HYERS-ULAM TYPE STABILITY FOR SOME DIFFERENTIAL EQUATIONS

A THESIS SUBMITTED IN PARTIAL FULFILLMENT FOR THE DEGREE

OF

DOCTOR OF PHILOSOPHY

IN THE

SCHOOL OF PHYSICAL AND APPLIED SCIENCES

GOA UNIVERSITY

By

VISHWAS PANDURANG SONALKAR

SCHOOL OF PHYSICAL AND APPLIED SCIENCES

GOA UNIVERSITY

AUGUST 2022

HYERS-ULAM TYPE STABILITY FOR SOME DIFFERENTIAL EQUATIONS

A THESIS SUBMITTED IN PARTIAL FULFILLMENT FOR THE DEGREE

OF

DOCTOR OF PHILOSOPHY

IN THE

SCHOOL OF PHYSICAL AND APPLIED SCIENCES

GOA UNIVERSITY

By

VISHWAS PANDURANG SONALKAR

SCHOOL OF PHYSICAL AND APPLIED SCIENCES

GOA UNIVERSITY

AUGUST 2022

DECLARATION

I, Mr. Vishwas Pandurang Sonalkar hereby declare that this thesis represents work which has been carried out by me and that it has not been submitted, either in part or full, to any other University or Institution for the award of any research degree.

Place: Taleigao Plateau.

Date : 26-08-2022 **Vishwas Pandurang Sonalkar**

CERTIFICATE

I/We hereby certify that the work was carried out under my/our supervision and may be placed for evaluation.

Dr. Y. S.Valaulikar,

Guide,

Ex. Faculty,

Department of Mathematics,

Goa University.

Acknowledgements

I express my sincerest gratitude and profound thanks to Dr. A. N. Mohapatra, for his guidance and for introducing me to this topic for my Ph. D. problem. I express my gartitude and profound thanks to my guide Dr. Y. S. Valaulikar, Ex. Faculty, Department of Mathematics, Goa University, for his persistent guidance, valuable suggestions, encouragement and diligent follow-ups for the completion of this research work.

I am thankful to Dr. M. Kunhanandan, faculty member and Prof. R S. Gad, Vice-Dean, SPAS, for their valuable suggestions during FRC presentations . I am also grateful to Dr. M. Tamba, Program Director, Department of Mathematics, SPAS, Goa University, Goa for his valuable suggestions and encouragement. I thank Prof. K.R. S. Priolkar, Dean, SPAS, Goa University, Goa for his suggestions and guidance.

I express my heartfelt gratitude to Dr. A. J. Jayanthan for his persistent guidance during the period of my study. I am also thankful to the faculty members Dr. Jessica Fernandez and Mr. Brandon Fernandes for their valuable suggestions during the period of this work. I wish to thank Mr. Nanda Gawas, Mrs. Manjiri Khaunte and the other non-teaching staff, for their support and co-operation. I am also grateful to Dr. S. K. Pawar, Head, Department of Mathematics, S. R. M. College, Kudal for interaction and discussion with them from time to time.

I am very thankful to Hon'ble President, H. H. Khemsawant Bhonsle (Rajesaheb), Hon'ble Chairman, H. H. Shubhdadevi Bhonsle (Ranisaheb) and Hon'ble Executive Trustee, H. H. Lakhamsawant Bhonsle (Yuvraj) of the

S. Z. S. P. Mandal's, Sawatwadi for their encouragement for my research. I place on record my sincere thanks to Principal, Dr. D. L. Bharmal and all staff members of S. P. K. Mahavidyalaya, Sawantwadi, for their constant support and inspiration for my Ph. D. work.

I find no words to thank my members of the family, my Father Pandurang Ganapati Sonalkar, Mother Lt. Malubai Pandurang Sonalkar, brothers Vijay and Sarjerao Sonalkar, wife Mrs. Shamal Sonalkar and daughter Radhika Sonalkar, for their incessant support and boosting my morale during my research work.

During the course of my study, I have encountered support and encouragement from many at varying stages, in various forms. I am greatly indebted to all these people whose names may not feature but their kindness, friendship, moral support and enthusiasm gave me the grace to reach here.

Finally, my sincere thanks are due to the authorities of Goa University, for providing the necessary facilities to carry out this research work.

This thesis is dedicated to my parents.

Contents

Chapter 1

INTRODUCTION

The study of differential Equations is a broad area of mathematical research. It has contributed significantly to the growth of Mathematics and its uses, particularly for the advancement of Physics, Chemistry and Engineering. From last two centuries, differential equations have been widely used in Biology, Information Technology, Social Sciences and Economics, to mention few outside the group of physical sciences. Different real world problems can be converted into mathematical problems (model) and analysed. Most of the mathematical models contain differential equations which are mainly non-linear and cannot be solved analytically. In the absence of method of solving these equations, the study of properties of solutions, without obtaining it, attains importance. Hence it is seen that one studies properties such as boundedness, oscillatory behavior, periodicity, stability, etc. of the solutions.

In 1940, the study of stability problems for various functional equations was sparked by the famous talk presented by Stanislaw Marcin Ulam [62], at the Mathematics Club of the University of Wisconsin. In this talk he discussed a number of important unsolved problems. Among them was one concerning the stability of group homomorphism.

In 1941, D.H. Hyers [19], solved this problem of Ulam for the case where G_1 and G_2 are Banach spaces. This gives a partial solution to Ulam's question in terms of stability of linear functional equations and opened up new avenues for research in the field of functional equations and differential equations. The result in [19], is now known as Hyers-Ulam (HU) stability for the additive Cauchy equation $f(x+y) = f(x) + f(y)$.

The various extensions of HU stability has been named with additional word. One such extension is Hyers-Ulam-Rassias (HUR) stability, published in 1978 by Th. M. Rassias [55]. In this paper the condition for stability is weakened and result is proved for Cauchy difference equation by making use of a direct method. After this remarkable HUR stability result, many mathematicians have looked into stability results for other types of differential equations. Since then, many results on HU and HUR stability of various functional equations, linear and non-linear ordinary differential equations, linear and non-linear partial differential equations, delay differential equations, fractional differential equations etc. have been studied. In 1993, Marta Obloza seems to be the first researcher who investigated the HU stability of linear differential equation.

