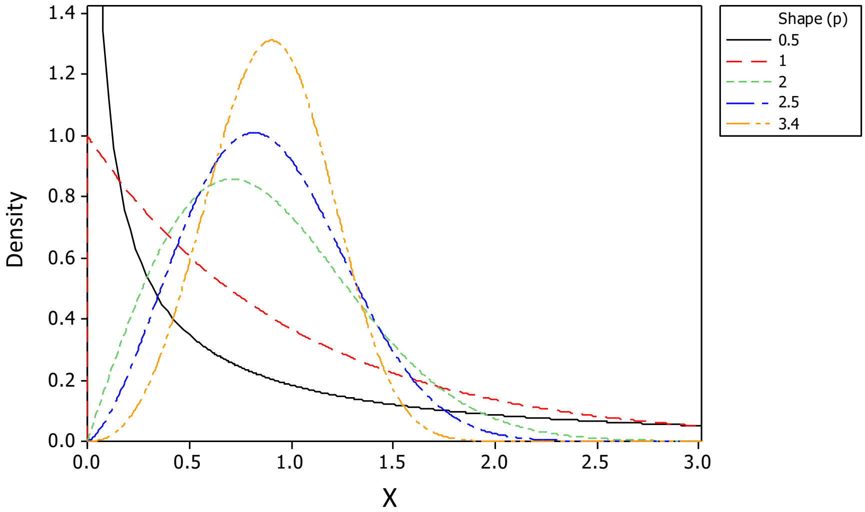


Figure 1.6 Weibull distribution plot for $\mu = 0, \sigma = 1,\ p = 0.5,\ 1,\ 2,\ 2.5,\ 3.4$

The density curve of the three-parameter Weibull distribution can take a wide variety of shapes by appropriately choosing the shape parameter. A curve of decreasing density function is obtained when σ takes the values from 0 to 1. When $\sigma$ = 1, the Weibull distribution is identical to the two-parameter exponential distribution; when $\sigma$ = 2, it becomes the Rayleigh distribution; when $\sigma$ = 2.5, it approximates the lognormal distribution. When $\sigma$ > 1, the curve is bell-shaped and is right-skewed. In many cases, a value of $\sigma$ = 3.4 is used to approximate the normal distribution [10]. The probability distribution plot for Weibull distribution with varying parameters is shown in Figure 1.6. Because of such flexibility, the Weibull distribution is one of the most widely used models in reliability studies and life testing. The Weibull distribution is also used in engineering sciences, medicine, agriculture and biology [11–14].

### 1.3.5 Gamma distribution

Gamma distribution is another widely used distribution because of its flexibility and relation to normal and exponential distribution. The pdf of gamma distribution can be given by

$$f(x) = \frac{1}{\beta^{\mathrm{k}}\ \Gamma\mathrm{k}}(x)^{\mathrm{k}-1}e^{-\frac{(x)}{\beta}} \qquad 1.14$$

and cdf is given by

$$F(x) = \frac{1}{\Gamma\mathrm{k}}\,\gamma\left(\mathrm{k}, \frac{x}{\beta}\right) \qquad 1.15$$

where x is a random variable following gamma distribution, k is the shape parameter and β is the scale parameter of gamma distribution. $\gamma\left(\mathrm{k}, \frac{x}{\beta}\right)$ is the lower incomplete gamma function. The distribution plot for gamma distribution is show in Figure 1.7.

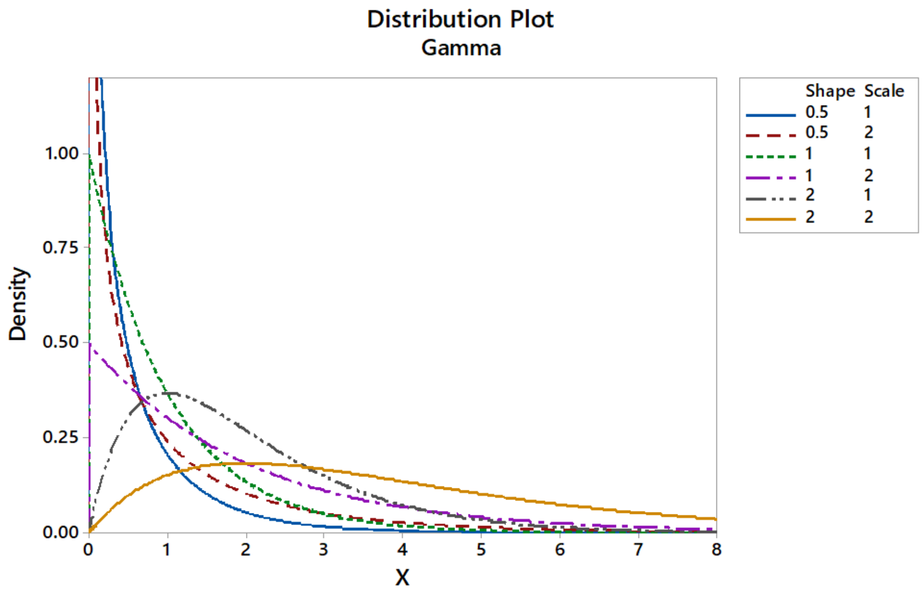


Figure 1.7 Gamma distribution plot

## 1.4 Stress-Strength Interference Theory

Stress-strength analysis finds many applications in the mechanical or structural systems. These systems differ from electrical or electronic systems in a number of ways. The electrical or electronic systems fail because of prolonged use due to deterioration. Also, these systems are mass-produced because of which a lot of data is available and the reliability for these systems is evaluated by considering time to failure as the random variable. On the other hand, the mechanical or structural systems are not mass produced; hence the data available is sometimes insufficient. Also, the failures in these cases occur because of small variations or difference in load and strength. Thus, the reliability of mechanical or structural systems is dealt with using interference theory. The stress-strength interference theory states that if stress and strength follow a particular distribution, their interference area gives the probability of failure. Figure 1.8 shows the interference with stress and strength following a particular distribution. The figure depicts two curves of the distribution of stress and the distribution of strength. The common area between the two curves which is hatched (interference area) gives the probability of failure. Apart from mechanical systems the interference theory can be used to compare the two variables which finds applications in other fields like medical, service, etc.

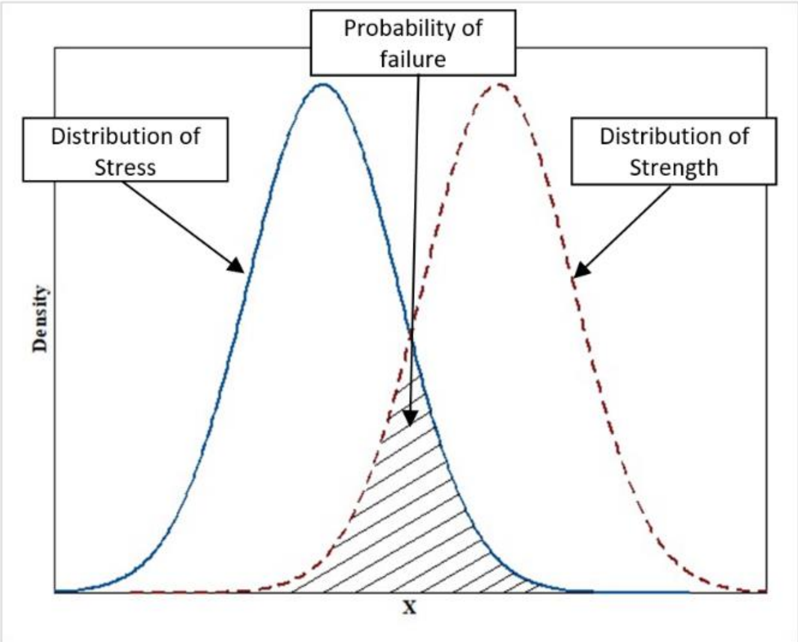


Figure 1.8 Stress-strength interference

The reliability can be found using stress-strength interference model which is given by:

$$R = P(X > Y) = \int_0^\infty f(x)\left(\int_0^x f(y)dy\right)dx \quad 1.16$$

where X is the random variable for strength and Y is the random variable for stress.

Alternatively, the reliability can also be found by using the equation

$$R = P(X > Y) = \int_0^\infty f(y)\left(\int_y^\infty f(x)dx\right)dy \quad 1.17$$

While using the above equation, one should note that the units of stress and strength in the above expression should be the same. The context of stress and strength can be used in various applications. For example, if the system under study is a structure subjected to some load, then the stress will be the maximum load acting, and the strength will be the yield strength of the structure. For a machine tool, the power required for machining operation will be considered as the stress, and the rated horsepower of the machine tool can be regarded as the strength. For aerospace applications, the maximum pressure generated by fuel ignition can be considered as the stress and the strength of the rocket chamber for the successful firing of the rocket. In medical applications, the theory can be used to compare the effect of one treatment over another. Similarly, the concept can also be used for comparing the strength of one material being greater than another. The theory can be used in non-mechanical applications as well. For example, in a hydrological system, the stress can be considered as the demand of water, and the strength can be considered as the available supply.

## 1.5 Aims and Objectives

Reliability is critical in predicting the functioning of components at a given time. Stress-strength interference theory which has been described in this chapter earlier is also a crucial concept in predicting the reliability of the components. There are many stress-strength interference models which have been developed. But there are distributions for which the interference models are not available in closed form. The first objective of this research work is to develop a reliability model for stress-strength interference having distributions such as Laplace, exponential, Weibull, gamma, etc., for which the model in close form is not available. As we know from the stress-strength interference theory, the reliability depends significantly on the parameters driving the stress and strength distributions. Weibull distribution is one of the most widely used distribution in reliability studies. This research aims to estimate the stress-strength reliability when stress and strength follow Weibull distribution. It also aims to study the variation in reliability with the variation in stress-strength parameters. In real life, a mechanical component's strength does not remain the same and degrades over time. Even in fatigue loading, the strength of the material decreases with the number of cyclic loads. Thus, the nature of strength can be considered to be dynamic. The final objective of the research is to evaluate reliability model for stress-strength interference taking strength degradation into consideration.

## 1.6 Organization of Report

This thesis has been divided into seven chapters. The first chapter deals with the introduction to the topic and explains the concepts of reliability-based design, various types of probability distributions that are used in the research work along with the distribution plots with varying parameters, and the stress-strength interference theory. The aims and objectives have been presented at the end of the chapter. The second chapter gives a detailed literature review of the existing research work in the area of stress-strength interference, reliability estimation, and time-dependent stress-strength reliability. The problem description and solution methodologies have been discussed in the third chapter. The fourth chapter includes the stress-strength interference models developed when stress and strength follow exponential and Laplace distributions and vice versa. Also, the models for evaluating stress-strength interference for Weibull and gamma distribution have been depicted in this chapter. The fifth chapter deals with the methodology of estimation and discusses the stress-strength reliability estimation for stress and strength first for two parameter Weibull distribution with common scale parameter and then for three-parameter Weibull distribution with common shape parameter. The sixth chapter depicts the methodology to develop stress-strength interference models in case of strength degradation where the strength is considered to deteriorate with time or in case of fatigue loading. A polynomial regression has been considered and the study has been carried out to see the effect of type of degradation on reliability. The flow of research work has been depicted in Figure 1.9. The final chapter states the conclusions, contributions and limitations of the research work. The future scope for the research has also been stated for the further work that can be carried out in this field of research.

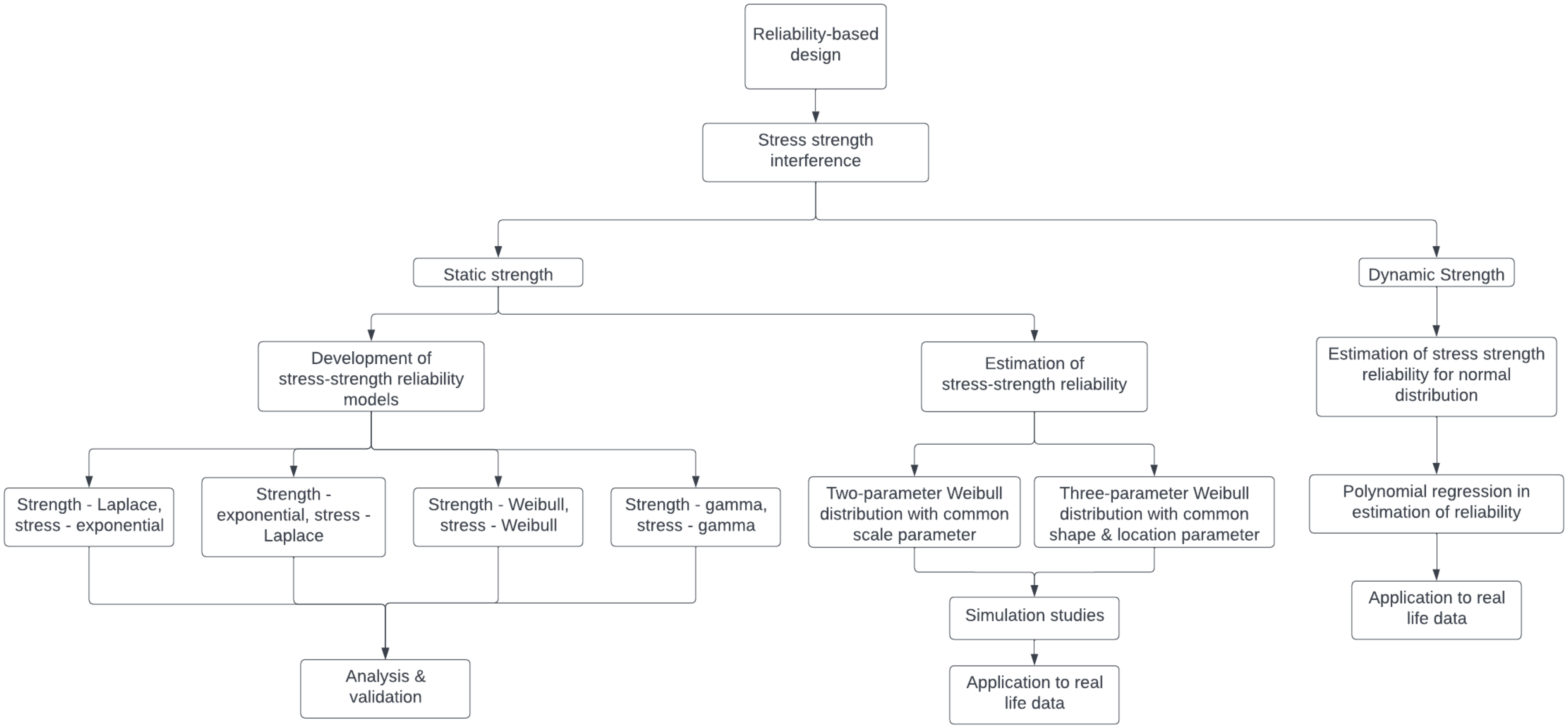


Figure 1.9 Flow of research work

# Chapter 2

# Literature Review

## 2.1 General

Reliability of the form P [X > Y] is used in cases of stress-strength interference [15]. Stress and strength are important properties of a material. These properties do not have a single fixed value because of the uncertainties present in the environment like temperature, humidity, etc. So, they can be considered to follow a certain distribution. According to the interference theory, if stress and strength follow a certain distribution, their interference area gives the probability of failure. The concept of stress-strength interference in evaluating reliability has been used by many researchers in their studies. Liu et al. [16] evaluated the reliability of automotive seat adjuster by using the stress-strength interference model. The finite element model of the seat adjuster was constructed, and the analysis was verified with the bench test. The theory has also been used in medical applications by Miller and Freivalds [17] to obtain the probability of failure of tendons in carpel tunnel syndrome.

## 2.2 Stress-Strength Interference

Reliability models have been developed when strength and stress are seen to be following normal, lognormal, exponential distribution and their interference [18].

### 2.2.1 Reliability when stress and strength follow normal distribution

Consider the strength (random variable X) and stress (random variable Y) follow normal distribution with pdf

$$f(x) = \frac{1}{\delta_x\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x - \mu_{nx}}{\delta_x}\right)^2\right) \quad 2.1$$

and

$$f(y) = \frac{1}{\delta_y\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y - \mu_{ny}}{\delta_y}\right)^2\right) \quad 2.2$$

respectively, where $\mu_{nx}$ is the mean of strength, $\delta_x$ is the standard deviation of strength, $\mu_{ny}$ is the mean of stress and $\delta_y$ is the standard deviation of strength.

As per the interference theory, the reliability of the system will be equal to [19]

$$R = \int_0^\infty \frac{1}{\delta_x\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x - \mu_{nx}}{\delta_x}\right)^2\right)\left(\int_0^x \frac{1}{\delta_y\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y - \mu_{ny}}{\delta_y}\right)^2\right) dy\right) dx \quad 2.3$$

On simplifying the above equation, the reliability can be obtained as

$$R = \phi\left(\frac{\mu_{nx} - \mu_{ny}}{\sqrt{\delta_x{}^2 + \delta_y{}^2}}\right) \quad 2.4$$

### 2.2.2 Reliability when stress and strength follow lognormal distribution

Consider that the strength (random variable X) and stress (random variable Y) follow lognormal distribution with pdf

$$f(x) = \frac{1}{x.\,\delta_x\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln(x) - \mu_{nx}}{\delta_x}\right)^2\right) \quad 2.5$$

and

$$f(y) = \frac{1}{y.\,\delta_y\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln(y) - \mu_{ny}}{\delta_y}\right)^2\right) \quad 2.6$$

respectively, where $\mu_{nx}$ is the mean and $\delta$ is the standard deviation of ln(X), $\mu_{ny}$ is the mean and $\delta_y$ is the standard deviation of ln(Y).

As per the interference theory, the reliability of the system will be equal to [20]

$$R = \int_0^\infty \frac{1}{x.\,\delta_x\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln(x) - \mu_{nx}}{\delta_x}\right)^2\right)\left(\int_0^x \frac{1}{y.\,\delta_y\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln(y) - \mu_{ny}}{\delta_y}\right)^2\right) dy\right) dx \quad 2.7$$

On simplifying the above equation, the reliability can be obtained as

$$R = \phi\left(\frac{\mu_{nx} - \mu_{ny}}{\sqrt{\delta_x{}^2 + \delta_y{}^2}}\right) \qquad 2.8$$

### 2.2.3 Reliability when stress and strength follow exponential distribution

Consider that the strength (random variable X) and stress (random variable Y) follow lognormal distribution with pdf

$$f(x) = \lambda_x\, e^{-\lambda_x x} \qquad 2.9$$

and

$$f(y) = \lambda_y\, e^{-\lambda_y y} \qquad 2.10$$

where $\lambda$ is the rate parameter.

As per the interference theory, the reliability of the system will be equal to

$$R = \int_0^\infty \lambda_x\, e^{-\lambda_x x}\left(\int_0^x \lambda_y\, e^{-\lambda_y y} dy\right) dx \qquad 2.11$$

On simplifying the above equation, the reliability can be obtained as [21]

$$R = \frac{\lambda_y}{\lambda_x + \lambda_y} \qquad 2.12$$

### 2.2.4 Reliability when strength follows normal distribution and stress follows exponential distribution

Consider that the strength follows normal distribution (random variable X) and stress follows exponential distribution (random variable Y) with pdf

$$f(x) = \frac{1}{\sigma_x\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x - \mu_x}{\sigma_x}\right)^2\right) \qquad 2.13$$

and

$$f(y) = \lambda_y\, e^{-\lambda_y y} \qquad 2.14$$

then the reliability of the system can be given as

$$R = \int_0^{\infty} \frac{1}{\sigma_x\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right)\left(\int_0^{x} \lambda_y\, e^{-\lambda_y y} dy\right) dx \qquad 2.15$$

On simplification, the closed form of reliability can be obtained as [22]

$$R = 1 - \phi\left(-\frac{\mu_s}{\sigma_s}\right) - exp\left[-\frac{1}{2}(2\mu_s\lambda_s - \lambda_s^2{\sigma_s}^2)\right]\left[1 - \phi\left(-\frac{\mu_s - \lambda_s\,{\sigma_s}^2}{\sigma_s}\right)\right] \qquad 2.16$$

### 2.2.5 Reliability when strength follows exponential distribution and stress follows normal distribution

Consider that the strength follows exponential distribution (random variable X) and stress follows normal distribution (random variable Y) with pdf

$$f(x) = \lambda_x\, e^{-\lambda_x x} \qquad 2.17$$

and

$$f(y) = \frac{1}{\sigma_y\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right) \qquad 2.18$$

then the reliability of the system can be given as

$$R = \int_0^{\infty} \lambda_x\, e^{-\lambda_x x}\left(\int_0^{x} \frac{1}{\sigma_y\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right) dy\right) dx \qquad 2.19$$

Equation 2.19 can be further simplified to obtain the closed form of reliability [23]. On simplification, the closed form of reliability can be obtained as

$$R = \phi\left(-\frac{\mu_s}{\sigma_s}\right) + exp\left[-\frac{1}{2}(2\mu_s\lambda_s - \lambda_s^2{\sigma_s}^2)\right]\left[1 - \phi\left(-\frac{\mu_s - \lambda_s\,{\sigma_s}^2}{\sigma_s}\right)\right] \qquad 2.20$$

### 2.2.6 Reliability when stress and strength follow Laplace distribution

Consider that the strength with random variable X and stress with random variable Y follow Laplace distribution. Their corresponding pdf will be as shown in equation 2.13, 2.14 and 2.15 for strength and 2.16, 2.17 and 2.18 for stress respectively.

$$f(x) = \frac{1}{2\phi_1} e^{-\frac{|x-\theta_1|}{\phi_1}} \tag{2.21}$$

$$f(x) = \frac{1}{2\phi_1} e^{-\frac{(x-\theta_1)}{\phi_1}} \quad x \geq \theta_1 \tag{2.22}$$

$$f(x) = \frac{1}{2\phi_1} e^{-\frac{(\theta_1-x)}{\phi_1}} \quad x < \theta_1 \tag{2.23}$$

$$f(y) = \frac{1}{2\phi_2} e^{-\frac{|y-\theta_2|}{\phi_2}} \tag{2.24}$$

$$f(y) = \frac{1}{2\phi_2} e^{-\frac{(y-\theta_2)}{\phi_2}} \quad y \geq \theta_2 \tag{2.25}$$

$$f(y) = \frac{1}{2\phi_2} e^{-\frac{(\theta_2-y)}{\phi_2}} \quad y < \theta_2 \tag{2.26}$$

where $\theta_1$ and $\phi_1$ are the location and scale parameters for strength, $\theta_2$ and $\phi_2$ are the location and scale parameters for stress.

As per the interference theory, the reliability of the system will be equal to

$$R = \int_0^\infty \frac{1}{2\phi_1} e^{-\frac{|x-\theta_1|}{\phi_1}} \left( \int_0^x \frac{1}{2\phi_2} e^{-\frac{|y-\theta_2|}{\phi_2}} dy \right) dx \tag{2.27}$$

On simplifying the above equation, the reliability can be obtained as [24]

$$R = \frac{{\phi_1}^2}{2({\phi_1}^2 - {\phi_2}^2)} exp\left(\frac{\theta_1-\theta_2}{\phi_1}\right) - \frac{{\phi_2}^2}{2({\phi_1}^2 - {\phi_2}^2)} exp\left(\frac{\theta_1-\theta_2}{\phi_2}\right) \text{ if } \theta_1 \leq \theta_2 \tag{2.28}$$

$$R = 1 + \frac{{\phi_1}^2}{2({\phi_2}^2 - {\phi_1}^2)} exp\left(\frac{\theta_2 - \theta_1}{\phi_1}\right) - \frac{{\phi_2}^2}{2({\phi_2}^2 - {\phi_1}^2)} exp\left(\frac{\theta_2 - \theta_1}{\phi_2}\right) \; if \; \theta_1 > \theta_2 \qquad 2.29$$

### 2.2.7 Reliability when stress and strength follow Weibull distribution

Consider that the strength (random variable X) and stress (random variable Y) follow Weibull distribution with pdf

$$f(x) = \frac{p_1}{{\sigma_1}^{p_1}}(x - \mu_1)^{p_1 - 1} \exp\left\{-\left(\frac{x - \mu_1}{\sigma_1}\right)^{p_1}\right\}, x > \mu_1, \sigma_1 > 0, p_1 > 0 \qquad 2.30$$

and

$$f(y) = \frac{p_2}{{\sigma_2}^{p_2}}(y - \mu_2)^{p_2 - 1} \exp\left\{-\left(\frac{y - \mu_2}{\sigma_2}\right)^{p_2}\right\}, y > \mu_2, \sigma_2 > 0, p_2 > 0 \qquad 2.31$$

where $\mu_1, \sigma_1$ $and$ $p_1$ are the location, scale and shape parameter respectively for strength and $\mu_2, \sigma_2$ $and$ $p_2$ are location, scale and shape parameter respectively for stress.
As per the interference theory, the reliability of the system will be equal to

$$R = \int_0^\infty \frac{p_1}{{\sigma_1}^{p_1}}(x - \mu_1)^{p_1 - 1} \exp\left\{-\left(\frac{x - \mu_1}{\sigma_1}\right)^{p_1}\right\}\left(\int_0^x \frac{p_2}{{\sigma_2}^{p_2}}(y - \mu_2)^{p_2 - 1} \exp\left\{-\left(\frac{y - \mu_2}{\sigma_2}\right)^{p_2}\right\} dy\right) dx \qquad 2.32$$

$$R = \int_0^\infty \frac{p_1}{{\sigma_1}^{p_1}}(x - \mu_1)^{p_1 - 1} \exp\left\{-\left(\frac{x - \mu_1}{\sigma_1}\right)^{p_1}\right\}\left(e^{-\frac{e^{p_2 \ln(-\mu_2)}}{{\sigma_2}^{p_2}}} - e^{-\frac{e^{p_2 \ln(x - \mu_2)}}{{\sigma_2}^{p_2}}}\right) dx \qquad 2.33$$

Equation 2.25 cannot be solved further and hence, the calculation of stress-strength reliability for Weibull distribution has to be solved using numerical or graphical methods [25]. This can sometimes lead to complications and is time consuming.

### 2.2.8 Reliability when stress and strength follow gamma distribution

Consider that the strength (random variable X) and stress (random variable Y) follow gamma distribution with pdf

$$f(x) = \frac{1}{{\beta_1}^{k_1} \Gamma k_1}(x)^{k_1 - 1} e^{-\frac{(x)}{\beta_1}} \qquad \beta_1 > 0, k_1 > 0 \qquad 2.34$$

and

$$f(y) = \frac{1}{{\beta_2}^{k_2}\,\Gamma k_2}(y)^{k_2-1}e^{-\frac{(y)}{\beta_2}} \qquad \beta_2 > 0, k_2 > 0 \tag{2.35}$$

where $\beta_1$ *and* $k_1$ scale and shape parameters respectively for strength and $\beta_2$ *and* $k_2$ are scale and shape parameters respectively for stress.

As per the interference theory, the reliability of the system will be equal to [26]

$$R = \int_0^\infty \frac{1}{{\beta_1}^{k_1}\,\Gamma k_1}(x)^{k_1-1}e^{-\frac{(x)}{\beta_1}}\left(\int_0^x \frac{1}{{\beta_2}^{k_2}\,\Gamma k_2}(y)^{k_2-1}e^{-\frac{(y)}{\beta_2}}\;dy\right)dx \tag{2.36}$$

Equation 2.28 cannot be solved entirely in order to obtain a model and thus is dependent on numerical or graphical techniques which is time-consuming and may not give accurate results.

Similarly, the reliability models have been developed for other distributions of stress and strength. S. Nadarajah, 2003 [27] developed stress-strength interference for stress and strength following lifetime distributions i.e. exponential and gamma distribution. He has also developed a reliability model for stress and strength following bivariate gamma distribution [24]. Patowary et al., 2013 [28] studied and proposed a mathematical model for stress-strength reliability for stress and strength following mixture of distributions. An inference on reliability was also drawn, stating standby redundancy aids in achieving high reliability. An et al., 2008 [29] developed a discrete stress-strength interference model based on universal generating function. K. Shen, 1992 [30] proposed a new empirical approach based on the subinterval probabilities of stress and strength in the interference region to compute the unreliability bounds. Kotz et al., 2003 [31] reviewed the stress-strength interference models and showed practical results in application of stress-strength interference concepts in industrial systems. Many studies have been carried out in developing stress-strength reliability models for various distributions. However, it has been identified that the reliability model for many distributions is not yet developed.

## 2.3 Stress-Strength Reliability Estimation

The technological developments in the field of aircraft, nuclear power plants, infrastructure, transportation, etc. have raised serious concerns with reliability and safety, as a small error in the design of applications in these sectors can cause a huge disaster. Novel challenges are posed every day and thus, the field of reliability and safety is gaining increasing importance. The designing and assessment of components or operating procedures in the above-mentioned industries based on reliability can be very effective in preventing failures or accidents [32,33].

Properties of a component like stress and strength require precise designing as these are vital in determining the safety of the component. In real life, we know that properties like stress and strength of mechanical components do not take a fixed value due to the various uncertainties in materials, loading conditions, environmental conditions, etc. Thus, they can be considered to follow a particular distribution which can be determined based on the application or prior data. Traditional methods may not be the best approach for designing mechanical components as they do not consider uncertainties. Hence, reliability-based design will be suitable in such cases which takes into account the probability of failure if the stress and strength takes various values within its range. In the context of stress and strength, reliability can be defined as the probability of strength being greater than stress [29,34–36].

In order to determine reliability, estimating the parameters of stress and strength distribution is crucial. A vast amount of research has been carried in estimation of reliability of components subjected to various distributions stress and strength.

### 2.3.1 Stress-strength reliability estimation for various distributions

Church and Harris, 1970 [37] studied the estimation of stress-strength reliability when stress and strength follow normal distribution. Kelley et al., 1976 [38] compared the maximum likelihood estimation method with Uniformly Minimum Variance Unbiased Estimator (UMVUE) in estimation of stress-strength reliability when stress and strength follow exponential distribution. MLE method was seen to be less complicated in estimation of reliability. Awad and Charraf, 1986 [39] carried out studies in estimation of reliability when the stress and strength are two independent Burr random variables.

Badr et al., 2019 [40] derived a closed form of stress-strength reliability models for exponentiated Frechet distribution using Maximum likelihood estimation, Bayes estimation and uniformly minimum variance unbiased estimator. Analysis was carried out using simulation to compare the results in terms of bias and mean squared error (MSE), and to see the effect of sample size and parameters on reliability estimation. It was observed that the MSE decreases with increase in sample size of either stress or strength, with other being constant. It was also observed that as the parameter of strength increases, the reliability estimation increases and if the parameter of stress increases, the reliability estimation decreases.

Kayal et al., 2020 [41] worked on evaluating multicomponent stress-strength reliability for Chen distribution using classical and Bayes estimation method. Similar studies have been conducted by many other researchers in the estimation of stress-strength reliability by analyzing different estimation methods for stress and strength following various distributions. Raqab and Kundu, 2005 [42] have presented a simple iterative procedure for estimating the stress-strength reliability using MLE for a scaled Burr type X distribution when the scale parameter is unknown. Chaudhary and Tomer, 2018 [43] estimated stress-strength reliability for Maxwell distribution by MLE and Bayes estimates using progressive type-II censored samples. Rezaei et al., 2015 [44] worked on estimation of stress-strength reliability for generalized Pareto distribution based on progressively censored samples using MLE and inferred that when the common scale parameter is unknown, the maximum likelihood estimator gives good results. They recommended the bootstrap percentile method for estimating the confidence intervals when the sample size is very small. Siju et al. 2020 [45] presented reliability estimation for the stress and strength following exponential distribution using MLE for three states namely, working, deteriorating and failed state. Pham, 2020 [46] applied the stress-strength reliability model for exponential distribution in human heart condition. The data was collected for average heart rate in 15 second period and the reliability of k out of n interval system was evaluated. A k-out-of-n interval system is a system with a series of n events in a given interval of time that successes if and only if at least k of the events succeed. Jha et al., 2020 [47] presented reliability estimation for a multicomponent setup when stress and strength follow unit Gompertz distribution with common scale parameter under progressive Type II censoring scheme using MLE.

Abravesh et al., 2018 [48] worked on estimating the reliability for stress-strength interference for distributions with power hazard function based on upper record values and the model was applied for steel specimen data. An et al. 2008 [29] treated stress and strength as discrete random variables and proposed a stress-strength interference model based on universal generating function. It was found that reducing the length of subinterval improved the estimation accuracy when the range of stress and strength is fixed. The advantage of the method is that the actual distributions of stress and strength need not be known in evaluating the reliability for this case. Gadde, 2017 [49] studied estimation of stress-strength reliability for erlang truncated exponential distribution with different shape parameters for a multicomponent setup. The method of MLE is used and simulation studies are carried out. It was observed that the average bias was negative when the shape parameter of strength was less than or equal to that of stress. The MSE was seen to be decreasing when the shape parameter of strength or stress increases.

### 2.3.2 Stress-strength reliability estimation for Weibull distributions

Kundu and Gupta, 2006 [50] studied the estimation of stress-strength reliability for two parameter Weibull distribution with common shape parameter. If X and Y follow Weibull distribution with $W(\sigma_1, p)$ and $W(\sigma_2, p)$ respectively, then the reliability can be given as

$$R = P(X > Y) = \frac{\sigma_1}{\sigma_1 + \sigma_2} \qquad 2.37$$

The maximum likelihood estimator of R was obtained by using a log likelihood function.

$$L(\sigma_1, \sigma_2, p) = (\mathrm{m} + \mathrm{n}) \ln(\mathrm{p}) - \mathrm{n}\ln(\sigma_1) - m ln(\sigma_2) + (p-1) \times \left( \sum_{i=1}^{n} \ln(x_i) + \sum_{j=1}^{m} \ln(y_j) \right) - \frac{1}{\sigma_1} \sum_{i=1}^{n} x_i{}^p - \frac{1}{\sigma_2} \sum_{j=1}^{m} y_j{}^p \qquad 2.38$$

The MLE of $\sigma_1, \sigma_2, p$ can be obtained as $\widehat{\sigma_1}$ , $\widehat{\sigma_2}, \hat{p}$ respectively as a solution of

$$\frac{\partial L}{\partial p} = \frac{m+n}{p} + \sum_{i=1}^{n} \ln(x_i) + \sum_{j=1}^{m} \ln(y_j) - \frac{1}{\sigma_1} \sum_{i=1}^{n} x_i{}^p \ln(x_i) - \frac{1}{\sigma_2} \sum_{j=1}^{m} y_j{}^p \ln(y_j) = 0 \qquad 2.39$$

$$\frac{\partial L}{\partial \sigma_1} = -\frac{n}{\sigma_1} + \frac{1}{\sigma_1{}^2} \sum_{i=1}^{n} x_i{}^p = 0 \qquad 2.40$$

$$\widehat{\sigma_1}(p) = \frac{1}{n} \sum_{i=1}^{n} x_i{}^p \qquad 2.41$$

$$\frac{\partial L}{\partial \sigma_2} = -\frac{m}{\sigma_2} + \frac{1}{\sigma_2{}^2} \sum_{j=1}^{m} y_j{}^p = 0 \qquad 2.42$$

$$\widehat{\sigma_2}(p) = \frac{1}{m} \sum_{j=1}^{m} y_j{}^p \qquad 2.43$$

$\hat{p}$ is obtained as a solution of non-linear equation of the form

$$h(p) = p \qquad 2.44$$

$$h(p) = \frac{m + n + \sum_{i=1}^{n} \ln(x_i{}^p) + \sum_{j=1}^{m} \ln\left(y_j{}^p\right)}{\dfrac{\sum_{i=1}^{n} x_i{}^p \ln(x_i)}{\frac{1}{n}\sum_{i=1}^{n} x_i{}^p} + \dfrac{\sum_{j=1}^{m} y_j{}^p \ln\left(y_j\right)}{\frac{1}{m}\sum_{j=1}^{m} y_j{}^p}} \qquad 2.45$$

The estimate of p is obtained as an iterative procedure when the difference between two consecutive iterations is sufficiently small. The iterative process may be complicated or time consuming and depends on the initial guess of p when starting the iterative process.

The reliability estimate then can be obtained as

$$\hat{R} = \frac{\widehat{\sigma_1}}{\widehat{\sigma_1} + \widehat{\sigma_2}} \qquad 2.46$$

The author in the end mentions that obtaining reliability considering varying shape parameters is quite complicated.

Valiollahi et al., 2013 [51] presented a different approach in estimating the stress-strength reliability for Weibull distribution with same scale parameters but different shape parameters under progressive type-II censoring scheme. A different form of two parameter Weibull distribution has been considered with X and Y following Weibull distribution W$(p_1, \sigma)$ and W$(p_2, \sigma)$ respectively with pdfs

$$f(x, p_1, \sigma) = \frac{p_1}{\sigma} x^{p_1 - 1} e^{-\frac{x^{p_1}}{\sigma}} \qquad x > 0, \qquad p_1, \sigma > 0 \qquad 2.47$$

and

$$f(y, p_2, \sigma) = \frac{p_2}{\sigma} y^{p_2 - 1} e^{-\frac{y^{p_2}}{\sigma}} \qquad x > 0, \qquad p_2, \sigma > 0 \qquad 2.48$$

The difference between this form and the one we have considered in our earlier description is that the scale parameter has been taken as $\sigma$ instead of $\sigma^p$.

The reliability is obtained as

$$R = P(X > Y) = 1 - H(p_1, p_2, \sigma) \qquad 2.49$$

where,

$$H(p_1, p_2, \sigma) = \int_0^{\infty} \frac{p_1}{\sigma} x^{p_1 - 1} e^{-\frac{1}{\sigma}(x^{p_1} + x^{p_2})}\, dx \qquad 2.50$$

The estimation of reliability is carried out using progressive Type II censored data on both variables. Considering X as a progressively Type II censored sample from W($p_1, \sigma$) with censored scheme r = ($r_1$, $r_2$, ….$r_n$) and Y as a progressively Type II censored sample from W($p_2, \sigma$) with censored scheme r = ($r'_1$, $r'_2$, ….$r'_n$), the likelihood function of $p_1, p_2, \sigma$ is given as

$$L(p_1, p_2, \sigma) = c_1 \prod_{i=1}^{n} f(x_i)\,[1 - F(x_i)]^{r_i} \text{ x } c_2 \prod_{j=1}^{m} f(y_j)\,[1 - F(y_j)]^{r'_j} \qquad 2.51$$

where,

$$c_1 = N(N - 1 - r_1)(N - 2 - r_1 - r_2) \ldots .(N - n + 1 - r_1 \ldots - r_{n-1}) \qquad 2.52$$

$$c_2 = M(M - 1 - r'_1)(M - 2 - r'_1 - r'_2) \ldots .(M - m + 1 - r'_1 \ldots - r'_{m-1}) \qquad 2.53$$

The log-likelihood function is obtained as

$$L(p_1, p_2, \sigma) \propto nln(p_1) + mln(p_2) - (n + m)ln(\sigma) + (p_1 - 1)\sum_{i=1}^{n} ln(x_i)$$

$$+ (p_2 - 1)\sum_{j=1}^{m} ln(y_j) \qquad 2.54$$

$$-\frac{1}{\sigma}\left[\sum_{i=1}^{n} (r_i + 1)\, x_i^{p_1} - \sum_{j=1}^{m} (r'_j + 1)\, y_j^{p_2}\right]$$

The MLE's of $p_1, p_2$ and $\sigma$ can be obtained as

$$\frac{\partial l}{\partial \sigma} = -\frac{n + m}{\sigma} + \frac{1}{\sigma^2}\left[\sum_{i=1}^{n} (r_i + 1)\, x_i^{p_1} + \sum_{j=1}^{m} (r'_j + 1)\, y_j^{p_2}\right] = 0 \qquad 2.55$$

$$\hat{\sigma} = \frac{1}{n + m}\left[\sum_{i=1}^{n} (r_i + 1)\, x_i^{\widehat{p_1}} + \sum_{j=1}^{m} (r'_j + 1)\, y_j^{\widehat{p_2}}\right] \qquad 2.56$$

$$\frac{\partial l}{\partial p_1} = \frac{n}{p_1} + \sum_{i=1}^{n} ln(x_i) - \frac{1}{\sigma}\sum_{i=1}^{n}(r_i + 1)\, x_i^{p_1} ln(x_i) = 0 \qquad 2.57$$

$$\frac{\partial l}{\partial p_2} = \frac{m}{p_2} + \sum_{j=1}^{m} ln(y_j) - \frac{1}{\sigma}\sum_{j=1}^{m}(r'_j + 1)\, y_j^{p_2} ln(y_j) = 0 \qquad 2.58$$

Substituting $\hat{\sigma}$ in equation 2.57 and 2.58, $\widehat{p_1}$ and $\widehat{p_2}$ can be obtained as a solution of following non linear equations.

$$\frac{n}{p_1} + \sum_{i=1}^{n} ln(x_i) - \frac{(n+m)\sum_{i=1}^{n}(r_i + 1)\, x_i^{p_1} ln(x_i)}{\sum_{i=1}^{n}(r_i + 1)\, x_i^{\widehat{p_1}} + \sum_{j=1}^{m}(r'_j + 1)\, y_j^{\widehat{p_2}}} = 0 \qquad 2.59$$

$$\frac{m}{p_2} + \sum_{j=1}^{m} ln(y_j) - \frac{(n+m)\sum_{j=1}^{m}(r'_j + 1)\, y_j^{p_2} ln(y_j)}{\sum_{i=1}^{n}(r_i + 1)\, x_i^{\widehat{p_1}} + \sum_{j=1}^{m}(r'_j + 1)\, y_j^{\widehat{p_2}}} = 0 \qquad 2.60$$

The equations 2.59 and 2.60 do not have explicit solutions and have to be solved with an iterative process. Once the estimates of parameters are obtained, the reliability estimate can be found by using the equation

$$\hat{R} = 1 - H(\widehat{p_1}, \widehat{p_2}, \hat{\sigma}) \qquad 2.61$$

Kundu and Raqab, 2009 [52] conducted research on estimating the stress-strength reliability for three parameter Weibull distribution with common location and shape parameter but different scale parameter using modified maximum likelihood estimation method (MMLE).

Consider X as ($X_1$, $X_2$, $X_3$, ……$X_n$) and Y as ($Y_1$, $Y_2$, $Y_3$, ….$Y_n$) are independent random variables in an ordered manner following Weibull distribution W($\mu, \sigma_1, p$) and W($\mu, \sigma_2, p$) respectively, then the likelihood function can be given as

$$l(\mu, \sigma_1, \sigma_2, p) \propto p^{m+n} \sigma_1{}^{-n} \sigma_2{}^{-m} \prod_{i=1}^{n} (x_i - \mu)^{p-1} + \prod_{j=1}^{m} (y_j - \mu)^{p-1} e^{-\frac{1}{\sigma_1}\sum_{i=1}^{n}(x_i-\mu)^{p-1}} e^{-\frac{1}{\sigma_2}\sum_{j=1}^{m}(y_j-\mu)^{p-1}} \times 1_{\{z > \mu\}} \quad 2.62$$

where z = min($x_1$, $y_1$) and $1_{\{z > \mu\}}$ has a value of 1 if z > μ or 0 if z < μ. The parameter μ is estimated to be $\hat{\mu}$ = z. Based on m+n-1 observations (ignoring the smallest value), the modified maximum likelihood estimation will then be equal to for $x_1 < y_1$

$$l(\hat{\mu}, \sigma_1, \sigma_2, p) \propto (m + n - 1) \ln(p) - (n - 1) \ln(\sigma_1) - m \ln(\sigma_2) + (p - 1) \sum_{i=2}^{n} \ln(x_i - \hat{\mu}) + (p - 1) \sum_{j=1}^{m} \ln(y_j - \hat{\mu}) - \frac{1}{\sigma_1} \sum_{i=2}^{n} (x_i - \hat{\mu})^p - \frac{1}{\sigma_2} \sum_{j=1}^{m} (y_j - \hat{\mu})^p \quad 2.63$$

and for $x_1 > y_1$

$$l(\hat{\mu}, \sigma_1, \sigma_2, p) \propto (m + n - 1) \ln(p) - n \ln(\sigma_1) - (m - 1) \ln(\sigma_2) + (p - 1) \sum_{i=1}^{n} \ln(x_i - \hat{\mu}) + (p - 1) \sum_{j=2}^{m} \ln(y_j - \hat{\mu}) - \frac{1}{\sigma_1} \sum_{i=1}^{n} (x_i - \hat{\mu})^p - \frac{1}{\sigma_2} \sum_{j=2}^{m} (y_j - \hat{\mu})^p \quad 2.64$$

Thus the parameter estimates $\widehat{\sigma_1}, \widehat{\sigma_2}$ and $\hat{p}$ can be obtained by maximizing the above likelihood equation with respect to $\sigma_1, \sigma_2$ *and* $p$. The estimates can be obtained as

$$\widehat{\sigma_1}(p) = \frac{\sum_{i=2}^{n} (x_i - \hat{\mu})^p}{n - 1} \quad and \quad \widehat{\sigma_2}(p) = \frac{\sum_{j=1}^{m} (y_j - \hat{\mu})^p}{m} \quad if\ x_1 < y_1$$

and

$$\widehat{\sigma_1}(p) = \frac{\sum_{i=1}^{n} (x_i - \hat{\mu})^p}{n} \quad and \quad \widehat{\sigma_2}(p) = \frac{\sum_{j=2}^{m} (y_j - \hat{\mu})^p}{m - 1} \quad if\ x_1 > y_1$$

$$h(p)=\frac{(m+n-1)+\sum_{i=2}^{n}\ln(x_i-\hat{\mu})^p+\sum_{j=1}^{m}\ln(y_j-\hat{\mu})^p}{\frac{\sum_{i=1}^{n}(x_i-\hat{\mu})^p\ln(x_i-\hat{\mu})}{\frac{1}{n-1}\sum_{i=2}^{n}(x_i-\hat{\mu})^p}+\frac{\sum_{j=1}^{m}(y_j-\hat{\mu})^p\ln(y_j-\hat{\mu})}{\frac{1}{m}\sum_{j=2}^{m}(y_j-\hat{\mu})^p}} \quad 2.65$$

The equation 2.25 can be solved using simple iterative procedure and p can be obtained when the difference between $p_k$ and $p_{k+1}$ is sufficiently small. Once the MMLEs $\hat{\mu}, \widehat{\sigma_1}, \widehat{\sigma_2}$ and $\hat{p}$ are obtained the MMLE of R can be obtained as

$$\hat{R}=\frac{\widehat{\sigma_1}}{\widehat{\sigma_1}+\widehat{\sigma_2}} \quad 2.66$$

Nadarajah and Jia, 2017 [53] studied the estimation of stress-strength reliability for two parameter Weibull distribution by taking a particular case of Fox-Wright function for obtaining the closed form of reliability. The MLE method has been used to obtain the estimates of shape and scale parameters.

Jia et al., 2017 [54] presented estimation of stress-strength reliability for two parameter Weibull distribution using Bayes estimation technique. For obtaining the closed form of R, a certain case of Fox Wright function was considered as in the previous case. For obtaining the Bayes estimates, prior distributions are assumed for Weibull parameters and posterior distributions are presented. A sampling method based on Monte Carlo Markov Chain (MCMC) is used to obtain the Bayes estimates for reliability.

Pobočíková and Sedliačková, 2014 [55] compared four methods namely least squares estimation, weighted least squares estimation, method of moments, and MLE for estimating the parameters of Weibull distribution. The analysis is carried out using Monte Carlo simulation. It has been shown that the MLE is the most suitable technique, while the weighted least squares method can be used for small sample sizes.

### 2.3.3 Application of metaheuristic techniques in estimation

Recently metaheuristic techniques of optimization are gaining vast importance in the field of estimation. Abbasi et al., 2006 [56] studied estimation of parameters of Weibull distribution using simulated annealing algorithm. The method of simulated annealing in optimization is framed in accordance with the annealing process in material science. The process of annealing is about heating a metal and then slowly cooling it to reduce the defects. Simulated annealing has the advantage of not being trapped in local minima. The algorithm is mainly used for the minimization problem. Apart from finding better minima for the objective function, the algorithm also accepts a higher value with a probability $p=e^{-\frac{\Delta}{T}}$ where Δ is the increase in the value of objective function T is the control parameter. The algorithm begins by initializing the control parameter and selecting an initial solution. The value of objective function is found using the initial solution. A solution is then generated in the neighbourhood of initial solution. If the new solution gives a lesser value of objective function (better minima) than the initial solution, the new solution is carried forward as the current best solution. If the new solution gives a greater value of objective function (worse minima) than the initial solution, then the new solution is not rejected totally. It is accepted with a certain probability. The process is carried on for the set number of iterations. The control parameter is reduced with each iteration using a suitable method. Then the final solution is compared with the initial solution which was first obtained. If the final solution is better, it is accepted as the new best solution. In estimation of Weibull parameters, the optimization of the maximum likelihood function is carried out in order to estimate the parameters.

Örkcü et al., 2015 [57] studied estimation of parameters of Weibull distribution using the differential evolution (DE) method. The method of DE has the advantages of simplicity in implementation, effective, robust, and reliable global optimization algorithm. The algorithm begins with initialization of DE parameters population size, crossover factor (crossover rate), mutation factor (scaling factor), and the population. Then the function value for the current population is calculated. Then the population is modified based on the random number generation, crossover factor, and mutation factor. The function value is then evaluated for the new population generated. If the new population gives a function value more optimum (maximum or minimum) than the function value by the previous population, the new population is accepted. The iterations are carried out till the termination criteria is met.

Abbasi et al., 2011 [58] merged the variable neighborhood search (VNS) algorithm with the simulated annealing algorithm in order to improve the optimization of parameters for Weibull distribution, calling it hybrid neighborhood search and simulated annealing (HNSA). The main aim of the VNS method is to explore different neighborhoods when a local optimum is obtained using the local search method. The algorithm for VNS works in two phases. In the first phase, a set of neighborhood structures is determined. Then an initial solution is obtained, and the termination criteria are set. In the second phase, a new solution is randomly generated in the neighborhood and a local optimum search is made in the current neighborhood to get a new optimum solution. If the new optimum solution gives a better function value (minimum or maximum) than the initial solution, then the new solution is set as the current best solution. This is repeated for all the structured neighborhoods to get the current best solution. In HNSA, the simulated annealing is carried out after the VNS process explained above to improve the solution if possible.

Örkcü et al., 2015 [59] worked on estimation of three parameters of Weibull distribution using particle swarm optimization (PSO) method. A maximum likelihood estimation technique was used to obtain the likelihood function which was maximized by using PSO. The method of PSO is inspired by flock of birds which move together in search of food. In PSO, an initial population that consists of set of solutions (particles) with different parameters is generated. All the particles move towards the best solution with each iteration keeping record of its current position. Each particle has a memory that helps to keep record of its previous best solution. If the new solution is better than the previous solution then the new solution is taken as the updated particle. Each particle also tracks the overall best among all the particles to which it is attracted and at each point compares its solution to the overall best solution. Acitas et al., 2019 [60] proposed a new approach in estimating the parameters of Weibull distribution using particle swarm optimization. The problem with particle swarm optimization is the initial search space. The researchers in this paper have shown a methodology in narrowing down the search space which improves the effectiveness of estimation. The proposed methodology has been applied to strength of glass fibres data to show the application.

Jaya Algorithm is one of the recent effective optimization technique that continuously works towards taking you closer to the best solution and away from the worst solution. Du et al., 2018 [61] used Jaya algorithm in solving optimization-based structural damage identification problem. The author presented the effectiveness of the technique even in high noise levels. R. V. Rao et al., 2017 [62] presented the application of the algorithm in optimizing the process parameters in machining while handling multiple objectives. Similarly, the technique has been used by other researchers for obtaining solutions in optimization problems with effectiveness and at a faster rate [63,64].

## 2.4 Time-dependent Stress-Strength Reliability

It is a known fact that reliability does not remain constant and decreases over a period of time as a result of degradation [65]. Alternatively, it can be said that reliability decreases over a number of cycles in case of fatigue stress [66–68]. Fatigue stress is a type of loading in which repetitive or cyclic loads act on a material for a period of time. Over a large number of loading cycles, the materials tend to fail at a stress level much lesser than the strength of that material. This is known as failure due to material fatigue. Several major accidents that can be remembered in history are as a result of fatigue failure and natural uncertainties [69,70]. These accidents could have been prevented by robust and reliable design. Thus, failure prediction and modeling is an important aspect in material design. Many studies have been carried out in assessing the dynamic reliability of components under various load conditions. Also, we are aware that automobile systems are exposed to severe environmental conditions during their lifetime. The automotive electronic circuits face the problem of shearing of solder joints over a period of time or over the distance traveled. In this paper, a simple method is exhibited for evaluating time-dependent reliability by degradation modeling using Jaya algorithm which is a recent metaheuristic technique of optimization. The above methodology has been applied to a data of solder strength of automotive circuits to show its effectiveness. A plot for reliability vs time has been obtained in order to see the failure behavior of the components over a period of time and a plot for the convergence behavior of Jaya algorithm has been obtained to show the performance of the algorithm over the number of iterations.

In mechanical components, failure occurs when the stress applied exceeds the strength of the material. In actual practice, the strength is not constant because of uncertainties and can be assumed to follow a particular distribution [19]. So, utmost care should be taken while designing the components so as to avoid the failure because of uncertainties in strength of the component. Bhuyan and Dewanji, 2015 [71] proposed two methods in estimation of reliability in case of strength degradation. The stress was considered to be accumulating due to shocks and the strength was considered to be undergoing random degradation. The phenomenon of strength degradation and stress accumulation has been accounted for in a single model. The method of inversion and simulation was presented in evaluating the reliability. The authors proved benefits of simulation method over other methods and concluded that the method of simulation is superior compared to the other methods in estimation of reliability. Park and Tang, 2006 [72] used the cycle counting approach to represent the time-dependent stress as several time-independent stress levels and developed a method to solve the problem by accumulated damage analysis with the first-order reliability analysis. Fatigue failure mechanism and reliability were considered as time dependent and that the properties of a component vary with time. A reliability factor determined by inverse reliability analysis which employs performance measure approach has been used. The study showcases a methodology for determining the fatigue life and reliability in dynamic conditions.

Lv et al., 2009 [73] developed a reliability model for gears with multiple failure modes using stress-strength interference concepts considering strength degradation. The results showed that the reliability of gears decreases gradually with time. Eryilmaz, 2013 [74] conducted stress-strength reliability studies for the strength degradation and stress remaining constant over time. Also, some results were provided for lifetimes of systems with same strength and subjected to different stresses. Stress-strength analysis was also carried out for the multicomponent form.

Finally, some examples have been taken in which strength is considered to undergo Weibull distribution to show the implementation of the methodology.

Liu and Frangopol, 2018 [75] presented reliability estimation studies when the strength is time dependent. In evaluating structural reliability, multiple failure modes should be considered. The failures may occur because of the cumulative effect of multiple shocks or hazards such as corrosion, pressure, etc. The methodology has been applied to ship structure for permanent set and buckling failure. In addition to the consideration of progressive damage, the damage due to shocks may have significant variation in safety assessment. The reliability reduces significantly while considering the hazards due to shocks. The reliability is also affected by the intensity and frequency of the shock hazards. Cumulative time approach in calculating the failure probability takes into account the strength degradation and the history of loading. Peng et al., 2019 [76] suggested a different approach in estimating the time dependent reliability because the evaluation of reliability using traditional approach of uncertain loads and design parameters is time consuming. The authors proposed a method in estimating the dynamic reliability of mechanical structure using surrogate modeling and data clustering technology. The methodology was based on observing the physics of failure with respect to time and building a surrogate model using BP neural network. Data clustering was used to identify least reliable domains. The extreme values of response in a particular time interval was identified using Genetic Algorithm and BP neural network was used to establish surrogate model at extreme values.

Gao et al., 2013 [77] studied evaluating the reliability in case of strength degradation of a mechanical structure. The loads acting on a mechanical structure may vary and thus the strength distribution is considered at each load point. The author mentions that this method leads to large errors in reliability calculation because of neglecting the correlation of the load and remaining strength in the degradation path. The method for quantitatively analysing the influence of material parameters on the reliability has been depicted. The traditional method of considering the large dispersion of strength in the initial stage resulting in lower reliability has been discussed by analysing that the dispersion of initial strength at different lifetimes having different effects on the reliability. Wang et al., 2015 [78] evaluated time dependent reliability when the uncertainty data is not sufficient. A non-probabilistic convex process model is proposed in estimating the reliability based on set theory and regularization treatment. Yadav et al., 2011 [79] developed a framework for optimization of reliability in case of strength degradation using multi objective optimization and quadratic quality loss concepts. The improvement in reliability comes at an initial loss in quality but it is beneficial over the products life cycle. The design using the proposed model will reduce repair/ warranty costs and increase customer satisfaction. Zhang et al., 2017 [65] proposed a stress-strength interference model for structural reliability in case of strength degradation. A time dependent model for stress was developed using quadratic response surface method and the strength degradation was developed considered to be linear or exponential. The method of estimation of structural reliability has been developed using Copula selection method.

Wu et al., 2011 [80] introduced a dynamic reliability model in case of strength deterioration. The stress was considered to be stochastic following Poisson distribution and the strength was modeled by gamma distribution which was considered to be dynamic following increments with gamma distribution having common scale parameters. Jose and Drisya, 2020 [81] presented studies on time dependent stress-strength models for random stress at random cycles of time. A model for stress-strength reliability was de introduced random cycle time following particular distribution and studied time-dependent stress-strength reliability for stress and strength following continuous phase-type distribution. The run time is considered to be a random variable following Weibull, gamma and exponential distribution. Simulation studies are carried out to see the effects of time on the stress-strength reliability. In call cases, the reliability was observed to be increasing with time. It was noted that the reliability decreases with increase in shape parameter in case of Weibull distribution. For gamma distribution, the reliability increases with increase in shape parameter value and decrease in rate parameter value of cycle distribution. In case of exponential distribution, the reliability increases with rate parameter value of cycle time distribution.

Lu and Meeker, 1993 [82] developed statistical methods using nonlinear mixed-effects model to obtain time to failure distribution for various degradation models. Reliability assessment was also carried out by finding point estimates and confidence intervals using Monte Carlo simulations. Wu et al., 1997 [83] presented various methods in estimation of fatigue damage and fatigue life in case of random loading. Chiodo and Mazzanti, 2006 [84] developed a model for estimating the reliability of electrical components in case of voltage surges which is considered as the stress. The general difficulty on modeling in such cases is because the stress data is easily available but the strength data is not sufficiently present. The proposed methodology is based on Bayesian approach and takes into account the degradation of components over its usage. The peak of stress in case of voltage surges is considered to be following Weibull distribution and strength is modeled by Weibull distribution with its aging based on inverse power law.

Rathod et al., 2011 [85] derived a degradation model in fatigue loading by the principle of damage accumulation considering normal distribution for strength with decreasing mean and linearly increasing variability over time along with reliability analysis. The methodology is developed based on linear damage accumulation model, probabilistic S-N curve and pdf transformation from fatigue life to damage accumulation. The damage accumulation was considered to follow a distribution with changing expected value and variance. The application of proposed methodology has been shown in case of single stress and multi stress level loading. Zhu et al., 2013 [86] carried out time-dependent reliability analysis for railway axle steels by evaluating the nonlinear damage accumulation model while considering the number of cycles following log-normal distribution.

The maximum likelihood estimation method has been used by many researchers for getting the best estimates of the fit model. Bhuyan and Dewanji, 2017 [87] mentioned that for a mechanical component in real life the strength degrades and the stress accumulates with time due to cumulative damage due to shocks and developed a model for estimating reliability in such cases. He proposed two sampling plans for estimation. The first sampling plan is by observing the time to failure and the number of shocks given up to failure. In second sampling plan the system is checked at specific time intervals and the number of shocks given up to that time. The condition of the system at each time point is checked to see the effect of damage accumulation. The MLE technique is used in estimation and simulation studies are carried out in order to check the performance of the proposed methodology. Feng et al., 2018 [88] presented a methodology to predict probabilistic S-N curves using the maximum likelihood method and carried out simulations to validate the results. D'Anna et al., 2017 [89] proposed a model for estimating the structural reliability considering the fatigue life to follow Birnbaum- Saunders model distribution and both the parameters depending on stress. The model parameters were estimated using the maximum likelihood estimation method and it was observed that the model gave a good fit for the fatigue failure data of aluminum coupons under cyclic stress obtained by accelerated life tests.

## 2.5 Summary

The literature review shows some of the stress-strength interference models which have been developed for stress and strength following various distributions. But there are still some models for which the interference models are not available yet. Also, some common distributions like Weibull, gamma, etc. do not have closed form of interference models and have computational difficulties. In the area of reliability estimation, a lot of work has been carried out in estimating the stress-strength reliability for various distributions. Some techniques have been shown that are used in estimating parameters of Weibull distribution since this distribution is widely used in mechanical systems. There is a scope to further enhance and improve the estimation methods using advanced techniques. Recently, some metaheuristic techniques have been used in estimation and are seen to perform quite well. In the last part of this chapter, a brief overview of the concept of strength degradation has been shown. The studies carried out in estimating the reliability in case of strength degradation has been discussed. A brief summary of literature review has been presented in Table 2.1.

Table 2.1 Summary of literature review

| | |
|---|---|
| Reliability-based Design | Reliability [29], [34-36]<br>Failure and uncertainties [1-4], [32-33]<br>Reliability-based design and applications [176] |
| Stress-strength interference | Stress strength interference theory [24]<br>Applications of stress strength interference theory [15-17]<br>Stress-strength reliability models [18-31] |
| Estimation of stress-strength reliability | Stress-strength reliability estimation for various distributions [37-49], [165-166]<br>Stress-strength reliability estimation for Weibull distributions [50-55] |
| Design of experiments | DOE in reliability [127-128]<br>Taguchi analysis [124-128], [130]<br>Response surface analysis [129-136] |
| Traditional methods of estimation | Maximum likelihood estimation [98-105], [144-148], [155-157]<br>Least squares estimation [106-107], [158-161], [168-169]<br>Weighted least squares estimation [108-111] |
| Optimization techniques in estimation | Application of various optimization techniques in estimation [56-64], [170-173]<br>Jaya algorithm in estimation [112-122], [174] |
| Time dependent stress-strength reliability | Strength degradation [65-68]<br>Regression in degradation [178], [179, 181]<br>Fatigue failure [69-70]<br>Estimation of reliability in strength degradation [71-89], [180] |

# Chapter 3

# Problem Description and Solution Methodologies

## 3.1 Introduction

Reliability based design is significant for mechanical systems where stress and strength is involved. There are many stress-strength interference models developed as can be seen in Chapter 2. But there are still some distributions for which the stress-strength interference does not have closed form. This research aims to develop models of stress-strength interference for some important distributions used in mechanical systems. Also, when it comes to estimation of reliability, a number of methods have been developed with various optimization techniques. This research will try to improvise on the existing methods to estimate the reliability of mechanical systems.

As we know that the property like strength does not remain fixed but changes over time or over number of cyclic loads. The existing methods of estimating reliability in such cases is complicated and time consuming. The interference model in such a case needs to be evaluated with a simple methodology.

## 3.2 Problem Statement

The problem statements considered for the present work are as follows:

1. Development of stress-strength interference models for the stress and strength following Laplace and exponential distribution respectively and vice versa; and to evaluate closed form interference models for the distributions like Weibull and gamma.

2. Improving the estimation of stress-strength reliability by integration of conventional estimation methods with advanced optimization technique.

3. Estimation of stress-strength reliability in case of non-linear strength degradation as it has a problem of slow convergence and non-convergence to real roots.

## 3.3 Motivation for Research Work

In the field of stress-strength interference, a number of models have been developed when stress and strength follow various distributions. Saralees Nadarajah, 2004 [24] mentioned that there are many distributions for which the stress-strength form of reliability has not been derived. Also, some frequently used distributions like Weibull, gamma, etc. do not have a closed form of reliability because of the issues in integration of stress-strength equation. In this research an attempt has been made to depict a methodology to obtain a closed form of reliability for the distributions like Weibull, gamma, etc in which solving the stress-strength integration equation is complicated and sometimes not possible. In estimation of stress-strength reliability, a number of methods have been used as shown in the literature review. There is a need to simplify and enhance the estimation technique in order to obtain precise estimates of the parameters under consideration.

## 3.4 General Assumptions

In developing stress-strength reliability models, the stress and strength are continuous and independent random variables. The analysis has been carried out for a specific range of parameters. The study can be extended to other values of parameters as per the data under observation.

In estimation of stress-strength reliability, the first case considers the data to be following two parameter Weibull distribution with common scale parameter and different shape parameter.

In the second case, the data is considered to be following three parameter Weibull distribution with common shape parameter and varying scale parameter. Bias and mean squared error are used as the basis for comparison of estimation. Kologomorov Smirnov test has been used to check the fit of the model to the data.

For reliability in case of strength degradation, the strength follows normal distribution with linearly and non-linearly varying mean and linearly varying standard deviation. The stress applied is deterministic in nature. In case of application to solder joints, the failure of circuit board is because of the failure of solder connections.

## 3.5 Scope of the Problem

Reliability-based design has gained vast importance in recent times as it takes care of the variation in the material properties which might affect the design and safety of a component. The application of reliability assessments with traditional design improves safety in order to give a robust design under the uncertainties considered [90]. Some of the most widely used methods in reliability analysis are First Order Reliability Method (FORM), Second Order Reliability Method (SORM), stress strength interference theory, Monte Carlo simulations, etc. The method of FORM is mainly used in structural reliability which involves design considering load and strength factors, reliability index and limit state function. The reliability index represents a relative measure of the ability to perform required function. The load factors include effects due to dead loads, live loads, wind loads, etc. and strength factors include the effects of variability in material property, dimensions, model error, etc. The values of load and stress factors in different conditions and reliability index for various materials used in structural industry are available in standard structural codebooks [91–93]. The major disadvantage of FORM is that gives precise solutions only if the limit state is linear and the basic variables are normally distributed [94]. But in general, variables do not necessarily follow normal distribution. The method of SORM is more accurate than FORM as it approximates quadratic polynomial for the limit state function. But the numerical evaluation for failure probability of quadratic polynomial is difficult and not very efficient [95]. Monte Carlo simulations are direct and simple but involve high computational cost for complex systems. This study deals with reliability-based design using stress strength interference theory which is flexible, simple and effective. Stress-strength interference theory explained in the previous chapter is one of the important concepts in reliability-based design. Many models have been developed for stress-strength reliability when stress and strength follow distributions like exponential, normal, lognormal, etc. and their interference. But there are many distributions like Weibull, gamma, interactions of Laplace and exponential, etc. for which the reliability models have not yet been developed. So, there is a broad scope to work in this area to develop models for which the closed form reliability is not developed yet. This will help researchers to optimize their work in encounter of stress-strength situations of these distributions. In estimation of reliability, studies are being carried out to evaluate the stress-strength reliability using various modern techniques. The method of estimation can be further enhanced by integration of such modern estimation and optimization techniques. A comparative study with existing estimation methods can be performed. Also, in case of material degradation, some research shows the effect of degradation on reliability. We know that degradation can take many forms. Not much research has been performed on varying the type of degradation and studying its effect on reliability. This thesis works out on some of these gaps and tries to propose effective solutions. The scope of the research work is outlined in Figure 3.1.

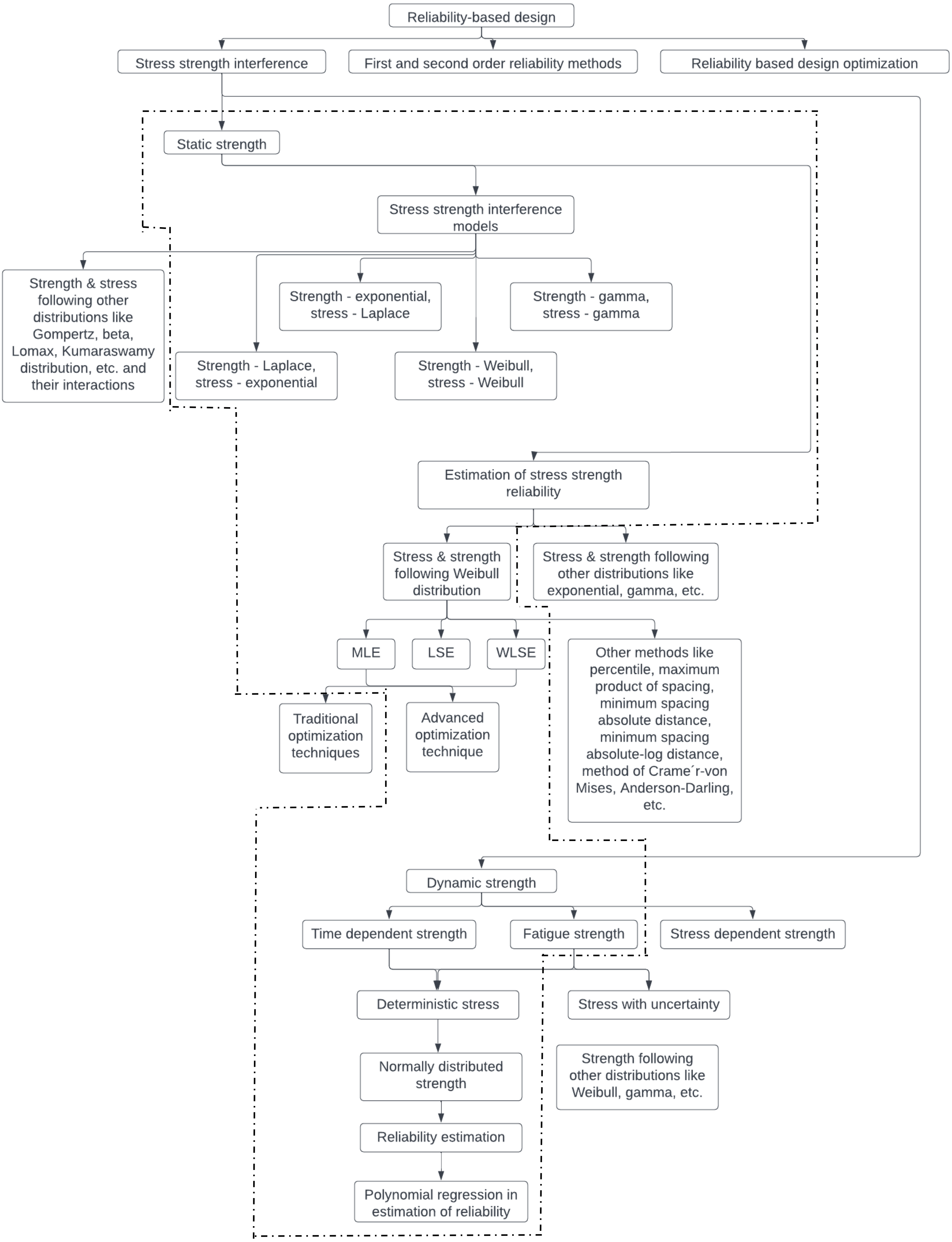


Figure 3.1 Scope of the research work

## 3.6 Solution Methodologies

The stress-strength interference reliability model has been developed for stress and strength following Laplace and exponential distribution and vice versa. Taguchi analysis has been used to show the variation in reliability with the change in parameters of stress and strength distribution. The response surface analysis has also been used to give the two-parameter interaction towards reliability. The stress-strength interference model for distributions like Weibull and gamma does not have closed form and the integral is unsolvable. In these cases, Design of Experiments (DOE) technique has been used and a methodology has been shown to develop a model for the parameters taking values in a particular range for a certain application. Taguchi analysis and response surface analysis has been used to study the variation of reliability with the variation of parameters.

For estimation of reliability, Weibull distribution has been considered since it is widely used in reliability studies. Two cases of Weibull distribution have been considered. First a two parameter Weibull distribution with stress and strength having common scale parameter but different shape parameter. Three estimation methods have been used namely maximum likelihood, least squares and weighted least squares estimation. Bayesian analysis and UMVUE are some of the other methods used in estimation of reliability for Weibull distribution but they have been excluded from this research. Bayesian analysis has some limitations like validity of the chosen prior distribution and high computational cost especially with large number of parameters [96]. UMVUE is inconsistent and the accuracy of the results vary depending on the values of distribution parameters [97]. Jaya algorithm has been used in optimization of functions obtained in respective estimation method. In the second case, a three parameter Weibull distribution has been considered with stress and strength having common shape parameter and different scale parameter. Maximum likelihood technique has been used in estimation. Jaya algorithm has been used to optimize the likelihood function.

The reliability model in case of strength degradation has been developed for polynomial degradation of mean considering the strength following normal distribution.

We know that a probability distribution is driven by its parameters. Given a data and knowing its distribution, it is sometimes very crucial to identify the parameters of the distribution. The parameter estimation methods help us in identifying the values of parameters from a sample data of the population. There are many parameter estimation methods. Some of the most widely used estimation techniques are maximum likelihood estimation, least squares estimation and weighted least squares estimation.

### 3.3.1 Maximum likelihood estimation

The method of maximum likelihood is a popular estimation technique and has been used by several authors in estimating the parameters of various distributions [98–100]. Chacko and Mohan, 2017 [101] used the MLE method in estimating the parameters of two-parameter Kumaraswamy-exponential distribution for progressive type-II censored samples. Tzavelas, 2009 [102] proposed estimation of parameters of three-parameter gamma distribution using MLE via reparameterization of function and predictor-corrector method. MLE method has also been used by Aggarwala and Balakrishnan, 2002 [103] in the estimation of scale and location parameters of Laplace distribution. Ng et al., 2009 [104] discussed estimating the parameters of three-parameter Weibull distribution for type II progressively censored samples using MLE and weighted MLE. Abushal, 2021 [105] applied MLE technique to estimate the unknown parameters and reliability characteristics for Akash distribution. Consider $x_1$, $x_2$ ..., $x_n$ is a random sample of size 'n' from a population of a particular distribution then the likelihood function can be given as:

$$L = \prod_{i=1}^{n} f(x_i) \tag{3.1}$$

By maximizing the likelihood function 3.1 using various traditional or advanced techniques, the estimates of parameters can be obtained.

### 3.3.2 Least squares estimation

The method of least squares aims to find the best estimates in order to minimize the sum of squares of residuals. The least squares estimation technique was used by Swain et al. [106] in Johnson's translation system for modeling glucose levels in diabetes, in the analysis of statistical models, and structural reliability. Ashour and Eltehiwy [107] proposed the application of the technique in estimation of parameters of exponentiated power Lindley distribution. In least squares estimation method, the parameters are adjusted in order to fit the best model. If $x_i$ is the independent variable and $y_i$ a dependent variable for data set with n number of points (i=1, 2, 3, …., n), then the model can be given by f(x, β) = $\beta_o + \beta_1 x$ (considering a straight line), where $\beta_o$ is the y intercept and $\beta_1$ is the slope. The parameters are found by measuring the residual that best fits the model. The residual is the difference between the observed value and the value predicted by the model.

$$r = y_i - \mathrm{f}(x_i, \beta) \tag{3.2}$$

The sum of squares will be

$$S = \sum_{i=1}^{n} r_i^2 \qquad 3.3$$

$$S = \sum_{i=1}^{n} (y_i - \mathrm{f}(x_i, \beta)) \qquad 3.4$$

By minimizing the sum of squared residuals using various techniques, the parameters can be calculated.

### 3.3.3 Weighted least squares estimation

Weighted least squares which is a modification of least squares estimation method considers the variances in the errors while evaluating the estimates. The weighted least squares estimation technique has been used in many applications [108]. The method was used by Wu et al., 1987 [109] in moving identification and found the suitability of the application in time-varying systems. The technique has also been used in estimation of parameters of multiplicative generalization of binomial distribution [110]. The method of weighted least squares estimation is similar to least squares estimation. In weighted least squares instead of minimizing the sum of squared residuals we minimize the weighted sum of squared. Benchiha et al., 2021 [111] used LSE and WLSE techniques in estimating the parameters of weighted generalized Quasi Lindley distribution.

$$S = \sum_{i=1}^{n} w_i \, (y_i - \mathrm{f}(x_i, \beta)) \qquad 3.5$$

The weights $w_i$ is inversely proportional to the variances of the measurement. The parameters can then be estimated by minimizing the weighted sum of squares function.

### 3.3.4 Jaya algorithm

Jaya algorithm is a recent heuristic optimization technique capable of solving a large number of optimization problems with high effectiveness and was first proposed by Rao and Waghmare, 2017 [112]. It is a gradient-free optimization algorithm which continuously moves closer to the best solution and away from the worst solution [113,114]. It has many applications in in optimizing the designs of heat exchangers, heat pipes and heat sinks, ice thermal energy storage system, machining processes, nano-finishing processes, casting processes and other disciplines of engineering and science [115]. The researchers have used the technique in several applications and found satisfactory results. Meshram et al., 2020 [116] carried out electrical discharge machining with around eight control variables and two responses. Taguchi's L12 orthogonal array was used in designing the experiment. The regression equations were taken as objective functions and Jaya algorithm was used in optimization, observing improvement in response variables. Caldeira and Gnanavelbabu, 2019 [63] presented the implementation of Jaya algorithm for effectively solving the flexible job-shop scheduling problem. Gupta et al., 2019 [117] discussed the superiority of Jaya algorithm over other similar metaheuristic techniques in optimizing standard functions for the application of workflow scheduling in cloud computing. Jin et al., 2019 [118] identified the parameters of wind turbine power models using Jaya algorithm and monitoring with multivariate control charts. Similarly, the algorithm has been used by many other researchers in such optimization problems [119–122].

Consider an objective function to be optimized for some unknown parameters. A random set of parameters with a certain population size are generated within the specified boundaries. The function value is found for each set of parameters in the population. The best and the worst function value are noted. Then the set of parameters are updated based on the following equation:

$$U'_{b,c,a} = U_{b,c,a} + r_{1,b,a} (U_{b,best,a} - |U_{b,c,a}|) - r_{2,b,a} (U_{b,worst,a} - |U_{b,c,a}|) \quad 3.6$$

where a is the iteration number, b is the parameter variable and c is the population size. $U_{b,c,a}$ is the value of parameter b for iteration a and population number c. $U'_{b,c,a}$ is the updated value for the same based on equation 6. If the new set of parameters for a given population number gives a better solution, then $U'_{b,c,a}$ is the new accepted set for the respective population. The updated set will then be taken as the input for next iteration. Figure 3.2 shows the flowchart of the Jaya algorithm.

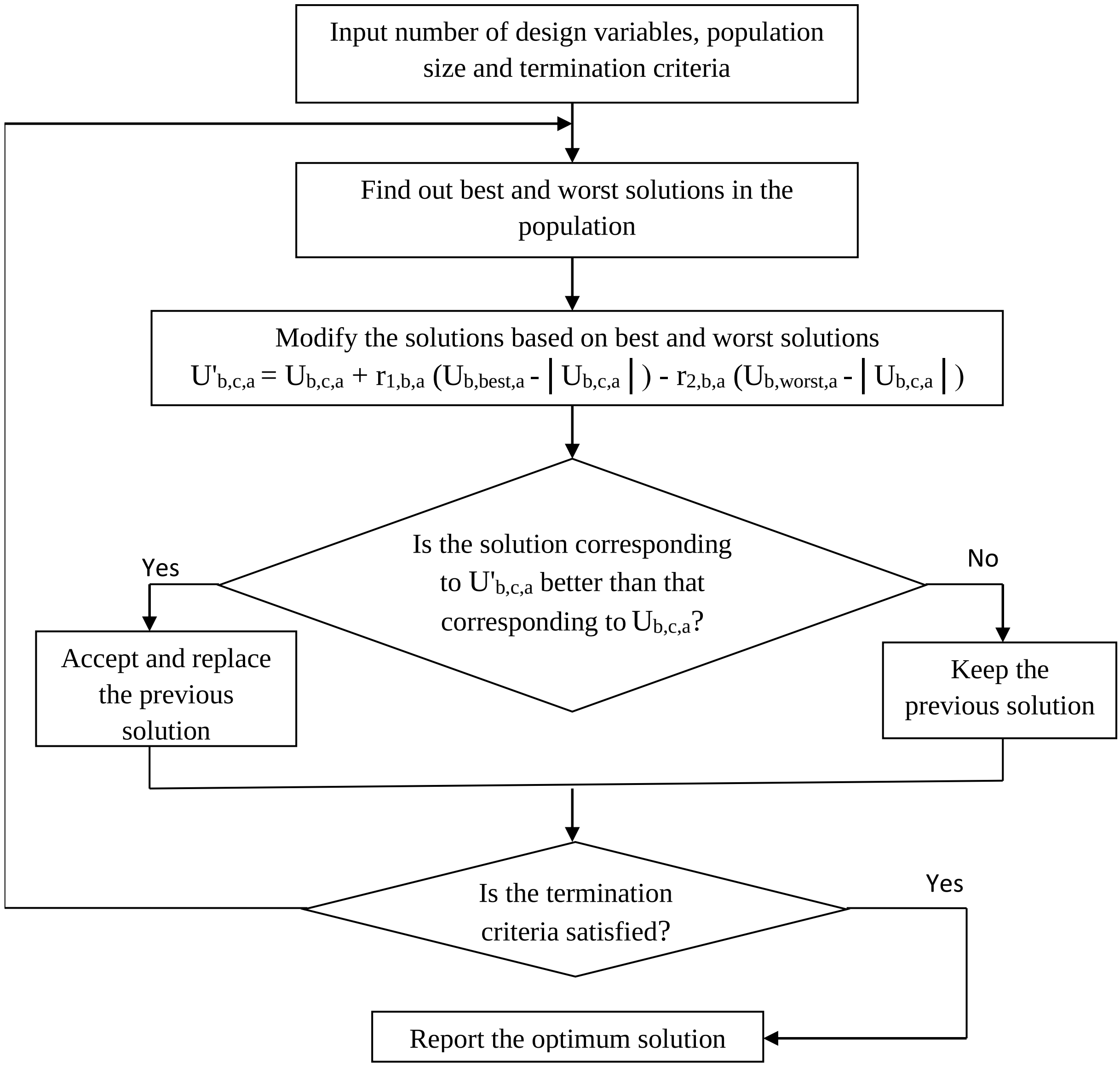


Figure 3.2 Flowchart of Jaya algorithm [123]

### 3.3.5 Design of experiments

Design of experiments (DOE) is a systematic tool to find the relations between the input variables and the response. DOE gives a significant experimental setup sufficient to find the relation between input variables and output response which helps in saving time, cost and resources. Taguchi method and response surface methodology (RSM) are some of the widely used techniques in DOE. Taguchi method is used to obtain a set of significant experiments and to find the most influential parameter towards the response. RSM is also used to obtain a set of significant parameters and analyze for the influencing parameters. Additionally, RSM gives a prediction model of input variables and the output response. Khare et al, 2018 [124] used DOE in optimizing the surface roughness of AA 6061 material in turning operation. The cutting speed, feed rate, depth of cut and rake angle were taken as the input parameters while the surface roughness was taken as the response. Taguchi's method was used for carrying out the DOE and analysis. It was found that the rake angle was the most influential parameter towards the surface roughness followed by cutting speed. The set of optimum parameters was also found. Similarly Taguchi analysis has been used by many researchers in their studies [125–128]. Laghari et al., 2018 [129] developed prediction models for tool wear and surface roughness in turning of Al/SiCp workpiece using response surface methodology (RSM). Cutting speed, feed rate and depth of cut were taken as the independent variables towards the response. The response surface methodology was proved to be effective in modeling the prediction equation. Ammeri et al., 2015 [130] combined Taguchi method and RSM in determining the optimal lot size for the manufactured product in supply chain. RSM has been used to develop models and carrying out analysis of parameters and response [131–134]. Nair et al., 2004 [135] used design of experiments for design of accelerated test experiments for reliability improvement. Rigdon et al., 2022 [136] studied on the use of design of experiments to understand and improve product reliability. A detailed description of the statistical distributions, methods to model reliability, various DOE methods that can be used and the analysis that can be carried out has been made in this research.

## 3.7 Summary

In this chapter, the problem statements have been identified based on the literature review. Also, the motivation for this research work and various assumptions considered in the research work have been summarized. This chapter further discusses the solution methodologies that have been adopted in this research work and an introduction to the solution techniques has been given along with some of the overview of the work that has been carried out in the respective fields.

# Chapter 4

# Stress-Strength Interference Models

Reliability has received utmost importance in today's world as people are continuously looking for reliable and safe mechanical systems. Thus, it is very essential to consider reliability right at the modelling and design stage of mechanical components and systems. In reliability based design, a large amount of work has been carried out in cases of stress-strength interference models of type P (X > Y) with various distributions. But there are some distributions for which the stress-strength models are yet to be developed. In this chapter an attempt has been made to develop stress-strength models for some common distributions of which the reliability models are not yet available.

## 4.1 Strength – Laplace Distribution, Stress - Exponential Distribution

Laplace distribution is used to fit the data which has more kurtosis than the normal distribution and is more concentrated towards the mean. The distributions which have heavier tails are usually modelled by this distribution. Exponential distribution has been used to in modelling of stress-strength data in various studies [137].

### 4.1.1 Model development

Consider strength (s) follows Laplace distribution and stress (l) follow exponential distribution. According to the interference theory, the reliability of the component can be evaluated using equation:

$$R = \int_0^\infty g(s) \left[ \int_0^s f(l)\, dl \right] ds \qquad 4.1$$

$$R = \int_0^\infty \frac{1}{2\phi} e^{-\frac{|s-\theta|}{\phi}} \left[ \int_0^s \lambda e^{-\lambda l}\, dl \right] ds \qquad 4.2$$

$$R = \int_0^\infty \frac{1}{2\phi} e^{-\frac{|s-\theta|}{\phi}} \left[1 - e^{-\lambda s}\right] ds \qquad 4.3$$

$$R = \int_0^\theta \frac{1}{2\phi} e^{-\frac{(\theta-s)}{\phi}} \left[1 - e^{-\lambda s}\right] ds + \int_\theta^\infty \frac{1}{2\phi} e^{-\frac{(s-\theta)}{\phi}} \left[1 - e^{-\lambda s}\right] ds \qquad 4.4$$

Solving the integral in equation (4) and further simplification gives us the reliability expression as:

$$R = 1 - \frac{\lambda\phi\, e^{\frac{-\theta}{\phi}}}{2(\lambda\phi - 1)} + \frac{e^{-\lambda\theta}}{(\lambda\phi - 1)(\lambda\phi + 1)} \qquad 4.5$$

Equation 4.5 can be used to find the reliability when the strength follows Laplace distribution and stress follows exponential distribution. The distribution plot for Laplace strength and exponential stress is shown in Figure 4.1. The dotted curve depicts Laplace distribution with parameters $\theta = 2.5$ and $\phi = 1$. The solid line depicts exponential distribution with parameter $\lambda = 1$.

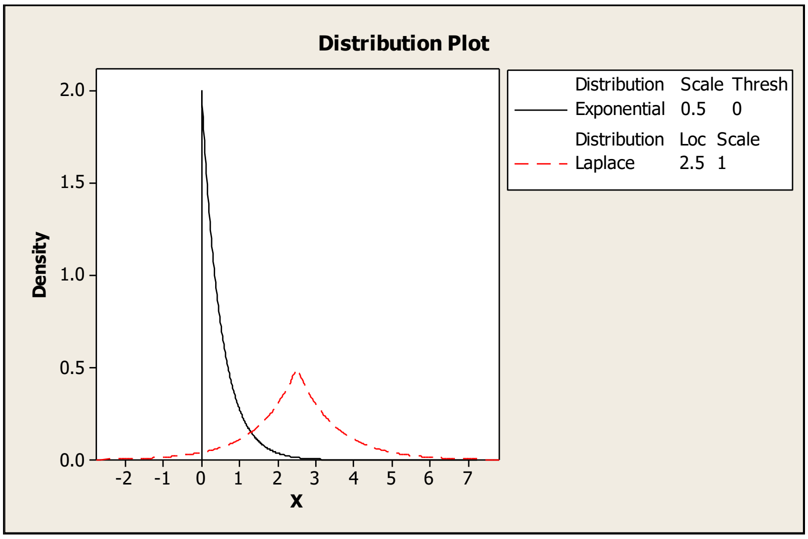


Figure 4.1 Distribution plot for Laplace strength and exponential stress

### 4.1.2 Results and analysis

Taguchi analysis shows the variation of reliability with change in parameters of stress and strength. Figure 4.2 shows the main effects plot for means of reliability when the strength follows Laplace distribution and stress follows exponential distribution. As can be seen in the the figure, the reliability increases when the rate parameter of the exponential distribution and the location parameter of Laplace distribution increases. It can also be noted that the reliability increases when the scale parameter of Laplace distribution decreases.