E. Ahmed et. al. [2], focused on HU stability applications in Biology and Economics. It is important to notice that there are many applications for HU stability in other topics e.g. in nonlinear analysis problems including differential equations and integral equations.

1.1. RESEARCH OBJECTIVE

Differential equations are used in many different areas of study, including physical and applied sciences. In most of the cases, these equations are non-linear and cannot be solved analytically. If there is no way to solve these equations, it is important to study the properties of the solution without obtaining it. One of the important property is stability of the solutions of these equations. The study of stability of solutions of the differential equations has developed considerably in past few decades. This stability theory of differential equations have motivated to take up the study on HU and HUR stability for differential equations.

The objective of this research work is to study HU and HUR stablility of different types of non-linear ordinary differential equations, linear partial differential equations and non-linear partial differential equations.

Our objectives are :

To study

- HU and HUR stablility of different linear and non-linear ordinary differential equations.
- HU and HUR stablility of different linear and nonlinear partial differential equations.
- Generalised HUR stability of different non-linear partial differential equations, by using various techniques.

1.2. ORGANIZATION OF THE THESIS

The thesis is divided into seven chapters. The outline of these chapters is as follows :

1. The first chapter gives an introduction to the topic and motivation for taking up the study. We give the objective of our work and chapter wise description of the work done.

2. Chapter 2 deals with the survey of the available literature on HU and HUR stability of different types of equations such as functional equations, difference equations and differential equations. The survey is divided into two parts. The first section gives the brief survey of the literature. Second section give some basic results used for our work.

3. Chapter 3 is devoted to the study of HUR stability of third order ordinary differential equation. In this chapter we study the HUR stability of following third order ordinary differential equation

   $$y'''(x) + p(x)y''(x) + q(x)y'(x) + r(x)y(x) = f(x), \tag{1.1}$$

   where $y \in C^3[a,b], p,q,r,f \in C[a,b]$ and $-\infty < a < b < \infty$.

   This stability result is proved by imposing certain integrability conditions on the coefficients functions $p(x), q(x)$ and $r(x)$ of the differential equation. An example is provided in support of the result.

4. In chapter 4, we establish the HUR stability for first and third order linear homogeneous partial differential equations. Further, we extend the method

to obtain the result for n^{th} order linear homogeneous partial differential equation of the form :

$$\frac{\partial u}{\partial t} = a^n \frac{\partial^n u}{\partial x^n}, \qquad t > 0, 0 < x < l, a > 0. \tag{1.2}$$

These results are proved by employing Laplace transform method and using the idea presented in [54].

5. Chapter 5 is on HUR stability of linear non-homogeneous partial differential equations. Here we prove the HUR stability of the second order partial differential equation of the type

$$\begin{aligned} r(x,t)u_{tt}(x,t) + p(x,t)u_{xt}(x,t) + q(x,t)u_t(x,t) + p_t(x,t)u_x(x,t) \\ -p_x(x,t)u_t(x,t) = g(x,t,u(x,t)). \end{aligned} \tag{1.3}$$

Further in the chapter, we have established the HUR stability for the third order non-homogeneous partial differential equation:

$$\begin{aligned} s(x,t)u_{ttt}(x,t) + r(x,t)u_{tt}(x,t) + p(x,t)u_{xt}(x,t) + q(x,t)u_t(x,t) \\ +p_t(x,t)u_x(x,t) - p_x(x,t)u_t(x,t) = g(x,t,u(x,t)). \end{aligned} \tag{1.4}$$

These results are proved by using Banach contraction principle and some results in [18].

6. The Chapter 6 deals with the HU stability of the nonlinear ordinary and partial differential equations :

$$u_x(x,t) + K(x,u(x,t)) = 0, \tag{1.5}$$

$$u_{xx}(x,t) + F(x,u)u_x(x,t) + H(x,u) = 0. \tag{1.6}$$

These results are proved by employing Banach's contraction principle.

Further in this chapter, we have established the HU stability for the second order non-linear ordinary and partial differential equations :

$$u_{xx}(x,t) = f(x,t,u(x,t),u_x(x,t)), \ \ 0 \leq x \leq a, 0 \leq t \leq b. \tag{1.7}$$

and

$$u_{xt}(x,t) = f(x,t,u(x,t),u_x(x,t)) \ \ 0 \leq x \leq a, 0 \leq t \leq b. \tag{1.8}$$

These results are proved by employing Grownwall type inequality and some integral inequalities.

7. Finally in Chapter 7, we discuss the generalised HUR stability of the following second order non-linear ordinary partial differential equation :

$$u_{xx}(x,t) = f(x,t,u(x,t),u_x(x,t)). \tag{1.9}$$

Then, we establish the generalised HUR stability for the following second order non-linear partial differential equation.

$$u_{xt}(x,t) = f(x,t,u(x,t),u_x(x,t),u_t(x,t),u_{xx}(x,t)). \tag{1.10}$$

These results are proved by employing Grownwall type inequality, some integral inequalities and using the result in [41].

At last, we give a brief summary of the results obtained in this thesis and present some problems for further study. The thesis ends with a complete bibliography.

Chapter 2

SURVEY OF LITERATURE

2.1 INTRODUCTION

This chapter deals with the survey of the literature on Hyers-Ulam (HU) and Hyers-Ulam-Rassias (HUR) type stability of various linear, nonlinear ordinary and partial differential equations. HU stability of differential equations has drawn much attention since Ulam's presentation [62] of the problem on stability of group homomorphism in 1940. Its various extensions have been named with additional word/s. One such extension is HUR stability. We plan to give review of HU and HUR stability results of ordinary and partial differential equations in section 1. Section 2 contains some basic theorems (results) used for the work undertaken.

2.2 REVIEW OF LITERATURE

In 1940, Stanislaw Marcin Ulam [62], presented a wide ranging talk to the Mathematics Club of the University of Wisconsin, where he discussed a number of important unsolved problems. Among them was one concerning the stability of group homomorphism, namely:

Let G_1 be a group and G_2 be a metric group with a metric d. For a given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$, for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x), H(x)) < \varepsilon$, for all $x \in G_1$?