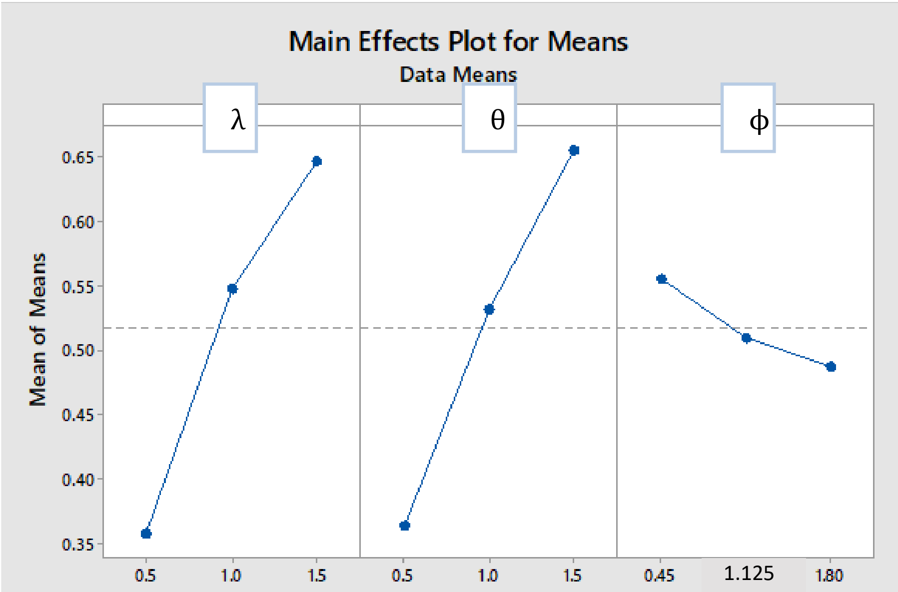


Figure 4.2 Taguchi analysis for Laplace strength and exponential stress

Table 4.1 gives the order of influence of each parameter on the response. It can be seen that when strength follows Laplace distribution and stress follows exponential distribution, the location parameter of Laplace distribution is most influential followed by the rate parameter of exponential distribution and then the scale parameter of Laplace distribution.

Table 4.1 Response tables for means for Laplace strength and exponential stress

| Level | λ | θ | ϕ |
|---|---|---|---|
| 1 | 0.3571 | 0.3636 | 0.5550 |
| 2 | 0.5477 | 0.5319 | 0.5092 |
| 3 | 0.6467 | 0.6561 | 0.4873 |
| Delta | 0.2897 | 0.2925 | 0.0677 |
| Rank | 2 | 1 | 3 |

The contour plot depicted in Figure 4.3 shows two parameter interaction towards the response(reliability). As can be seen in the figure, when the value of λ is held at 1, the reliability values greater than 0.7 are obtained when parameter θ has values greater than 1.25 and parameter ϕ has values lesser than 0.75. When the value of θ is held at 1, the reliability values greater than 0.7 are obtained when λ has values greater than 1 and ϕ has values on the lower side. When the value of ϕ is held at 1.125, higher reliability values can be obtained when θ is greater than 1.25 and ϕ is greater than 1.2.

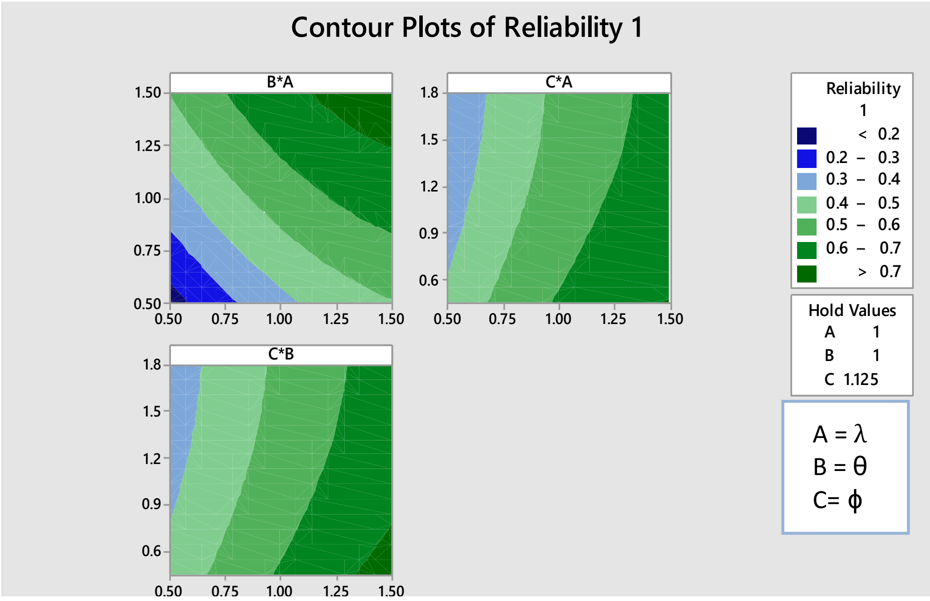


Figure 4.3 Contour plot for reliability in case of Laplace strength and exponential stress

### 4.1.3 Validation of the model

Simulation studies are conducted for validation of the model. Sets of random numbers of sample size 100, 1000, 10000 each were generated for exponential and Laplace distribution with parameters $\lambda = 0.5$, $\theta = 2.5$, $\phi = 1$ and $\lambda = 1.5$, $\theta = 3$, $\phi = 1$ respectively using Matlab statistical software. Reliability was evaluated with the random numbers using the principle P(Y> X). Reliability is also calculated using the model proposed in this paper. Both the results are shown in Table 4.2 which includes the parameter set, the sample size, reliability using simulation, reliability using the proposed methodology, bias and the error. The reliability obtained from the proposed model for the above sets of parameters considered is equal to 0.659 and 0.9342 respectively. Simulation studies show that as the sample size increases the reliability using simulation moves closer to the reliability obtained with proposed model. The bias and the error decreases with the increase in sample size for both sets of parameters. Figure 4.4 shows the corresponding plot for error. For a sample size of 10000, the error for the two sets of parameters considered is 0.0197% and 0.0321% respectively, which shows the precision of the proposed model in evaluating reliability.

Table 4.2 Validation of the model for Laplace strength and exponential stress

| Sr No. | Parameters | Sample Size | Reliability using Simulation | R estimated as per proposed model | Bias | Error (%) |
|---|---|---|---|---|---|---|
| 1 | $\lambda = 0.5$, | 100 | 0.652 | 0.659 | -0.007 | 1.0736 |
| | $\theta = 2.5$, $\phi = 1$ | 1000 | 0.6598 | 0.659 | -0.0008 | 0.1215 |
| | | 10000 | 0.65913 | 0.659 | -0.00013 | 0.0197 |
| 2 | $\lambda = 1.5$, | 100 | 0.926 | 0.9342 | 0.0082 | 0.8855 |
| | $\theta = 3$, $\phi = 1$ | 1000 | 0.9303 | 0.9342 | 0.0039 | 0.4189 |
| | | 10000 | 0.9339 | 0.9342 | 0.0003 | 0.0321 |

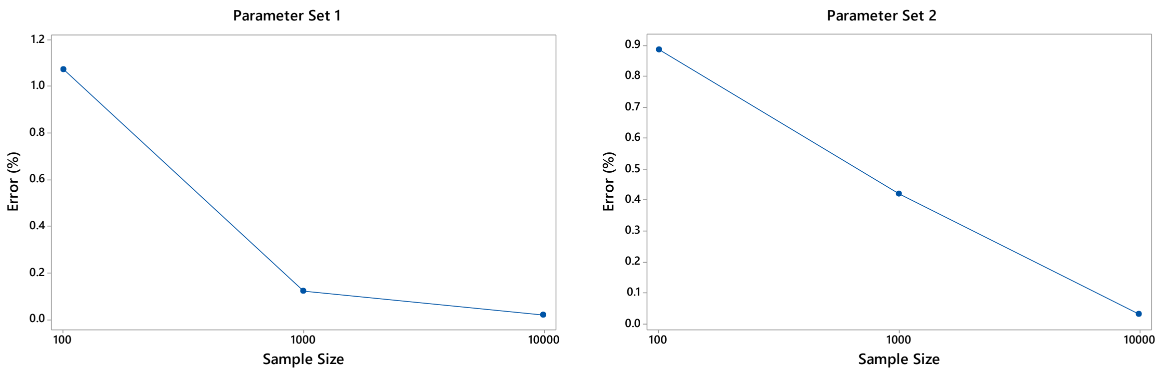


Figure 4.4 Plot of error for validation experiment for Laplace strength and exponential stress

# 4.2 Strength – Exponential Distribution, Stress - Laplace Distribution

### 4.2.1 Model development

This model considers that strength (s) follows exponential distribution and stress (l) follows Laplace distribution. The reliability can be computed using the equation as derived below:

$$R = \int_0^\infty g(l) \left[ \int_l^\infty f(s)\, ds \right] dl \qquad 4.6$$

$$R = \int_0^\infty \frac{1}{2\phi} e^{-\frac{|l-\theta|}{\phi}} \left[ \int_l^\infty \lambda e^{-\lambda s}\, ds \right] dl \qquad 4.7$$

$$R = \int_0^{\infty} \frac{1}{2\phi} e^{-\frac{|l-\theta|}{\phi}} \left[ \int_l^{\infty} \lambda e^{-\lambda s} \, ds \right] dl \qquad 4.8$$

$$R = \int_0^{\infty} \frac{1}{2\phi} e^{-\frac{|l-\theta|}{\phi}} [e^{-\lambda l}] \, dl \qquad 4.9$$

$$R = \int_0^{\theta} \frac{1}{2\phi} e^{-\frac{(\theta-l)}{\phi}} [e^{-\lambda l}] \, dl + \int_{\theta}^{\infty} \frac{1}{2\phi} e^{-\frac{(l-\theta)}{\phi}} [e^{-\lambda l}] \, dl \qquad 4.10$$

Solving the integral in equation (7) and simplification, equation obtained for reliability is:

$$R = \frac{\lambda\phi \, e^{\frac{-\theta}{\phi}}}{2(\lambda\phi - 1)} - \frac{e^{-\lambda\theta}}{(\lambda\phi - 1)(\lambda\phi + 1)} \qquad 4.11$$

Equation 4.11 can be used to find the reliability when the strength follows exponential distribution and stress follows Laplace distribution. The distribution plot for exponential strength and Laplace stress is shown in Figure 4.5. The curve for exponential distribution is depicted by a solid line with parameters $\lambda = 1.2$, whereas the Laplace distribution is depicted by a dotted line with parameters $\theta = 0.2$ and $\phi = 1$.

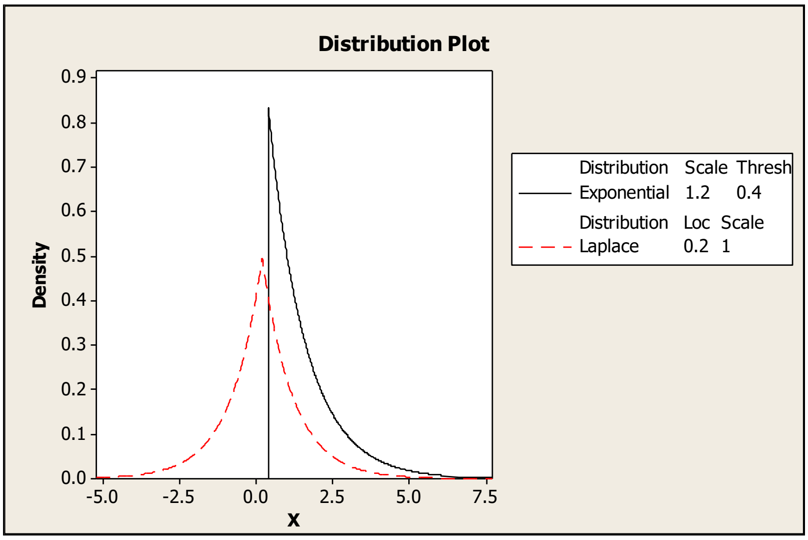


Figure 4.5 Distribution plot for Laplace stress and exponential strength

### 4.2.2 Results and analysis

The main effects plot for means of response (reliability) is shown in Figure 4.6 when the strength follows exponential distribution and stress follows Laplace distribution. As can be seen in the figure, reliability decreases when λ increases. It can also be seen that as θ and ϕ increases, the probability of failure increases (reliability decreases). The order of influence of each parameter towards the response can be seen in Table 4.3 which shows that λ is the most influential parameter towards the response followed by ϕ and θ.

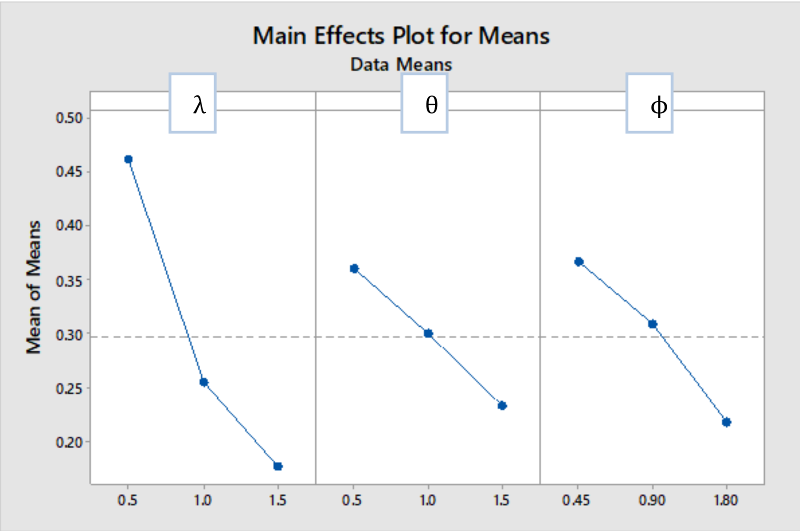


Figure 4.6 Taguchi analysis for Laplace stress and exponential strength

Table 4.3 Response tables for means Laplace stress and exponential strength

| Level | λ | θ | ϕ |
|---|---|---|---|
| 1 | 0.4607 | 0.3597 | 0.3662 |
| 2 | 0.2551 | 0.2996 | 0.3088 |
| 3 | 0.1775 | 0.2341 | 0.2184 |
| Delta | 0.2832 | 0.1256 | 0.1478 |
| Rank | 1 | 3 | 2 |

The two-parameter interaction towards the response(reliability) can be seen in the contour plot depicted in Figure 4.7. As seen in the figure, when the value of λ is set at 1, higher levels of reliability are obtained when θ is lesser than 1 and ϕ is lesser than 1.1 for the considered parameter range. When the value of θ is set to 1, the reliability can be seen lying in higher ranges with λ value lesser than 0.75 and ϕ value in the lower range. When the value of ϕ is held at 1.125, higher reliability can be obtained with λ less than 0.75 and θ less than 1.2.

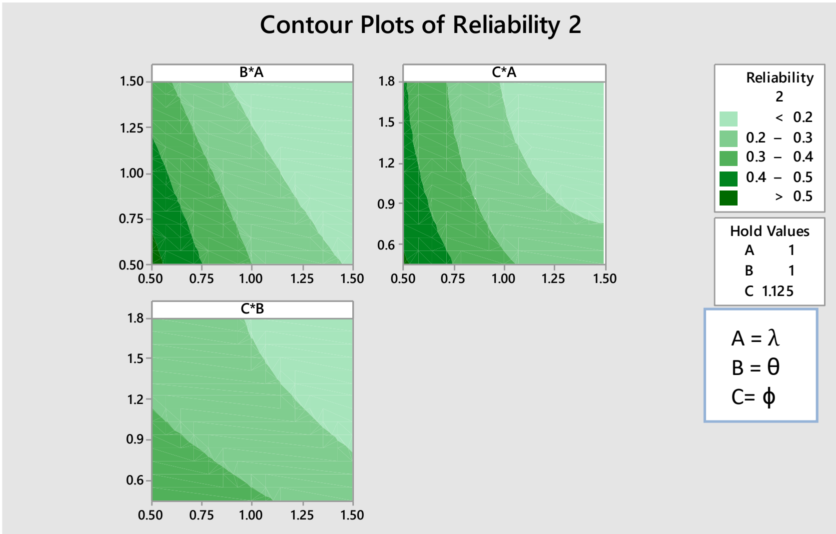


Figure 4.7 Contour plot for reliability in case of Laplace stress and exponential strength

### 4.2.3 Validation of the model

Validation is carried out by simulation studies using simulated number generation. Random numbers of sample size 100, 1000, 10000 each were generated for Laplace and exponential distribution with parameters λ = 0.2, θ = 1.2, ϕ = 0.4 and λ = 0.8, θ = 1.5, ϕ = 1 respectively. Reliability was evaluated using simulation and was also found out using the model proposed in the research. Results are shown in Table 4.4. The reliability obtained for the above set of parameters are 0.3903 and 0.7895 respectively. It can be observed that as the sample size increases, the bias decreases, error decreases and the reliability obtained from simulation moves closer to the reliability obtained from the proposed model. Figure 4.8 shows the corresponding plot of error. For sample size of 10000, the error for the two sets of parameters are 0.09233% and 0.0076% respectively.

Table 4.4 Validation of the model for Laplace stress and exponential strength

| Sr No. | Parameters | Sample Size | Reliability using Simulation | R estimated as per proposed model | Bias | Error (%) |
|---|---|---|---|---|---|---|
| 1 | $\lambda = 0.8$, $\theta = 1.5$, $\phi = 1$ | 100 | 0.385 | 0.3903 | 0.0053 | 1.3766 |
| | | 1000 | 0.3875 | 0.3903 | 0.0028 | 0.7226 |
| | | 10000 | 0.38994 | 0.3903 | 0.00036 | 0.09232 |
| 2 | $\lambda = 0.2$, $\theta = 1.2$, $\phi = 0.4$ | 100 | 0.7825 | 0.7895 | 0.007 | 0.8946 |
| | | 1000 | 0.78815 | 0.7895 | 0.00135 | 0.1713 |
| | | 10000 | 0.78944 | 0.7895 | 0.00006 | 0.0076 |

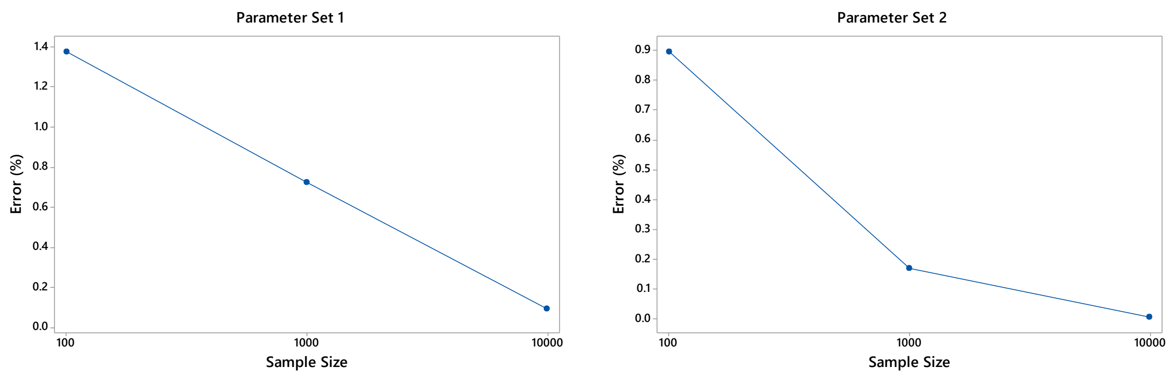


Figure 4.8 Plot of error for validation experiment for exponential strength and Laplace stress

# 4.3 Strength – Weibull Distribution, Stress - Weibull Distribution

### 4.2.1 Model development

Weibull distribution is most commonly used to describe mechanical systems. The interference model of reliability when the stress and strength follow Weibull distribution is given as:

$$R = \int_0^\infty \frac{p_2}{\sigma_2}\left(\frac{s-\mu_2}{\sigma_2}\right)^{p_2-1} e^{-\frac{(s-\mu_2)}{\sigma_2}} \left[\int_0^s \frac{p_1}{\sigma_1}\left(\frac{l-\mu_1}{\sigma_1}\right)^{p_1-1} e^{-\frac{(l-\mu_1)}{\sigma_1}} dl\right] ds \qquad 4.12$$

The integration of the above model is complicated and does not have a closed form. This study attempts to obtain the closed form of stress-strength reliability using design of experiments (DOE) when the stress and strength follow Weibull distribution. Minitab 16 was the software used to conduct design of experiments and analysis. The design chosen was L27 for 6 factors of 3 level each. The parameters chosen for DOE are shown in Table 4.5. The results of design of experiments with response are displayed in Table 4.6. The stress-strength equation was partly solved manually and partly using Wolfram Mathematica software.

Table 4.5 Factors and levels for stress and strength following Weibull distribution

| Distribution | Factors | Levels |
|---|---|---|
| Stress | Shape Parameter: $p_1$ | 0.5, 1.5, 2.5 |
| | Scale Parameter: $\sigma_1$ | 1, 1.5, 2 |
| | Location Parameter: $\mu_1$ | 0, 0.5, 1 |
| Strength | Shape Parameter: $p_2$ | 1.5, 2.5, 3.5 |
| | Scale Parameter: $\sigma_2$ | 1, 1.5, 2 |
| | Location Parameter: $\mu_2$ | 1.5, 2, 2.5 |

Table 4.6 Design of experiments for stress and strength following Weibull distribution

| Stress | | | Strength | | | Reliability |
|---|---|---|---|---|---|---|
| $p_1$ | $\sigma_1$ | $\mu_1$ | $p_2$ | $\sigma_2$ | $\mu_2$ | R |
| 0.5 | 1 | 0 | 1.5 | 1 | 1.5 | 0.78155 |
| 0.5 | 1 | 0 | 1.5 | 1.5 | 2 | 0.83273 |
| 0.5 | 1 | 0 | 1.5 | 2 | 2.5 | 0.86709 |
| 0.5 | 1.5 | 0.5 | 2.5 | 1 | 1.5 | 0.67035 |
| 0.5 | 1.5 | 0.5 | 2.5 | 1.5 | 2 | 0.74267 |
| 0.5 | 1.5 | 0.5 | 2.5 | 2 | 2.5 | 0.79106 |
| 0.5 | 2 | 1 | 3.5 | 1 | 1.5 | 0.56323 |
| 0.5 | 2 | 1 | 3.5 | 1.5 | 2 | 0.65846 |
| 0.5 | 2 | 1 | 3.5 | 2 | 2.5 | 0.72007 |
| 1.5 | 1 | 0.5 | 3.5 | 1 | 2 | 0.97066 |
| 1.5 | 1 | 0.5 | 3.5 | 1.5 | 2.5 | 0.996097 |
| 1.5 | 1 | 0.5 | 3.5 | 2 | 1.5 | 0.979634 |
| 1.5 | 1.5 | 1 | 1.5 | 1 | 2 | 0.722195 |
| 1.5 | 1.5 | 1 | 1.5 | 1.5 | 2.5 | 0.8851964 |
| 1.5 | 1.5 | 1 | 1.5 | 2 | 1.5 | 0.743035 |
| 1.5 | 2 | 0 | 2.5 | 1 | 2 | 0.815387 |
| 1.5 | 2 | 0 | 2.5 | 1.5 | 2.5 | 0.91893406 |
| 1.5 | 2 | 0 | 2.5 | 2 | 1.5 | 0.85073635 |
| 2.5 | 1 | 1 | 2.5 | 1 | 2.5 | 0.99773487 |
| 2.5 | 1 | 1 | 2.5 | 1.5 | 1.5 | 0.91938117 |
| 2.5 | 1 | 1 | 2.5 | 2 | 2 | 0.993511113 |
| 2.5 | 1.5 | 0 | 3.5 | 1 | 2.5 | 0.99883877 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 2.5 | 1.5 | 0 | 3.5 | 1.5 | 1.5 | 0.97996489 |
| 2.5 | 1.5 | 0 | 3.5 | 2 | 2 | 0.998682329 |
| 2.5 | 2 | 0.5 | 1.5 | 1 | 2.5 | 0.87792976 |
| 2.5 | 2 | 0.5 | 1.5 | 1.5 | 1.5 | 0.6762277 |
| 2.5 | 2 | 0.5 | 1.5 | 2 | 2 | 0.87038164 |

$$R = 0.879 + 0.1346\, p_1 - 0.6489\, \sigma_1 - 0.3621\, \mu_1 + 0.2496\, p_2 + 0.5987\, \sigma_2 - 0.4066\, \mu_2 - 0.04577\, p_1{*}p_1 + 0.0502\, \sigma_1{*}\, \sigma_1 + 0.0215\, \mu_1{*}\, \mu_1 - 0.01542\, p_2{*}\, p_2 - 0.0504\, \sigma_2{*}\, \sigma_2 + 0.0009\, \mu_2{*}\, \mu_2 - 0.00085\, p_1{*}\, \sigma_2 + 0.04883\, p_1{*}\mu 2 - 0.0652\, \sigma_1{*}\, \sigma_2 + 0.2211\, \sigma_1{*}\, \mu_2 - 0.0930\, \mu_1{*}\, \sigma_2 + 0.1933\, \mu_1{*}\mu_2 - 0.0925\, p_2{*}\, \sigma_2 \quad 4.13$$

Equation 4.13 is the reliability model evaluated by DOE analysis and can be used for reliability prediction for the parameters within the considered range. The R-sq value for the above equation is 99.82% which shows that the equation can predict reliability with significantly less variability.

**4.3.2 Response surface analysis**

The response surface analysis was carried out to study the two-parameter interaction towards the reliability. Figure 4.9 shows the main effects plot for reliability. It can be seen that the reliability increases with increase in location and scale parameter of strength, and decrease with increase in location and scale parameter of stress. A notable observation that can be made is that reliability increases with increase in shape parameter of both the stress and strength distribution. Figure 4,10 shows the interaction plot for reliability. Figure 4.11 shows contour plot of two parameter interaction towards reliability for stress and strength following Weibull distribution. When the parameters are held at middle values from the levels considered, a high reliability greater than 0.9 is obtained when parameter set lies in a region inscribed by the origin and the following as shown in the figure: $p_1$ greater than 1.4 and $\sigma_1$ on the minimum side preferably lesser than 1.4 in $p_1$ x $\sigma_1$ interaction, $\mu_1$ lesser than 0.25 and $p_1$ greater than 1.7 in $\mu_1$ x $p_1$ interaction, $\mu_2$ greater than 2.2 and $p_1$ greater than 1.7 in $\mu_2$ x $p_1$ interaction, $\mu_1$ lesser than 0.5 and $\sigma_1$ lesser than 1.4 in $\mu_1$ x $\sigma_1$ interaction, $p_2$ greater than 2.4 and $\sigma_1$ on the minium side in $p_2$ x $\sigma_1$ interaction, $\sigma_2$ greater than 1.5 and $\sigma_1$ lesser than 1.25 in $\sigma_2$ x $\sigma_1$ interaction, $\sigma_1$ close to 1 in $\mu_2$ x $\sigma_1$ interaction, and $\sigma_2$ greater than 1.75 and $\mu_1$ lesser than 0.2 $\sigma_2$ x $\mu_1$ interaction.

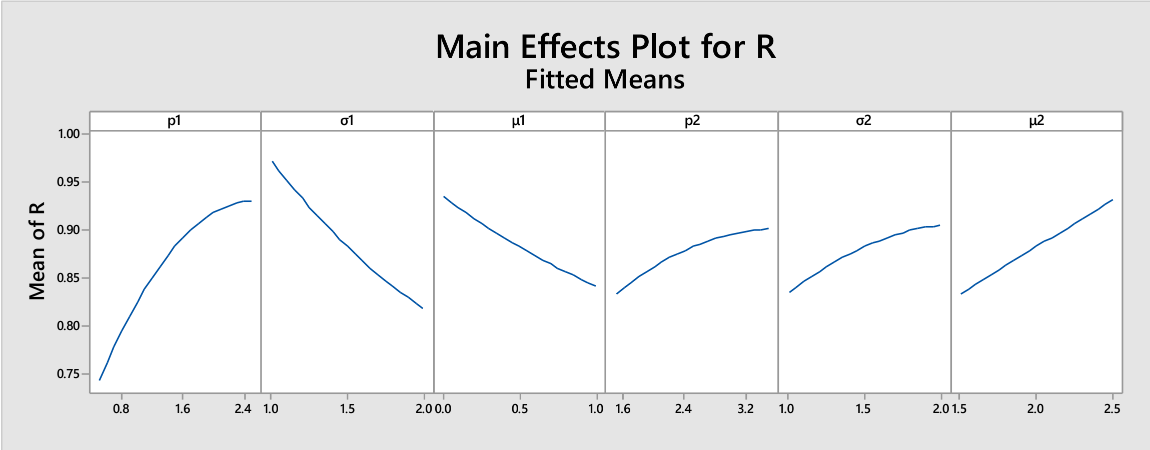


Figure 4.9 Main effects plot for reliability for Weibull stress and strength

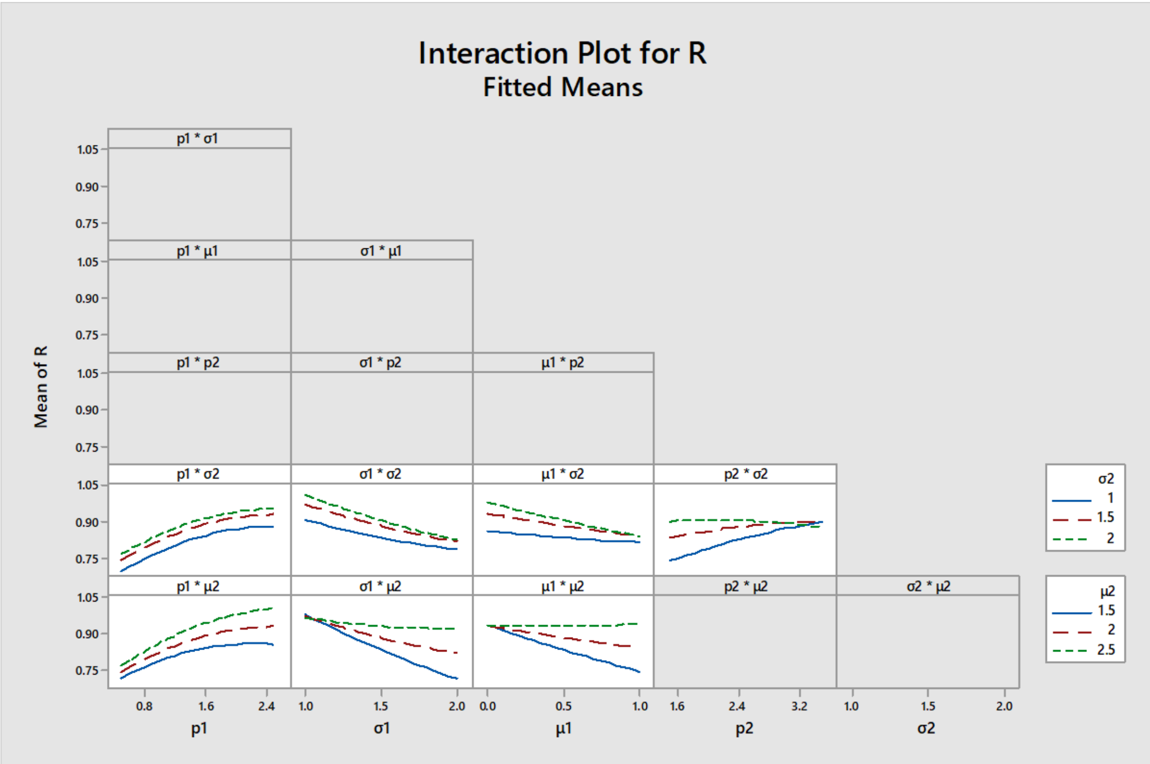


Figure 4.10 Interaction plot for reliability for Weibull stress and strength

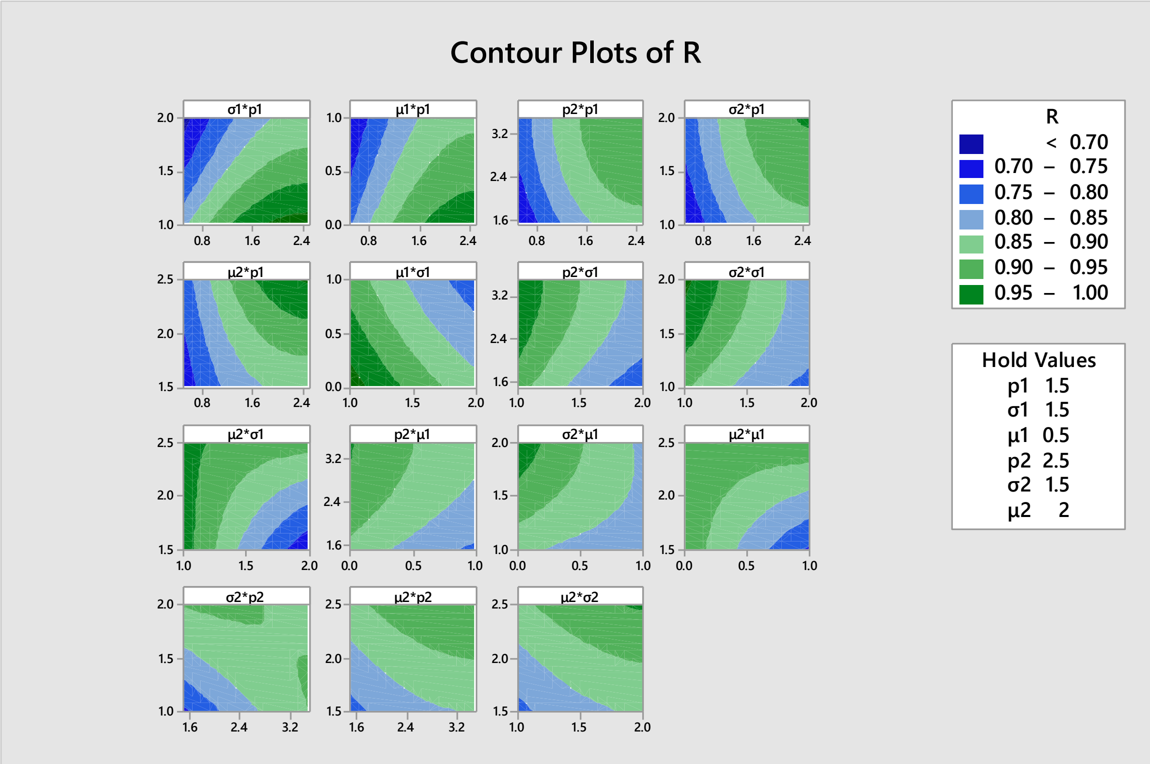


Figure 4.11 Contour plot for reliability in case of stress and strength following Weibull distribution

### 4.3.3 Validation experiment

Two parameter sets are considered for the validation experiment. The parameter values are as shown below:

1) $p_1 = 2.5$ $\sigma_1 = 1$ $\mu_1 = 0$ $p_2 = 3.5$ $\sigma_2 = 2$ $\mu_2 = 2$
2) $p_1 = 0.5$ $\sigma_1 = 1$ $\mu_1 = 0$ $p_2 = 2.5$ $\sigma_2 = 2$ $\mu_2 = 2.5$

The results of the validation experiment have been depicted in Table 4.7. A set of 100, 1000 and 10000 numbers were generated for both the stress and strength distributions. Simulation was carried out using Matlab software. The distribution plot of corresponding stress-strength interference is shown in Figure 4.12. The error in probability obtained from the model and that obtained from random number generation is below 1% and is depicted in Figure 4.14. The optimization of equation 4.13 was carried out using Jaya algorithm and the reliability obtained was 0.99999 with the parameter set $p_1 = 2.5$, $\sigma_1 = 1$, $\mu_1 = 0$, $p_2 = 1.5$, $\sigma_2 = 1$, $\mu_2 = 2.5$. The distribution plot with the optimum set of parameters is shown in Figure 4.14.

Table 4.7 Results of validation experiment for stress and strength following Weibull distribution

| Sr No. | Parameters | Sample Size | Reliability using Simulation | R estimated as per proposed model | Bias | Error (%) |
|---|---|---|---|---|---|---|
| 1 | $p_1 = 2.5$, $\sigma_1 = 1$ | 100 | 0.9876 | 0.99633 | 0.00873 | 0.88396 |
| | $\mu_1 = 0$, $p_2 = 3.5$ | 1000 | 0.99143 | 0.99633 | 0.0049 | 0.49424 |
| | $\sigma_2 = 2$, $\mu_2 = 2$ | 10000 | 0.99612 | 0.99633 | 0.00021 | 0.02108 |
| 2 | $p_1 = 0.5$, $\sigma_1 = 1$ | 100 | 0.8732 | 0.8687 | -0.0045 | 0.5153 |
| | $\mu_1 = 0$, $p_2 = 2.5$ | 1000 | 0.87078 | 0.8687 | -0.0021 | 0.24116 |
| | $\sigma_2 = 2$, $\mu_2 = 2.5$ | 10000 | 0.86973 | 0.8687 | -0.0010 | 0.11843 |

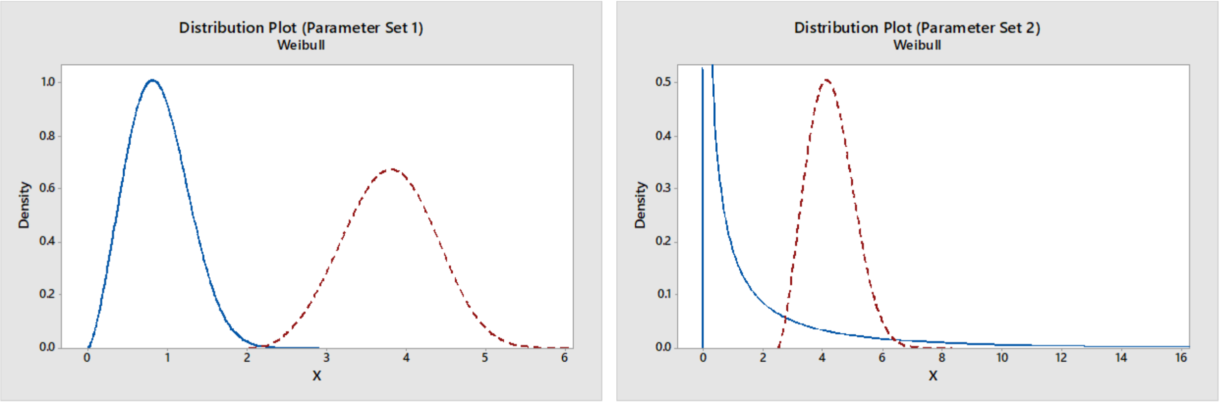


Figure 4.12 Distribution plots for stress and strength following Weibull distribution

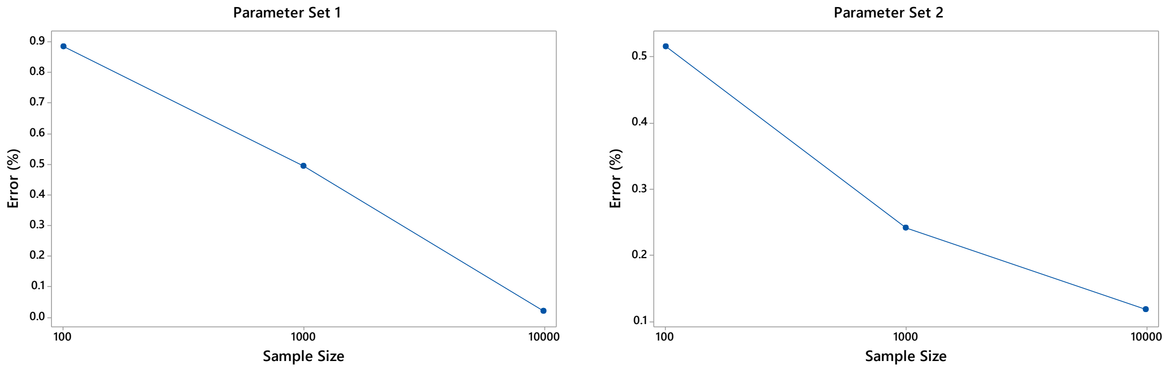


Figure 4.13 Plot of error in estimation of stress-strength reliability for Weibull distribution

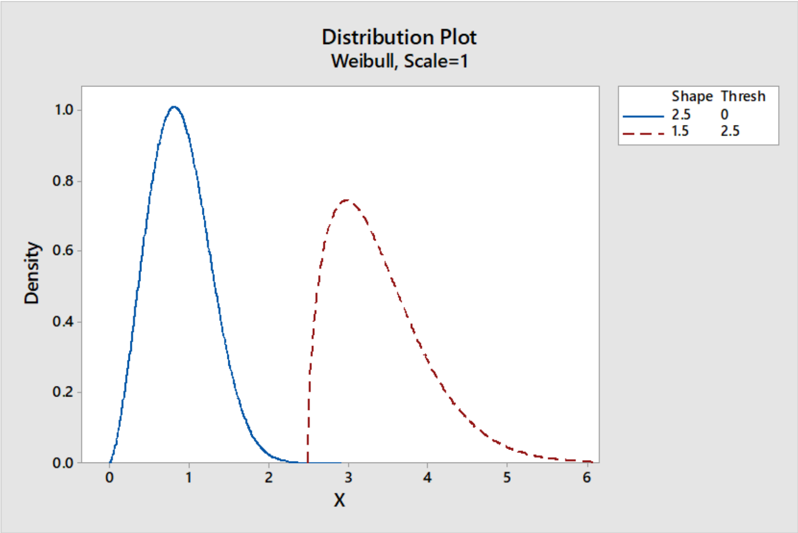


Figure 4.14 Stress-strength interference with optimum set of parameters for Weibull distribution

## 4.4 Strength – Gamma Distribution, Stress - Gamma Distribution

### 4.4.1 Model development

Gamma distribution is most commonly used in life testing, including mechanical systems. The interference model of reliability when the stress and strength follow gamma distribution is given as:

$$R = \int_0^{\infty} \frac{1}{\beta_2{}^{k2}\,\Gamma k_2}(s)^{k_2-1}e^{-\frac{(s)}{\beta_2}}\left[\int_0^{s} \frac{1}{\beta_1{}^{k_1}\,\Gamma k_1}(l)^{k_1-1}e^{-\frac{(l)}{\beta_1}}dl\right]ds \qquad 4.14$$

where $k_1$ and $\beta_1$ are the shape and scale parameters respectively of stress and $k_2$ and $\beta_2$ are the shape and scale parameters respectively of strength. The integration of the above model is complicated as it leads to a form of incomplete gamma function. The reliability, in this case, can be found by numerical integration or by graphical approach which is time-consuming.

In this research, an attempt was made to develop a model for stress-strength interference was developed using response surface methodology when the stress and strength follow gamma distribution. The software used to conduct design of experiment and analysis was Minitab 16. The design chosen was L9 for 4 factors of 3 level each. The parameters chosen for DOE are as shown in Table 4.8. The design of experiments along with the response is shown in Table 4.9. The reliability was calculated partly by manual integration and partly using Wolfram Mathematica software.

Table 4.8 Factors and levels for stress and strength following gamma distribution

| Distribution | Factors | Levels |
|---|---|---|
| Stress | Shape Parameter: $k_1$ | 1, 2, 3 |
| | Scale Parameter: $\beta_1$ | 1, 1.25, 1.5 |
| Strength | Shape Parameter: $k_2$ | 4, 5, 6 |
| | Scale Parameter: $\beta_2$ | 1, 1.25, 1.5 |

Table 4.9 Design of experiments for stress and strength following Weibull distribution

| Stress Factors | | Strength Factors | | Reliability |
|---|---|---|---|---|
| $k_1$ | $\beta_1$ | $k_2$ | $\beta_2$ | R |
| 1 | 1 | 4 | 1 | 0.9375 |
| 1 | 1.25 | 5 | 1.25 | 0.96875 |
| 1 | 1.5 | 6 | 1.5 | 0.984375 |
| 2 | 1 | 5 | 1.5 | 0.95904 |
| 2 | 1.25 | 6 | 1 | 0.892195 |
| 2 | 1.5 | 4 | 1.25 | 0.750538 |
| 3 | 1 | 6 | 1.25 | 0.916646 |
| 3 | 1.25 | 4 | 1.5 | 0.737166 |
| 3 | 1.5 | 5 | 1 | 0.580096 |

$$R = 0.515115 + 0.162159*k_1 + 0.227822*\beta_1 + 0.0914395*k_2 - 0.0436033*\beta_2 - 0.0210245*k_1*k_1 - 0.216925*\beta_1*\lambda_1 - 0.0078125*k_2*k_2 - 0.034456*\beta_2*\beta_2 - 0.199411*k_1*\beta_1 + 0.013921k1*k_2 + 0.015164*\beta_1*k_2 + 0.266573*\beta_1*\beta_2 - 0.0189620*k_2*\beta_2 \qquad 4.15$$

Equation 4.15 is the reliability model obtained from DOE studies and can be used in reliability prediction for the parameters lying within the considered range. The R-sq value is 99.35% which shows significantly less variability in the reliability prediction.

### 4.4.2 Response surface analysis

The main effects plot is depicted in Figure 4.15. It can be observed that the reliability decreases if the shape and scale parameter of stress increases while the reliability increases with increase in shape and scale parameter of strength. Figure 4.17 and 4.18 shows the interaction plot and contour plot respectively for two parameter interaction. If the parameters are set at their mid values, then the reliability greater than 0.9 can be obtained in the region inscribed by origin and the parameters as follows: $\beta_1$ less than 4 and $k_1$ lesser than 2 for $\beta_1$ x $k_1$ interaction, $k_2$ greater than 4.5 and $k_1$ lesser than 1.5 in $k_2$ x $k_1$ interaction, $\beta_2$ greater than 1.1 and $k_1$ less than 1.5 in $\beta_2$ x $k_1$ interaction, $k_2$ greater than 5 and $\beta_1$ less than 1.5 in $k_2$ x $\beta_1$ interaction, $\beta_2$ greater than 1.25 and $\beta_1$ on the minimum side close to 1 in $\beta_2$ x $\beta_1$ interaction.


Figure 4.15 Main effects plot for stress and strength following gamma distribution

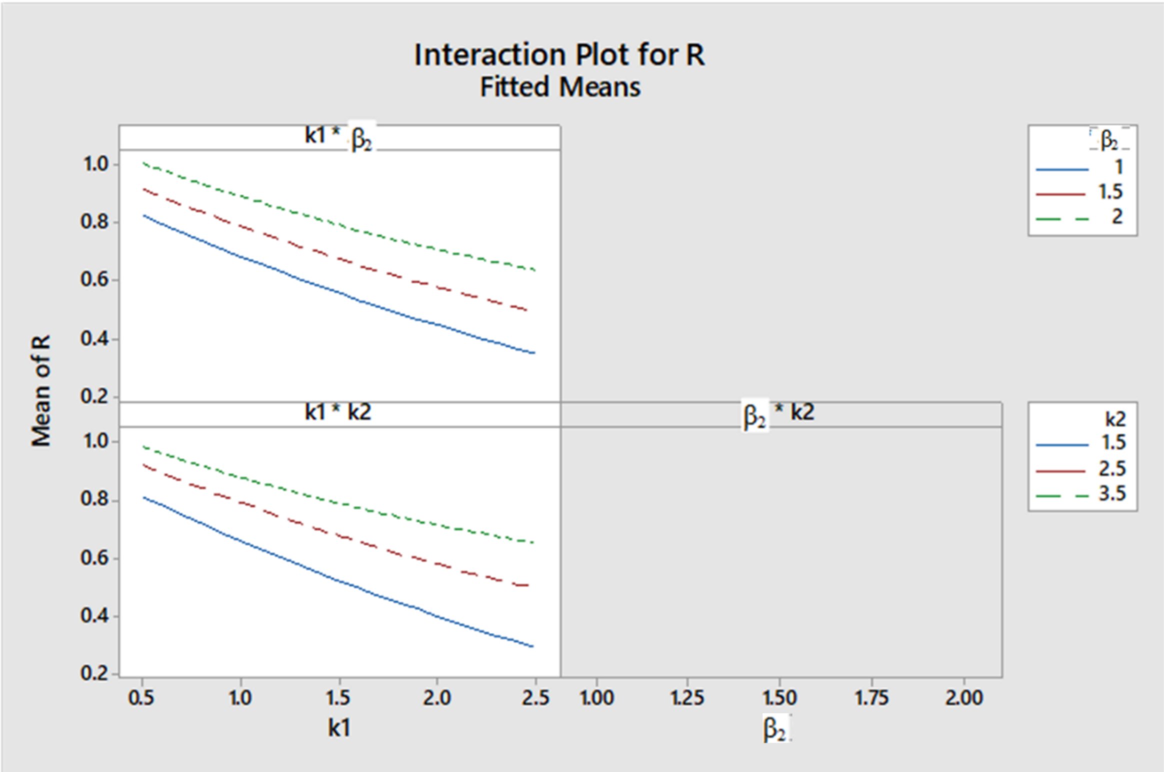


Figure 4.16 Interaction plot for stress and strength following gamma distribution

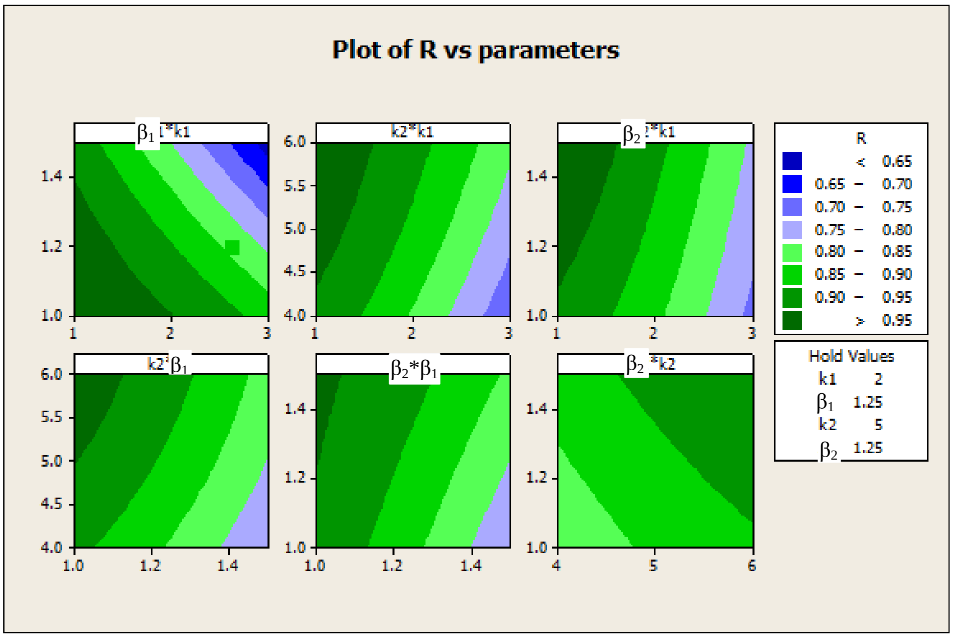


Figure 4.17 Contour plot for reliability in case of stress and strength following gamma distribution

### 4.4.3 Validation experiment

The parameters chosen for the validation experiment are as shown below:

1) $k_1 = 1$ $\beta_1 = 1$ $k_2 = 5$ $\beta_2 = 1.25$
2) $k_1 = 2$, $\beta_1 = 1.25$, $k_2 = 4$, $\beta_2 = 1.5$

The results of the validation experiment have been depicted in Table 4.10. Sets of 100, 1000 and 10000 numbers were generated for both the distributions. The simulation analysis was carried out using Matlab software. The distribution plots for both parameter sets are shown in Figure 4.18. The error in probability obtained from the model and that obtained from random number generation is below 2%. For a sample size of 10000, the error for parameter set 1 and 2 are 0.0407% and 1.0186% respectively. The corresponding plot of error is shown in Figure 4.19. The optimization of equation 4.15 was carried out using Jaya algorithm and the reliability obtained was 0.98735 with the parameter set $k_1 = 1.2252$, $\beta_1 = 1.4459$, $k_2 = 6$, $\beta_2 = 1.5$. The distribution plot with the optimum set of parameters is shown in Figure 4.20.

Table 4.10 Results of validation experiment for stress and strength following gamma distribution

| Sr No. | Parameters | Sample Size | Reliability using Simulation | R estimated as per proposed model | Bias | Error (%) |
|---|---|---|---|---|---|---|
| 1 | $k_1 = 1, \beta_1 = 1,$ | 100 | 0.983 | 0.9814 | -0.0016 | 0.1628 |
| | $k_2 = 5, \beta_2 = 1.25$ | 1000 | 0.98271 | 0.9814 | -0.00131 | 0.1333 |
| | | 10000 | 0.9818 | 0.9814 | -0.0004 | 0.0407 |
| 2 | $k_1 = 2, \beta_1 = 1.25,$ | 100 | 0.8564 | 0.8737 | 0.0173 | 2.0201 |
| | $k_2 = 4, \beta_2 = 1.5$ | 1000 | 0.8631 | 0.8737 | 0.0106 | 1.2281 |
| | | 10000 | 0.86489 | 0.8737 | 0.00881 | 1.0186 |

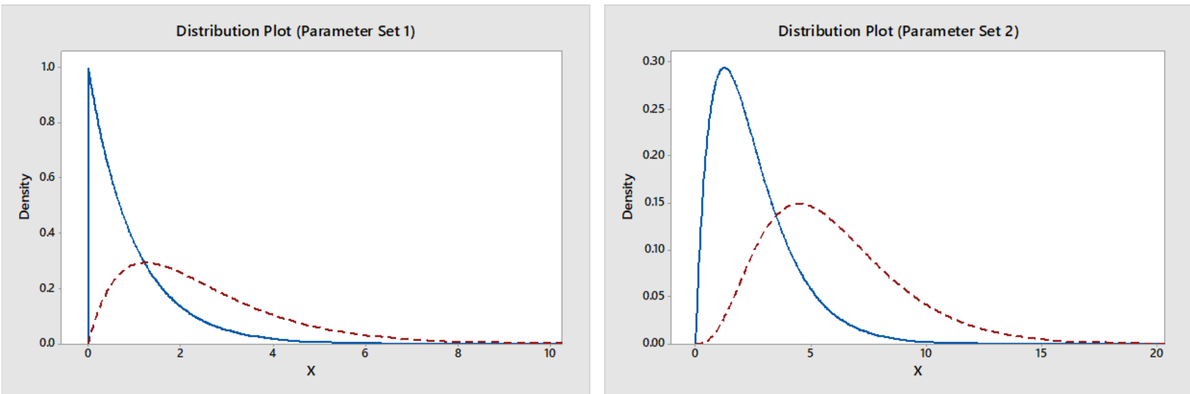


Figure 4.18 Distribution plot for stress and strength following gamma distribution

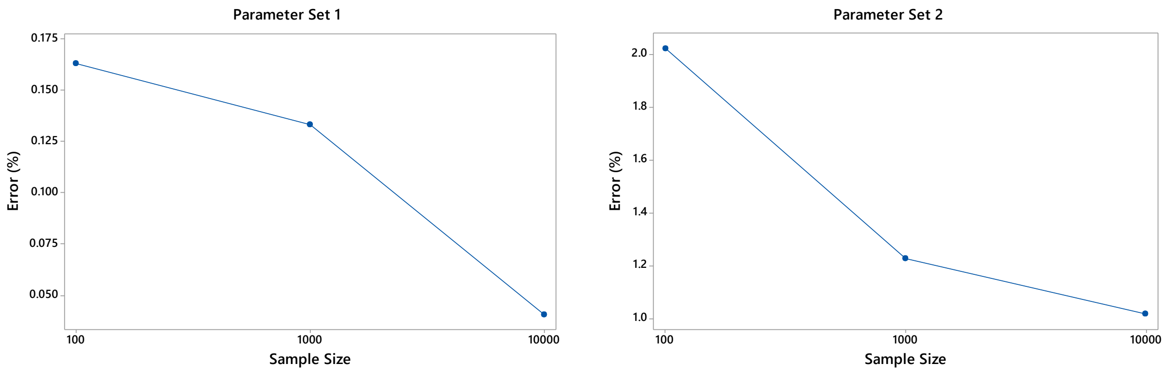


Figure 4.19 Plot of error in estimation of stress-strength reliability for gamma distribution

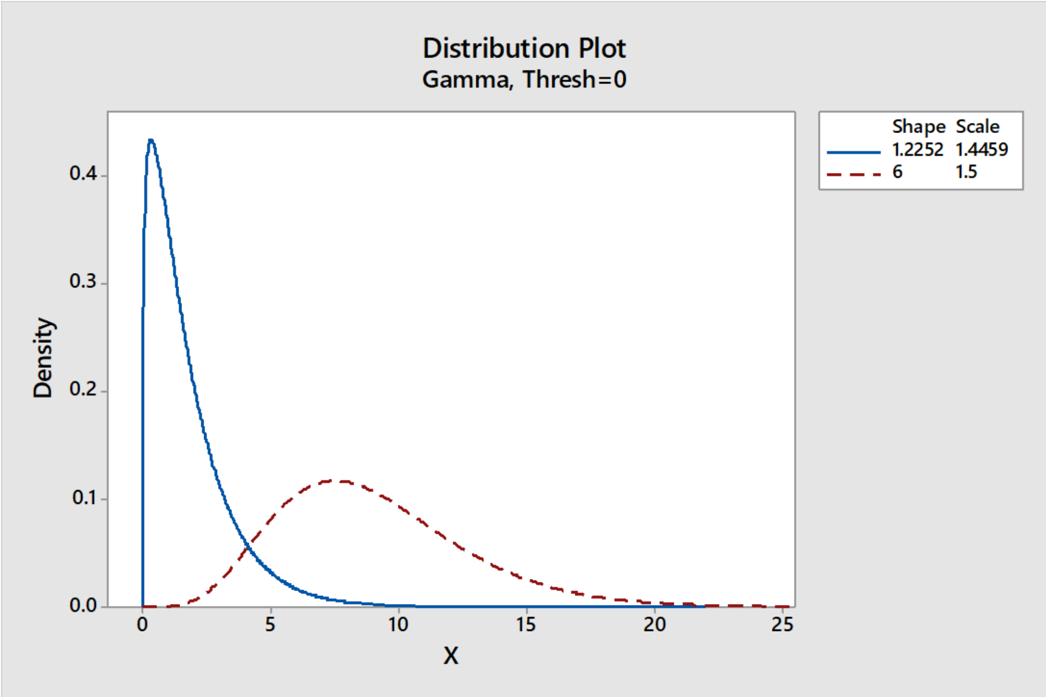


Figure 4.20 Stress-strength interference with optimum set of parameters for gamma distribution

The models derived in equations 4.5, 4.11, 4.13 and 4.15 can be compared with the estimation model of well-established normal distribution depicted in equation 2.4. The models evaluated in equation 4.5 and equation 4.11 for strength Laplace distribution and stress exponential distribution and vice versa are complicated to solve and will be more efficient using advanced numerical calculators. The stress strength reliability model for normal distribution is comparatively simpler and easier to solve. The models evaluated in 4.13 and 4.15 for Weibull and gamma distribution respectively are lengthy and hence may take some time to calculate or feed into the calculators. Also, these models can estimate reliability only when the range of parameters are known. The estimate of normal distribution on the other hand has fewer terms and is flexible for the data with infinite range. Another limitation of the models evaluated in the above section need some analytical software in estimation of distribution parameters for a given data set. The estimation of parameters in case of normal distribution is manageable even using manual calculation for data of small sample size. Even for a large data, the parameters can be estimated using central limit theorem resulting in precise calculations. Thus, the normal distribution can be first tried to obtain a fit for a given data compared to the other distributions mentioned above because of the simplicity of the model and ease in estimation of parameters. If normal distribution does not give a good fit, then other suitable distributions can be tried and worked upon using above evaluated equations.

## 4.5 Summary

In this chapter, two interference models for reliability have been developed when stress follows exponential distribution and strength follows Laplace distribution and vice versa. Additionally, a methodology to derive a model for Weibull and gamma distribution has been proposed. Analysis has been carried out to show the influence of parameters on reliability. Validation for all the models have been conducted using simulation. Interference plots and error plots have been shown for the parameter sets considered for validation. The models have been optimized to obtain the parameters which maximize the reliability and their interference plots have been depicted.

# Chapter 5

# Estimation of Stress-Strength Reliability

## 5.1 Estimating the Parameters of Weibull Distribution

Predicting crucial data such as the strength of materials or lifespan of a component requires real-life statistics in order to fit a distribution model. Various statistical distributions like normal, Weibull, beta, gamma, etc., have been used to model the data. For precise estimation, the distribution should fit the data samples. Thus, in order to obtain the distribution properties, accurate parameter estimation of the related function is required. Determining the statistical distribution is one of the main problems in real-time forecasting since the improper selection of distribution will result in incorrect estimation.

The estimation of parameters of Weibull distribution is essential for its proper implementation and can sometimes be a strenuous task. There are many methods for estimating the parameters of Weibull distribution. Some of the common methods used are maximum likelihood (ML), modified maximum likelihood (MML), moment estimators, method of least squares, etc.[138–141]. Teimouri et al., 2013 [142] and Akram and Hayat (2014) [143] carried out a comparison of methods of estimation for parameters of Weibull distribution. ML is the most widely used because of its good statistical properties like consistency, efficiency, lack of bias, normality, etc. These properties are satisfied when regularity conditions are met [144–146]. It should be noted that the ML method does not provide precise estimators of the parameters directly in some cases. Therefore, we use numerical methods. However, the drawbacks of numerical methods are slow convergence, non-convergence of iterations and convergence to wrong roots [147,148]. Using effective metaheuristics in such cases can help in improving estimation quality with lesser time. A brief overview of the use of metaheuristics in estimation of parameters has been presented in Chapter 2. In this paper, we have focused on estimating the parameters of three-parameter Weibull distribution via maximum likelihood estimation using Jaya algorithm.

### 5.1.1. Likelihood function for Weibull distribution

The method of maximum likelihood is a popular estimation technique and has been used by several authors in estimating the parameters of various distributions[98–100]. Also, MLE is one of the most preferred methods in estimation of parameters for Weibull distribution[52]. Consider $x_1, x_2 ..., x_n$ is a random sample of size 'n' from a population of Weibull distribution W(μ, $\sigma$, $p$), then the likelihood function can be given as:

$$L = \prod_{i=1}^{n} f(x_i) \quad 5.1$$

$$L = \prod_{i=1}^{n} \frac{p}{\sigma^p} (x_i - \mu)^{p-1} \exp\left\{-\left(\frac{x_i - \mu}{\sigma}\right)^p\right\} \quad 5.2$$

$$\ln L = n \ln p - n \ln \sigma + (p-1) \sum_{i=1}^{n} \ln\left(\frac{x_i - \mu}{\sigma}\right) - \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma}\right)^p \quad 5.3$$

The likelihood function (5.3) has to be maximized in order to obtain the best estimates of parameters μ, $\sigma$ and $p$. In this study, Jaya algorithm is used to maximize the likelihood function and attain the estimates for the parameters of Weibull distribution.

### 5.1.2. Implementation of Jaya algorithm to Weibull parameter estimation

For analyzing the performance of Jaya algorithm, three sets of parameters (2, 2, 2), (4, 3, 2) and (5, 2, 3) have been considered. The number of variables is taken as 3 considering the three parameters to be estimated and the population size is taken as 10. The sample sizes considered are 100, 500, and 1000. Search space is chosen as (0,100) for all the parameters. The beauty of this algorithm is that the estimated values of parameters approach the real values even if the search space is large. However, as the search space increases, the number of iterations and time taken to approach the real values increases. The number of iterations is considered as the stopping criteria. In this case, the number of iterations is considered as 200 and total 10 independent experiments are performed to check the repeatability of the algorithm. The compilation has been carried out using Matlab R2018a software on a processing unit of 1.7 GHz, Intel (R) i3, 4.00 GB RAM, 64-bit Operating System. The proposed steps for the implementation of Jaya algorithm to estimation of parameters is given in Table 5.1.

Table 5.1 Steps for implementation of Jaya algorithm

**Input:** $p$ – Population size or number of candidates

$v$ – Number of design variables or generators

MI – Maximum number of iterations

$A$ – Sample from a population

min – Minimum value of search space

max – Maximum value of search space

$f$ – fitness function

**Output:** $max(f)$ – Best solution (maximum) of the fitness function

$min(f)$ – Worst solution (minimum) of the fitness function

$U_{best}$ – Best candidate of fitness function value

$U_{worst}$ – Worst candidate of fitness function value

**Start:**

For $k$ = 1 to $p$ **do** (i.e., population size)

For $i$ = 1 to $v$ **do** (i.e., design variables)

Initialize population sample $X_{i,\mathrm{k}}$

End

End

Evaluate $X_{best,1}$ and $X_{worst,1}$ (Based on solution of fitness function)

Evaluate $max(f)$ = function value for corresponding to candidate $X_{best,i}$

$min(f)$ = function value for corresponding to candidate $X_{worst,i}$

Set $l$ = 1 (Initialize iteration number)

While maximum number of iterations (*maxGen*) is not met

For $k$ := 1 to *pop* **do** (i.e., population size)

For $i$ := 1 to *var* **do** (i.e., design variables)

Set $r_{1,i,l}$ = a random number between [0,1]

Set $r_{2,i,l}$ = a random number between [0,1]

Update $X'_{i,k,l} = X_{i,k,l} + r_{1,i,l}\,(X_{i,best,l} - X_{i,k,l}) - r_{2,i,l}\,(X_{i,worst,l} - X_{i,k,l})$

End

If solution $X'_{i,k,l}$ better than $X_{i,k,l}$ (Based on the solution(maximum) of fitness function)

Set $X_{i,k,l+1} = X'_{i,k,l}$

Else

Set $X_{i,k,l+1} = X_{i,k,l}$

End

End

Set $l = l+1$

Update $X_{best,i}$ and $X_{worst,i}$

Update *max(f)* and *min(f)*

End While *maximum iterations or termination criterion are satisfied*

print: $X_{best,i}$ , $X_{worst,i}$ , *max(f)* , *min(f)*

**End**

A comparative study has also been carried out for the performance of estimation using Jaya algorithm with that of simulated annealing (SA), differential evolution (DE) and hybrid neighborhood search with simulated annealing (HNSA)[56–58], which are some common metaheuristic estimation techniques. Table 5.2 – 5.4 gives the simulation results for Jaya algorithm in comparison to the other methods from the literature. In these examples, $\hat{\mu}$, $\hat{\sigma}$ and $\hat{p}$ denote the estimated parameter values of location, scale and shape parameter respectively, $\hat{f}$ denotes likelihood function at estimated values of parameters, $f$ denotes likelihood function at real values of parameters and $t$ denotes run time in seconds. The results show that in some cases of estimation with HNSA and DE, the log-likelihood function value with the real parameters has a better maximum value than that of the estimated parameters for the same set of generated data. This means that there is a better solution available. The method of SA gives fair results, but the compilation time is too long compared to other algorithms. The Jaya algorithm always moves towards the best estimates as it can be seen that in all the cases, the likelihood function values using estimated parameters are always greater than the likelihood function values using real parameters. Moreover, the compilation time for Jaya algorithm is very less as it gives results within a few seconds depending on the sample size. As the heuristic algorithms are random in nature, their convergence behavior is crucial to understand the approach of the algorithm towards the optimum solution. Figure 5.1 – 5.3 shows the likelihood function behavior of Jaya algorithm which exhibits the values taken by the likelihood function with respect to the iterations during the process of maximization. It can be seen from the graph that the Jaya algorithm values converge after around 25-70 iterations.

Table 5.2 Results for estimation of Weibull parameters for (μ, σ, p) = (2, 2, 2)

| Sample Size | Parameters | SA | HNSA | DE | Jaya algorithm |
|---|---|---|---|---|---|
| n=100 | $\hat{\mu}$ | 2.25181 | 1.9493 | 2.052 | 2.1534 |
| | $\hat{\sigma}$ | 1.6363 | 2.0991 | 1.684 | 2.1645 |
| | $\hat{p}$ | 1.8392 | 2.0962 | 2.04 | 1.9393 |
| | $\hat{f}$ | -134.6563 | -127.606 | -114.883 | -132.3427 |
| | $f$ | (-137.2584) | (-127.5285) | (-110.494) | (-132.7944) |
| | $t_c$ | 72.2188 | 16.6547 | 0.5911 | 0.402844 |
| n=500 | $\hat{\mu}$ | 2.0718 | 1.9741 | 1.984 | 1.9928 |
| | $\hat{\sigma}$ | 1.9306 | 2.0989 | 2.016 | 2.0546 |
| | $\hat{p}$ | 1.8329 | 2.2154 | 2.237 | 2.063 |
| | $\hat{f}$ | -615.3697 | -636.6678 | -609.516 | -642.85 |
| | $f$ | (-618.3805) | (-635.3678) | (-613.856) | (-643.4578) |
| | $t_c$ | 89.6094 | 21.0010 | 1.502 | 1.618632 |
| n=1000 | $\hat{\mu}$ | 2.0354 | 1.9405 | 2.031 | 1.9782 |
| | $\hat{\sigma}$ | 1.9051 | 2.0998 | 2.001 | 2.0175 |
| | $\hat{p}$ | 1.9511 | 2.029 | 2.087 | 2.084 |
| | $\hat{f}$ | -1301.8 | -1277.8 | -1235.975 | -1269.4 |
| | $f$ | (-1305.5) | (-1274.2) | (-1240.203) | (-1270.4) |
| | $t_c$ | 96.2500 | 27.9880 | 2.6507 | 2.830968 |

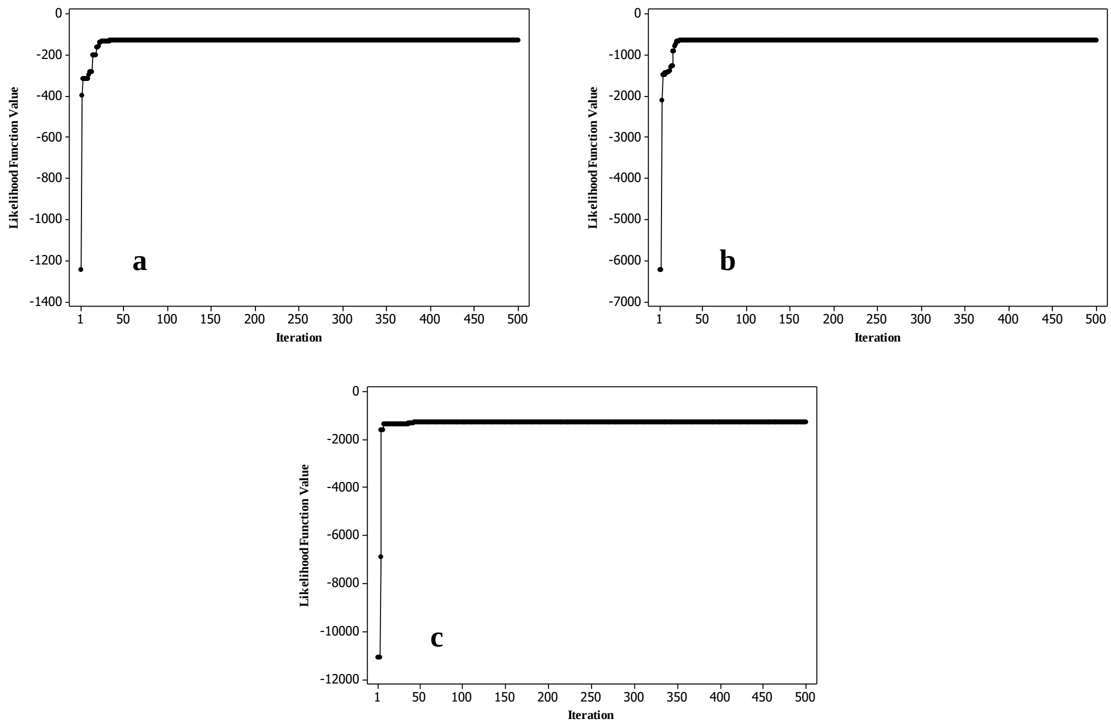


Figure 5.1 Jaya algorithm approach towards ML function maximum for (μ, σ, p) = (2, 2, 2): (a) sample size = 100, (b) sample size = 500, (c) sample size = 1000

Table 5.3 Results for estimation of Weibull parameters for (μ, σ, p) = (4, 3, 2)

| Sample Size | Parameters | SA | HNSA | DE | Jaya algorithm |
|---|---|---|---|---|---|
| n=100 | $\hat{\mu}$ | 4.0707 | 3.8148 | 4.108 | 4.1323 |
| | $\hat{\sigma}$ | 3.0067 | 2.8295 | 2.797 | 2.8295 |
| | $\hat{p}$ | 1.8974 | 1.9289 | 2.228 | 1.9445 |
| | $\hat{f}$ | -93.1223 | -144.782 | -153.881 | -166.1462 |
| | $f$ | (-93.4527) | (-143.5801) | (-156.601) | (-166.4581) |
| | $t_c$ | 86.7500 | 17.6824 | 0.586 | 0.45026 |
| n=500 | $\hat{\mu}$ | 4.1451 | 4.1685 | 4.03 | 3.9927 |
| | $\hat{\sigma}$ | 2.917 | 3.1096 | 2.856 | 2.9517 |
| | $\hat{p}$ | 1.8267 | 1.9905 | 2.07 | 2.046 |
| | $\hat{f}$ | -458.2988 | -743.32 | -810.039 | -831.9281 |
| | $f$ | (-461.2994) | (-733.2223) | (-813.217) | (-832.6032) |
| | $t_c$ | 87.2969 | 23.0984 | 1.502 | 1.595823 |
| n=1000 | $\hat{\mu}$ | 3.9295 | 3.9321 | 4.003 | 4.0163 |
| | $\hat{\sigma}$ | 3.2164 | 2.9165 | 2.893 | 2.962 |

| | | | | |
|---|---|---|---|---|
| $\hat{p}$ | 2.094 | 2.0896 | 2.043 | 1.9615 |
| $\hat{f}$ | -974.2441 | -1489.3 | -1640.7 | -1694.7 |
| $f$ | (-974.9306) | (-1465.0) | (-1644.208) | (-1694.9) |
| $t_c$ | 90.6250 | 31.5077 | 2.694 | 2.820331 |

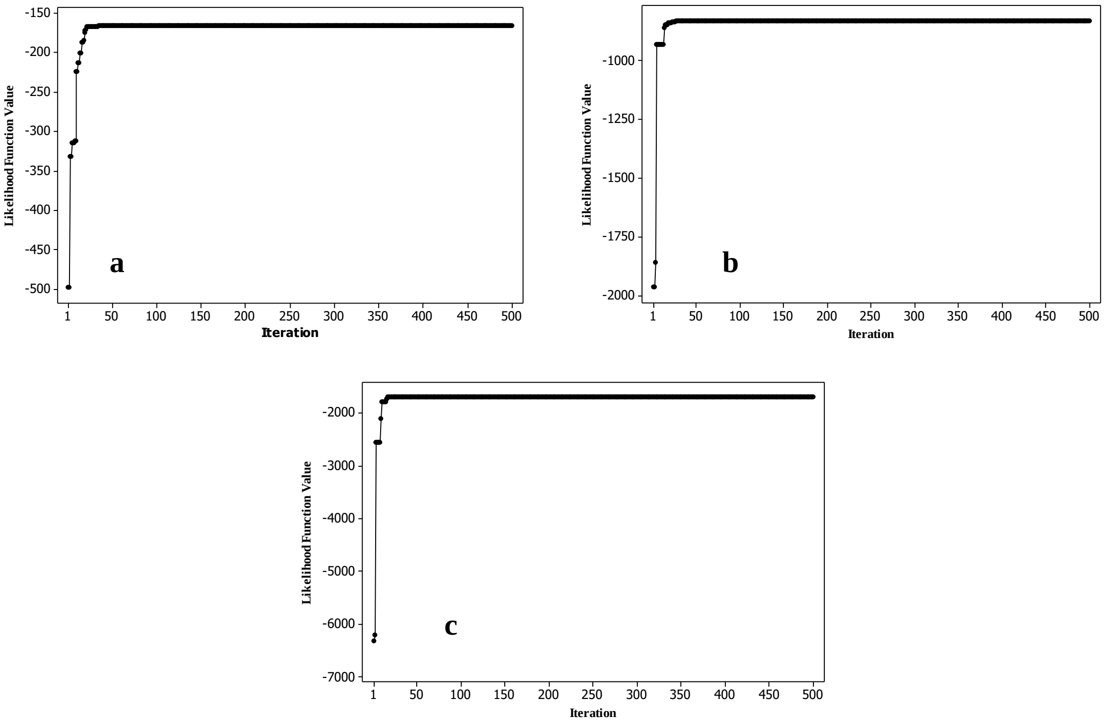


Figure 5.2 Jaya algorithm approach towards ML function maximum for (μ, σ, p) = (4, 3, 2): (a) sample size = 100, (b) sample size = 500, (c) sample size = 1000

Table 5.4 Results for estimation of Weibull parameters for (μ, σ, p) = (5, 2, 3)

| Sample Size | Parameters | SA | HNSA | DE | Jaya algorithm |
|---|---|---|---|---|---|
| n=100 | $\hat{\mu}$ | 5.2287 | 4.9713 | 5.072 | 4.9945 |
| | $\hat{\sigma}$ | 1.7596 | 2.4079 | 1.873 | 1.9955 |
| | $\hat{p}$ | 2.6096 | 2.9966 | 3.395 | 3.2589 |
| | $\hat{f}$ | -165.1076 | -213.2846 | -84.700 | -89.9672 |
| | $f$ | -166.0052 | -211.4428 | -81.134 | -90.5908 |
| | $t_c$ | 83.7031 | 16.8213 | 0.585 | 0.41031 |
| n=500 | $\hat{\mu}$ | 5.0398 | 4.7445 | 4.949 | 4.9799 |
| | $\hat{\sigma}$ | 1.9971 | 2.0171 | 2.059 | 2.0635 |
| | $\hat{p}$ | 2.9469 | 3.1137 | 3.509 | 3.1713 |

| | | | | | |
|---|---|---|---|---|---|
| | $\hat{f}$ | -838.5784 | -855.4235 | -441.658 | -484.3014 |
| | $f$ | -838.9136 | -836.7576 | -447.947 | -485.582 |
| | $t_c$ | 86.7813 | 22.6629 | 1.500 | 1.5326 |
| n=1000 | $\hat{\mu}$ | 4.9547 | 4.9012 | 5.054 | 4.9738 |
| | $\hat{\sigma}$ | 1.9814 | 2.1581 | 1.937 | 2.0215 |
| | $\hat{p}$ | 3.1251 | 3.1997 | 3.016 | 3.1029 |
| | $\hat{f}$ | -1743.1 | -1759.2 | -941.907 | -961.63 |
| | $f$ | -1746.6 | -1748.7 | -943.630 | -962.1128 |
| | $t_c$ | 91.8125 | 24.3881 | 2.646 | 2.9228 |

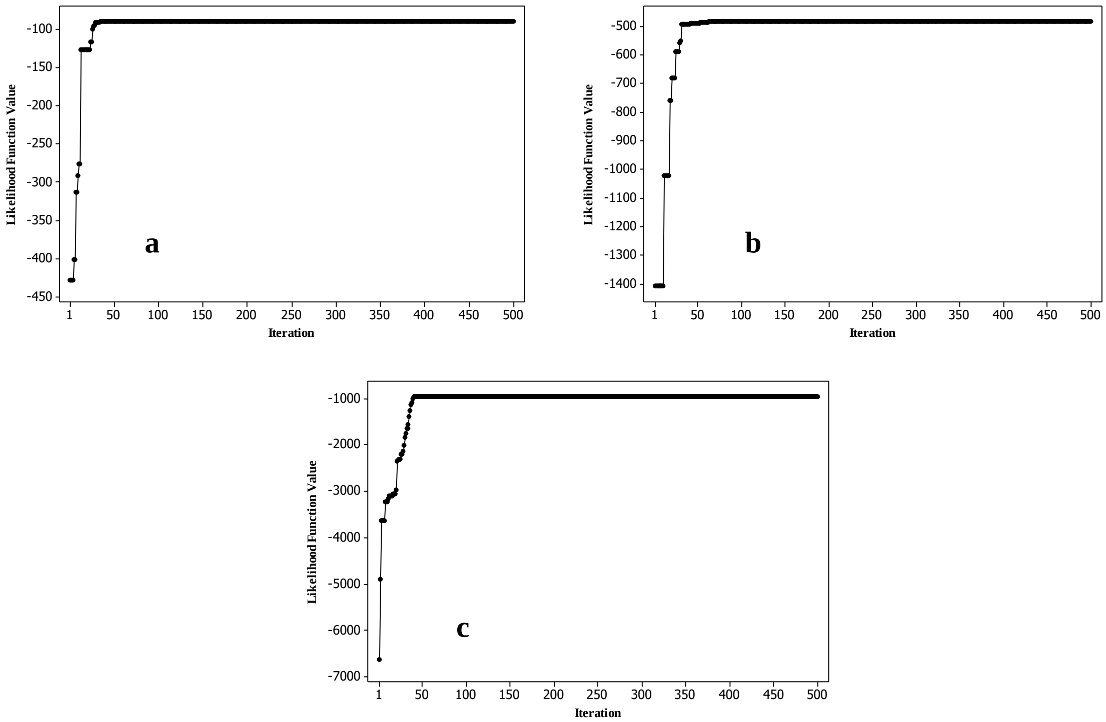


Figure 5.3 Jaya algorithm approach towards ML function maximum for (μ, σ, p) = (5, 2, 3): (a) sample size = 100, (b) sample size = 500, (c) sample size = 1000

### 5.1.3 Application to strength of glass fibres

In order to consider the implementation of the proposed methodology, the fibre glass data which is widely used in statistical studies is considered. Appendix I consists of breaking strength for 63 glass fibres of length 1.5 cm first studied by Smith and Naylor, 1987 [149]. The data was modeled using the proposed method and the results are presented in Table 5.5 along with a comparison of modeling the data by recent literature using particle swarm optimization (PSO) [60]. The parameters $\hat{\mu}$, $\hat{\sigma}$ and $\hat{p}$ are estimated to be -1.5672, 3.2087 and 11.7605 respectively. It is evident from the results that the Jaya algorithm gives a better maximum likelihood function value (-14.2853) compared to that with PSO algorithm (-14.2860). Hence it can be concluded in this case that Jaya algorithm gives better estimates than particle swarm optimization algorithm. Figure 5.4 shows the histogram with fitted densities and probability plot for the strength of glass fibre data with estimated parameters. It can be seen that the Weibull distribution with estimated parameters using Jaya algorithm provides a very good fitting for the data.

Table 5.5 Results for estimation of Weibull parameter for strengths of glass fibre data

| Sample Size | Parameters | PSO with PSS | PSO with SS1 | PSO with SS2 | PSO with SS3 | Jaya algorithm with SS3 |
|---|---|---|---|---|---|---|
| n=63 | μ | -1.0058 | -1.9987 | 0 | Not included because of worst performance | -1.5672 |
| | σ | 3.1420 | 3.6384 | 1.6281 | | 3.2087 |
| | p | 11.5076 | 13.4402 | 5.7807 | | 11.7605 |
| | $\hat{f}$ | -14.2860 | -14.3030 | -15.2068 | | -14.2853 |

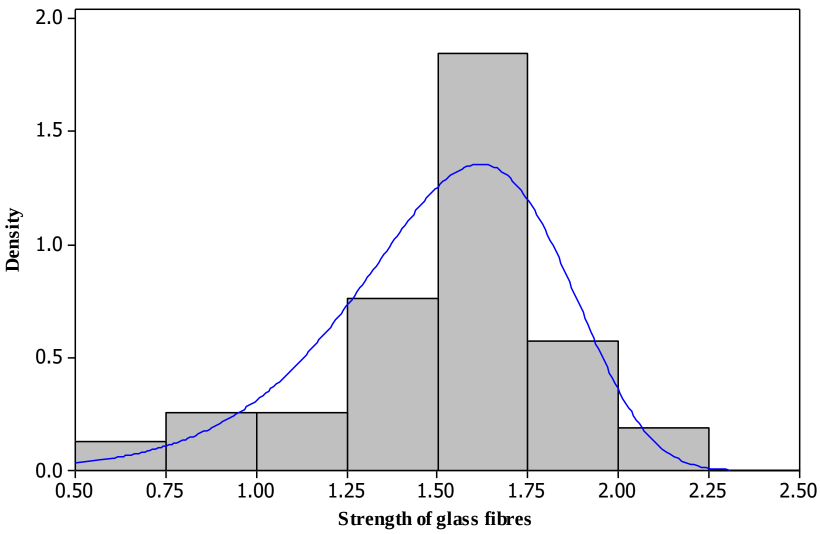


Figure 5.4 The histogram and fitted density for glass fibre data

## 5.2 Estimation of Reliability for Stress and Strength following Weibull Distribution with Common Scale Parameter

Designing and assessing mechanical or structural components based on reliability can be very effective in preventing failures or accidents. Properties of a component like stress and strength require precise designing as these are vital in determining the component's safety. In real life, we know that properties like stress and strength of mechanical components do not take a fixed value due to the various uncertainties in materials, loading conditions, environmental conditions, etc.

Thus, they can be considered to follow a particular distribution which can be determined based on the application or prior data. Traditional methods may not be the best approach for designing of mechanical components as it does not take uncertainties into consideration. Hence, reliability-based design will be suitable in such cases which takes into account the probability of failure if the stress and strength takes various values within its range. Reliability of the form P [X > Y] is used in cases of stress-strength interference. According to the interference theory, if stress and strength follow a particular distribution, then their interference area gives the probability of failure. The concept of stress-strength interference in evaluating reliability has been used by many researchers in their studies.

Weibull distribution has been widely used by researchers in their study as it is capable of fitting large data types [150–152]. If x and y are the random variables following Weibull distribution W($\sigma, p_1$) and W($\sigma, p_2$) respectively then their pdf can be given as:

$$f(x;\sigma,p_1) = \frac{p_1}{\sigma^{p_1}}(x)^{p_1-1}\exp\left\{-\left(\frac{x}{\sigma}\right)^{p_1}\right\}, x > 0, \sigma > 0, p_1 > 0 \qquad 5.4$$

and

$$f(y;\sigma,p_2) = \frac{p_2}{\sigma^{p_2}}(y)^{p_2-1}\exp\left\{-\left(\frac{y}{\sigma}\right)^{p_2}\right\}, y > 0, \sigma > 0, p_2 > 0 \qquad 5.5$$

respectively. The corresponding cumulative distribution function (cdf) for strength and stress is given by

$$F(x;\sigma,p_1) = 1 - \exp\left\{-\left(\frac{x}{\sigma}\right)^{p_1}\right\} \qquad 5.6$$

and

$$F(y;\sigma,p_2) = 1 - \exp\left\{-\left(\frac{y}{\sigma}\right)^{p_2}\right\} \qquad 5.7$$

where $p_1$ and $p_2$ are shape parameters for strength and stress respectively, and $\sigma$ is the common scale parameter. It is difficult to evaluate the stress-strength reliability model when both parameters of Weibull distribution are different. The purpose of the present study is to show the effectiveness of Jaya algorithm in the estimation of reliability for Weibull distribution. Hence, it has been assumed that the scale parameter for stress and strength distribution remains the same. Stress-strength Weibull distribution with common scale parameter has been used in estimating the reliability for strength of carbon fibers [51].

### 5.2.1. Reliability estimation

If X and Y denote the strength and stress distribution with common scale parameter but different shape parameter then according to interference theory the reliability can be given as

$$R = P\ (X > Y) = \int_0^{\infty} \left( f(x;\sigma,p_1) \int_0^{x} f(y;\sigma,p_2) dy \right) dx \qquad 5.8$$

$$R = P\ (X > Y) = \int_0^{\infty} \begin{pmatrix} \frac{p_1}{\sigma^{p_1}} (x)^{p_1-1} \exp\left\{-\left(\frac{x}{\sigma}\right)^{p_1}\right\} . \\ \int_0^{x} \frac{p_2}{\sigma^{p_2}} (y)^{p_2-1} \exp\left\{-\left(\frac{y}{\sigma}\right)^{p_2}\right\} dy \end{pmatrix} dx \qquad 5.9$$

$$R = P\ (X > Y) = \int_0^{\infty} \begin{pmatrix} \frac{p_1}{\sigma^{p_1}} (x)^{p_1-1} \exp\left\{-\left(\frac{x}{\sigma}\right)^{p_1}\right\} . \\ \left\{1 - \exp\left\{-\left(\frac{x}{\sigma}\right)^{p_2}\right\}\right\} \end{pmatrix} dx \qquad 5.10$$

$$R = P\ (X > Y) = 1 - \int_0^{\infty} \frac{p_1}{\sigma^{p_1}} (x)^{p_1-1} \exp\left\{-\left(\left(\frac{x}{\sigma}\right)^{p_1} + \left(\frac{x}{\sigma}\right)^{p_2}\right)\right\} dx \qquad 5.11$$

If $\widehat{p_1}$, $\widehat{p_2}$ and $\hat{\sigma}$ are the estimated parameters of Weibull distribution, then the estimated reliability $\hat{R}$ can be given as

$$\hat{R} = 1 - \int_0^{\infty} \frac{\widehat{p_1}}{\hat{\sigma}^{\widehat{p_1}}} (x)^{\widehat{p_1}-1} \exp\left\{-\left(\left(\frac{x}{\hat{\sigma}}\right)^{\widehat{p_1}} + \left(\frac{x}{\hat{\sigma}}\right)^{\widehat{p_2}}\right)\right\} dx \qquad 5.12$$

In this study, some of the most widely used estimation methods are implemented namely maximum likelihood estimation, least squares estimation, and weighted least squares estimation. Louzada et al., 2016 [153] used these methods in estimating the parameters of extended exponential geometric distribution for medical data. Datsiou and Overend, 2018 [154] presented a comparison of various methods including MLE and LSE in the estimation of parameters for Weibull distribution applied to a data of strength of glass fibers by evaluating the fitness of the parameters using Anderson Darling goodness of fit test. The above estimation methods are simple and easy to evaluate.