In 1941, D.H. Hyers [19] solved this problem of Ulam for the case where G_1 and G_2 are Banach spaces. He proved that, if $f : G_1 \longrightarrow G_2$ be such that $f(tx)$ is a continuous function in t for each fixed $x \in G_1$, where $t \in (-\infty, \infty)$ and if $\exists\ \varepsilon > 0$ such that $||f(x+y) - f(x) - f(y)|| \leq \varepsilon$, for all $x, y \in G_1$, then there exists a linear map $T : G_1 \longrightarrow G_2$ such that $||f(x) - T(x)|| \leq \delta$, for all $x \in G_1$.

This is now known as Hyers-Ulam (HU) stability for the additive Cauchy equation $f(x+y) = f(x) + f(y)$.

In 1945, D. H. Hyers and S. M. Ulam [20], established the existence of a true isometry $U(x)$, which approximates given ε-isometry $T(x)$. An ε-isometry is a transformation $T(x) : E \to E'$ such that, $|\rho(x, y) - \rho(T(x), T(y))| < \varepsilon$, for some $\varepsilon > 0$, where E and E' are metric spaces. Precisely, they proved the existence of a constant $k > 0$ depending only on the metric spaces E and E' such that $||(T(x), U(x))|| < k\varepsilon$, for all $x \in E$.

In 1978, Th. M. Rassias [55] weakened the condition for Cauchy difference equation and proved the following result by making use of a direct method.

Theorem 2.1.1 : [55]

Consider G_1 and G_2 to be two Banach spaces, and let $f : G_1 \longrightarrow G_2$ be a mapping such that $f(tx)$ is a continuous function in t for each fixed x. Assume that there exist $\varepsilon \geq 0$ and $p \in [0, 1)$ such that $||f(x+y) - f(x) - f(y)|| \leq \varepsilon(||x||^p + ||y||^p)$, for all $x,y \in G_1$. Then there exists a unique linear mapping $T : G_1 \longrightarrow G_2$ such that $||f(x) - T(x)|| \leq \frac{2\varepsilon}{2-2^p}(||x||^p)$, for any $x \in G_1$.

This stability criterion is now called Hyers-Ulam-Rassias (HUR) stability. This result of Rassias stability has influenced a number of Mathematicians to investigate the stability problems for various functional equations (see [7], [19],[23],[25], [31],[33],[36],[38],[40],[56] and [59]). Since then, this type of stability criteria have been applied to various other kinds of equations (see Jung [33], Sahoo and Khannapan [59]).

S. Czerwik [15], established the following result of the HUR stability of quadratic functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$, by applying fixed point Method.

Theorem 2.1.2 : [15]

Let E_1 and E_2 be a normed space and a Banach space respectively. If there is a function $f : E_1 \rightarrow E_2$ satisfying the inequality

$||f(x+y)+f(x-y)-2f(x)-2f(y)|| \leq \delta + \theta((||x||^p + ||y||^p)$, for some $\delta, \theta \geq 0$ and $p < 2$ and for all $x,y \in E_1 \setminus \{0\}$, then there exists a unique quadratic function $Q : E_1 \rightarrow E_2$ such that $||f(x) - Q(x)|| \leq (\frac{1}{3})(\delta + c) + 2(4-2^p)^{-1}\theta||x||^p$, for any $x \in E_1 \setminus \{0\}$, where $c = ||f(0)||$.

Next, we have the following result giving the direct method.

Theorem 2.1.3 : [8]

Let a and b be nonnegative real numbers with $\alpha = a+b > 0$. Let $H : [0,\infty)^2 \rightarrow [0,\infty)$ be a function, for which, there exists a positive number $k < \alpha$ such that

$H(\alpha s, \alpha t) \leq \alpha H(s,t)$, for all $s,t \in [0,\infty)$. Given a real normed space E_1 and a real Banach space E_2, assume that a function $f : E_1 \rightarrow E_2$ satisfies the inequality

$||f(ax+by) - af(x) - bf(y)|| \leq H(||x||, ||y||)$, $\forall x,y \in E_1$. Then there exists a unique function $A : E_1 \to E_2$ such that $A(ax+by) = aA(x)+bA(y)$, for any x and y in E_1 and $||f(x)-A(x)|| \leq (\alpha-k)^{-1}H(||x||,||x||)$, $\forall x \in E_1$.

This result was proved by using additive function A which is explicitely constructed from the given function f by $A(x) = \lim_{n\to\infty} 2^n f(2^{-n}x)$. This method is known as direct method. It's an important and the most powerful tool for study of stability for various functional equations.

L. Losonczi [40] has proved the stability of the Hosszu's functional equation $f(x+y-xy) = f(x)+f(y)-f(xy)$. Result is given below.

Theorem 2.1.4 : [40]

Let Y be a Banach space and suppose that a function $f : \mathbb{R} \to Y$ satisfies the functional inequality $||f(x+y-xy)-f(x)-f(y)+f(xy)|| \leq \varepsilon$, for some $\varepsilon > 0$ and for all $x, y \in \mathbb{R}$. Then there exist a unique additive function $A : \mathbb{R} \to E$ and a unique constant $b \in Y$ such that $||f(x)-A(x)-b|| \leq 20\varepsilon$, for all $x \in \mathbb{R}$.

This result is proved by using local stability, in HU sense, for Cauchy's functional equation. Further paper contains a local stability theorem for additive functions in Banach space settings.

In 1997, S. M. Jung [23] has proved some results on HU and HUR stability of the gamma functional equation. Following results are from [23].

Theorem 2.1.5 :

If a mapping $f : (0,\infty) \to \mathbb{R}$ satisfies the inequality $|f(x+1)-xf(x)| \leq \delta$, $\delta > 0$, $\forall x > n_0$, n_0 is a given non-negative integer, then there exists a unique solution $F : (0,\infty) \to \mathbb{R}$ of the gamma functional equation $f(x+1) = xf(x)$ with $|F(x)-f(x)| \leq \frac{3\delta}{x}$, $\forall x > n_0$.

In the following theorem, let $\delta, \varepsilon > 0$ be given and $\alpha(x), \beta(x)$ be functions defined as follows:

$$\alpha(x) = \Pi_{i=0}^{\infty}[1-\delta(x+i)^{-(1+\varepsilon)}],\ \beta(x) = \Pi_{i=0}^{\infty}[1+\delta(x+i)^{-(1+\varepsilon)}],$$

for any $x > \delta^{1/(1+\varepsilon}$, and $n_0 \in \mathbb{Z}$.

Theorem 2.1.6 :

If a mapping $f:(0,\infty)\to(0,\infty)$ satisfies the inequality $|\frac{f(x+1)}{xf(x)}-1| \leq \frac{\delta}{x^{1+\varepsilon}}$, $\forall x > n_0$, then there exists a unique solution $F:(0,\infty)\to[0,\infty)$ of the gamma functional equation with $\alpha(x) \leq F(x)/f(x) \leq \beta(x)$, for any $x > \max\{n_0, \delta^{1/(1+\varepsilon)}\}$.

This is called general or modified HUR stability.

In [24], Jung has investigated the stability problem of the quadratic functional equation of Pexider type viz. $f(x+y)+f(x-y) = 2f(x)+2f(y)$. This result generalizes the result in [15]. Using the same ideas, Jung and Sahoo [35] proved the HU stability of a quadratic functional equation of Pexider type, viz. $f_1(x+y)+f_2(x-y) = f_3(x)+f_4(y)$.

Next, Jung has discussed HU stability of logarithmic functional equation $f(x+y)-f(x)-f(y) = f(x^{-1}+y^{-1})$. Following is the result.

Theorem 2.1.7 : [25]

If a function $f:\mathbb{R}\to\mathbb{R}$ satisfies the functional inequality $|f(x+y)-f(x)-f(y)-f(x^{-1}+y^{-1})| \leq \delta$, for some $0 \leq \delta < log(2)$ and for all $x,y \in R\setminus\{0\}$, then there exists a unique logarithmic function $l: R\setminus\{0\}\to\mathbb{R}$ such that $|f(x)-l(x)| \leq 5\delta - \frac{11}{2}\,log(2-e^{\delta})$, for each $x \in R\setminus\{0\}$.

In 2001, J. Chmieli'nski and S. M. Jung [13], established the HU Stability of the Wigner Equation $|<f(x)|f(y)>| = |<x|y>|$, for all $x,y \in E$, where E is a real or complex Hilbert space with the inner product and the associated norm denoted by $<\cdot|\cdot>$ and $||\cdot||$ respectively. Let $1 \neq c > 0$ and $d \geq 0$ be given constants. D be the set defined by

$$D = \begin{cases} \{x \in E : ||x|| \geq d\}, & \text{if } 0 < c < 1, \\ \{x \in E : ||x|| \leq d\}, & \text{if } c > 1. \end{cases}$$

The case $D = \{0\}$ is trivial and not considered. Let $\phi : E \times E \to [0, \infty)$ be a function satisfying the property

$$\lim_{m+n\to\infty} c^{m+n}\phi(c^{-m}x, c^{-n}y) = 0, \forall x, y \in D.$$

We have the following result from [13].

Theorem 2.1.8 :

If $f : E \to F$ satisfies the property $|| < f(x)|f(y) > | - | < x|y > || \leq \phi(x,y)$, for all $x, y \in D$, then there exists a unique (up to a phase equivalent function) mapping $I : E \to F$ satisfying the Wigner equation and such that $||f(x) - I(x)|| \leq \sqrt{\phi(x,x)}, \forall x \in D$.

Next, we have a result that gives HUR stability of the Beta functional equation by construction of a Cauchy Sequence.

Theorem 2.1.9: [38]

Let $F : (0,\infty) \times (0,\infty) \to (0,\infty)$ be a mapping that satisfies the inequality $| \frac{xy}{(x+y)(x+y+1)} \frac{F(x,y)}{F(x+1,y+1)} - 1 | \leq \ \psi(x,y),$ for all $x, y > n_0$, where $n_0 \in \mathbb{Z}$ and $\psi : (0,\infty) \times (0,\infty) \to (0,1)$ is a mapping such that

$\alpha(x,y) := \sum_{i=0}^{\infty} log(1 - \psi(x+i, y+i))$ and $\beta(x,y) := \sum_{i=0}^{\infty} log(1 + \psi(x+i, y+i))$

are bounded for $x, y > n_0$. Then there exists a unique solution $T : (0,\infty) \times (0,\infty) \to (0,\infty)$ of the beta functional equation

$$F(x+1, y+1)^{-1} = \frac{(x+y)(x+y+1)}{xy} F(x,y)^{-1},$$

with $\quad e^{\alpha(x,y)} \leq \frac{F(x,y)}{T(x,y)} \leq e^{\beta(x,y)}.$

In 2002, G. H. Kim, Bing XU and W. Zhang [37], proved the generalized HUR stability of the generalized gamma functional equation $g(x+p) = a(x)g(x)$, by using Ratio Test and under some conditions of convergence of the series. They proved the following result.

Theorem 2.1.10 : [37]

Consider the approximate solution $f:(0,+\infty)\to\mathbb{R}$ of $g(x+p)=a(x)g(x)$, which satisfy the inequality $|f(x+p)-a(x)f(x)|\leq\Psi(x)$, for all $x>n_0$, where $\Psi:(0,+\infty)\to(0,+\infty)$ is a fixed function and n_0 is a non-negative constant. If $\underset{k\to\infty}{lim\,inf}\ \frac{\Psi(x+p(k-1))}{\Psi(x+pk)}\times a(x+pk)>1$, for all $x>n_0$, then the equation $g(x+p)=a(x)g(x)$ has the generalized Hyers-Ulam-Rassias stability.

In 2006, S. M. Jung and P. K. Sahoo [36], established the HUR type stability for a Davison functional equation $f(xy)+f(x+y)=f(xy+x)+f(y)$, for a class of functions from a Ring into a Banach space. They proved the result by using the Direct method.