### 5.2.2 Maximum likelihood estimation in estimation of reliability

Maximum likelihood estimation is one of the common and effective methods in the estimation of parameters[155–157]. Also, MLE is one of the most preferred methods in estimation of parameters for Weibull distribution [52]. Let $x_1$, $x_2$, $x_3$ …$x_n$ be a random sample of size n drawn from W($\sigma, p_1$) and $y_1$, $y_2$, $y_3$, …., $y_n$ be the random sample of size m from W($\sigma, p_2$). Then the likelihood function can be given as:

$$L = \prod_{i=1}^{n} f(x_i) \prod_{j=1}^{m} f(y_j) \tag{5.13}$$

$$L = \prod_{i=1}^{n} \frac{p_1}{\sigma^{p_1}} (x_i)^{p_1 - 1} \exp\left\{-\left(\frac{x_i}{\sigma}\right)^{p_1}\right\} . \prod_{j=1}^{m} \frac{p_2}{\sigma^{p_2}} (y_j)^{p_2 - 1} \exp\left\{-\left(\frac{y_j}{\sigma}\right)^{p_2}\right\} \tag{5.14}$$

$$\begin{aligned} \ln L = n \ln p_1 + m \ln p_2 - np_1 \ln \sigma - mp_2 \ln \sigma \\ + (p_1 - 1) \sum_{i=1}^{n} \ln (x_i) + (p_2 - 1) \sum_{j=1}^{m} \ln\left(y_j\right) \\ - \frac{1}{\sigma^{p_1}} \sum_{i=1}^{n} (x_i)^{p_1} - \frac{1}{\sigma^{p_2}} \sum_{\mathrm{j=1}}^{\mathrm{m}} \left(y_j\right)^{p_2} \end{aligned} \tag{5.15}$$

The log-likelihood function 5.15 is to be maximized in order to obtain the best estimates of parameters.

### 5.2.3 Least squares estimation in estimation of reliability

The least squares estimation technique was used by many researchers in the field of estimation and has been presented in the Literature review. The method of least square and weighted least square have a property of unbiased estimation for large number of observations and have been used in estimating stress-strength reliability for various distributions including Weibull distribution [158–161].

Consider $x_1$, $x_2$, $x_3$, …, $x_n$ is the random sample in ascending order of size n following Weibull distribution W($\sigma$, $p_1$) and $y_1$, $y_2$, $y_3$, …, $y_m$ is the random sample in ascending order of size m following Weibull distribution W($\sigma$, $p_2$). Then the least squares criterion can be obtained as [55]

$$\begin{aligned} S = \sum_{i=1}^{n} \left( \ln\left( \ln\left( \frac{1}{1 - \widehat{F(x_i)}} \right) \right) - p_1 \ln(x_i) + p_1 \ln(\sigma) \right)^2 \\ + \sum_{j=1}^{m} \left( \ln\left( \ln\left( \frac{1}{1 - \widehat{F(y_j)}} \right) \right) - p_2 \ln\left(y_j\right) + p_2 \ln(\sigma) \right)^2 \end{aligned} \tag{5.16}$$

The estimates of parameters can be obtained by minimizing function 5.16. The estimate values of F(x) and F(y) can be obtained by mean rank as

$\widehat{F(x_i)} = \frac{i}{n+1}$ and $\widehat{F(y_j)} = \frac{j}{m+1}$

### 5.2.4 Weighted least squares estimation in estimation of reliability

Weighted least squares which is a modification of least squares estimation method has been used in many applications. The criterion to be minimized for weighted least squares can be given as

$$S_w = \sum_{i=1}^{n} w_i \left( \ln\left( \ln\left( \frac{1}{1-\widehat{F(x_i)}} \right) \right) - p_1 \ln(x_i) + p_1 \ln(\sigma) \right)^2 + \sum_{j=1}^{m} w_j \left( \ln\left( \ln\left( \frac{1}{1-\widehat{F(y_j)}} \right) \right) - p_2 \ln(y_j) + p_2 \ln(\sigma) \right)^2 \qquad 5.17$$

where $w_i = \left(1-\widehat{F(x_i)}\right) ln\left(1-\widehat{F(x_i)}\right)^2$ and $w_j = \left(1-\widehat{F(y_j)}\right) ln\left(1-\widehat{F(y_j)}\right)^2$

The estimates of parameters using weighted least squares method can be obtained by minimizing equation 5.17.

Equations 5.15, 5.16 and 5.17 discussed above are optimization problems. Solutions to these equations using numerical computation do not yield precise results. It also has problems of slow convergence and non-convergence to real roots. So, these methods have to be assisted with a suitable optimization technique in order to improve their effectiveness. In this case, Jaya algorithm is used to optimize these functions.

### 5.2.5 Jaya algorithm in estimation of stress-strength reliability

The application of metaheuristic techniques in optimization problems has seen increasing importance in modern times [162,163]. In this section, Jaya algorithm has been used in order to estimate the stress-strength reliability for Weibull distribution. The detailed steps in using Jaya algorithm in estimation of reliability are as follows:

1. Specify the population size and number of design variables.
2. Set the boundary conditions.
3. Generate a random set of parameters with the number of sets equal to population size and the number of parameters equal to the number of design variables.
4. Trim the generated set as per boundary conditions.
5. Calculate the function value for each set based on the objective function 5.15, 5.16 and 5.17 for MLE, LSE and WLSE respectively.
6. Identify the best and the worst function value.

7. Update the parameter set based on equation (3.6) within the boundary conditions. Calculate the updated function value and identify the best and worst function values for the parameter sets.
8. If the updated function value of a set is better than the earlier function value of the respective set, replace the earlier set of the design parameters with the updated parameter set. This completes the first iteration.
9. The iteration number can be considered as the termination criteria.
10. Note the estimated parameters and calculate reliability based on equation 5.12.

The flowchart for application of Jaya algorithm in estimation of stress strength reliability has been depicted in Figure 5.5.

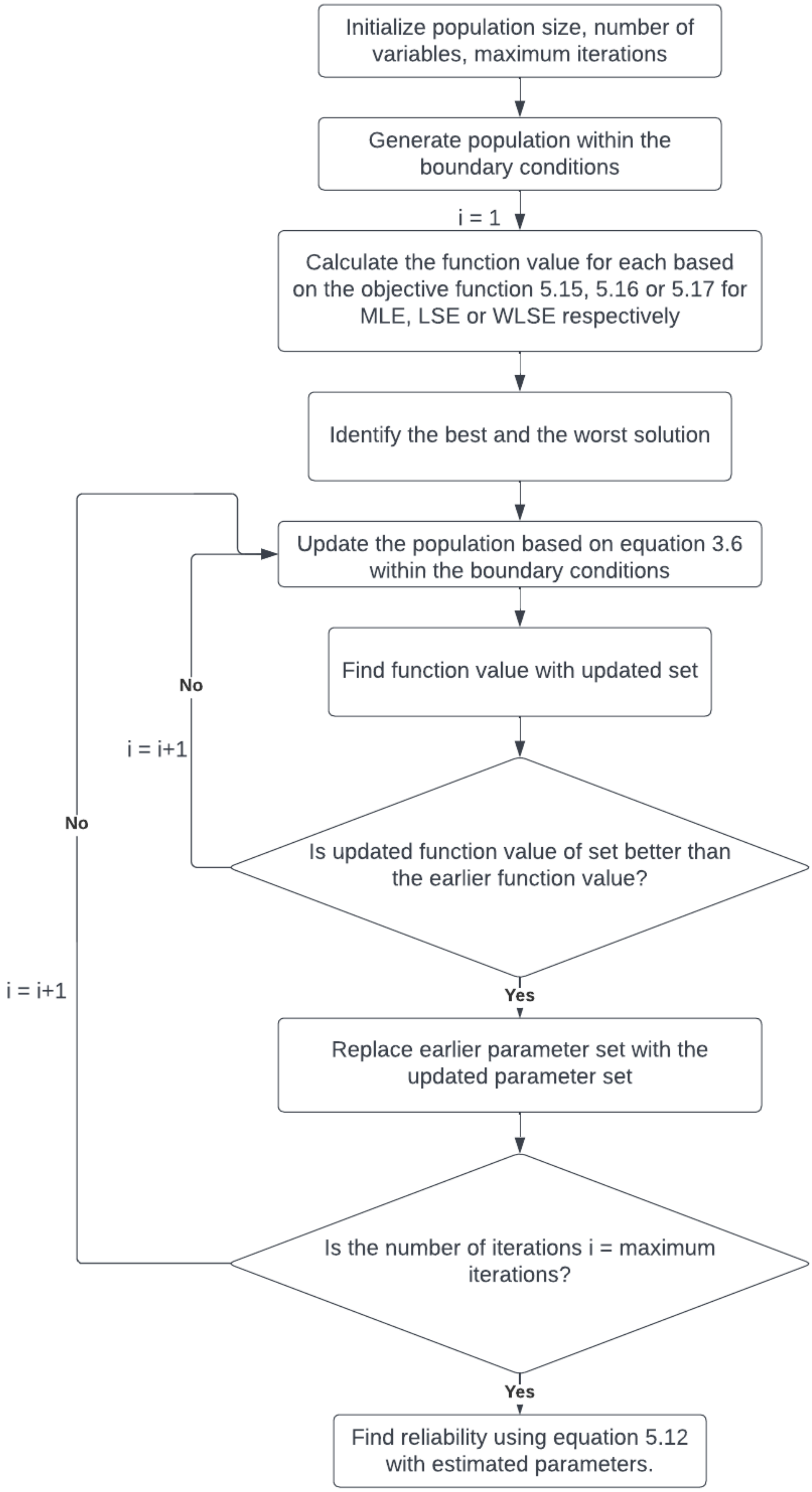


Figure 5.5 Flowchart for Jaya algorithm in estimation of stress strength reliability

### 5.2.6 Simulation studies and comparison of estimation methods

Random numbers were generated with shape parameters for strength, shape parameter for stress, and common scale parameter ($p_1$, $p_2$, $\sigma$) taken as (1.5, 2, 1), (2, 2, 1), (2.5, 2, 1) and (2.5, 2, 2). The sample sizes taken were (25, 25), (50, 50), (100, 100) and (500, 500). Total 500 experiments were conducted to check the repeatability of the estimation method. The parameters were estimated using the proposed methodology for MLE, LSE, and WLSE methods. The reliability was evaluated along with bias and mean squared error. The results of simulation studies are presented in Table 5.6 – 5.8. The estimation using proposed methodology gives very good results with reliability estimates close to the actual reliability. It can be noted that the accuracy of estimation increases with increase in sample size. The trend is strongly followed by MLE method compared to the other two. However, as the sample size increases, the time taken for compilation also increases. Another fact that can be observed is that if the shape parameter for strength increases in comparison to that of stress, the reliability increases. Also, the reliability decreases with an increase in common scale parameter. Figure 5.5 – 5.7 shows the box plots in estimation of reliability for 500 experiments across the sample sizes for considered different sets of parameters. It can be seen that the accuracy of the estimation increases as the sample size increases. For example, all the estimates of 500 experiments are very close to the actual reliability values in case of sample size (500, 500), whereas the spread increases with a decrease in sample size. Though the spread is observed to be more in case of a smaller sample size, the mean of reliability estimate is close to the actual reliability values. Also, the values for bias and MSE are lesser as compared to the other estimation methods in the literature. A notable observation can be made of many outliers wide away from the actual reliability in case of estimation with LSE and WLSE. Figure 5.8 – 5.10 shows the convergence behavior of Jaya algorithm for different sample sizes using MLE, LSE, and WLSE respectively. It can be noted that the algorithm converges to real roots after around 40 to 60 iterations for MLE, 80 to 100 iterations for LSE, and around 80-120 for WLSE. Figure 5.11 – 5.14 shows comparative graphs of bias and mean squared error (MSE) for the three estimation methods. It can be seen that the algorithm with MLE gives lesser bias and MSE in almost all the cases. This shows that Jaya algorithm with MLE is superior as compared to the other two estimation methods.

Table 5.6 Simulation results of 500 experiments for MLE using Jaya algorithm

| ($p_1$, $p_2$, σ) | R | (n, m) | $\hat{R}$ | Bias | MSE | T(s) |
|---|---|---|---|---|---|---|
| (1.5, 2, 1) | 0.48063 | (25, 25) | 0.48173 | 0.0011 | 0.000214321 | 12.791 |
| | | (50, 50) | 0.481418 | 0.000788 | 0.000103129 | 21.257 |
| | | (100, 100) | 0.480968 | 0.000338 | 4.2815E-05 | 39.671 |
| | | (500, 500) | 0.48077 | 0.00014 | 9.90769E-06 | 191.624 |
| (2, 2, 1) | 0.5 | (25, 25) | 0.498637 | -0.00136 | 0.000204486 | 12.378 |
| | | (50, 50) | 0.500416 | 0.000416 | 0.000114916 | 21.639 |
| | | (100, 100) | 0.499798 | -0.0002 | 0.0000509602 | 40.261 |
| | | (500, 500) | 0.500074 | 0.0000736 | 0.00000896665 | 198.089 |
| (2.5, 2, 1) | 0.5151 | (25, 25) | 0.516159 | 0.001059 | 0.000206911 | 12.4769 |
| | | (50, 50) | 0.515342 | 0.000242 | 0.000109094 | 23.1445 |
| | | (100, 100) | 0.51516 | 0.0000597 | 0.0000482781 | 39.5029 |
| | | (500, 500) | 0.515126 | 0.0000258 | 0.0000101923 | 188.5496 |
| (2.5, 2, 2) | 0.515049 | (25, 25) | 0.515432 | 0.000383 | 0.00021713 | 12.642 |
| | | (50, 50) | 0.514981 | -0.000068 | 0.00010362 | 21.849 |
| | | (100, 100) | 0.515108 | 0.0000589 | 0.0000512349 | 38.137 |
| | | (500, 500) | 0.515019 | -0.00003 | 0.00000925878 | 186.955 |

Table 5.7 Simulation results of 500 experiments for LSE using Jaya algorithm

| ($p_1$, $p_2$, σ) | R | (n, m) | $\hat{R}$ | Bias | MSE | T(s) |
|---|---|---|---|---|---|---|
| (1.5, 2, 1) | 0.48063 | (25, 25) | 0.483988 | 0.003358 | 0.000374892 | 8.208396 |
| | | (50, 50) | 0.482671 | 0.002041 | 0.000296409 | 13.715322 |
| | | (100, 100) | 0.480902 | 0.000272 | 0.000120576 | 24.107568 |
| | | (500, 500) | 0.481391 | 0.000761 | 0.0000338445 | 116.482948 |
| (2, 2, 1) | 0.5 | (25, 25) | 0.501092 | 0.001092 | 0.000709956 | 8.772598 |
| | | (50, 50) | 0.500238 | 0.000238 | 0.000227603 | 13.440006 |
| | | (100, 100) | 0.500839 | 0.000839 | 0.000243236 | 23.763826 |
| | | (500, 500) | 0.500048 | 0.0000482 | 0.0000223605 | 107.937992 |
| (2.5, 2, 1) | 0.5151 | (25, 25) | 0.515538 | 0.000438 | 0.000568429 | 8.1625 |
| | | (50, 50) | 0.517007 | 0.001907 | 0.000734081 | 13.4845 |
| | | (100, 100) | 0.515227 | 0.000127 | 0.000108498 | 23.929722 |
| | | (500, 500) | 0.514984 | -0.00012 | 0.0000234756 | 118.805165 |
| (2.5, 2, 2) | 0.515049 | (25, 25) | 0.517451 | 0.002402 | 0.001412757 | 8.75954 |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | (50, 50) | 0.517467 | 0.002418 | 0.000708432 | 13.9196 |
| | | (100, 100) | 0.515638 | 0.000589 | 0.000117623 | 23.839686 |
| | | (500, 500) | 0.515736 | 0.000686798 | 0.000178473 | 119.150790 |

Table 5.8 Simulation results of 500 experiments for WLSE using Jaya algorithm

| ($p_1$, $p_2$, σ) | R | (n, m) | $\hat{R}$ | Bias | MSE | T(s) |
|---|---|---|---|---|---|---|
| (1.5, 2, 1) | 0.48063 | (25, 25) | 0.484014 | 0.003384 | 0.001492711 | 10.315126 |
| | | (50, 50) | 0.481896 | 0.001266 | 0.000637294 | 18.109618 |
| | | (100, 100) | 0.481325 | 0.000695 | 0.000666559 | 30.799488 |
| | | (500, 500) | 0.482667 | 0.002037 | 0.000730472 | 143.485757 |
| (2, 2, 1) | 0.5 | (25, 25) | 0.502422 | 0.002422 | 0.001433974 | 10.609115 |
| | | (50, 50) | 0.503042 | 0.003042 | 0.00127458 | 18.112464 |
| | | (100, 100) | 0.502669 | 0.002669 | 0.001481735 | 31.145245 |
| | | (500, 500) | 0.503678 | 0.003678 | 0.001585831 | 143.482368 |
| (2.5, 2, 1) | 0.5151 | (25, 25) | 0.516737 | 0.001637 | 0.001899504 | 11.188098 |
| | | (50, 50) | 0.518774 | 0.003674 | 0.001496135 | 17.097244 |
| | | (100, 100) | 0.517434 | 0.002334 | 0.00137011 | 31.350793 |
| | | (500, 500) | 0.517283 | 0.002183 | 0.000599416 | 143.034958 |
| (2.5, 2, 2) | 0.515049 | (25, 25) | 0.517838 | 0.002789 | 0.002217398 | 10.456831 |
| | | (50, 50) | 0.516697 | 0.001648 | 0.000536541 | 17.376978 |
| | | (100, 100) | 0.519082 | 0.004033 | 0.001620499 | 32.245491 |
| | | (500, 500) | 0.5165 | 0.001451149 | 0.000401925 | 143.257339 |

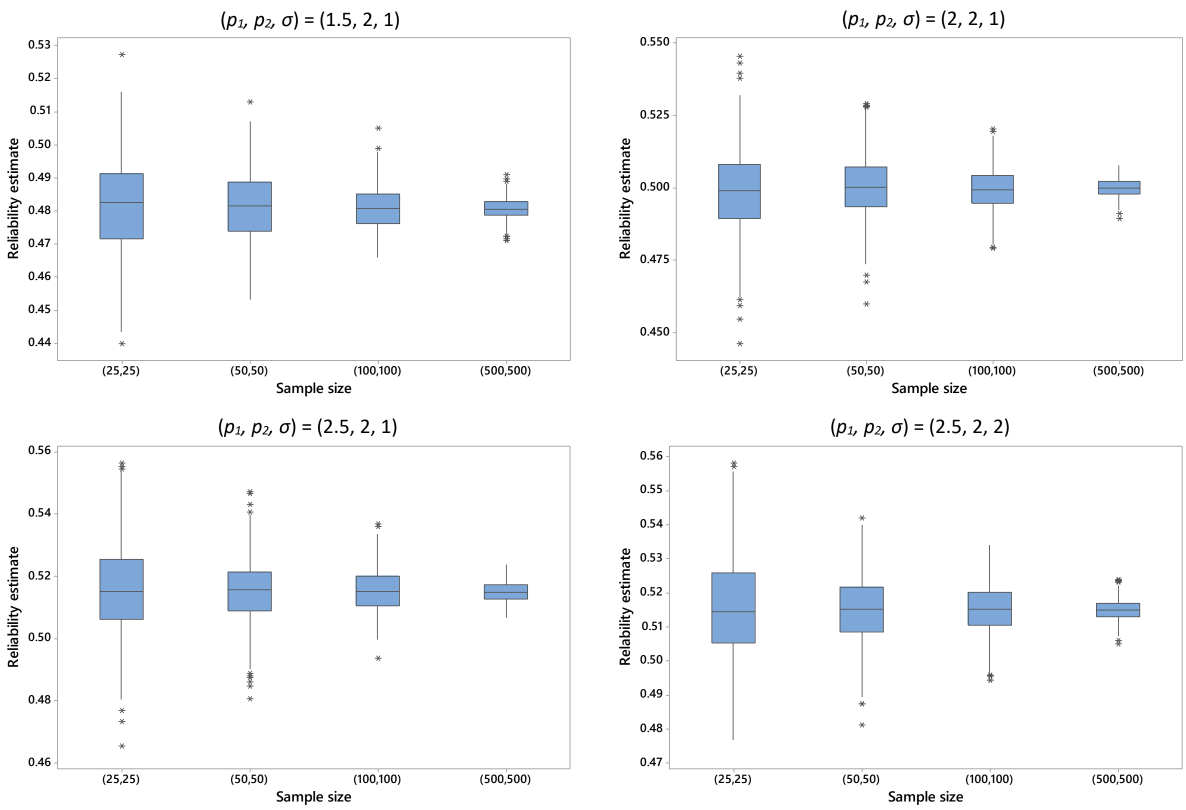


Figure 5.6 Box plots for reliability estimates across different sample sizes with MLE

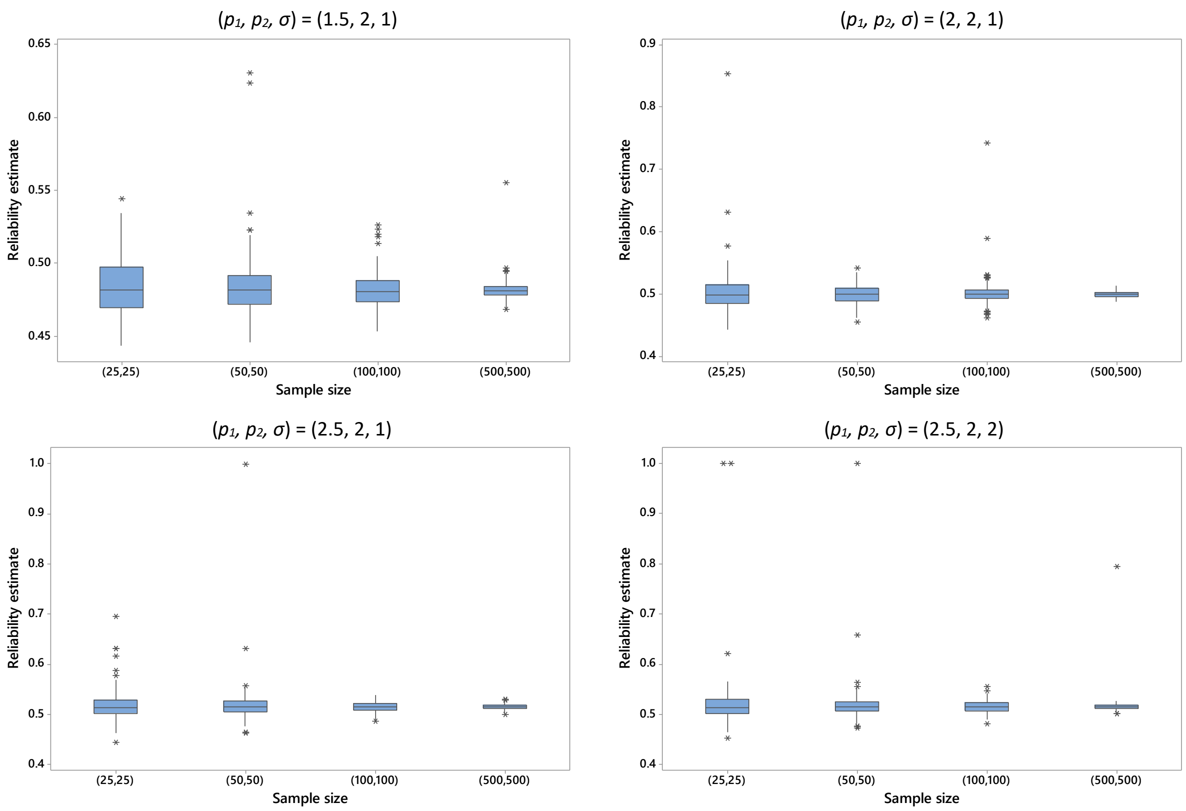


Figure 5.7 Box plots for reliability estimates across different sample sizes with LSE

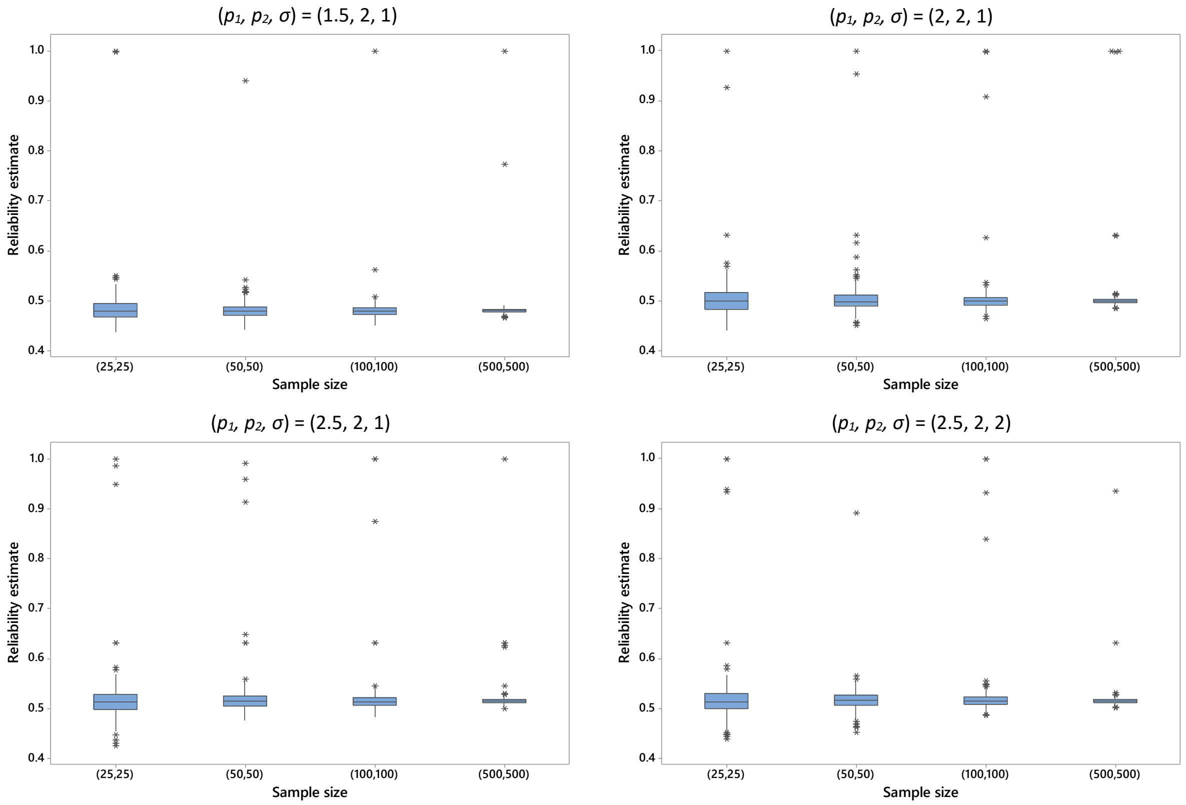


Figure 5.8 Box plots for reliability estimates across different sample sizes with WLSE

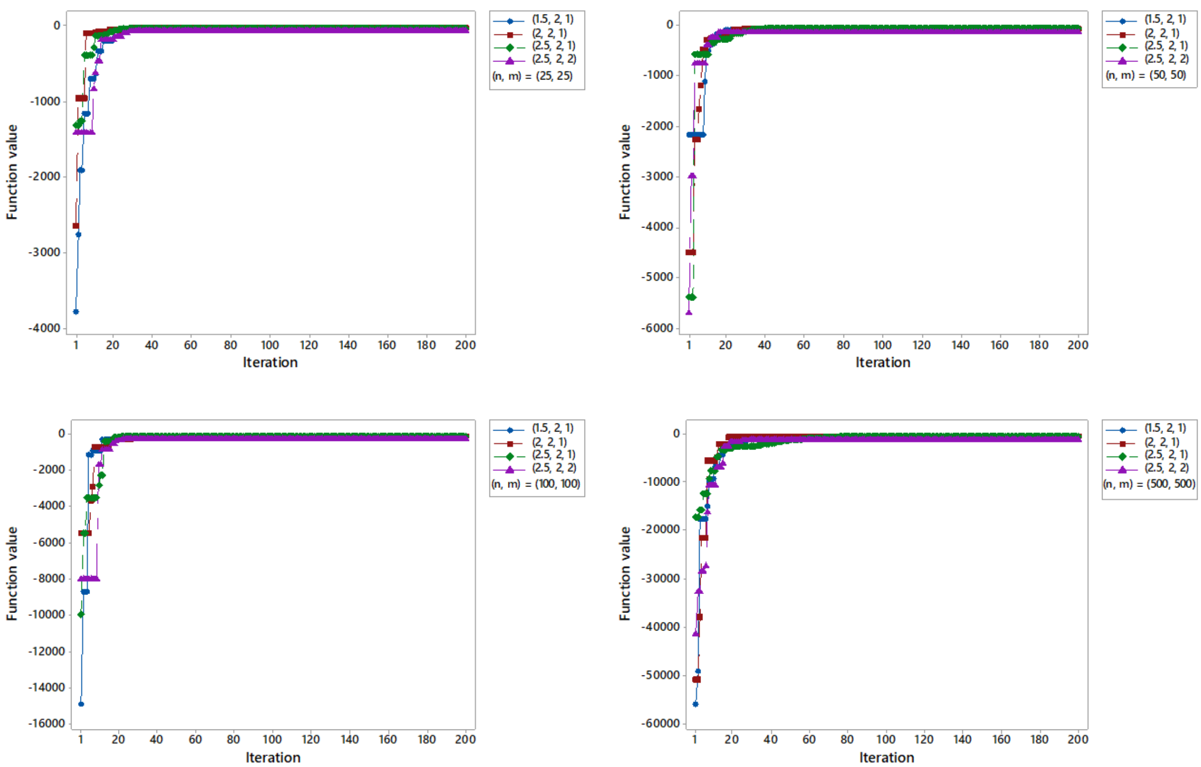


Figure 5.9 Convergence behavior of Jaya algorithm for different sample sizes with MLE

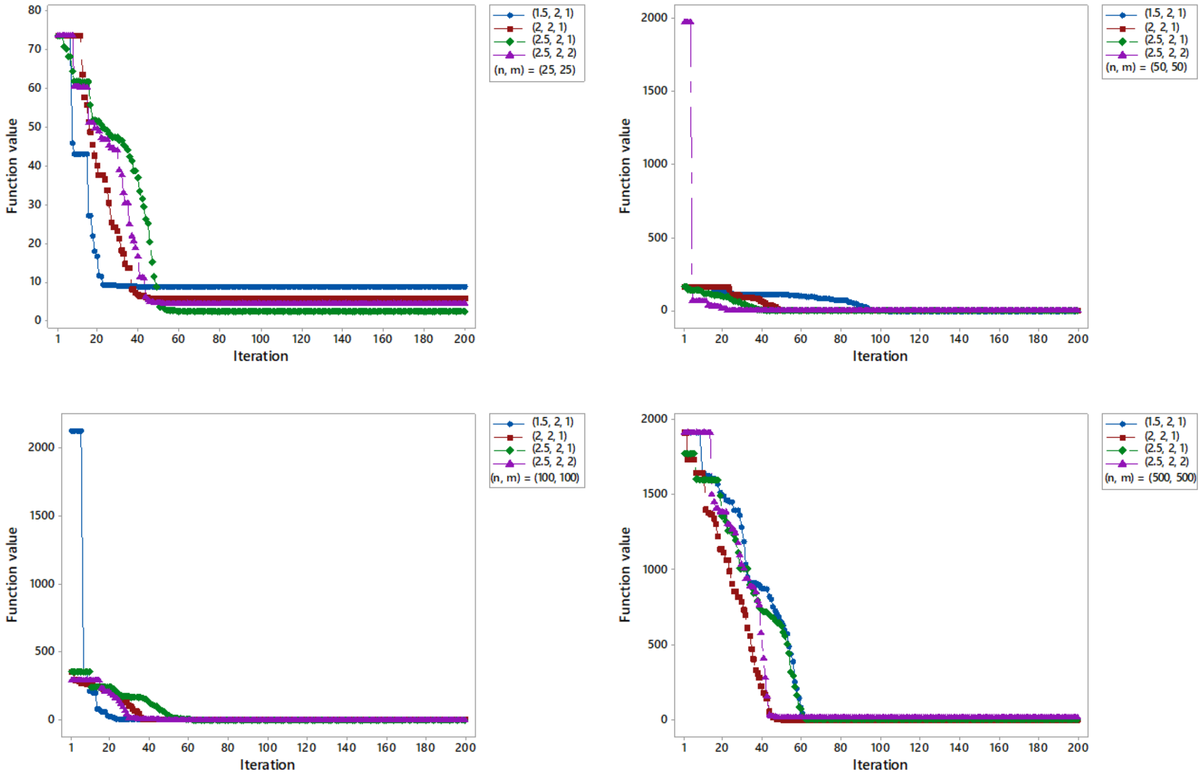


Figure 5.10 Convergence behavior of Jaya algorithm for different sample sizes with LSE

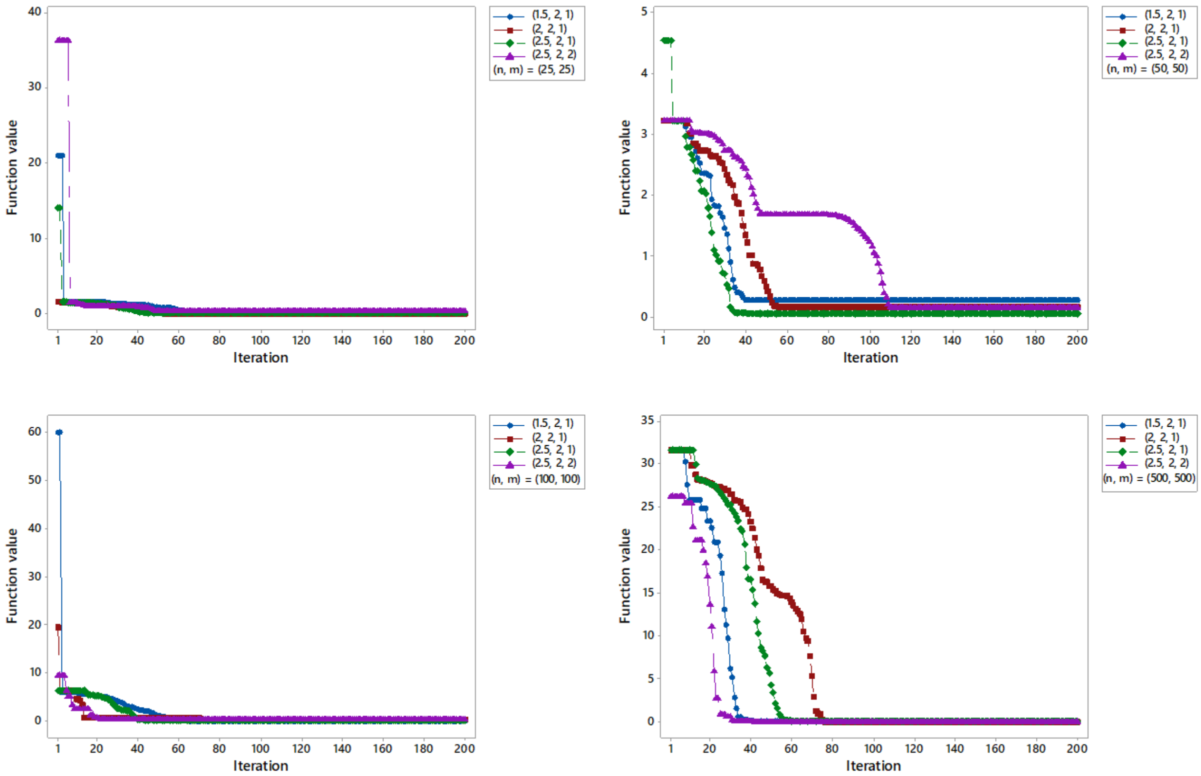


Figure 5.11 Convergence behavior of Jaya algorithm for different sample sizes with WLSE

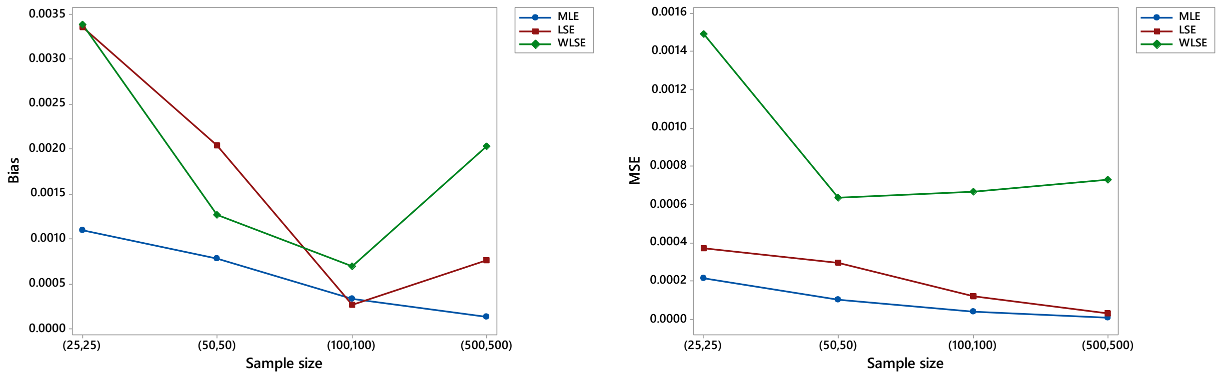


Figure 5.12 Comparison of bias and MSE for estimation methods using Jaya algorithm for ($p_1$, $p_2$, $\sigma$) = (1.5, 2, 1)

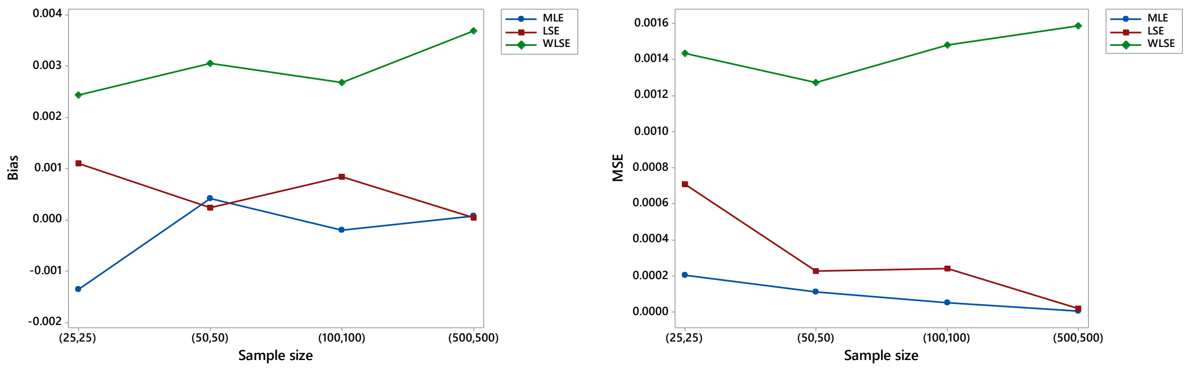


Figure 5.13 Comparison of bias and MSE for estimation methods using Jaya algorithm for ($p_1$, $p_2$, $\sigma$) = (2, 2, 1)

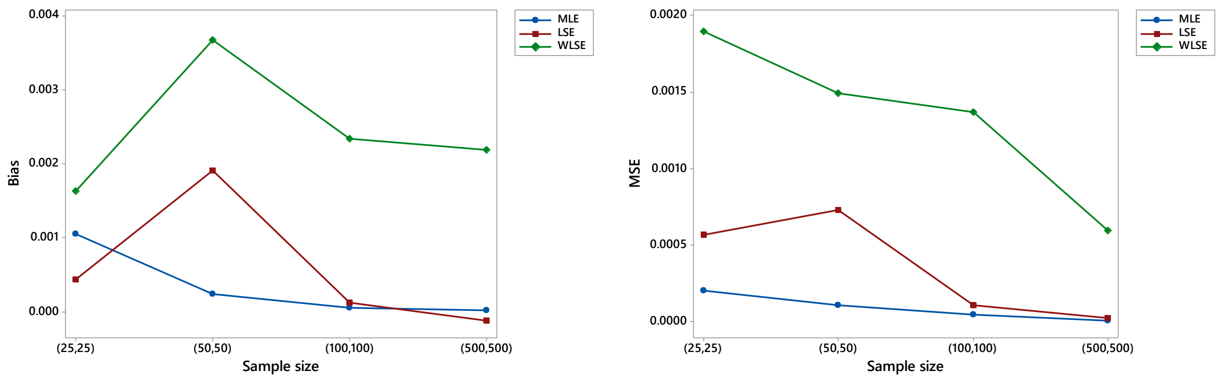


Figure 5.14 Comparison of bias and MSE for estimation methods using Jaya algorithm for ($p_1$, $p_2$, $\sigma$) = (2.5, 2, 1)

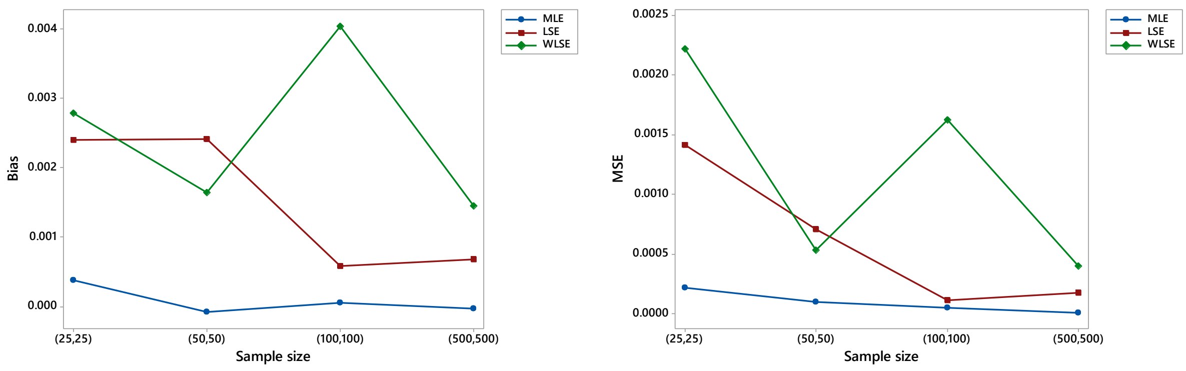


Figure 5.15 Comparison of bias and MSE for estimation methods using Jaya algorithm for ($p_1$, $p_2$, σ) = (2.5, 2, 2)

### 5.2.7 Application to real life data

The methodology has been applied to real-life data of strength of carbon fibers of gauge length 10mm and 20mm first studied by Badar and Priest [41] and then transformed by Valiollahi *et al.* [7] to fit for a common scale parameter. The transformed data sets are shown in Tables 5.9 and 5.10. Using the proposed methodology, the estimated parameters $(\widehat{p_1}, \widehat{p_2}, \widehat{\sigma})$ are obtained as (5.5061, 5.0514, 0.9999) using MLE, (5.5647, 5.7374, 0.9982) using LSE and (5.6584, 4.9353, 0.9880) using WLSE method. The Kolmogorov-Smirnov test was used to check the fit of the estimated Weibull model to the data sets. The K-S statistic, p-value, and estimated reliability using the three methods are given in Table 5.11. Figure 5.15 – 5.20 shows the fitted pdf and probability plot with estimated parameters using various methods for data sets I and II. The proposed methodology gives rapid results in a very short time compared to other common estimation methods using metaheuristic techniques [56,57,59,60]. The Akaike information criterion was used to find the best fit model for the given data among MLE, LSE and WLSE. The results are displayed in Table 5.12. It can be seen that the minimum value for AIC is obtained with MLE and the maximum value is obtained for LSE. Thus, it can be inferred that the proposed methodology using MLE gives the best fit model and LSE gives the worst fit model for the given data sets.