In the following result, L. C Adariu and V. Radu [11], proved the stability of Jensen's functional equation. This result was proved by using fixed point Method.

Theorem 2.1.11 : [11]

Let E_1 and E_2 be a (real or complex) vector space and a Banach space, respectively. Assume that a function $f:E_1\to E_2$ satisfies $f(0)=0$ and the inequality $||2f(\frac{x+y}{2})-f(x)-f(y)||\leq\phi(x,y)$, for all $x,y\in E_1$, where $\phi:E_1^2\to[0,\infty)$ is a given function. Moreover, assume that there exists a positive constant $L<1$ such that $\phi(x,0)=Lq_i\phi(\frac{x}{q_i},0)$, where $q_0=2$ and $q_1=\frac{1}{2}$. If ϕ satisfies $\underset{n\to\infty}{lim}\ q_i^{-n}\phi(q_i^nx,q_i^ny)=0$, for all $x,y\in E_1$, then there exists a unique additive function $A:E_1\to E_2$ such that, $||f(x)-A(x)||\leq\frac{L^{1-i}}{1-L}\phi(x,0)$, for any $x\in E_1$.

In 2009, S. M. Jung [31] proved the HU stability of generalized Fibonacci functional equation $f(x)=pf(x-1)-qf(x-2)$, in the class of functions $f:\mathbb{R}\to X$, where X is a real (or complex) Banach space.

Theorem 2.1.12 : [31]

If a function $f:\mathbb{R}\to X$ satisfies the inequality $||f(x)-pf(x-1)+qf(x-2)||\leq\varepsilon$, for all $x\in\mathbb{R}$ and for some $\varepsilon\geq 0$, then there exists a unique solution function $F:\mathbb{R}\to X$ of the functional equation $f(x)=pf(x-1)-qf(x-2)$ such that $||f(x)-F(x)||\leq\frac{|a|-|b|}{|a-b|}\frac{\varepsilon}{(|a|-1)(1-|b|)}$, for all $x\in\mathbb{R}$.

Note that the estimate obtained in above theorem is better than the estimate obtained for the equations of higher order in [9]. In 2011, P. K. Sahoo and P. Kannapan [59], proved the result for the HU stability of Abel functional equation $f(x+y)=g(xy)+h(x-y)$. We state the result below.

Theorem 2.1.13 : [59]

If functions $f,g,h:\mathbb{R}\to\mathbb{R}$ satisfy the functional inequality $|f(x+y)-g(xy)$ $-h(x-y)|\leq\varepsilon$, for some $\varepsilon\geq 0$ and for all $x,y\in\mathbb{R}$, then there exists a unique additive function $A:\mathbb{R}\to\mathbb{R}$ such that $|f(x)-A(\frac{x^2}{4})-f(0)|\leq 22\varepsilon$, $|g(x)-A(x)-f(0)+h(0)|\leq 21\varepsilon$ and $|h(x)-A(\frac{x^2}{4})-h(0)|\leq 22\varepsilon$, for all $x\in\mathbb{R}$.

Now we shall have a look at the results on HU type stability of differential equations. HU stability for differential equations is defined as follows.

Definition 2.1.1 : The differential equation

$$\phi(f(t),y(t),y'(t),\cdots,y^{(n)}(t))=0 \tag{2.1}$$

is said be Hyers-Ulam stable on an interval I, if for a given $\varepsilon>0$ and a function y_1 such that

$$|\phi(f(t),y_1(t),y_1'(t),\cdots,y_1^{(n)}(t))|\leq\varepsilon, \tag{2.2}$$

there exist a solution y_2 of (2.1) such that

$$|y_1(t)-y_2(t)|\leq k(\varepsilon), \tag{2.3}$$

where $k(\varepsilon) \to 0$ as $\varepsilon \to 0$ and does not depend on y_1 and y_2.

A differential equation (2.1) is said to be Hyers-Ulam-Rassias stable with a preassigned function $\varepsilon(t)$, if ε is replaced by $\varepsilon(t)$ in the inequality (2.2) and $k(\varepsilon)$ is replaced by $\psi(t)$ in the inequality (2.3)

In a similar way one can define HU and HUR stablility for partial differential equation,

$$\phi(g(\underline{x}), u(\underline{x}), u_{x_1}(\underline{x}), u_{x_2}(\underline{x}), \cdots, u_{x_n}(\underline{x}), u_{x_1x_2}(\underline{x}), \cdots, u_{x_1x_2\cdots x_n}(\underline{x})) = 0,\ \underline{x} = (x_1, \ldots, x_n)$$

on a domain $\Omega \subseteq R^n$.

M. Obloza (see [48], [49]) seems to be the first researcher who investigated the HU stability of linear differential equations in 1993. The author has established the following results.

Theorem 2.1.14 : [48]

Let $I \subset R$ be a bounded interval and suppose that g is a continuous real valued function defined on I such that $\int_I |g(t)|dt < \infty$. Then the equation $x'(t) + g(t)x(t) = p(t)$ is stable in sense of Hyers, where p is a continuous real valued function defined on interval I.

In the same paper [48], following two results are proved.

Theorem 2.1.15 :

Let I denote the interval $[A, +\infty)$, for some $A > 0$. Suppose that there exits $C > 0$ and $T \leq A$ such that $g(t) \leq C$, for $t \geq T$. Then the equation $x'(t) + g(t)x(t) = p(t)$ is stable in the sense of Hyers.

Theorem 2.1.16 :

Let $I = [A, +\infty)$, for some $A > 0$. Suppose that there exits a positive number $T \geq A$ such that $g(t) \leq \frac{1}{\sqrt{t}}, \forall t \geq T$. Then the equation $x'(t) + g(t)x(t) = p(t)$ is stable in the sense of Hyers.

Further in 1997, Obloza established the connection between HU stability and Lyapunov stability. Following results are proved in [49].

Theorem 2.1.17 :

Let $\delta > 0$ and let us assume the following:

(i) $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function Lipschitzian with respect to the second variable with constant L .

(ii) x_1, x_2 are solutions of equation $x'(t) = f(t, x(t))$ defined on $\mathbb{R}$.

(iii) There exists $\tau \in \mathbb{R}$ such that $|x_1(\tau) - x_2(\tau)| \leq \delta$, then there exists a $d > 0$ (independent of δ)such that the inequality $|x_1(t) - x_2(t)| \leq 2\delta$ holds for all $t \in [\tau - d, \tau + d]$. This result is used to prove the following:

Theorem 2.1.18 :

Let us assume the following:

(i) $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function Lipschitzian with respect to the second variable.

(ii) The equation $x' = f(t, x), f \in (C, Lip)$ is stable in the sense of Hyers.

Then it is stable in the sense of Lyapunov.

The converse of the above theorem is not true and an illustrative example is given in the same paper. In 1998, Alsina and Ger (see [5]), proved the stability for first order linear differential equation . They proved the following result.

Theorem 2.1.19 :

Given $\varepsilon > 0$, if $f : I \to R$ is a differentiable function satisfying $|f'(x) - f(x)| \leq \varepsilon$, for all $x \in I$, then there exists a function g satisfying $g' = g$ such that $|f(x) - g(x)| \leq 3\varepsilon$, for all $x \in I$.

After this paper ([5]), many researchers have investigated the HU and HUR stability of different types of differential equations. For more results on HU and HUR stability of ordinary and partial differential equations, one can refer [[1], [14], [16], [17], [26], [27], [28], [29], [32], [34], [43], [44], [45], [46], [47], [52], [53], [54], [55], [57] and references therein].

In 2002, Takahasi et al. [60] extended the work for Banach space valued differential equation $y'(t) = \lambda y$ over an interval I. Defining $m(I,\lambda) = \inf\{e^{-Re\lambda t}; t \in I\}$ and $M(I,\lambda) = \sup\{e^{-Re\lambda t}; t \in I\}$, their main result is stated as follows:

Theorem 2.1.20 : [60]

For $\varepsilon > 0$ and $\phi : I \to X$, a strongly differentiable function such that $||\phi'(t) - \lambda\phi(t)|| \leq \varepsilon$, for all $t \in I$, following assertions are true :

a) If $Re\ \lambda \neq 0$, then there exists an element $x_\phi \in$ X such that $||\phi(t) - e^{\lambda t}x_\phi|| \leq |Re\ \lambda|^{-1}\{1 - \frac{m(I,\lambda)}{M(I,\lambda)}\}\varepsilon$, for all $t \in I$. In particular, if $m(I,\lambda) = 0$, then x_ϕ with the property that $||\phi(t) - e^{\lambda t}x_\phi|| < \infty$ is unique.

b) If $Re\ \lambda = 0$ and the diameter $\delta(I)$ of I is finite, then there exists an $x_\phi \in X$ such that $||\phi(t) - e^{\lambda t}x_\phi|| \leq \varepsilon\delta(I)$, for all $t \in I$.

c) If $Re\ \lambda = 0$ and $\delta(I) = \infty$, then the HU stability of the differential equation $y' = \lambda y$ does not hold.

In 2003 , the above result was extended to Banach space valued first order differential equation of the form $u' + hu = v$. They proved the following result.

Theorem 2.1.21 : [43]

Let $h:\mathbb{R}\to C$ be a continuous function and $T_h : C^1(\mathbb{R},X)\to C(\mathbb{R},X)$ be the linear operator defined by $(T_hu)(t) = u'(t)+h(t)u(t)$, for all $u\in C^1(\mathbb{R},X)$ and $t\in\mathbb{R}$. Suppose that one of $C_h \overset{def}{=} \sup_{t\in\mathbb{R}} \frac{1}{|\tilde{g}(t)|}\int_t^{\infty}|\tilde{g}(s)|ds$, $D_h \overset{def}{=} \sup_{t\in\mathbb{R}} \frac{1}{|\tilde{h}(t)|}\int_{-\infty}^{t}|\tilde{h}(s)|ds$ and $E_h \overset{def}{=} \sup_{t\in\mathbb{R}} \frac{1}{|\tilde{h}(t)|}|\int_0^t|\tilde{h}(s)|ds|$ is finite. Then T_h has the Hyers - Ulam stability with Hyers Ulam stability constants C_h, D_h and E_h respectively. Moreover if, $C_h<\infty$ or $D_h<\infty$ then for each $v\in C(\mathbb{R},X)$ and $u\in C^1(\mathbb{R},X)$ satisfying $||T_h(u)-v||_\infty<\infty$, there exists an element $u_0\in C^1(\mathbb{R},X)$ with the condition $T_hu_0 = v$ and $||u-u_0||_\infty < \infty$ is uniquely determined.

In 2003, T. Miura, S. Miyajima and S. Takahasi [43], proved the converse of the above theorem. They have proved the following.

Theorem 2.1.22 :

Let h: $\mathbb{R}\to\mathbb{C}$ be a complex valued continuous function and let $T_h : C^1(\mathbb{R},X)\to C(\mathbb{R},X)$ be the linear operator defined by $(T_hu)(t) = u'(t)+h(t)u(t)$, for all $u\in C^1(\mathbb{R},X)$ and $t\in\mathbb{R}$. Suppose that T_h has the Hyers - Ulam stability. Then the following assertions held.

a) If $\inf_{t\in(0,\infty)}|\tilde{h}(t)| = 0$, then $C_h<\infty$. Moreover, C_h is the HUS constant for T_h.

b) If $\inf_{t\in(-\infty,0)}|\tilde{h}(t)| = 0$, then $D_h<\infty$. Moreover, D_h is the HUS constant for T_h.

c) If $\inf_{t\in\mathbb{R}}|\tilde{h}(t)|>0$, then $E_h<\infty$.

d) Either $\inf_{t\in(-\infty,0)}|\tilde{h}(t)|$ or $\inf_{t\in(0,\infty)}|\tilde{h}(t)|$ is positive.