Table 5.9 Data of gauge length 20mm (Data set I)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0.495 | 0.496 | 0.558 | 0.585 | 0.641 | 0.68 | 0.702 | 0.704 | 0.733 | 0.739 |
| 0.742 | 0.753 | 0.757 | 0.762 | 0.765 | 0.775 | 0.778 | 0.791 | 0.807 | 0.822 |
| 0.839 | 0.845 | 0.85 | 0.856 | 0.857 | 0.858 | 0.868 | 0.868 | 0.89 | 0.899 |
| 0.899 | 0.915 | 0.918 | 0.919 | 0.935 | 0.939 | 0.947 | 0.948 | 0.956 | 0.963 |
| 0.968 | 0.97 | 0.976 | 0.992 | 0.993 | 0.997 | 0.999 | 1.013 | 1.017 | 1.028 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1.045 | 1.046 | 1.056 | 1.06 | 1.063 | 1.064 | 1.074 | 1.086 | 1.114 | 1.136 |
| 1.157 | 1.163 | 1.166 | 1.168 | 1.18 | 1.22 | 1.295 | 1.352 | 1.352 | |

Table 5.10 Data of gauge length 10mm (Data set II)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0.573 | 0.643 | 0.665 | 0.672 | 0.681 | 0.709 | 0.712 | 0.723 | 0.723 | 0.738 |
| 0.74 | 0.746 | 0.76 | 0.761 | 0.762 | 0.764 | 0.777 | 0.789 | 0.789 | 0.79 |
| 0.792 | 0.802 | 0.807 | 0.826 | 0.827 | 0.862 | 0.88 | 0.883 | 0.886 | 0.886 |
| 0.898 | 0.904 | 0.914 | 0.943 | 0.947 | 0.949 | 0.971 | 0.972 | 0.976 | 0.978 |
| 0.985 | 0.987 | 0.994 | 1.005 | 1.009 | 1.019 | 1.028 | 1.036 | 1.054 | 1.056 |
| 1.067 | 1.072 | 1.074 | 1.094 | 1.162 | 1.168 | 1.172 | 1.198 | 1.214 | 1.215 |
| 1.274 | 1.326 | 1.514 | | | | | | | |

Table 5.11 Comparison of MLE, LSE and WLSE in data fit and estimation of reliability

| Estimation method | $\widehat{p_1}$ | $\widehat{p_2}$ | $\hat{\sigma}$ | K-S | p-value | Reliability | Compilation time(s) |
|---|---|---|---|---|---|---|---|
| MLE | 5.5061 | 5.0514 | 0.9999 | 0.0564 (DS I)<br>0.0881 (DS II) | 0.9773 (DS I)<br>0.6929 (DS II) | 0.505824 | 0.079715 |
| LSE | 5.5647 | 5.7374 | 0.9982 | 0.0513 (DS I)<br>0.1096 (DS II) | 0.9920 (DS I)<br>0.4145 (DS II) | 0.497935 | 0.888459 |
| WLSE | 5.6584 | 4.9353 | 0.9880 | 0.0523 (DS I)<br>0.1002 (DS II) | 0.9900 (DS I)<br>0.5295 (DS II) | 0.509234 | 1.359188 |

Table 5.12 Model comparison based on AIC

| Estimation method | No. of estimated parameters | Log Likelihood | AIC | Delta AIC | Rank |
|---|---|---|---|---|---|
| MLE | 3 | 31.2502 | -56.5004 | 0 | 1 |
| LSE | 3 | 29.9176 | -53.8352 | 2.6653 | 3 |
| WLSE | 3 | 30.8663 | -55.7326 | 0.2714 | 2 |

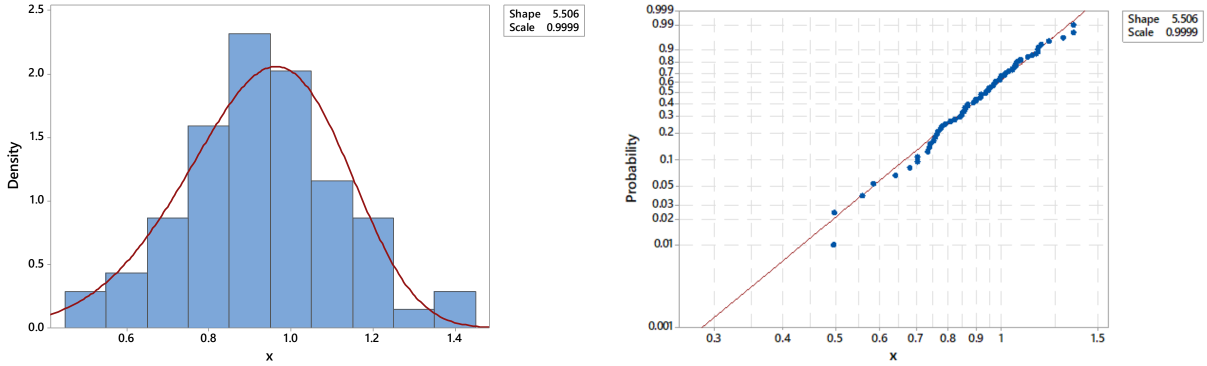


Figure 5.16 The fitted pdf and probability plot for Data set I with MLE

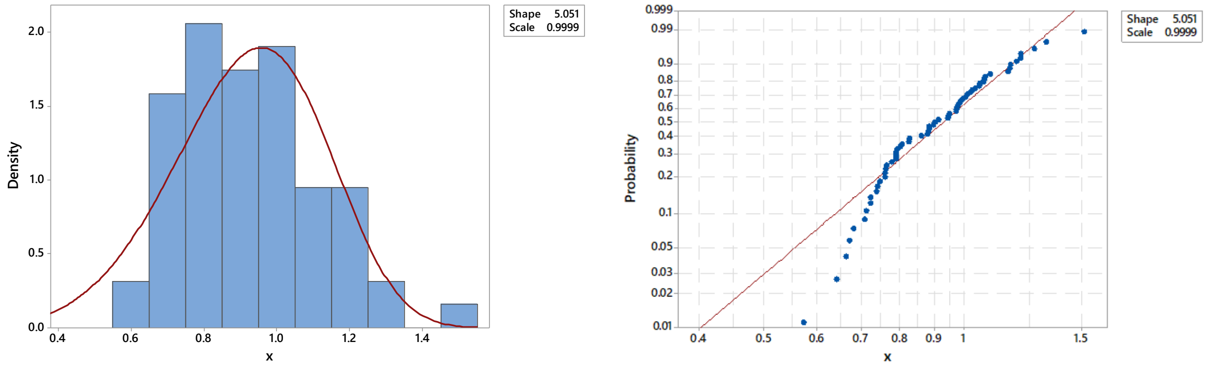


Figure 5.17 The fitted pdf and probability plot for Data set II with MLE

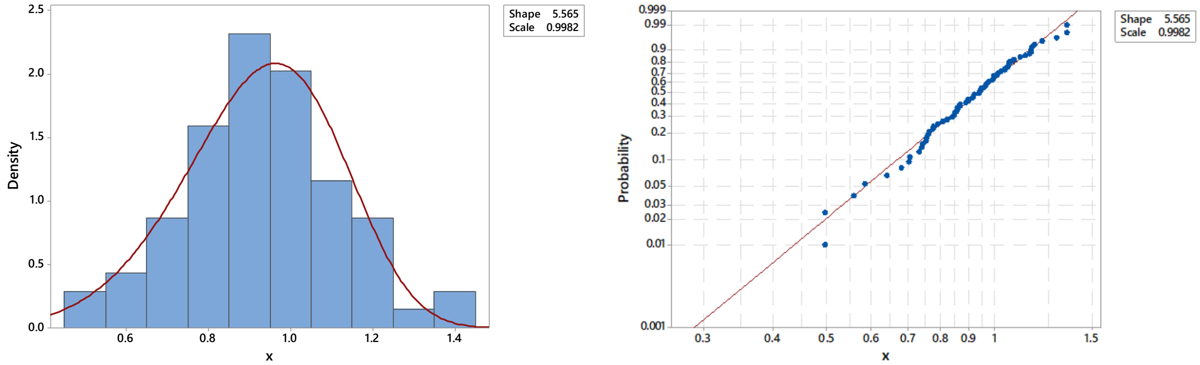


Figure 5.18 The fitted pdf and probability plot for Data set I with LSE

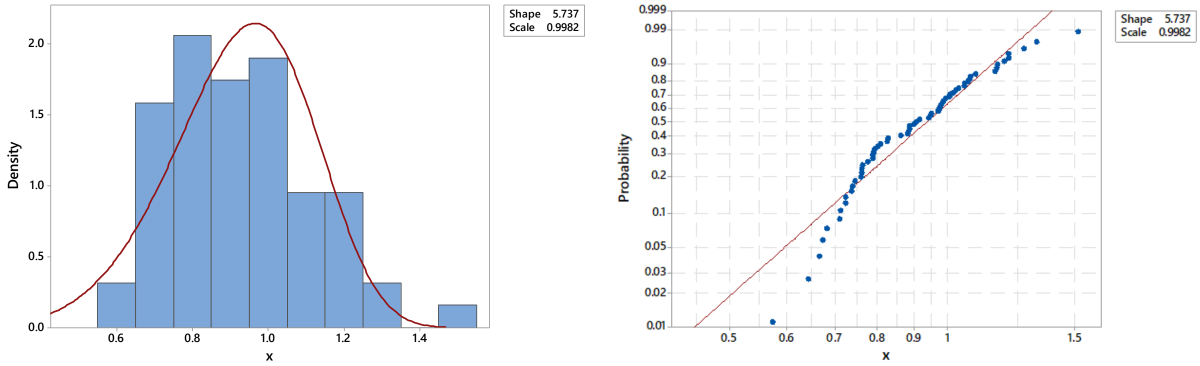


Figure 5.19 The fitted pdf and probability plot for Data set II with LSE

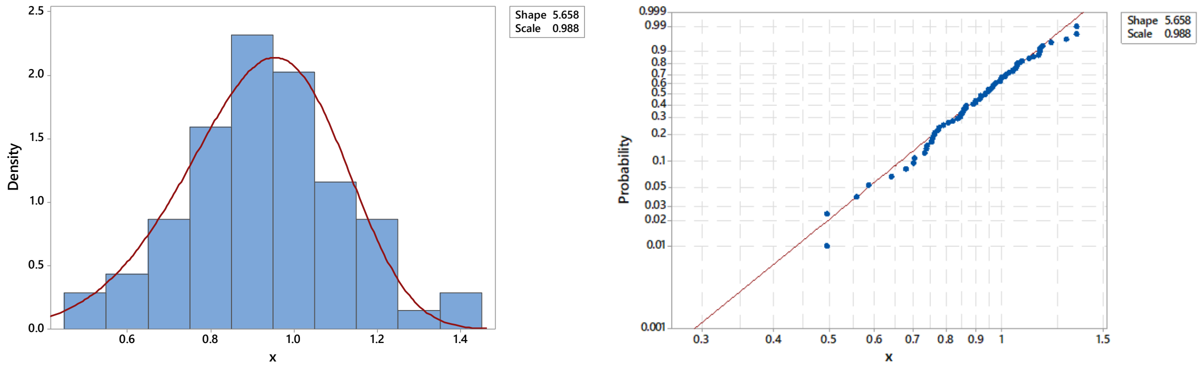


Figure 5.20 The fitted pdf and probability plot for Data set I with WLSE

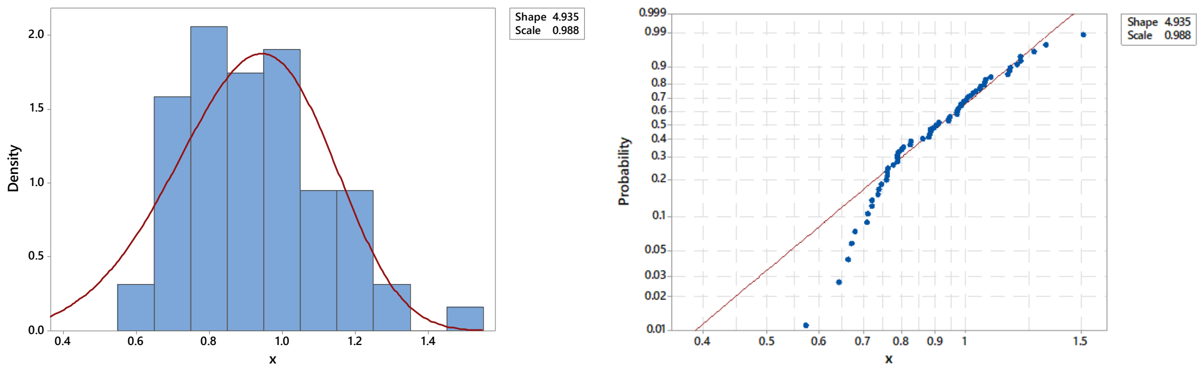


Figure 5.21 The fitted pdf and probability plot for Data set II with WLSE

## 5.3 Estimation of Reliability for Stress and Strength following Weibull Distribution with Common Shape Parameter

In the context of stress and strength, reliability can be defined as the probability of strength being greater than stress. If stress and strength follow Weibull distribution, then the interference area of the two distributions give the probability of failure as can be seen in Figure 5.21.

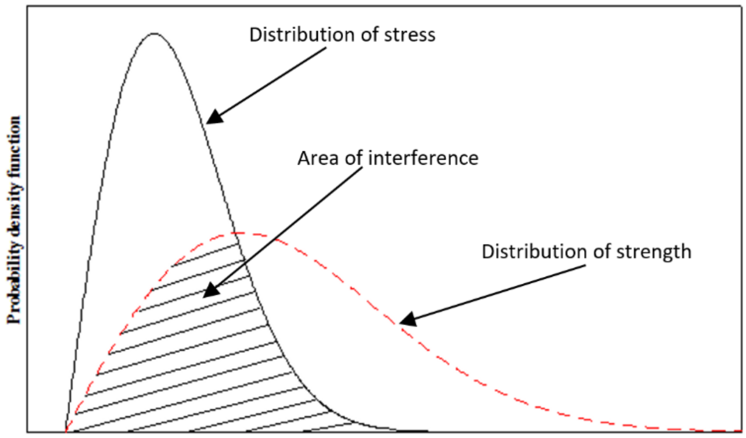


Figure 5.22 Stress-Strength interference for Weibull stress & strength distribution

In order to determine reliability, estimating the parameters of stress and strength distribution is crucial. Vast research has been carried in estimation of reliability of components subjected to various distributions of stress and strength. [165,166]. Recent studies have shown increasing usage of metaheuristics in estimation of parameters. This section deals with estimation of parameters of three-parameter Weibull distribution with common shape parameter and different scale parameter.

Weibull distribution is widely used in reliability studies because of its flexibility and ability to fit a wide range of data [167]. Extensive research has been carried out in estimating the reliability when stress and strength follow Weibull distribution. If X and Y denote the Weibull random variables for the strength and stress respectively, having common shape (μ), location parameter (p) and different scale parameters (σ1 for strength and σ2 for stress), then their probability density function (pdf) is given by

$$f(x; \mu, \sigma_1, p) = \frac{p}{\sigma_1}(x-\mu)^{p-1} \exp\left\{-\frac{1}{\sigma_1}(x-\mu)^p\right\}, x > \mu, \sigma_1 > 0, p > 0 \qquad 5.18$$

and

$$f(y; \mu, \sigma_2, p) = \frac{p}{\sigma_2}(y-\mu)^{p-1} \exp\left\{-\frac{1}{\sigma_2}(y-\mu)^p\right\}, y > \mu, \sigma_2 > 0, p > 0 \qquad 5.19$$

respectively. The corresponding cumulative distribution function (cdf) for strength and stress is given by

$$F(x;\mu,\sigma_1,p) = 1 - \exp\left\{-\frac{1}{\sigma_1}(x-\mu)^p\right\} \quad 5.20$$

$$F(y;\mu,\sigma_2,p) = 1 - \exp\left\{-\frac{1}{\sigma_2}(y-\mu)^p\right\} \quad 5.21$$

respectively, where x > μ, y > μ, $\sigma_1 > 0$, $\sigma_2 > 0$ and p > 0.

### 5.3.1 Maximum likelihood estimation in estimation of reliability

One of the classical and efficient method used in estimation of parameters is maximum likelihood estimation [168,169]. The method is considered to be simple, effective and can get accurate results if assisted with proper computational techniques. Let x be a random sample of size n drawn from W(μ, $\sigma_1$, p) and y be the random sample of size m from W(μ, $\sigma_2$, p). Then the likelihood function can be given as:

$$L = \prod_{i=1}^{n} f(x_i) \prod_{j=1}^{m} f(y_j) \quad 5.22$$

$$L = \prod_{i=1}^{n} \frac{p}{\sigma_1}(x_i-\mu)^{p-1} \exp\left\{-\frac{1}{\sigma_1}(x-\mu)^p\right\} . \prod_{j=1}^{m} \frac{p}{\sigma_2}(y_j-\mu)^{p-1} \exp\left\{-\frac{1}{\sigma_2}(y-\mu)^p\right\} \quad 5.23$$

$$\begin{aligned}\ln L = {} & (m+n)\ln p - n\ln\sigma_1 - m\ln\sigma_2 \\ & + (p-1)\left[\sum_{i=1}^{n} \ln(x_i-\mu) + \sum_{j=1}^{m} \ln(y_j-\mu)\right] \\ & - \frac{1}{\sigma_1}\sum_{i=1}^{n}(x_i-\mu)^p - \frac{1}{\sigma_2}\sum_{j=1}^{m}(y_j-\mu)^p\end{aligned} \quad 5.24$$

The purpose is to maximize the log-likelihood function (5.24) i.e. the parameter values at which the log-likelihood function attains it's maximum. Solving likelihood equations involving nonlinear functions using numerical methods can be difficult because of the problems associated with it like non convergence, slower convergence and convergence to wrong values. Hence using a heuristic technique can be a good choice in solving the likelihood equations. In this paper, Jaya algorithm is used to maximize the above likelihood equation.

### 5.3.2 Reliability estimation using Jaya algorithm

Metaheuristics have been used for a long time for solving optimization problems and are found to be effective in converging to real roots [170–173]. Ant colony optimization, genetic algorithm, particle swarm optimization, differential evolution, etc. are some of the metaheuristic techniques which have been used widely by researchers in optimization problems. Jaya algorithm is one such recently developed metaheuristic for solving optimization problems effectively. The specialty of the algorithm is that it constantly tries to move towards success and away from failure with each iteration. The algorithm has been used by many researchers in solving optimization problems [112,174]. For the cases involving stress-strength interference, the reliability can be given as P (X > Y) which is equal to

$$R = P\ (X > Y) = \int_0^\infty \left( f(x; \mu, \sigma_1, p) \int_0^x f(y; \mu, \sigma_2, p) dy \right) dx \qquad 5.25$$

$$R = P\ (X > Y) = \int_0^\infty \begin{pmatrix} \frac{p}{\sigma_1}(x-\mu)^{p-1} \exp\left\{-\frac{1}{\sigma_1}(x-\mu)^p\right\} . \\ \int_0^x \frac{p}{\sigma_2}(y-\mu)^{p-1} \exp\left\{-\frac{1}{\sigma_2}(y-\mu)^p\right\} dy \end{pmatrix} dx \qquad 5.26$$

On simplification, the reliability can be obtained as [175]

$$R = \frac{\sigma_1}{\sigma_1 + \sigma_2} \qquad 5.27$$

The log likelihood function for estimated values of parameters can be given as:

$$\begin{aligned} \ln L = (m+n)\ln \hat{p} - n \ln \widehat{\sigma_1} - m \ln \widehat{\sigma_2} \\ + (\hat{p}-1)\left[\sum_{i=1}^{n} \ln (x_i - \hat{\mu}) + \sum_{j=1}^{m} \ln \frac{1}{\widehat{\sigma_2}}(y_j - \hat{\mu})\right] \\ - \frac{1}{\widehat{\sigma_1}} \sum_{i=1}^{n} (x_i - \hat{\mu})^{\hat{p}} - \frac{1}{\widehat{\sigma_2}} \sum_{j=1}^{m} (y_j - \hat{\mu})^{\hat{p}} \end{aligned} \qquad 5.28$$

where $\hat{\mu}$ and $\hat{p}$ are common estimated values of location and shape parameter. $\widehat{\sigma_1}$ is the scale parameter for strength and $\widehat{\sigma_2}$ is the scale parameter for stress. The parameters are to be estimated in such a manner to maximize the likelihood function (5.28). Hence it becomes an optimization problem. In this study, Jaya algorithm has been used to maximize the likelihood function (5.28). Table 5.13 shows the detailed steps for evaluating reliability using Jaya algorithm. The number of design variables taken are 4 considering the four parameters to be estimated and the population size is taken as 10. Number of iterations is considered as the termination criteria. The initial population is randomly generated within the specified range and constraints. The best (maximum) and the worst (minimum) solution is calculated based on the log-likelihood function. The population is then updated based on equation (3.6). If the new population gives a better maximum value for equation (5.28) than the previous one, then the new population is accepted and the next iteration begins with the updated population. If the new population does not give a better maximum than the previous one, the next iteration will be proceeded with the previous population. After the fixed number of iterations are completed and no variation in the convergence is observed, the final variables obtained in the population are the optimum parameter estimates. Run the above steps for a number of times to find the best population which gives the best maximum.

If $\widehat{\sigma_1}$ and $\widehat{\sigma_2}$ are the estimates of scale parameters for strength and stress respectively, then the reliability estimate, bias and mean squared error can be calculated as:

$$\hat{R} = \frac{\widehat{\sigma_1}}{\widehat{\sigma_1} + \widehat{\sigma_2}} \tag{5.29}$$

$$\text{Bias} = \frac{1}{N}\sum_{i=1}^{N} \hat{R} - R \tag{5.30}$$

$$\text{MSE} = \frac{1}{N}\sum_{i=1}^{N} (\hat{R} - R)^2 \tag{5.31}$$

Table 5.13 Steps for reliability estimation using Jaya algorithm

1. Generate data of 500 samples of Weibull distribution with real parameters
2. Input the population size = 10 and number of design variables as 4
3. Set the maximum number of iterations for each sample
4. Specify the boundaries for the variables
5. Generate a random population within the constraints
6. Compute the maximum likelihood function value from equation (5.28)
7. Update the population based on equation (1.16)
8. If the updated population gives a better maximum, then accept the new population; else reject. This completes one iteration.

9. Go for the next iteration and similarly run the program till the maximum iteration number is reached.
10. The final set of variables is the best solution of estimated parameters for the current experiment.
11. Compute reliability from equation (5.29).
12. Run the program for 500 experiments.
13. Compute bias and MSE based on equation (5.30) and (5.31) respectively.

### 5.3.3 Simulation studies

Simulations are performed in order to study the effectiveness of the proposed methodology. The data is randomly generated for strength and stress using Weibull distribution with shape parameter 1.5 and location parameter 2. The study has been carried out by varying the scale parameter and analyzing its effect on reliability. The scale parameter for strength is taken as 2, 2.5 and that for stress is considered as 1, 1.5, 2, 2.5 and 3. The sample sizes selected are (25, 25), (50, 50), (100, 100) and (500, 500). Total 500 independent experiments are conducted to check the repeatability of the proposed methodology. The number of iterations of 200 is set as the termination criteria. Table 5.14 depicts the estimated values of reliability using the proposed methodology for different sample sizes when $\mu = 2$, $\sigma1 = 2$, $p = 1.5$ and $\sigma2$ taking values 1, 1.5, 2, 2.5, 3. Figure 5.22 – 5.25 illustrates the estimated values of reliability for the simulation of 500 experiments. A general trend that can be observed in the figure and is evident from interference theory that as the scale parameter of stress, $\sigma2$ increases, the reliability decreases. Also, the accuracy of reliability estimation increases with increase in sample size. The reliability estimates using the proposed methodology are compared with the reliability using Monte Carlo simulations, R (MCS) for estimated parameters. It can be observed that the results of reliability estimates using the proposed methodology are very close to the reliability using Monte Carlo simulations. Comparison is also made by calculating bias, mean standard error and compilation time. Table 5.15 and Figure 5.26 – 5.29 shows these results for the same set of parameters with $\sigma1 = 2.5$. The computational code for Jaya algorithm was compiled using MATLAB 2018a software. The estimated reliability values obtained in all the cases are very close to the real reliability values. Also, as the sample size increases, the mean squared error reduces and the estimated reliability moves closer to the real reliability. But simultaneously, as the sample size increases, the time taken for compilation also increases. So, optimization for reliability is a trade-off between the accuracy of reliability value and the compilation time. Figure 5.30 shows the convergence behavior of the proposed algorithm for different sample sizes. It can be seen that the Jaya algorithm values converge towards real roots after at most 50 iterations in all cases.

Table 5.14 Results of reliability estimate, bias, mean square error (MSE) and compilation time (t) when μ = 2, $\sigma_1$ = 2, $p$ = 1.5

| (n, m) | | $\sigma_2$ = 1 ($R$ = 0.66667) | $\sigma_2$ = 1.5 ($R$ = 0.57143) | $\sigma_2$ = 2 (R = 0.5000) | $\sigma_2$ = 2.5 (R = 0.44444) | $\sigma_2$ = 3 (R = 0.4) |
|---|---|---|---|---|---|---|
| (25, 25) | $\hat{R}$ | 0.670476 | 0.576460 | 0.498645 | 0.441998 | 0.393946 |
| | Bias | 0.003809 | 0.005030 | -0.00135 | -0.00245 | -0.00605 |
| | MSE | 0.005173 | 0.002481 | 0.002880 | 0.002352 | 0.002320 |
| | t (s) | 37.32877 | 37.31540 | 37.43130 | 37.53155 | 37.40354 |
| | $R$ (MCS) | 0.674064 | 0.576148 | 0.501425 | 0.443737 | 0.398676 |
| (50, 50) | $\hat{R}$ | 0.670214 | 0.572828 | 0.502343 | 0.441395 | 0.398309 |
| | Bias | 0.003547 | 0.001398 | 0.002343 | -0.00305 | -0.00169 |
| | MSE | 0.000945 | 0.001070 | 0.001256 | 0.001094 | 0.001081 |
| | t (s) | 64.15220 | 65.14447 | 64.16610 | 66.30122 | 65.09571 |
| | $R$ (MCS) | 0.667521 | 0.573944 | 0.495442 | 0.441390 | 0.400954 |
| (100, 100) | $\hat{R}$ | 0.668761 | 0.571466 | 0.499421 | 0.443630 | 0.399801 |
| | Bias | 0.002091 | 0.000036 | -0.00058 | -0.00081 | -0.00020 |
| | MSE | 0.000471 | 0.000524 | 0.000576 | 0.000565 | 0.000497 |
| | t (s) | 118.1140 | 118.1314 | 119.1136 | 118.6877 | 119.5847 |
| | $R$ (MCS) | 0.670214 | 0.573284 | 0.497913 | 0.441619 | 0.398044 |
| (500, 500) | $\hat{R}$ | 0.667812 | 0.571279 | 0.500259 | 0.444384 | 0.400183 |
| | Bias | 0.001142 | -0.00015 | 0.000259 | -0.00006 | 0.000183 |
| | MSE | 0.000077 | 0.000096 | 0.000112 | 0.000109 | 0.000089 |
| | t (s) | 550.1336 | 551.3976 | 551.2057 | 550.0018 | 552.2750 |
| | $R$ (MCS) | 0.666857 | 0.572780 | 0.502463 | 0.444391 | 0.400233 |

Table 5.15 Results of reliability estimate, bias, mean square error (MSE) and compilation time (t) when μ = 2, $\sigma_1$ = 2.5, $p$ = 1.5

| (n, m) | | $\sigma_2$=1 ($R$ = 0.714286) | $\sigma_2$=1.5 ($R$ = 0.625) | $\sigma_2$=2 (R = 0.5556) | $\sigma_2$=2.5 (R = 0.5) | $\sigma_2$=3 (R = 0.4545) |
|---|---|---|---|---|---|---|
| (25, 25) | $\hat{R}$ | 0.721895 | 0.627828 | 0.554906 | 0.504770 | 0.454122 |
| | Bias | 0.007609 | 0.002828 | -0.00069 | 0.004770 | -0.00038 |
| | MSE | 0.002197 | 0.005127 | 0.003727 | 0.003775 | 0.003993 |
| | t (s) | 37.38062 | 37.44218 | 37.30677 | 37.72256 | 37.48941 |
| | $R$ (MCS) | 0.722723 | 0.624677 | 0.557381 | 0.497248 | 0.453166 |
| (50, 50) | $\hat{R}$ | 0.719250 | 0.627199 | 0.557145 | 0.497970 | 0.454175 |
| | Bias | 0.004964 | 0.002199 | 0.001545 | -0.00203 | -0.00033 |
| | MSE | 0.000790 | 0.001007 | 0.001086 | 0.001220 | 0.001062 |
| | t (s) | 64.84338 | 65.14443 | 65.03510 | 65.00493 | 65.67913 |
| | $R$ (MCS) | 0.710195 | 0.627185 | 0.556801 | 0.501515 | 0.455153 |

| | | | | | | |
|---|---|---|---|---|---|---|
| (100, 100) | $\hat{R}$ | 0.717133 | 0.626887 | 0.555828 | 0.499655 | 0.455104 |
| | Bias | 0.002847 | 0.001887 | 0.000228 | -0.00035 | 0.000604 |
| | MSE | 0.000350 | 0.000519 | 0.000579 | 0.000685 | 0.000545 |
| | t (s) | 119.8956 | 120.6025 | 120.1363 | 120.2996 | 120.6354 |
| | $R$ (MCS) | 0.716262 | 0.621087 | 0.557070 | 0.500421 | 0.005419 |
| (500, 500) | $\hat{R}$ | 0.714634 | 0.626094 | 0.555492 | 0.500190 | 0.454426 |
| | Bias | 0.000348 | 0.001094 | -0.00011 | 0.000190 | 0.000074 |
| | MSE | 0.000067 | 0.000086 | 0.000103 | 0.000113 | 0.000109 |
| | t (s) | 637.9747 | 626.1736 | 626.3683 | 631.7630 | 635.9884 |
| | $R$ (MCS) | 0.715995 | 0.624330 | 0.552340 | 0.500700 | 0.453526 |

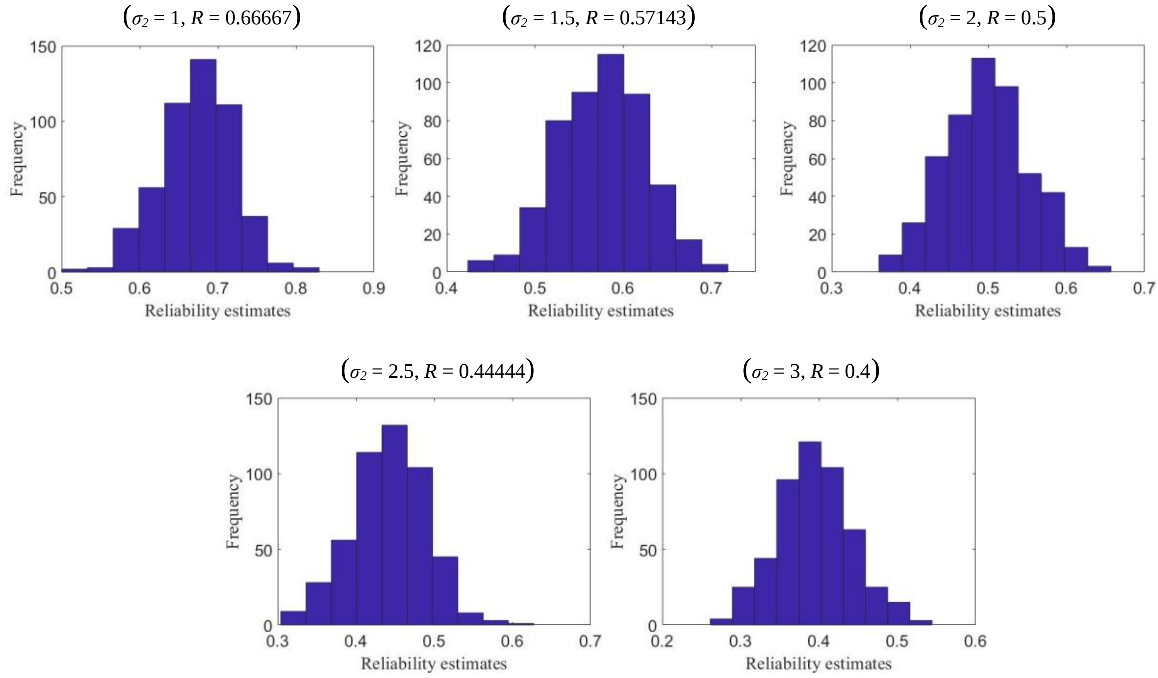


Figure 5.23 Histogram of 500 experiments for μ = 2, $\sigma_1$ = 2, $p$ = 1.5 and sample size (10, 10)

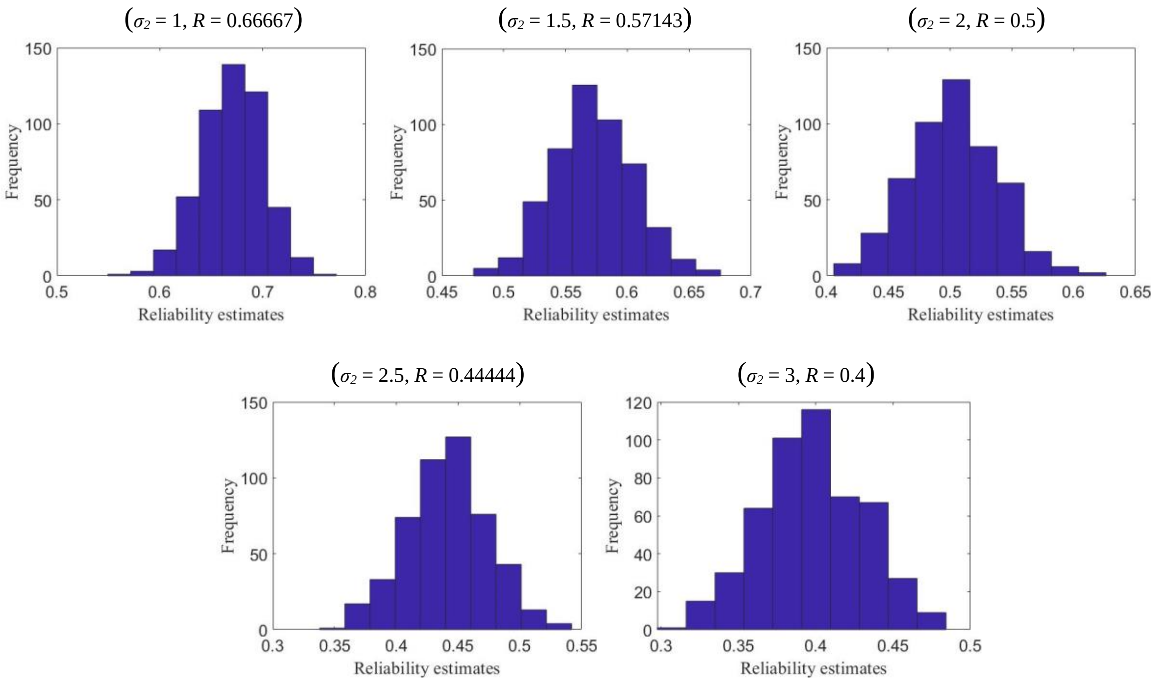


Figure 5.24 Histogram of 500 experiments for μ = 2, $\sigma_1$ = 2, $p$ = 1.5 and sample size (50, 50)

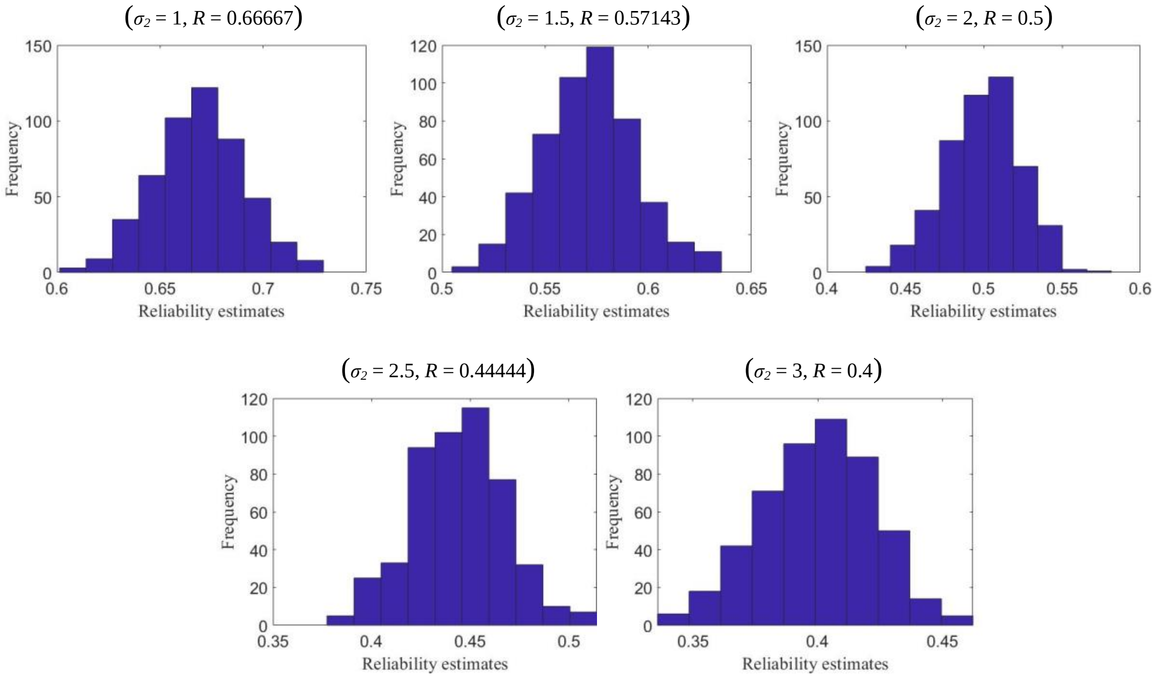


Figure 5.25 Histogram of 500 experiments for μ = 2, $\sigma_1$ = 2, $p$ = 1.5 and sample size (100, 100)

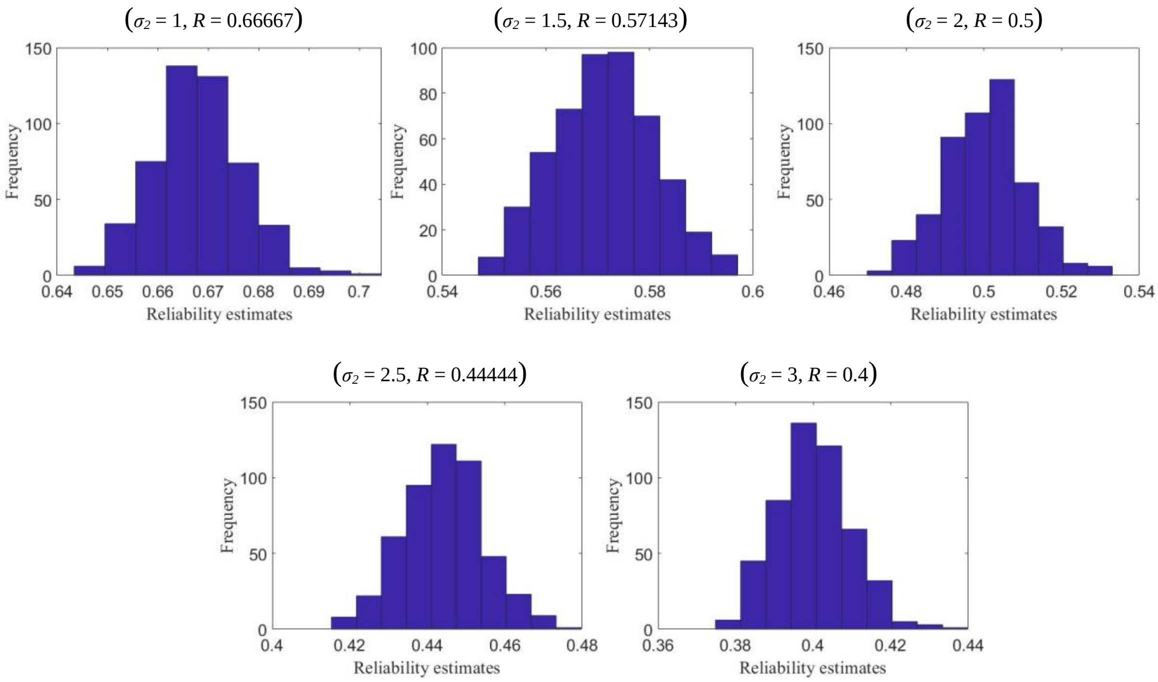


Figure 5.26 Histogram of 500 experiments for μ = 2, $\sigma_1$ = 2, $p$ = 1.5 and sample size (500, 500)

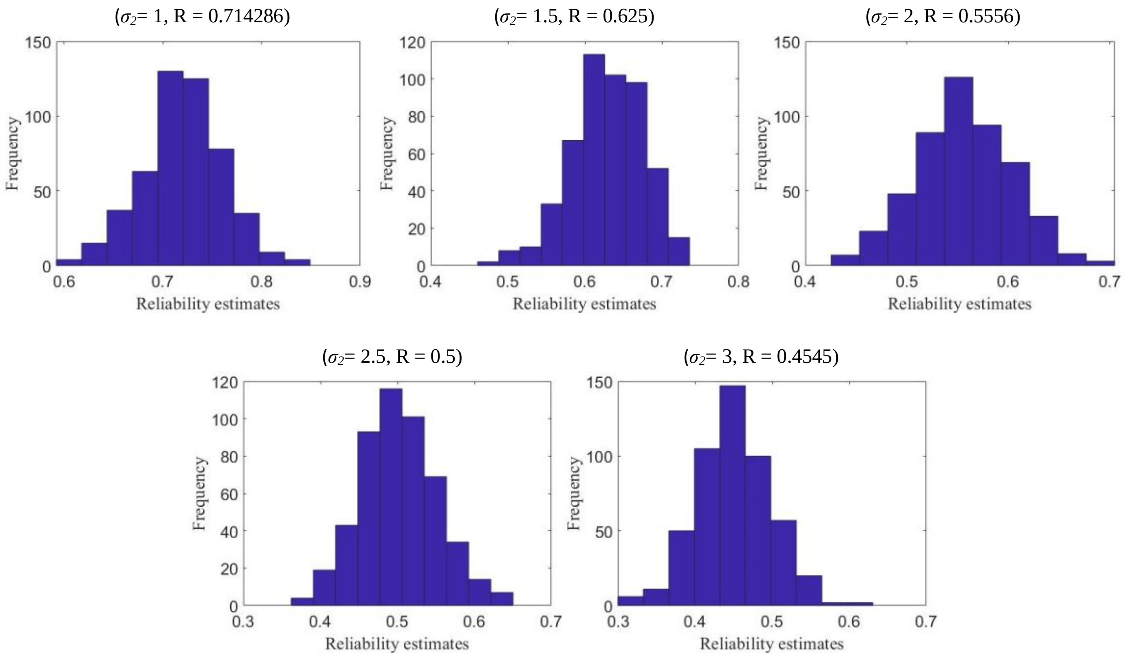


Figure 5.27 Histogram of 500 experiments for μ = 2, $\sigma_1$ = 2.5, $p$ = 1.5 and sample size (25, 25)

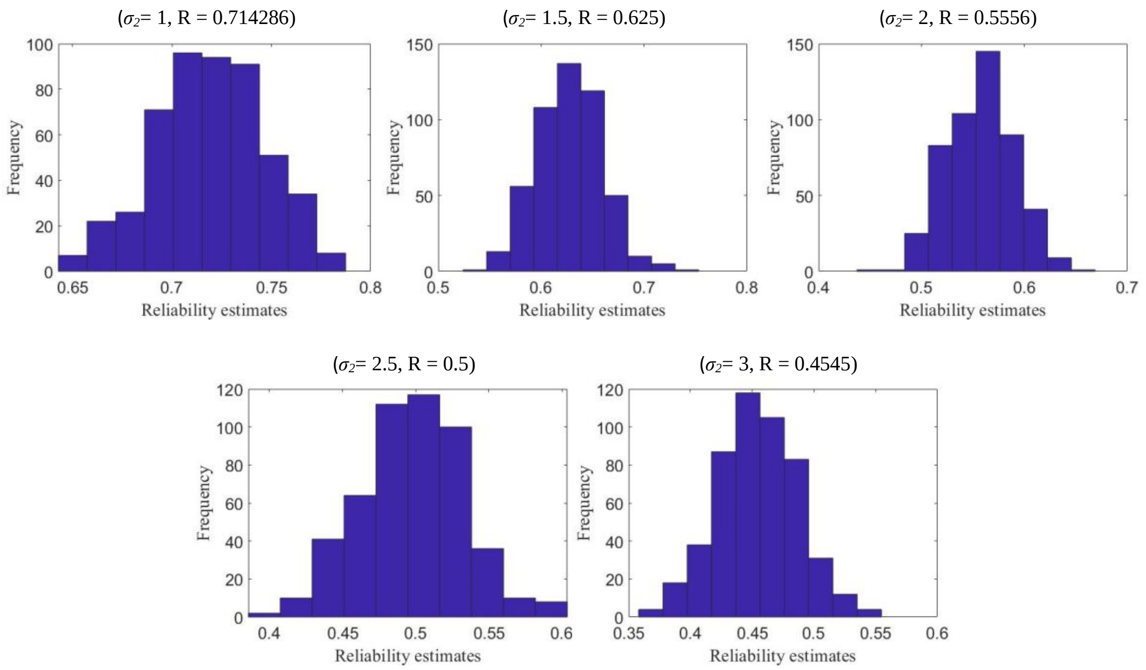


Figure 5.28 Histogram of 500 experiments for μ = 2, $\sigma_1$ = 2.5, $p$ = 1.5 and sample size (50, 50)