In [61], the authors completely characterised the HU-stability of the above first order linear Banach space valued differential equation in terms of C_h, D_h and E_h. HU stability of linear differential equation of second order with constant coefficients for compact intervals were studied by Y. Li. et al. [39]. HU stability for Banach space valued n^{th} order linear differential equations with constant coefficients was studied by Cimpean and Popa [14] by using the arguments provided in [39]. In this paper authors obtained stability of the linear differential equation in Aoki -Rassias sense. We state the definition.

Definition 2.1.2 : Let $(X,||.||)$ be a Banach space over $\mathbb{C}$ and $I=(a,b)$, $a,b\in\mathbb{R}\cup\{\pm\infty\}, a<b$ and $\phi: I\to[0,\infty)$ be a given mapping. The equation $y^{(n)}(x)-\sum_{j=0}^{n-1}a_jy^{(j)}(x)=f(x), x\in I$ is said to be stable in Aoki-Rassias sense if there exists a mapping $\psi: I\to[0,\infty)$ such that for every function $y\in C^n(I,X)$, satisfying the relation $||y^{(n)}(x)-\sum_{j=0}^{n-1}a_jy^{(j)}(x)-f(x)||\leq\phi(x)$, $\forall x\in I$, there exists a solution $u\in C^{(n)}(I,X))$ of the equation such that $||y(x)-u(x)||\leq\Psi(x), \forall x\in I$.

Note that when ϕ and ψ are constants, the equation is said to be HU stable. Here ψ depends on ϕ. As a consequence we obtain the HU stability of the equation. A connection with the dynamical system is established.

Let $\Re(\lambda)$ denote the real part of the complex number λ and let $f\in C(I,X)$. Define $L_\lambda(h)(x)=\begin{cases} e^{\Re(\lambda)x}\int_x^b h(t)e^{-\Re(\lambda)t}\,dt, \text{ if } e^{\Re(\lambda)x}\geq 0, \\ e^{\Re(\lambda)x}\int_a^x h(t)e^{-\Re(\lambda)t}\,dt, \text{ if } e^{\Re(\lambda)x}<0, \end{cases}$ for every h for which integrals converge.

The main result is given below:

Theorem 2.1.23 : [14]

Let $\varepsilon: I\to[0,\infty)$ be a continuous function and suppose that $L_{r_k}oL_{r_{k-1}}o\cdots oL_{r_1}(\varepsilon)$ are integrable on every interval $[c,b)$ if $R(r_{k+1})\geq 0$, respectively on every interval $(a,c]$ if $R(r_{k+1})<0, 1\leq k\leq n-1$. Then for every mapping $y\in C^n(I,X)$ satisfying the inequality $||y^{(n)}(x)-\sum_{j=0}^{n-1}a_jy^{(j)}(x)-f(x)||\leq\varepsilon(x)$, $\forall x\in I$, there exist a unique solution $u\in C^n(I,X)$ of equation $y^{(n)}(x)-\sum_{j=0}^{n-1}a_jy^{(j)}(x)=f(x)$ $\forall x\in I=(a,b)$ such that $||y(x)-u(x)|| \leq L_{r_n} o L_{r_{n-1}} o \cdots o L_{r_1}(\varepsilon)(x), \forall\, x \in I.$

Jung [28] proved the HUR stability for first order non-homogeneous linear differential equation with variable coefficients, where the coefficient function satisfy certain integrability condition. These ideas were applied to second order non-homogeneous equations by Javadian et al. [22]. This was then extended to n^{th} order linear equations in [21]. Also using Grownwall's inequality HU stability for second order linear differential equations was established in [4].

HUR stability for linear differential operators of n^{th} order with non constant coefficients were studied in [51] and [45]. In 2012, D. Popa and I. Rosa [51], proved the HUR stability under the assumption that the n^{th} order equation can be factorized into n first order equations [12]. This factorisation methods could be applied to non linear equations such as Riccati and Lienard equations.

In [52], the authors proved the following HU stability results.

Theorem 2.1.24 : [52]

Suppose that $|\alpha(x)| \leq L < 1,$ for all $x \geq x_0$ and that

$y \in C^2(I)$, $I = [x_0, x] \subseteq \mathbb{R}$, $x_0 > 0$. is such that it satisfies the inequality

$|y'' + y - \alpha(x)y| \leq \varepsilon$, with the initial conditions $y(x_0) = 0 = y'(x_0)$.

Then the equation $y'' + y = \alpha(x)y$ has the HU stability.

In the same paper, they have obtained the following result.

Theorem 2.1.25 : [52]

Let $I = [x_0, x] \subseteq \mathbb{R}$, $x_0 > 0$. Suppose $|h(x)| \leq A$, for all $x \geq x_0$ and that $y \in C^2(I)$, is such that it satisfies the inequality $|y'' + y - h(x)y^{\beta}| \leq \varepsilon$, $\beta \in (0,1)$ with the initial conditions $y(x_0) = 0 = y\prime(x_0)$. If $A < \frac{(\beta+1)}{2}\{\max_{x \geq x_0} |y(x)|\}^{-\beta}$, for $x \geq x_0$, then the equation $y'' + y = h(x)y^{\beta}$, $\beta \in (0,1)$, has the HU stability.

Using this result the authors have proved the HU stability for the nonlinear equation

$z'' + pz' + qz = h(x)z^{\beta} e^{(\frac{\beta-1}{2})} e^{\int p(t)dt}$, $\beta \in (0,1)$,

where $h, p \in C^1(I)$, $p(x) > 0$, $x_0 > 0$, $q \in C(I)$, $I = [x_0, x]$ and h is bounded for all sufficiently large $x \in \mathbb{R}$.

The results in this paper are supported by illustrations. Some special case of the equation under study is discussed.

In [3], the authors proved the HUR stability of linear differential equation of second order and a nonlinear differential equation of second order with initial condition. They prove these results by using same arguments as used in [52].

In [53], the authors proved the HU stability of nonlinear differential equation

$y'' - F(x, y(x)) = 0$ on $[a, b]$, with initial condition at $y(a) = 0$. Here the $\varepsilon-$approximate solution z is assumed to satisfy

$|F(x, z(x))| \leq A|z|^{\alpha}$, $\alpha > 0$, $|z(x)| \leq |z'(x)|$ and $0 < A < \{\max_{x\in[a,b]} |z(x)|\}^{1-\alpha}$.