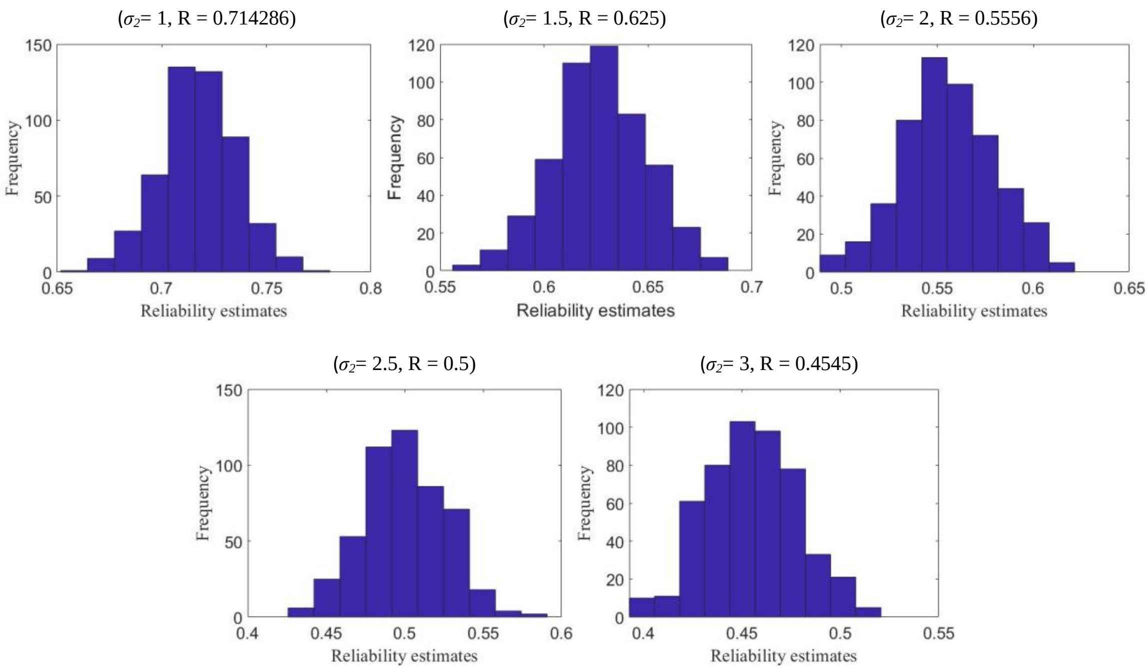


Figure 5.29 Histogram of 500 experiments for μ = 2, σ1 = 2.5, p = 1.5 and sample size (100, 100)

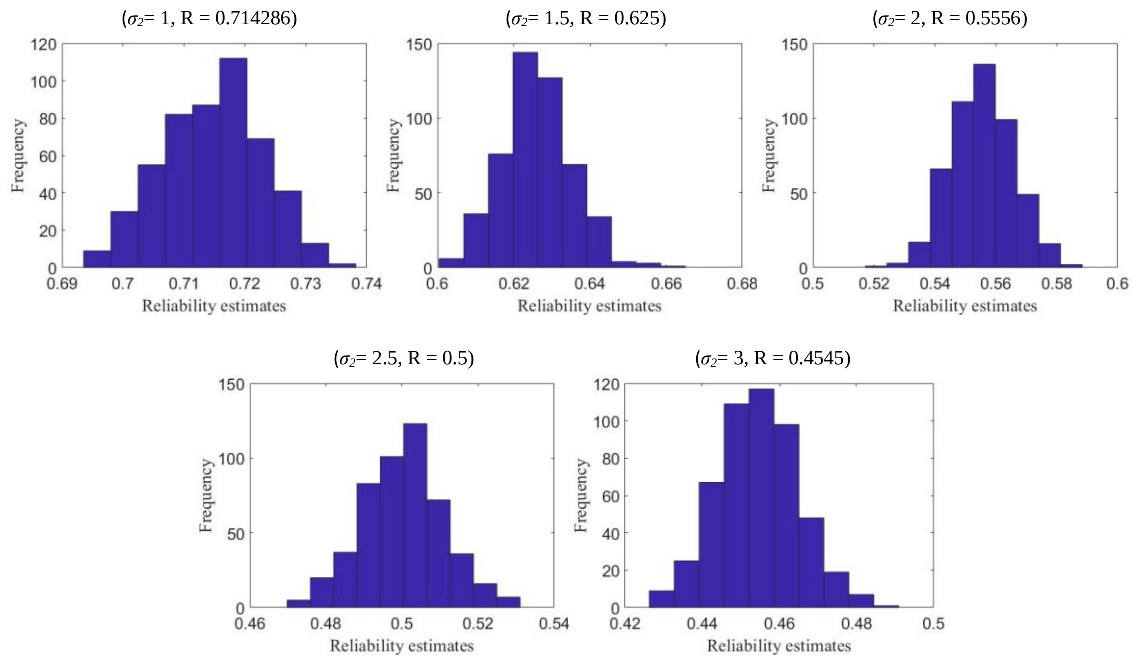


Figure 5.30 Histogram of 500 experiments for μ = 2, $\sigma_1$ = 2.5, $p$ = 1.5 and sample size (500, 500)

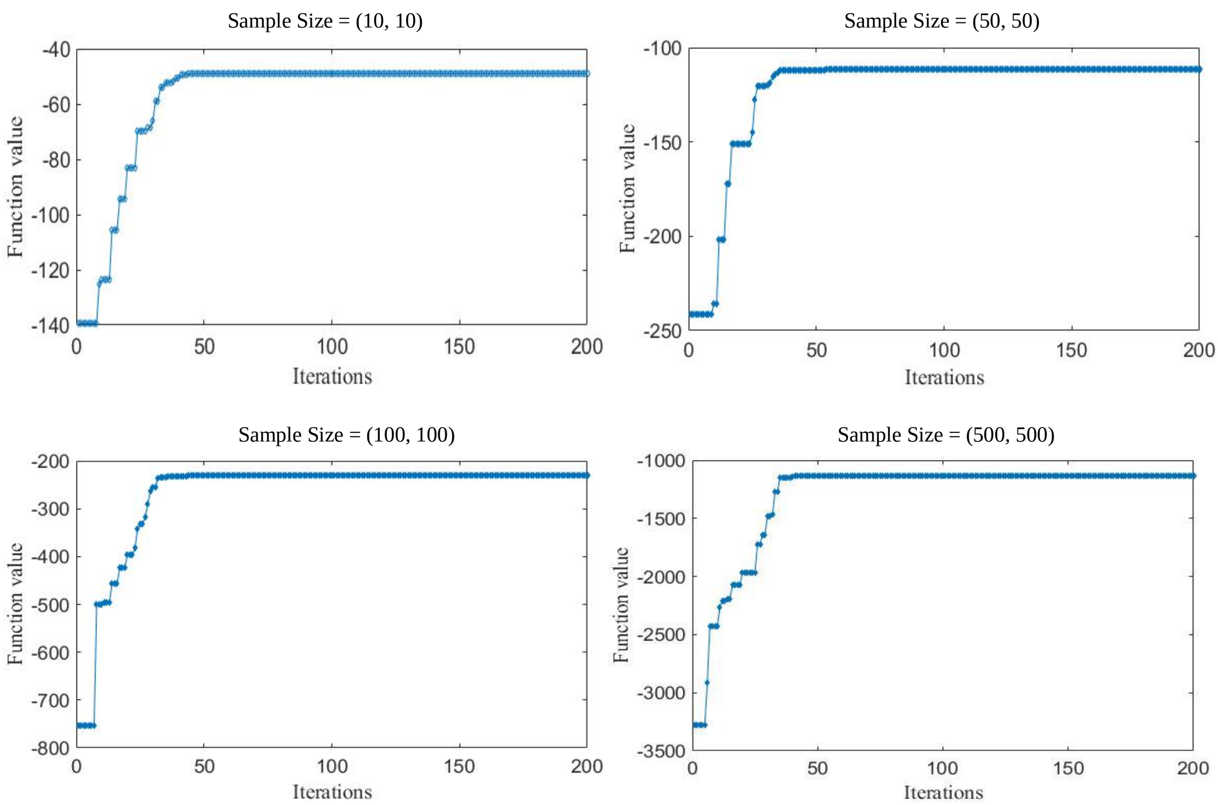


Figure 5.31 Convergence behavior of Jaya algorithm for different sample sizes

### 5.3.4 Application to real life data

The proposed methodology has been applied to the strength data shown in Appendix II previously used by many researchers in their studies [54,164]. The data depicts the strength of single glass fibers for 63 samples of length 10mm and 69 samples of length 20mm. The reliability in this context is evaluated based on the strength of gauge length 10mm being greater than the strength of gauge length 20mm.

The scale parameter for data of gauge length 10 mm is estimated to be 16.1367 and that for gauge length 20 mm is estimated to be 5.1950. The location and shape parameters are considered common for both the data and are found to be 0.9980 and 3.3837 respectively. The Kolmogorov-Smirnov test was conducted to check the fitness of estimated parameters with the data. For the data of gauge length 10mm, the Kolmogorov-Smirnov statistic was found to be 0.0410, p-value 0.998 and log-likelihood function value as -59.4464. The Kolmogorov-Smirnov statistic, p-value and log-likelihood function value for data of gauge length 20mm are calculated to be 0.0783, 0.8196 and -49.0843 respectively. This shows that the Weibull distribution with estimated parameters gives a good fit for both the data sets. The maximum value of log-likelihood function in equation (5.28) is obtained as -108.5307. The estimated reliability is 0.7564. The interference of distributions with the estimated parameters can be seen in Figure 5.31.

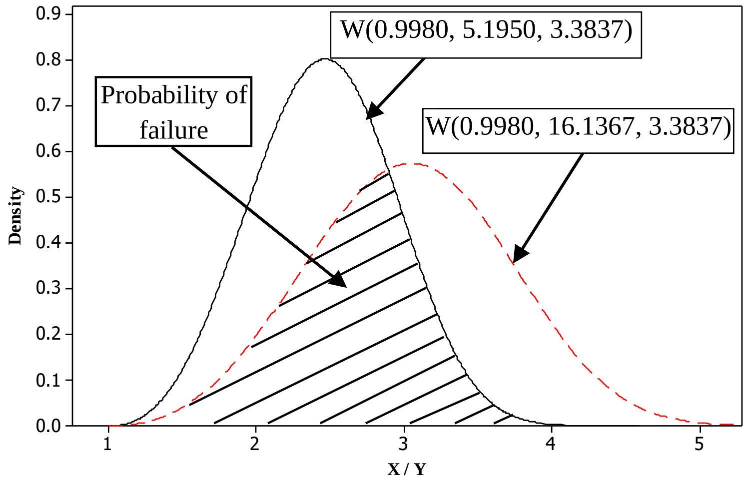


Figure 5.32 Weibull plots for gauge length data with estimated parameters

Table 5.16 shows the results of the proposed methodology to that of the existing literature [50–53]. The proposed methodology gives a better log-likelihood value compared to the ones in the literature. The reliability obtained using the proposed methodology is close to the results obtained by Kundu and Gupta, 2006 and Kundu and Raqab, 2009. The results for reliability estimate is higher compared to the estimate by Valiollahi et al., 2013 and slightly on the lower side compared to the results of Nadarajah and Jia, 2017.

Table 5.16 Comparison of results for proposed methodology

| | Data Set | Shape Parameter | Scale Parameter | Location Parameter | K-S | p-value | Log likelihood value | Reliability |
|---|---|---|---|---|---|---|---|---|
| Proposed Methodology | I | 3.3837 | 16.1367 | 0.9980 | 0.0410 | 0.998 | -59.4464 | 0.7564 |
| | II | 3.3837 | 5.1950 | 0.9980 | 0.0783 | 0.8196 | -49.0843 | |
| Kundu and Gupta, 2006 | I | 3.8770 | 37.2333 | - | 0.0800 | 0.8154 | -60.1524 | 0.7624 |
| | II | 3.8770 | 11.6064 | - | 0.0461 | 0.9985 | -48.8703 | |
| Kundu and Raqab, 2009 | I | 4.6344 | 86.9579 | 1.312 | 0.0767 | 0.8525 | - | 0.7406 |
| | II | 4.6344 | 248.3652 | 1.312 | 0.0464 | 0.9984 | - | |
| Nadarajah and Jia, 2017 | I | 5.049 | 0.12547 | - | 0.088 | 0.719 | -39.438 | 0.8000 |
| | II | 6.725 | 0.01046 | - | 0.050 | 0.997 | -68.149 | |
| Valiollahi et al., 2013 | I | 5.049 | 424.574 | - | 0.0867 | 0.7197 | - | 0.5002 |
| | II | 5.505 | 214.131 | - | 0.0578 | 0.9658 | - | |

## 5.4 Summary

This chapter deals with studies on stress-strength reliability estimation for Weibull distribution. In the first part of the chapter, a new methodology has been proposed for estimating the parameters of Weibull distribution. Simulation studies are carried out and the methodology has been implemented to the real-life data of strength of glass fibres. The results have been compared with other similar methodologies in the literature. In the second part, a methodology has been proposed in estimating the stress-strength reliability for Weibull distribution. First, the methodology has been applied to estimate the stress-strength reliability when stress and strength follow two Weibull distribution with common scale parameter and different shape parameter. Three estimation techniques have been compared and application to real life data has been presented. Then, the methodology has been applied to three Weibull distribution with common shape parameter and different scale parameter in estimation of stress-strength reliability. The technique has been applied to real life data and a comparative study has been carried out for the same data with other estimation techniques in the literature.

# Chapter 6

# Estimation of Reliability considering Strength Degradation

Reliability is the probability that a component or a product will perform its functions under given conditions for a specific period. With increasing developments in the sectors of transportation, infrastructure, energy generation, etc., there is a need for robust design in these areas taking into account the uncertainties present in nature. Reliability studies consider these uncertainties of nature, giving a rigid and dependable analysis. Thus studies on reliability modeling have been given much importance in recent times [176]. As discussed earlier, evidences in a few cases have been seen that the reliability does not remain constant over time and it undergoes deterioration as a result of degradation of material properties like strength. This chapter deals with estimating reliability in case of strength degradation. Also, a discussion has been made on whether the type of degradation affects reliability.

## 6.1 Stress-Strength Reliability Model for Time Dependent and Fatigue Strength

The main objective of this study is to develop a simple yet effective methodology to evaluate the reliability of a component subjected to stress over a period of time taking strength degradation with time into consideration. The proposed method will seek to give close results with less compilation time. The methodology will be applied to the shear strength of solders data in automotive circuits and a plot for reliability prediction over time will be presented. A lot of research has been conducted in estimating the reliability in case of dynamic strength.

Most of the mechanical components are subjected to fatigue due to varying/cyclic loads during their lifetime. The failure of many mechanical components has been traced down to fatigue as the major cause. Thus, it is very important to predict the failure of such components and improve the product life by reliability-based design. Even though a large amount of work has been carried out in the field of fatigue, many challenges can still be seen in this area. The current study carries out reliability studies for a component whose strength decreases with time or number of cycles. Consider the conceptual monotonically decreasing degradation path depicted in Figure 6.1 [177]. The presumed distribution of strength is depicted with increasing time i.e., at $t_1$, $t_2$ and $t_3$ respectively. As can be seen, the distribution of strength is degrading over a period of time which in this case is considered to be nonlinear. At any given time, the degradation of strength will follow a particular distribution. In this research, the distribution is assumed to be normal with a non-linearly varying mean and linearly varying standard deviation. In Figure 6.1, it can be seen that the mean of strength decreases and the standard deviation increases over time. A failure will occur when the strength reaches below the threshold value D. Note that this may not be the case for all failure mechanisms, but this behavior is often observed in mechanical components. For any particular time, reliability can then be estimated as the probability that the degradation measure is greater than a critical threshold value.

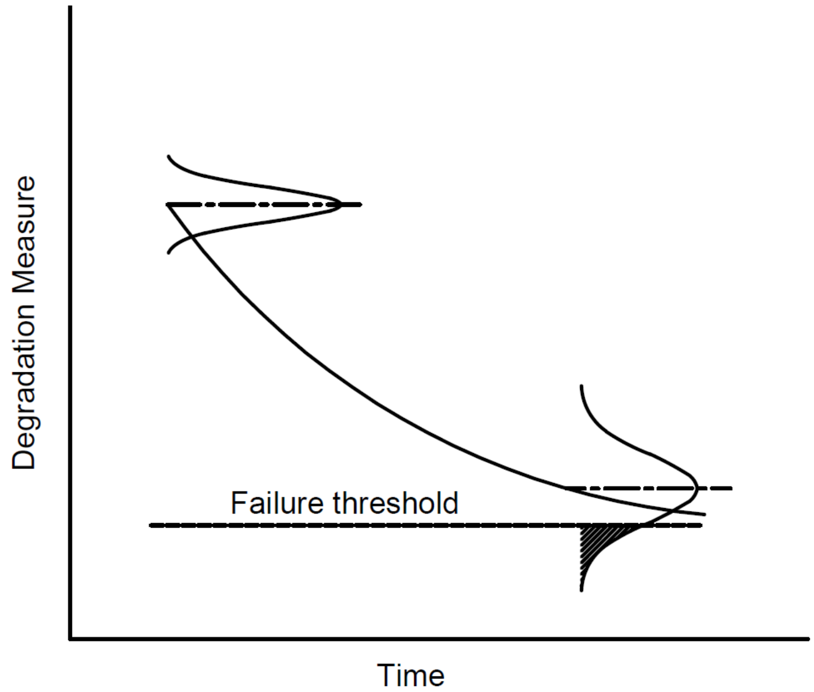


Figure 6.1 Strength degradation path example

Consider the strength with a random variable for X following normal distribution.

$$f(x/t) = \frac{1}{\delta\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu_n(t)}{\delta(t)}\right)^2\right) \quad 6.1$$

$$F_x(x/t) = \int_0^{\infty} \frac{1}{\delta\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{u-\mu_n(t)}{\delta(t)}\right)^2\right) du \qquad 6.2$$

$$F_x(x/t) = \phi\left(\frac{x-\mu(t)}{\delta(t)}\right) \qquad 6.3$$

where $f(x/t)$ and $F_x(x/t)$ are defined as the probability density function and cumulative distribution function for random strengths conditional on time or number of cycles. The terms $\mu_n(t)$ and $\delta(t)$ denotes mean and standard deviation of the distribution which are dependent on time.

### 6.1.1 Likelihood function for dynamic strength

The method of maximum likelihood has been used by researchers to find the best estimates of the parameters of a probability distribution by maximizing the likelihood function. The methodology has been found to be very effective in estimation of parameters. The likelihood function can be obtained as:

$$L(a/t) = \prod_{i=1}^{m} \prod_{j=1}^{n_i} f_x(x_{ij}/t_i) \qquad 6.4$$

$$L(a/x,t) = \prod_{i=1}^{m} \prod_{j=1}^{n_i} \frac{1}{(a_4+a_5t_i)\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x_{ij}-(a_0-a_1t_i-a_2{t_i}^2)}{a_4+a_5t}\right)^2\right) \qquad 6.5$$

$$\log(L) = n\sum_{i=1}^{m} \log\left(\frac{1}{a_4+a_5t_i}\right) + mn\log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n_i}\left(\frac{x_{ij}-(a_0-a_1t_i-a_2{t_i}^2)}{a_4+a_5t_i}\right)^2 \qquad 6.6$$

### 6.1.2 Jaya algorithm in estimation of dynamic reliability

If N components are connected in series, the failure time of a component, T, is the minimum of N individual components failure times.

$$T_i = \text{minimum}\{t; x_i(t) \le D\}\; i = 1,2,3, \ldots\ldots N$$

$$T = \text{minimum}\{T_1, T_2, T_3 \ldots\ldots T_N\}$$

Reliability can be given as

$$R(t) = 1 - F_T(t) = \Pr(T > t) = \Pr\{\text{minimum}\, T_i > t\}$$

$$= \Pr(T_1 > t \cap T_2 > t \cap T_3 > t \ldots \cap T_N > t)$$

$$= \Pr(T_1 > t)\Pr(T_2 > t)\; \Pr(T_3 > t)\ldots\; \Pr(T_N > t)$$

$$=\big(1 - F_x(D/t)\big)\big(1 - F_x(D/t)\big)\; \big(1 - F_x(D/t)\big)\ldots.\; \big(1 - F_x(D/t)\big)$$

$$R = \big(1 - F_x(D/t)\big)^N \qquad 6.7$$

As studied for various data of SN curves, it has been observed that the quadratic model gives a better fit than a linear model [178,179]. The parameters μ(t) and σ(t) which are dependent on time are modeled as

$$\mu(t) = a_1 - a_2 t - a_3 t^2 \qquad 6.8$$

$$\sigma(t) = a_4 + a_5 t \qquad 6.9$$

The reliability thus becomes time-dependent and can be given as

$$R(t) = \Pr(y > D)$$

$$= 1 - F_y(D/t) \qquad \text{- for a single component}$$

$$= 1 - F_y(D/t)^N \qquad \text{-for N components}$$

$$R= \left(1 - \phi\left(\frac{D-(a_0 - a_1 t - a_2 t^2)}{a_4 + a_5 t}\right)\right) \quad \text{- for a single component} \qquad 6.10$$

$$R= \left(1 - \phi\left(\frac{D-(a_0 - a_1 t - a_2 t^2)}{a_4 + a_5 t}\right)\right)^N \quad \text{- for N components} \qquad 6.11$$

For the normal distribution example, the conditional probability density function and reliability can be evaluated by estimating the parameters $a_1$, $a_2$, $a_3$ and $a_4$. If normal distribution is not suitable, other distributions (e.g., Weibull, gamma, exponential, etc.) can be considered depending on the application and past data.

Jaya algorithm has been used to maximize the likelihood function in order to obtain the estimates of parameters. The algorithm is simple and is seen to perform very effectively. In this study, Jaya algorithm has been used to maximize the likelihood function 11. The number of design variables is taken as 5 and the population size is taken as 10. The iteration number of 500 is considered as the termination criteria.

### 6.1.3. Application to shearing of solder joints in automotive circuits

We know that automotive systems have to operate under different conditions like temperature, humidity, etc., in its lifetime. The automotive electrical circuits tend to deteriorate over the number of miles run as a result of working in severe environmental conditions. Thus, the solders in automotive electronic circuits tend to shear over a period of time thus undergoing a failure because of fatigue, creep, etc. A solder connection failure will occur when its strength (measured by a shear strength test) reduces to a critical threshold. The circuit card assembly consisting of a number of solder connections fails when any one of the solder connections deteriorates to the critical level. Therefore, reliability can be defined as the probability that the minimum shear strength, at a particular time, exceeds the critical value. Failure time of circuit card is the minimum of N individual solder connection failure times. Prediction based on reliability is crucial in degradation modeling and thus creating suitable designs for safety and quality. Degradation modeling is based on the probability that the predicted distribution of strength intersects the predefined failure threshold limit. The proposed methodology has been applied to the data for shearing of solder joints in automotive circuits with 7 solder resistors in series which was first mentioned by Coit et al., 2005 [180] which is shown in Appendix III. The time $t_i$ is considered as the number of miles the car travels before the occurrence of a failure and $x_{ij}$ is the shear strength. It was also shown that the normal distribution gives a suitable fit for the strength data of individual circuit boards.

Using the proposed methodology, the estimates for the parameters obtained are shown in Table 6.1. It can be noted that at time $t_i$=0, the mean equals to $a_0$, i.e., 6.029325. Parameters $a_1$, $a_2$, $a_4$ and $a_5$ are found out to be 0.0000067502, 4.675394 x $10^{-11}$, 0.479823 and 4.56611 x $10^{-06}$ respectively. The maximum function value is obtained as -156.8523 and the time taken for the compilation is 0.78237 seconds. It can be observed that the function converges to real roots after around 80 iterations. The scatter plot of the data along with the fitted line plot is shown in Figure 6.2. As can be seen in the figure, as the mileage increases, the mean of strength decreases. Also, it can be observed that the spread increases with an increase in mileage. The convergence behavior of Jaya algorithm is shown in Figure 6.3.

Table 6.1 Estimation of parameters using proposed methodology for dynamic strength

| | |
|---|---|
| $a_0$ | 6.029325 |
| $a_1$ | 0.0000067502 |
| $a_2$ | $4.675394 \times 10^{-11}$ |
| $a_4$ | 0.479823 |
| $a_5$ | $4.56611 \times 10^{-06}$ |
| fmax | -156.8523 |
| Compilation time (s) | 0.78237 |

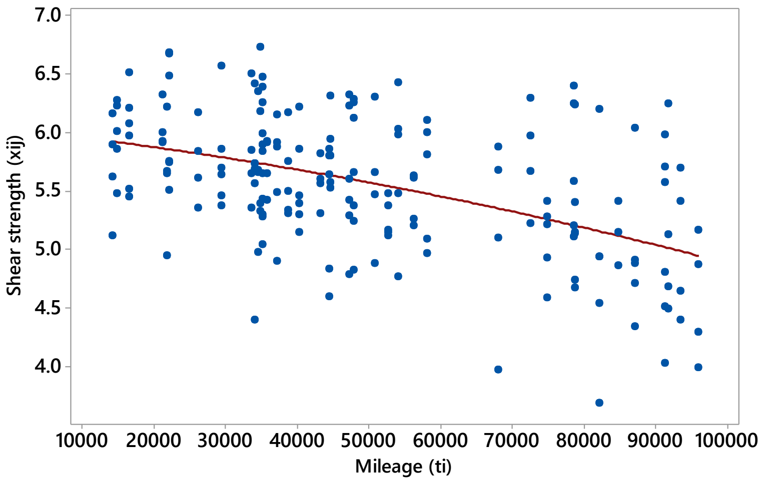


Figure 6.2 Scatter and fitted line plot for shearing strength vs time to shear

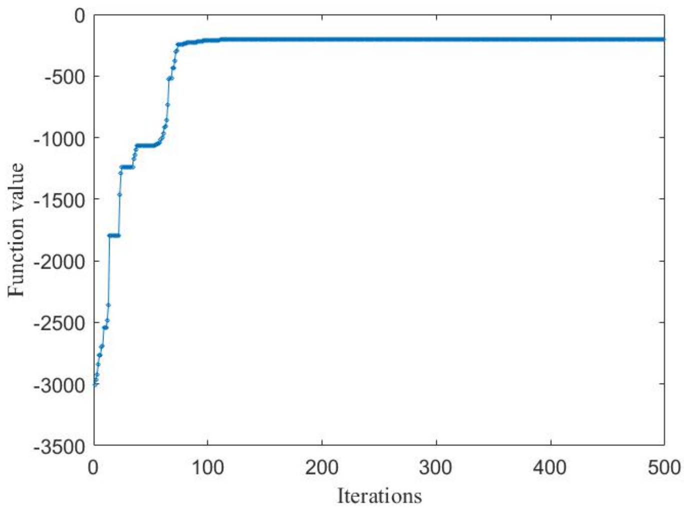


Figure 6.3 Convergence behaviour of Jaya algorithm

As per the literature [180] by experimental examination of the failed modules, a failure threshold of 4.0 was determined. The time-dependent reliability thus can be given as

$$R(t) = \left(1 - \phi\left(\frac{4 - (6.029325 - 0.0000067502t - 4.675394 \text{ x } 10^{-11}t^2)}{0.479823 + 4.56611 \text{ x } 10^{-6}t}\right)\right)^7 \qquad 6.12$$

Equation 6.12 is the reliability model for dynamic strength considering strength degradation. The plot for reliability vs. mileage is shown in Figure 6.4. The reliability starts decreasing significantly after around 20000 cycles. This is mainly due to degradation of material properties over time and the variation in data because of the driving conditions. A steep fall can be observed for reliability after around 50000 cycles. This can be because of the increase in variation of data after 50000 cycles as can be seen in Figure 6.2 in addition to the continuing degradation in material properties. This shows that the reliability in the existing literature may be overestimated.

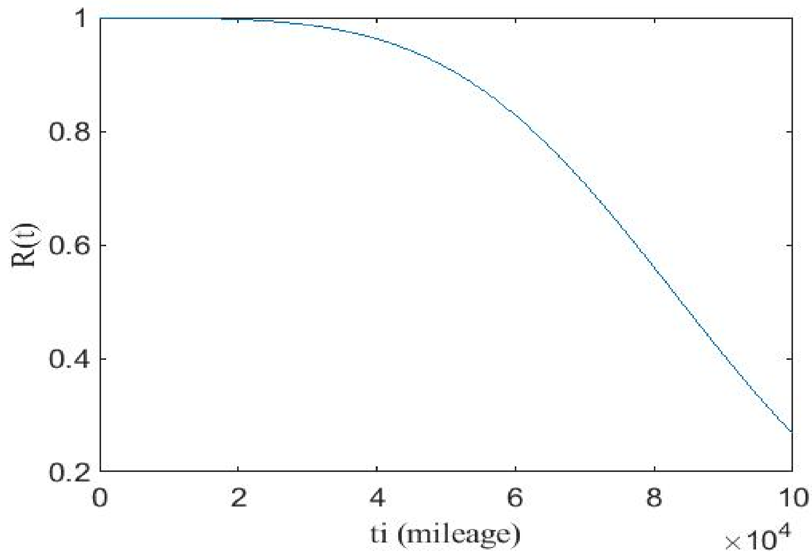


Figure 6.4 Reliability vs mileage

## 6.2 Polynomial Regression in Evaluating Reliability

In recent years many studies carried out conclude that the strength of a material does not remain same and changes over a period of time. The concept is well known as strength degradation which has been explained in the previous section. In this study, the reliability model has been applied to real-life data to see its impact on reliability calculations.

### 6.2.1 Regression in degradation

If stress and strength are considered to be following a certain distribution at a particular time, then as per the degradation principle its mean and standard deviation will both change over time. In this study, three cases for degradation of mean are considered namely linear degradation, quadratic degradation and cubic degradation in order to see its effect on reliability. The standard deviation is assumed to be increasing linearly.

Considering the mean $\mu(t)$ to be undergoing linear degradation and the standard deviation $\sigma(t)$ increasing linearly, the reliability and the log-likelihood function can be derived as follows:

$$\mu(t) = a_1 - a_2 t \tag{6.13}$$

$$\sigma(t) = a_4 + a_5 t \tag{6.14}$$

$$R = \left(1 - \phi\left(\frac{D - \mu_{ns}(t)}{\sigma_S}\right)\right) \tag{6.15}$$

$$R = \left(1 - \phi\left(\frac{D - (a_0 - a_1 t)}{(a_4 + a_5 t)}\right)\right) \tag{6.16}$$

$$\log(L) = n\sum_{i=1}^{m} \log\left(\frac{1}{a_4 + a_5 t_i}\right) + mn\log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n_i}\left(\frac{x_{ij} - (a_0 - a_1 t_i)}{a_4 + a_5 t_i}\right)^2 \tag{6.17}$$

Equation 6.16 can be used in order to find the reliability when the strength undergoes linear degradation. If the degradation is considered to be quadratic, then the equation can be derived as follows:

$$\mu(t) = a_1 - a_2 t - a_3 t^2 \tag{6.18}$$

$$\sigma(t) = a_4 + a_5 t \tag{6.19}$$

$$R = \left(1 - \phi\left(\frac{D - (a_0 - a_1 t - a_2 t^2)}{(a_4 + a_5 t)}\right)\right) \tag{6.20}$$

$$\log(\mathrm{L}) = \mathrm{n}\sum_{i=1}^{m}\log\left(\frac{1}{\mathrm{a_4 + a_5 t_i}}\right) + \mathrm{mnlog}\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n_i}\left(\frac{\mathrm{x_{ij} - (a_0 - a_1 t_i - a_2 t_i{}^2)}}{\mathrm{a_4 + a_5 t_i}}\right)^2 \quad 6.21$$

Equation 6.19 gives the reliability considering the mean of the distribution to be undergoing quadratic degradation and the standard deviation increasing linearly. If the degradation is cubic then the reliability can be derived as follows:

$$\mu(\mathrm{t}) = \mathrm{a_1 - a_2 t - a_3 t^2 - a_4 t^3} \quad 6.22$$

$$\sigma(\mathrm{t}) = \mathrm{a_4 + a_5 t} \quad 6.23$$

$$R = \left(1 - \phi\left(\frac{D - (\mathrm{a_0 - a_1 t - a_2 t^2 - a_3 t^3})}{(a_4 + a_5 t)}\right)\right) \quad 6.24$$

$$\log(\mathrm{L}) = \mathrm{n}\sum_{i=1}^{m}\log\left(\frac{1}{\mathrm{a_4 + a_5 t_i}}\right) + \mathrm{mnlog}\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n_i}\left(\frac{\mathrm{x_{ij} - (a_0 - a_1 t_i - a_2 t_i{}^2 - a_3 t_i{}^3)}}{\mathrm{a_4 + a_5 t_i}}\right)^2 \quad 6.25$$

Equation 6.22 gives reliability when the strength undergoes cubic degradation and the standard deviation increases linearly. In further sections these reliability models are applied to real life data.

### 6.2.2 Polynomial regression applied to data of solder joints in automotive circuits

The models derived above have been applied to the data of shearing of solder joints in automotive circuits, the example which has been taken in section 6.1. Jaya algorithm has been used in order to optimize the likelihood functions 6.17, 6.21 and 6.25 respectively and estimate the function parameters. The results can be seen in Table 6.2. For linear degradation, the parameters $a_0$ , $a_1$, $a_4$ and $a_5$ are obtained as 6.1307635, 1.163556, 0.4788977 and 4.604448 respectively. The function value is obtained as -157.0919 and the time taken for compilation is just 1.3112 seconds.

The mean squared error is 0.259611. In the case of quadratic degradation, the parameters $a_0$ , $a_1$, $a_2$, $a_4$ and $a_5$ are obtained as 6.029325, 0.67502, 4.675394, 0.479823 and 4.56611 respectively. The function value is -156.8523 and the MSE is 0.258672 which are slightly lesser than that obtained with linear degradation. But the time taken is 3.2671 which is more than double that of linear degradation. For the case of cubic degradation, the parameters $a_0$ , $a_1$, $a_2$, $a_3$, $a_4$ and $a_5$ are obtained as 6.197852, 2.023697, 25.942755, -19.873713, 0.484025 and 4.459873 respectively. The function value and MSE are -156.6676 and 0.257742 respectively which are lesser than that obtained with linear and quadratic degradation. The time taken is 4.375 seconds which is higher compared to the other two models. Figure 6.5, 6.7 and 6.9 shows the model fit for the data in linear, quadratic and cubic degradation respectively. Figure 6.6, 6.8 and 6.10 shows the analysis for model fit to data for linear, quadratic and cubic degradation respectively. In all the cases, the residuals are normally distributed and do not follow any specific pattern. The reliability for the three models have been calculated using equation 6.16, 6.19 and 6.24 respectively. The failure threshold value is taken as 4 which is determined by the literature mentioned in section 6.1. Figure 6.11 shows the comparison of dynamic reliability for polynomial regression. It can be observed that the reliability follows almost similar degradation for linear, quadratic and cubic degradation up to around 40000 miles. Thereafter, the linear degradation takes a steeper downfall compared to the other two up to around 80000 miles. The reliability with cubic degradation falls rapidly after 85000 miles.

Table 6.2 Results of polynomial regression in estimation of parameters for shearing of automotive circuits data

| **Model fit** | **Linear** | **Quadratic** | **Cubic** |
|---|---|---|---|
| **a0** | 6.1307635 | 6.029325 | 6.197852 |
| **a1 (x $10^{-5}$)** | 1.163556 | 0.67502 | 2.023697 |
| **a2 (x $10^{-11}$)** | - | 4.675394 | -25.942755 |
| **a3 (x $10^{-16}$)** | - | - | 19.873713 |
| **a4** | 0.4788977 | 0.479823 | 0.484025 |
| **a5 (x $10^{-6}$)** | 4.604448 | 4.56611 | 4.459873 |
| **f** | -157.0919 | -156.8523 | -156.6676 |
| **$t_c$ (s)** | 1.3112 | 3.2671 | 4.375 |
| **SSE** | 49.8454 | 49.6650 | 49.4864 |
| **MSE** | 0.259611 | 0.258672 | 0.257742 |
| **RMSE** | 0.509521 | 0.508598 | 0.507683 |

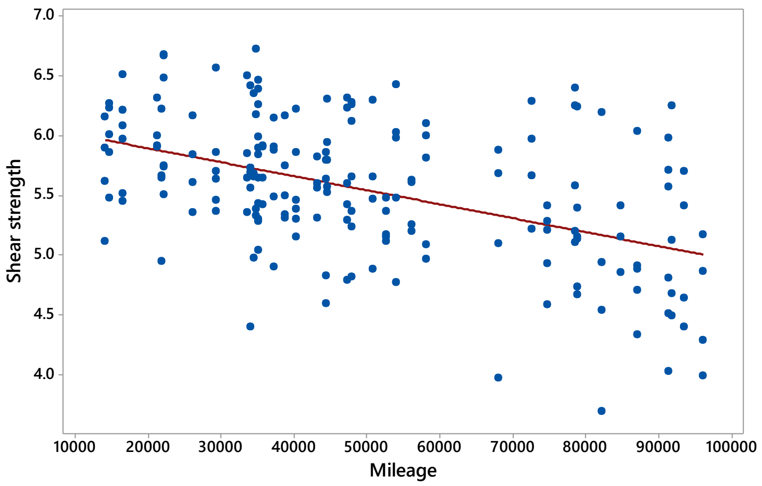


Figure 6.5 Model fit for the data considering linear degradation (Automotive circuits data)

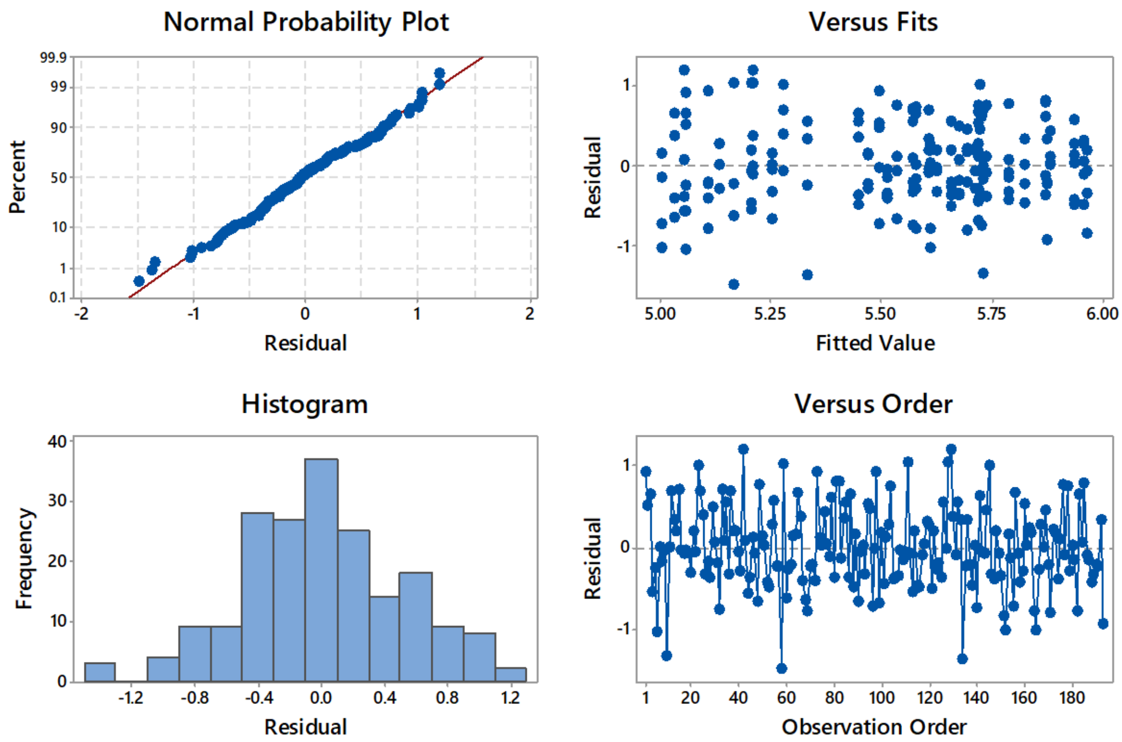


Figure 6.6 Analysis of model fit for linear degradation (Automotive circuits data)

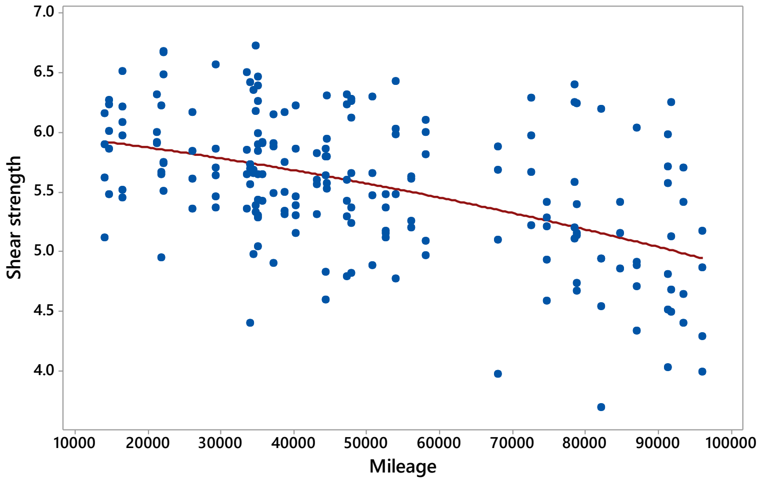


Figure 6.7 Model fit for the data considering quadratic degradation (Automotive circuits data)

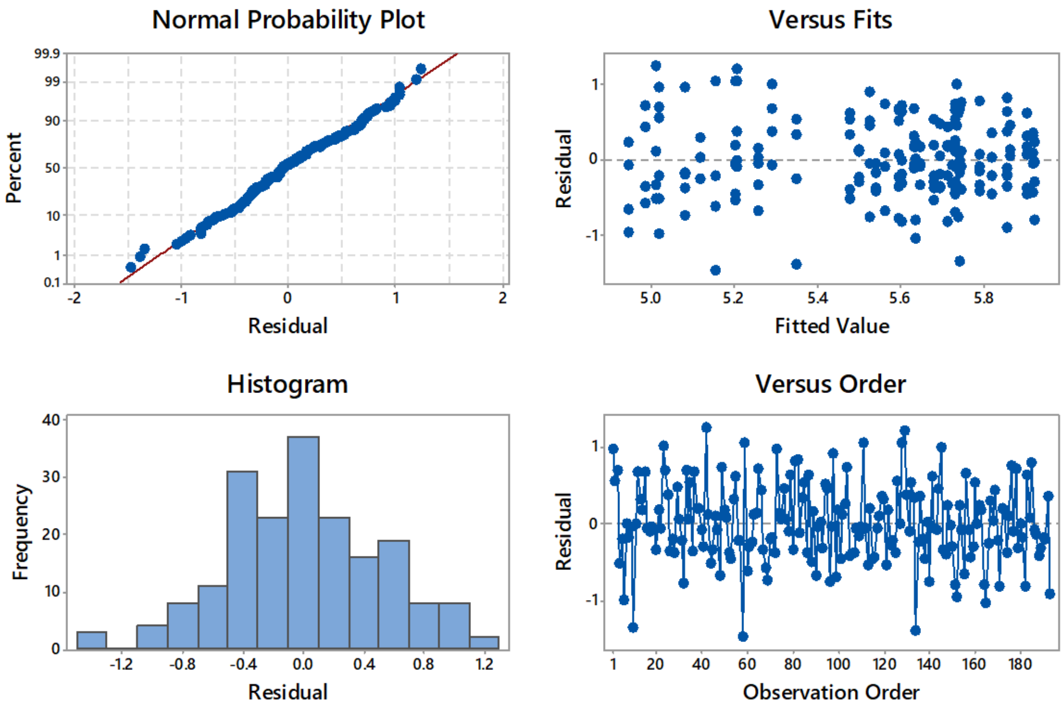


Figure 6.8 Analysis of model fit for quadratic degradation (Automotive circuits data)

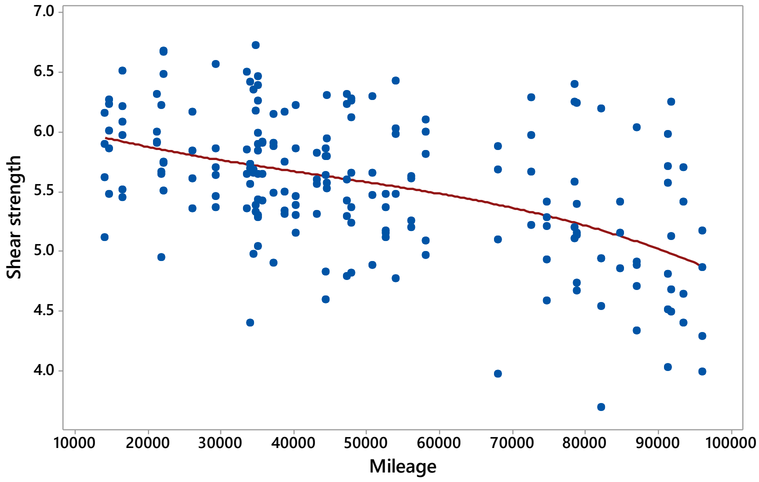


Figure 6.9 Model fit for the data considering cubic degradation (Automotive circuits data)

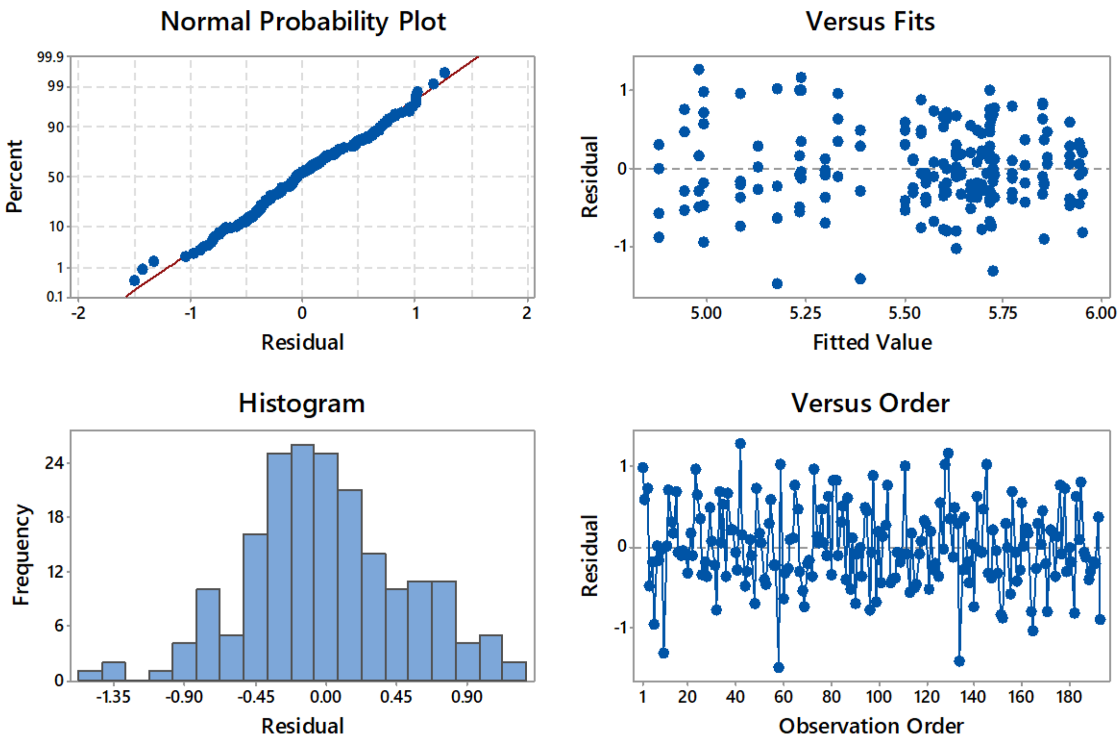


Figure 6.10 Analysis of model fit for cubic degradation (Automotive circuits data)

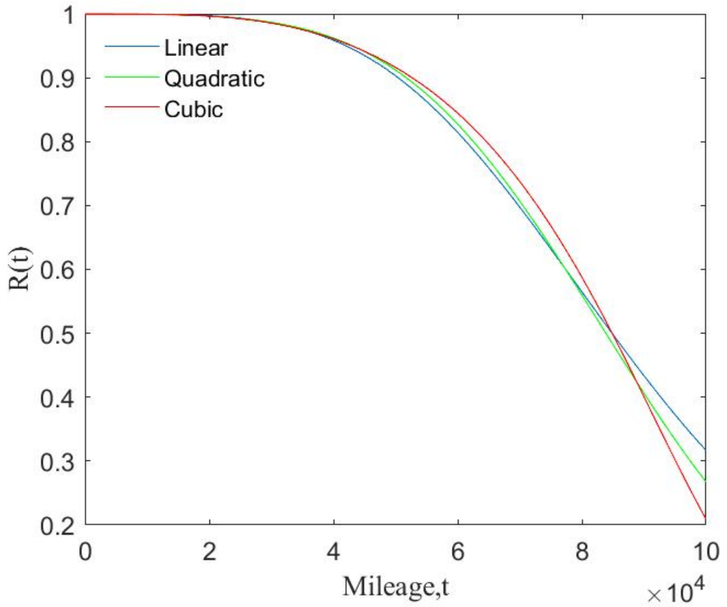


Figure 6.11 Comparison of reliability for polynomial regression (Automotive circuits data)

### 6.2.3 Application to SCM440 steel strength data

Yoshiyuki Furuya studied the degradation of SCM440 steel strength due to fatigue mechanism and compared a few models to fit the data [181]. In the current section, the methodology discussed in section 6.2.2 has been applied to the data of SCM440 steel strength to see the effect of degradation on reliability. The results of estimation have been depicted in Table 6.3. In case of a linear degradation the parameters $a_0$ , $a_1$, $a_4$ and $a_5$ are obtained as 1438.763475, 87.493161, 102.30078 and 6.94 x $10^{-15}$. The function value is obtained as -492.7328 and MSE is 5295.02. The compilation time is 6.135829 seconds. For quadratic degradation, the parameters $a_0$ , $a_1$, $a_2$, $a_4$ and $a_5$ are obtained as 1740.69486, 176.250131, -6.376601, -101.46551 and 0.02783907 respectively. The function value and MSE are -492.0498 and 5231.8 respectively, which are lesser than the ones obtained with linear degradation. The compilation time is 6.92 seconds which is close to the time for linear degradation. In case of cubic degradation, the parameters $a_0$ , $a_1$, $a_2$, $a_3$, $a_4$ and $a_5$ are obtained as 2547.84, 530.12, -57.067, 2.3737, 96.4391 and 0.565694 respectively. As can be observed, the equation parameters widely differ from each other depending on the type of degradation.  The function value is obtained as -491.5448 and MSE is 5205.96 which are lesser compared to linear and quadratic degradation. The compilation time is 7.66 seconds which is not very high compared to the time taken for other degradations. It can be inferred from the function value that a cubic degradation gives better fit for the considered data SCM440 steel followed by quadratic and then linear degradation.

Figure 6.12 and Figure 6.13 shows the model fit and the analysis for the polynomial regression respectively considering linear degradation. Figure 6.14 and Figure 6.15 shows the model fit along and the analysis for the polynomial regression respectively considering quadratic degradation. Figure 6.16 and Figure 6.17 shows the model fit along and the analysis for the polynomial regression respectively considering cubic degradation. In all the cases, the residuals follow normal distribution and does not have a specific pattern. The reliability is calculated using equations 6.16, 6.19 and 6.24 for linear, quadratic and cubic degradation respectively. The comparative plot of reliability for polynomial regression is shown in Figure 6.18. The threshold stress is taken as 700MPa. As can be seen, for all the three cases, reliability remains constant (high) upto around $10^3$ cycles. In cubic degradation, the reliability takes a sudden fall after $10^3$ cycles. The reliability in case of quadratic degradation decreases suddenly after $10^4$ cycles and in case of linear degradation after $10^5$ cycles. Thus, it can be inferred from the graph that the type of degradation has a strong influence on reliability and should be taken into account during the design stages.

Table 6.3 Results of polynomial regression in estimation of parameters for SCM440 steel strength data

| **Model fit** | **Linear** | **Quadratic** | **Cubic** |
|---|---|---|---|
| **a0** | 1438.763475 | 1740.69486 | 2547.84 |
| **a1** | 87.493161 | 176.250131 | 530.12 |
| **a2** | - | -6.376601 | -57.067 |
| **a3** | - | - | 2.3737 |
| **a4** | 102.30078 | 101.46551 | 96.4391 |
| **a5** | $6.94 \times 10^{-15}$ | 0.02783907 | 0.565694 |
| **f** | -492.7328 | -492.0498 | -491.5448 |
| **$t_c$ (s)** | 6.135829 | 6.92 | 7.66 |
| **SSE** | 444782 | 439470.132 | 437300 |
| **MSE** | 5295.02 | 5231.8 | 5205.96 |
| **RMSE** | 72.7669 | 72.3311 | 72.1523 |

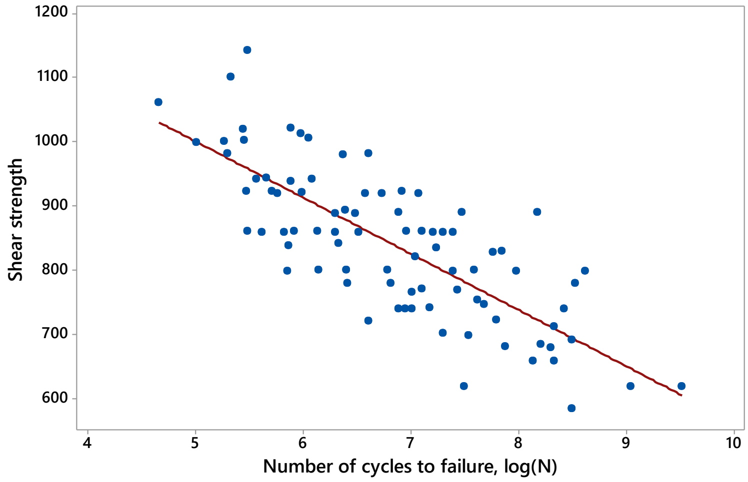


Figure 6.12 Model fit for the data considering linear degradation (SCM440 data)

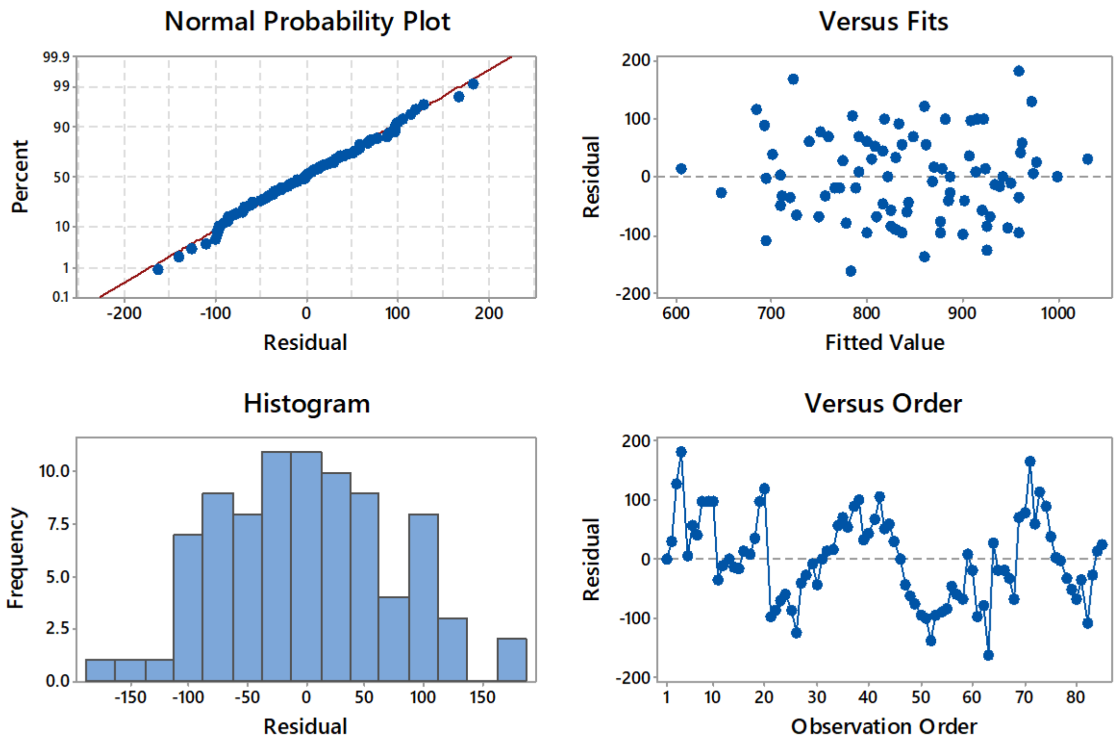


Figure 6.13 Analysis of model fit for linear degradation (SCM440 data)

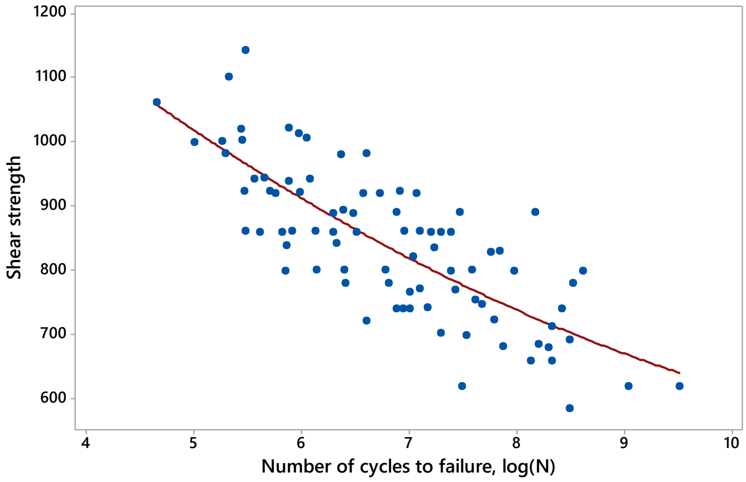


Figure 6.14 Model fit for the data considering quadratic degradation (SCM440 data)

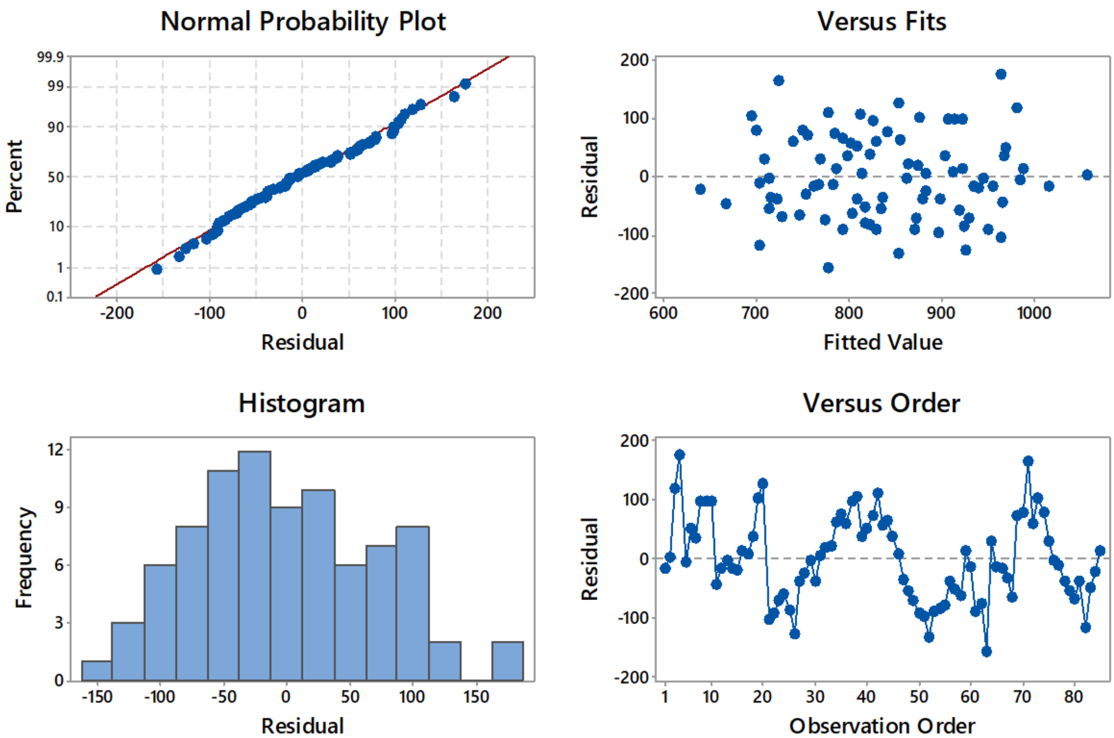


Figure 6.15 Analysis of model fit for quadratic degradation (SCM440 data)

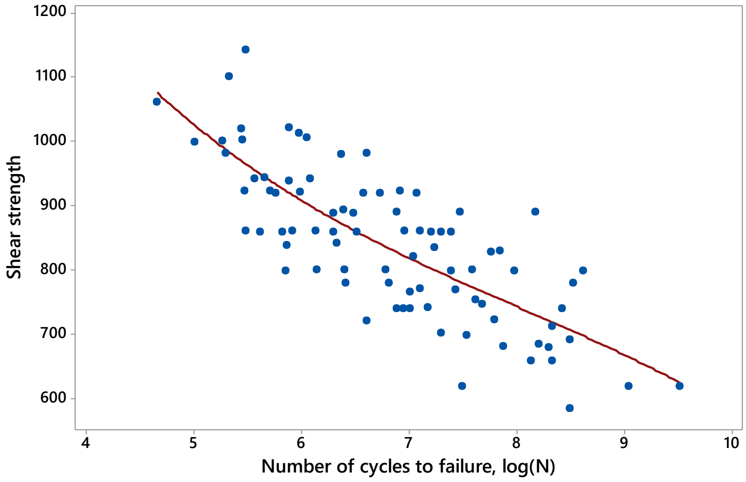


Figure 6.16 Model fit for the data considering cubic degradation (SCM440 data)

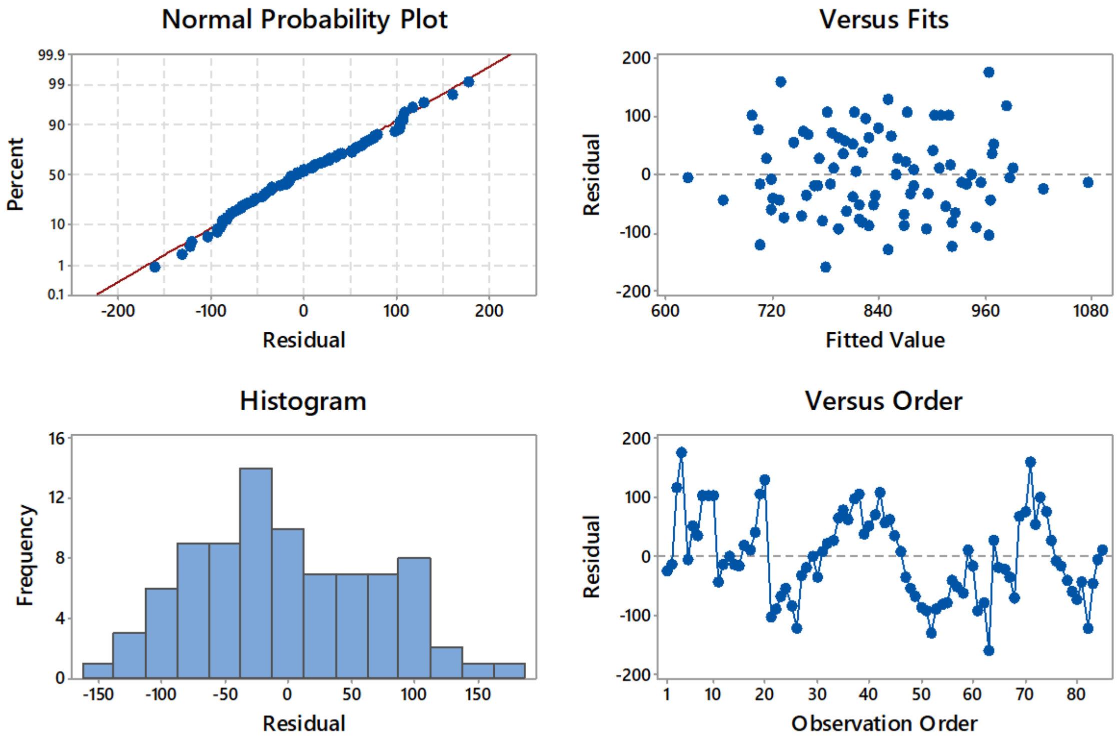


Figure 6.17 Analysis of model fit for quadratic degradation (SCM440 data)

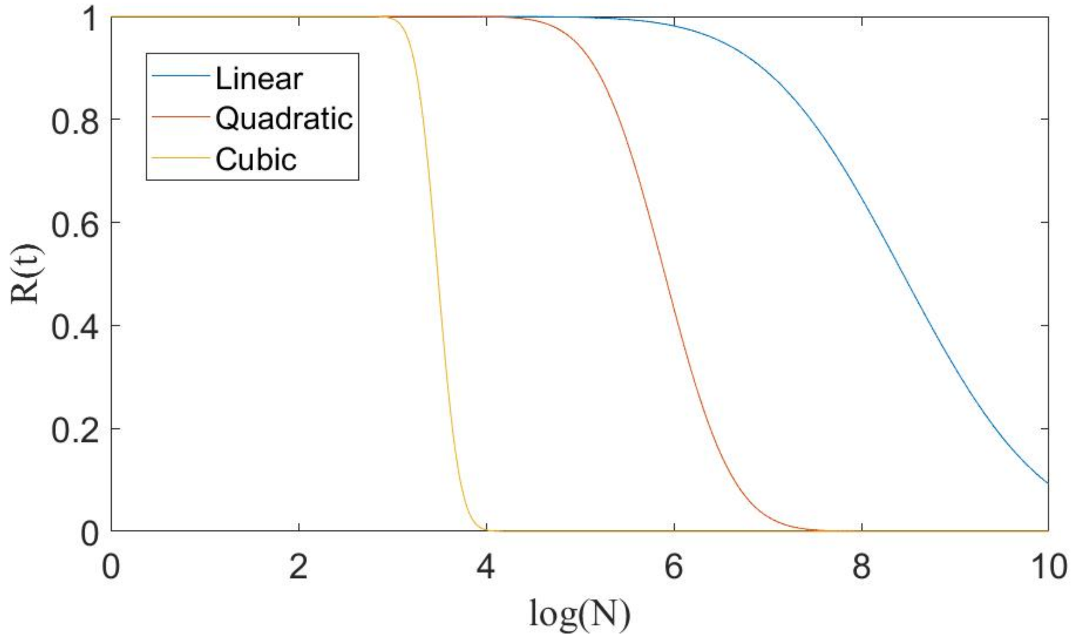


Figure 6.18 Comparison of reliability for polynomial regression (SCM440 steel strength data)

## 6.3 Summary

This chapter discusses the reliability evaluation considering strength degradation with time or number of cycles. A model has been developed for reliability considering quadratic degradation and a methodology has been proposed for solving and computing the reliability. The methodology has been applied to the data of shearing of solder joints in automotive circuits. In the next section, a study is carried out to see whether the type of degradation affects reliability by considering polynomial degradation i.e., linear, quadratic and cubic. The methodology has been applied to automotive circuits data and the data of steel strengths for SCM440 steel. The comparative plot for reliability for both the applications have been obtained and the conclusions have been drawn.

# Chapter 7

# Summary and Conclusions

This final chapter discusses the conclusions of the work done in this thesis under the title "Investigative Studies in Reliability Based Design Approach for Mechanical Systems." The contributions and limitations of the research work have been expressed, and the future scope for the additional or further work that can be carried out in this field has been presented.

## 7.1 Discussion

The work done in the thesis can mainly be categorized into 3 phases. In phase 1, the stress-strength reliability models are obtained for distributions Laplace, exponential, Weibull and gamma for which the closed form of reliability was not present. The models were analyzed and validated. In phase 2, the stress-strength reliability model was evaluated and a methodology to obtain reliability estimates was proposed. Two cases of stress-strength Weibull distribution were considered. First, two-parameter Weibull distribution with common scale parameter and different shape parameter and next, three parameter Weibull distribution with common shape parameter but different scale parameter. In phase 3, the strength was considered to be dynamic with time or number of cycles in cyclic loading. Normal distribution was considered to model reliability and polynomial regression for degradation was studied.

## 7.2 Conclusions

The conclusions of this research work are listed below:

1. In the present work, two reliability models based on stress-strength interference theory are developed. The first model gives reliability when strength follows Laplace distribution and stress follows exponential distribution. The second model gives reliability when strength follows exponential distribution and stress follows Laplace distribution. Taguchi analysis gives the influence of parameters on the response. When strength follows Laplace distribution and stress follows exponential distribution, $\theta$ is observed to be the most

influential parameter followed by λ and then ϕ. If θ, λ increases and ϕ decreases, reliability is seen to be increasing. When strength follows exponential distribution and stress follows Laplace distribution, λ is observed to be the most influential parameter followed by ϕ and then θ. In this case, increase in reliability is observed if all the parameters λ, ϕ, θ are increasing. The validation experiment shows that the reliability obtained by simulation is almost equal to the reliability obtained from the proposed model. Also, as the size of random numbers generated increases, the reliability estimates move closer to the reliability obtained from the model. Thus, the proposed models give fairly accurate reliability with stress and strength following Laplace and exponential distributions and vice versa. The study can be extended to the cases such as degradation of strength over time (Xie L.  and Wang Z.2009), considering interdependency among stress and strength (Huang H., Member S., and An Z.2009) and fatigue design considerations (Menan F., Adragna P. A. and François M.2015).

2. A methodology proposed for developing the stress-strength reliability model for Weibull and gamma distributions for which the closed form models are not available. The main effects plot, interaction plot and contour plot show the influence of parameters on reliability. The validation experiments show the accuracy of reliability prediction within the considered range of parameters.
3. Weibull distribution is an important model and estimation of its parameters is encountered in many real-life problems including reliability and life testing. In this article, Jaya algorithm has been used along with the MLE technique in order to estimate the parameters of three-parameter Weibull distribution. Simulation studies carried out show that the algorithm gives close estimates of parameters to real values and shows a rapid convergence behaviour. As the sample size increases, better estimates are obtained but the compilation time increases with increase in sample size. A comparative study with SA, HNSA, DE and PSO shows that Jaya algorithm gives better results in estimating the three parameters of Weibull distribution. The proposed method using Jaya algorithm was also applied to real-life data of strength of glass fibres and the results show that it outperforms the methodology applied in literature for a similar study. Also, a histogram with fitted curve and probability plot shows that the estimated parameters using Jaya algorithm gives a very good fit for the data.
4. In this research, the estimation of reliability for stress-strength interference has been carried out using Jaya algorithm via maximum likelihood estimation. It was considered that the stress and strength follow Weibull distribution with common location and shape parameter but different scale parameter. The methodology has been applied to simulated data sets of different sample sizes and different scale parameters. The results show that the estimated reliability values using the proposed methodology are very close to the real reliability values

and the reliability using Monte Carlo simulations for estimated parameters. Also, as the sample size increases, the estimated reliability moves closer to the real reliability and mean squared error also decreases supporting the general trend of estimated values approaching closer to real values. An application to real life data is also shown along with the estimated reliability and interference graph. The inclusion of location parameter in the study significantly affects the reliability results. A better maximum log-likelihood value is obtained compared to the ones existing in the literature.

5. This research deals with estimation P (X > Y) for X and Y following Weibull distribution with different shape parameters and same scale parameters. The estimation methods used are maximum likelihood estimation, least squares estimation, and weighted least squares estimation. Jaya algorithm has been used in optimizing the estimation functions. The reliability estimate equation has been presented and simulation studies are carried out in order to validate the model and compare the performance of the algorithm with the above estimation methods. Box plots showed the increasing accuracy of estimation with an increase in sample size. Jaya algorithm shows a consistent convergence towards the real roots. It was observed that the algorithm with maximum likelihood estimation outperforms the other two techniques studied with respect to the bias and mean squared error. The technique was applied to real-life data and it was observed that the estimated models with the proposed methodology give a very good fit for all the three estimation methods which were confirmed by the Kolmogorov-Smirnov test. The proposed methodology using MLE gives the best fit followed by WLSE and then LSE for the real-life data of strength of carbon fibers. There are many methods for estimating the parameters via various optimization techniques. But the proposed methodology gives highly accurate results with faster compilation time compared to most of these methods. Further studies can be carried out using the proposed methodology considering the location parameter and investigating its effects on reliability calculation. Also, the methodology can be applied to X and Y values following other distributions like gamma, exponential, Laplace, etc.
6. In this study, a methodology has been proposed for evaluating the time-dependent reliability of a component subjected to deterministic stress. A nonlinear model for the mean and linear model for standard deviation has been considered. The maximum likelihood estimation via Jaya algorithm has been used to obtain the estimates of the parameters. The method has been applied to the shear strength data of automotive circuits and SCM440 steel strength data and a plot for reliability vs. time has been obtained. It has been observed that the proposed methodology is simple, fast and gives very good results. The reliability varies significantly

with the type of degradation considered. So, the degradation model is crucial in anticipating the reliability of the components.

## 7.3 Contributions

The contributions of this research work are as follows:

1. A stress-strength interference model when strength follows Laplace distribution and stress follows exponential distribution and vice versa. The model can be used in mechanical systems in order to find the reliability of the system when the strength and stress follow Laplace and exponential distribution respectively and vice versa.
2. A methodology to evaluate the closed form of stress-strength interference model when stress and strength follow Weibull distribution. A similar methodology is shown for cases when stress and strength follow gamma distribution. The model is suitable if the range of parameters driving the distribution is known.
3. A methodology to determine the parameters of Weibull distribution with high accuracy, faster compilation time and its application to real life data.
4. A methodology to determine the stress-strength reliability for stress and strength following Weibull distribution with high accuracy and faster compilation time. Similar procedure can also be applied to find stress-strength reliability for stress and strength following other distributions.
5. An improved methodology to estimate the stress-strength reliability in case of dynamic strength. The model can be applied if the strength is time dependent and even for fatigue strength where the strength varies with cyclic loading.

## 7.4 Limitations of the Work

This research is conducted mainly to conduct a few investigations, develop some important models and propose novel methodologies to get improvised results. The research has completed its aims and objectives. More problems as cited below can be further explored and investigated. A few limitations of the research are as follows:

1. The methodology proposed for developing a closed form of stress-strength reliability model for Weibull and gamma distribution will be applicable only if the parameter range for the distributions is known. The predictions for reliability from the model can be made only in the parameter range. Also, the variation of parameters in predicting reliability is considered to be linear.

2. The methodology for stress-strength reliability estimation has been shown for Weibull distribution. The methodology has to be explored for other distributions where in new challenges may occur. It was also assumed that one of the parameters remains the same for simplicity in reliability calculations.
3. This work mainly deals with point estimation. The methods shown in this study can be further used to compute confidence intervals in estimation of reliability using various techniques available in the literature.
4. The methodology proposed in strength degradation considers linear, quadratic and cubic strength degradation of the mean. Various other degradations may be possible and needs more investigation. The standard deviation is considered to be varying linearly, whereas in real life the standard deviation also can follow another type of variation.
5. The degradation model is considered for strength following normal distribution. The methodology can be studied for more distributions for further investigations. Also, the methodology for degradation considers stress to be deterministic. However, in real life, the stress may also follow a particular distribution.

## 7.5 Future Scope

The limitations discussed in the previous section offer scope for the future. The following work could be considered as possible extensions and scope for future work.

1. More stress-strength interference models can be developed for stress and strength following other distributions such as Gompertz, beta, Lomax, Kumaraswamy distribution, etc. and their interactions. Also, reliability models for interactions of Weibull, gamma with other common distributions can be developed. The methodology can further be explored considering non-linear relations between the parameters and reliability.
2. The methodology of stress-strength reliability estimation can be applied to other common distributions like gamma, Laplace, Cauchy, etc. Further research can be carried out by varying the location and shape parameters and analyzing its influence on reliability. Studies on confidence interval estimation of reliability can be carried out using the proposed methodology in assistance to various estimation techniques in literature. Also, studies can be carried out to enhance the performance of Jaya algorithm by some modifications and narrowing/ proposing a search space for faster convergence to the optimum values.
3. More common degradations can be studied other than the ones studied in this research. Also, nonlinear variations in standard deviation can be considered to see its effects on reliability evaluation.

4. The methodology of degradation can be applied for strength following other distributions like Weibull, gamma, etc. Also, the studies can be further explored for non-deterministic nature of acting loads by considering the uncertainty in stress as well.

# Appendix I: Strength of Glass Fibres Data

| $n$ | $x_n$ |
|---|---|
| 1 | 0.55 |
| 2 | 0.93 |
| 3 | 1.25 |
| 4 | 1.36 |
| 5 | 1.49 |
| 6 | 1.52 |
| 7 | 1.58 |
| 8 | 1.61 |
| 9 | 1.64 |
| 10 | 1.68 |
| 11 | 1.73 |
| 12 | 1.81 |
| 13 | 2 |
| 14 | 0.74 |
| 15 | 1.04 |
| 16 | 1.27 |
| 17 | 1.39 |
| 18 | 1.49 |
| 19 | 1.53 |
| 20 | 1.59 |
| 21 | 1.61 |
| 22 | 1.66 |
| 23 | 1.68 |
| 24 | 1.76 |
| 25 | 1.82 |
| 26 | 2.01 |
| 27 | 0.77 |
| 28 | 1.11 |
| 29 | 1.28 |
| 30 | 1.42 |
| 31 | 1.5 |
| 32 | 1.54 |
| 33 | 1.6 |
| 34 | 1.62 |
| 35 | 1.66 |
| 36 | 1.69 |
| 37 | 1.76 |
| 38 | 1.84 |

| $n$ | $x_n$ |
|---|---|
| 39 | 2.24 |
| 40 | 0.81 |
| 41 | 1.13 |
| 42 | 1.29 |
| 43 | 1.48 |
| 44 | 1.5 |
| 45 | 1.55 |
| 46 | 1.61 |
| 47 | 1.62 |
| 48 | 1.66 |
| 49 | 1.7 |
| 50 | 1.77 |
| 51 | 1.84 |
| 52 | 0.84 |
| 53 | 1.24 |
| 54 | 1.3 |
| 55 | 1.48 |
| 56 | 1.51 |
| 57 | 1.55 |
| 58 | 1.61 |
| 59 | 1.63 |
| 60 | 1.67 |
| 61 | 1.7 |
| 62 | 1.78 |
| 63 | 1.89 |

# Appendix II: Strength Data of Gauge Length 10 mm and 20 mm

Strength data of gauge length 10mm (Data Set I)

| | | | | | | |
|---|---|---|---|---|---|---|
| 1.901 | 2.132 | 2.203 | 2.228 | 2.257 | 2.35 | 2.361 |
| 2.396 | 2.397 | 2.445 | 2.454 | 2.474 | 2.518 | 2.522 |
| 2.525 | 2.532 | 2.575 | 2.614 | 2.616 | 2.618 | 2.624 |
| 2.659 | 2.675 | 2.738 | 2.74 | 2.856 | 2.917 | 2.928 |
| 2.937 | 2.937 | 2.977 | 2.996 | 3.03 | 3.125 | 3.139 |
| 3.145 | 3.22 | 3.223 | 3.235 | 3.243 | 3.264 | 3.272 |
| 3.294 | 3.332 | 3.346 | 3.377 | 3.408 | 3.435 | 3.493 |
| 3.501 | 3.537 | 3.554 | 3.562 | 3.628 | 3.852 | 3.871 |
| 3.886 | 3.971 | 4.024 | 4.027 | 4.225 | 4.395 | 5.02 |

Strength data of gauge length 20mm (Data Set II)

| | | | | | | |
|---|---|---|---|---|---|---|
| 1.312 | 1.314 | 1.479 | 1.552 | 1.7 | 1.803 | 1.861 |
| 1.865 | 1.944 | 1.958 | 1.966 | 1.997 | 2.006 | 2.021 |
| 2.027 | 2.055 | 2.063 | 2.098 | 2.14 | 2.179 | 2.224 |
| 2.24 | 2.253 | 2.27 | 2.272 | 2.274 | 2.301 | 2.301 |
| 2.359 | 2.382 | 2.382 | 2.426 | 2.434 | 2.435 | 2.478 |
| 2.49 | 2.511 | 2.514 | 2.535 | 2.554 | 2.566 | 2.57 |
| 2.586 | 2.629 | 2.633 | 2.642 | 2.648 | 2.684 | 2.697 |
| 2.726 | 2.77 | 2.773 | 2.8 | 2.809 | 2.818 | 2.821 |
| 2.848 | 2.88 | 2.809 | 2.818 | 2.821 | 2.848 | 2.88 |
| 2.954 | 3.012 | 3.067 | 3.084 | 3.09 | 3.096 | 3.128 |
| 3.233 | 3.433 | 3.585 | 3.585 | | | |

# Appendix III:  Shearing Data of Solder Joints in Automotive Circuits

| Board | $t_i$ | $x_{ij}$ | Board | $t_i$ | $x_{ij}$ | Board | $t_i$ | $x_{ij}$ | Board | $t_i$ | $x_{ij}$ | Board | $t_i$ | $x_{ij}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A1 | 91392 | 5.98 | A8 | 35855 | 5.65 | A16 | 22233 | 6.48 | A23 | 14811 | 6.01 | A32 | 35143 | 5.65 |
| A1 | 91392 | 5.57 | A8 | 35855 | 5.42 | A16 | 22233 | 5.51 | A23 | 14811 | 6.27 | A32 | 35143 | 5.3 |
| A1 | 91392 | 5.71 | A9 | 91898 | 6.25 | A16 | 22233 | 6.67 | A23 | 14811 | 6.23 | A32 | 35143 | 5.43 |
| A1 | 91392 | 4.51 | A9 | 91898 | 5.13 | A16 | 22233 | 6.68 | A24 | 40351 | 5.15 | A32 | 35143 | 6.26 |
| A1 | 91392 | 4.81 | A9 | 91898 | 4.49 | A16 | 22233 | 5.74 | A24 | 40351 | 5.86 | A33 | 44464 | 5.64 |
| A1 | 91392 | 4.03 | A9 | 91898 | 4.68 | A17 | 58178 | 5.81 | A24 | 40351 | 5.39 | A33 | 44464 | 5.86 |
| A2 | 34155 | 5.73 | A10 | 50917 | 5.66 | A17 | 58178 | 6 | A24 | 40351 | 5.46 | A33 | 44464 | 5.8 |
| A2 | 34155 | 5.56 | A10 | 50917 | 5.47 | A17 | 58178 | 5.09 | A24 | 40351 | 5.3 | A33 | 44464 | 4.83 |
| A2 | 34155 | 5.69 | A10 | 50917 | 4.88 | A17 | 58178 | 6.1 | A24 | 40351 | 6.22 | A33 | 44464 | 4.6 |
| A2 | 34155 | 4.4 | A10 | 50917 | 6.3 | A17 | 58178 | 4.97 | A25 | 78689 | 5.2 | A34 | 84897 | 4.86 |
| A2 | 34155 | 5.73 | A11 | 16579 | 6.08 | A18 | 74882 | 5.41 | A25 | 78689 | 6.25 | A34 | 84897 | 5.41 |
| A2 | 34155 | 6.42 | A11 | 16579 | 5.97 | A18 | 74882 | 4.59 | A25 | 78689 | 6.4 | A34 | 84897 | 5.15 |
| A3 | 44654 | 5.94 | A11 | 16579 | 5.52 | A18 | 74882 | 5.21 | A25 | 78689 | 5.58 | A35 | 37268 | 6.15 |
| A3 | 44654 | 5.8 | A11 | 16579 | 5.45 | A18 | 74882 | 5.28 | A25 | 78689 | 5.11 | A35 | 37268 | 5.49 |
| A3 | 44654 | 6.31 | A11 | 16579 | 6.21 | A18 | 74882 | 4.93 | A26 | 68171 | 5.88 | A35 | 37268 | 4.9 |
| A3 | 44654 | 5.57 | A11 | 16579 | 6.51 | A19 | 54152 | 6.03 | A26 | 68171 | 5.68 | A35 | 37268 | 5.91 |
| A3 | 44654 | 5.53 | A12 | 82278 | 4.94 | A19 | 54152 | 5.98 | A26 | 68171 | 3.97 | A35 | 37268 | 5.88 |
| A4 | 43284 | 5.6 | A12 | 82278 | 4.94 | A19 | 54152 | 4.77 | A26 | 68171 | 5.1 | A36 | 33627 | 5.36 |
| A4 | 43284 | 5.56 | A12 | 82278 | 3.69 | A19 | 54152 | 5.48 | A27 | 26170 | 6.17 | A36 | 33627 | 5.85 |
| A4 | 43284 | 5.31 | A12 | 82278 | 6.2 | A19 | 54152 | 6.43 | A27 | 26170 | 5.61 | A36 | 33627 | 6.5 |
| A4 | 43284 | 5.82 | A12 | 82278 | 4.54 | A20 | 35233 | 5.04 | A27 | 26170 | 5.36 | A36 | 33627 | 5.65 |
| A5 | 72641 | 5.22 | A13 | 56255 | 5.2 | A20 | 35233 | 5.9 | A27 | 26170 | 5.84 | A37 | 47320 | 6.32 |
| A5 | 72641 | 6.29 | A13 | 56255 | 5.26 | A20 | 35233 | 5.28 | A28 | 34648 | 4.98 | A37 | 47320 | 5.29 |
| A5 | 72641 | 5.97 | A13 | 56255 | 5.61 | A20 | 35233 | 5.84 | A28 | 34648 | 6.35 | A37 | 47320 | 5.6 |
| A5 | 72641 | 5.67 | A13 | 56255 | 5.63 | A20 | 35233 | 5.99 | A28 | 34648 | 5.68 | A37 | 47320 | 5.42 |
| A6 | 38830 | 5.34 | A13 | 93534 | 5.7 | A20 | 35233 | 6.47 | A28 | 34648 | 5.66 | A37 | 47320 | 4.79 |
| A6 | 38830 | 5.5 | A13 | 93534 | 5.41 | A21 | 52757 | 5.12 | A29 | 34949 | 6.18 | A37 | 47320 | 6.23 |
| A6 | 38830 | 5.31 | A13 | 93534 | 4.64 | A21 | 52757 | 5.15 | A29 | 34949 | 6.73 | A38 | 29420 | 5.86 |
| A6 | 38830 | 6.17 | A13 | 93534 | 4.4 | A21 | 52757 | 5.17 | A29 | 34949 | 5.39 | A38 | 29420 | 6.57 |
| A6 | 38830 | 5.75 | A14 | 87170 | 4.34 | A21 | 52757 | 5.48 | A29 | 34949 | 5.33 | A38 | 29420 | 5.7 |
| A7 | 47913 | 5.37 | A14 | 87170 | 4.88 | A21 | 52757 | 5.37 | A30 | 14214 | 6.16 | A38 | 29420 | 5.64 |
| A7 | 47913 | 4.82 | A14 | 87170 | 4.91 | A22 | 78878 | 5.15 | A30 | 14214 | 5.9 | A38 | 29420 | 5.37 |
| A7 | 47913 | 6.28 | A14 | 87170 | 4.71 | A22 | 78878 | 6.24 | A30 | 14214 | 5.62 | A38 | 29420 | 5.46 |
| A7 | 47913 | 5.66 | A14 | 87170 | 6.04 | A22 | 78878 | 5.14 | A30 | 14214 | 5.12 | A39 | 21899 | 5.67 |
| A7 | 47913 | 6.12 | A15 | 21265 | 6 | A22 | 78878 | 4.67 | A31 | 96128 | 3.99 | A39 | 21899 | 5.65 |
| A7 | 47913 | 5.24 | A15 | 21265 | 5.91 | A22 | 78878 | 5.4 | A31 | 96128 | 5.17 | A39 | 21899 | 6.22 |
| A7 | 47913 | 6.26 | A15 | 21265 | 6.32 | A22 | 78878 | 4.74 | A31 | 96128 | 4.87 | A39 | 21899 | 4.95 |
| A8 | 35855 | 5.91 | A15 | 21265 | 5.92 | A23 | 14811 | 5.48 | A31 | 96128 | 4.29 | | | |
| A8 | 35855 | 5.92 | A16 | 22233 | 5.75 | A23 | 14811 | 5.86 | A32 | 35143 | 6.39 | | | |

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# List of Publications

**International Journals**

[1] "Jaya algorithm in estimation of P[X>Y] for two parameter Weibull distribution" published in SI: Optimization Algorithms in Dynamical Systems, *AIMS Mathematics, Volume 7, Issue 2, 2820–2839, 2022* **(SCIE/SCOPUS indexed)**

[2] "Reliability of Stress Strength model for three parameter Weibull distribution using Jaya algorithm" submitted to *Quality and Reliability Engineering International* **(SCIE/SCOPUS indexed)** (Under review)

[3] "Estimation of parameters of Weibull distribution using Jaya Algorithm" submitted to a journal *Indian Journal of Pure and Applied Mathematics* **(SCIE/SCOPUS indexed)** (Under review)

[4] "A novel methodology in developing stress-strength reliability model for Weibull distribution using design of experiments" (Ready for communication)

[5] "Polynomial Regression in evaluating reliability for dynamic strength using Jaya algorithm: An application to SCM 440 steel strength data" (Ready for communication)

**International Conferences**

[1] "Reliability Models of Stress-Strength Interference with Combination of Laplace and Exponential Distribution" at *International Conference in Industrial Engineering (ICIE 2019), Surat, 12-14 December 2019*

[2] "Stress Strength Reliability of time-dependent strength using Jaya algorithm: An application to automotive circuits" at *First Industrial Engineering and Operations Management (IEOM ) India conference, Bangalore 16-18 August 2021* ***(SCOPUS indexed)***