The idea used here is similar to that used in [52]. Using this they proved the HU stability of Emden - Fowler nonlinear differential equation with zero initial conditions at $x = a$. Following is the result.

Theorem 2.1.26 : [53]

Suppose that $z : [a, b] \to \mathbb{R}$ is a twice differentiable function. If $\frac{L(b-a)^2}{2} < 1$, then the equation $z'' = \phi(x, z(x))$, with the initial condition $z(a) = z\prime(a) = 0$, is stable in the Hyers-Ulam sense, where $z \in C^2(I), I = [a, b], -\infty < a < b < \infty$ and $\phi(x, z(x))$ is continuous for $x \in I, x \in \mathbb{R}$.

In [4], the authors proved the HU stability of nonlinear differential equation of second order of the form $u''(t) + F(t, u(t)) = 0$, where $F : [t_0, \infty) \times R \to (0, \infty)$ with $t_0 \geq 0$ and $u : [t_0, \infty) \to [0, \infty)$ is a twice continuously differentiable function with $u(t_0) = 0 = u'(t_0)$ and $\int_{t_0}^{\infty} |u'(t)|dt \leq L, L > 0$ by using a variant of Grownwall's inequality. They proved the following.

Theorem 2.1.27 : [4]

Given constants $L > 0$, $t_0 \geq 0$, assume that $F : [t_0, \infty) \times R \to (0, \infty)$ is a function satisfying $\frac{F'(t,u(t))}{F(t,u(t))} > 0$ with F(t,0) =1, for all $t \geq t_0$ and $u \in U(L,t_0)$. If a function $u : [t_0, \infty) \to (0, \infty)$ satisfies $u \in U(L,t_0)$ and the inequality $|u''(t) + F(t,u(t))| \leq \varepsilon$, for all $t \geq t_0$ and for some $\varepsilon > 0$, then there exists a solution $u_0 : [t_0, \infty) \to [0, \infty)$ of the differential equation $u''(t) + F(t,u(t)) = 0$ such that $|u(t) - u_0(t)| \leq L\varepsilon$, for any $t \geq t_0$.

In the same paper, they have proved the following result.

Theorem 2.1.28 : [4]

Given constants $L > 0$, $t_0 \geq 0$, assume that $h : [t_0, \infty) \to (0, \infty)$ is a differentiable function. Let α be an odd integer larger than 0. If a function $u : [t_0, \infty) \to [0, \infty)$ satisfies $u \in U(L,t_0)$ and the inequality $|u''(t) + h(t)u(t)^{\alpha}| \leq \varepsilon$, for all $t \geq t_0$ and for some $\varepsilon > 0$, then there exists a solution $u_0 : [t_0, \infty) \to [0, \infty)$ of the differential equation $u''(t) + h(t)u(t)^{\alpha} = 0$ such that $|u(t) - u_0(t)| \leq (\frac{\beta L\varepsilon}{h(t_0)})^{\frac{1}{\beta}}$, for any $t \geq t_0$, where $\beta = \alpha + 1$.

In 2003, T. Miura, S. Miyajima and S. Takahasi [44], proved that, if $P(z)$ is a polynomial of degree n with complex coefficients and $D = \frac{d}{dx}$, then the differential equation $P(D)f = 0$ is HU stable if and only if the equation $P(Z) = 0$ has no pure imaginary solution. This work has been extended in [10] to $n \times n$ complex linear system $x' = Ax$, where A is $n \times n$ complex matrix. It is shown here that this linear system is HU stable iff A is dichotomic i.e. the spectrum of A does not intersect with the imaginary axis.

The stability for partial differential equations have been investigated in [18],

[30], [42] and [54]. In [18], authors have proved the HU stability of the first and second order partial differential equations of the forms :

$y_x(x,t) = f(x,t,y(x,t))$, $ay_x(x,t) + by_t(x,t) = f(x,y,f(x,t))$,

$p(x,t)y_{xt}(x,t) + q(x,t)y_t(x,t) + p_t(x,t)y_x(x,t) - p_x(x,t)y_t(x,t) = f(x,t,y(x,t))$

and $p(x,t)y_{xx}(x,t) + q(x,t)y_x(x,t) = f(x,t,y(x,t))$

by using Banach's contraction principle.

In [30], Jung proved the HU stability for linear partial differential equation of first order of the form : $au_x(x,y) + bu_y(x,y) + g(y)u(x,y) + h(y) = 0,\ a \le 0, b > 0$. They proved the result by using change of axes and the result concerning the HU stability of a linear differential equation of first order established in [28].

In 2012, N. Lungu and D. Popa [42], proved the HU stability of first order partial differential equation of the form : $p(x,y)u_x(x,y) + q(x,y)u_y(x,y) = p(x,y)r(x)u + f(x,y)$. They proved the stability by using change of co-ordinates and using the following result.

Theorem 2.1.29 :

Let $\phi : [a,b) \to R$ be a solution of the differential equation $y' = \frac{q(x,y)}{p(x,y)}$. Then u is a solution of equation $p(x,y)u_x(x,y) + q(x,y)u_y(x,y) = p(x,y)r(x)u + f(x,y)$ if and only if there exists a function $F \in C^1(I,X)$ such that

$u(x,y) = e^{-L(x)}\{ \int_a^x \frac{f(\theta,\phi(\theta)+y-\phi(x))}{p(\theta,\phi(\theta)+y-\phi(x))} e^{L(\theta)} d\theta + F(y-\phi(x))\}$, for every $(x,y) \in D$,

where $L(x) = -\int_a^x r(\theta)d\theta, x \in [a,b)$, $I = \{y-\phi(x) : (x,y) \in D\}$, X is a Banach space over K (K is one the fields $\mathbb{R}$ or $\mathbb{C}$) , $D = [a,b) \times \mathbb{R}, a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$, $p,q \in C(D,K), f \in C(D,X)$ and $r \in C([a,b),\mathbb{R})$.

In [54], HUR stability for the heat equation have been studied. In this paper author have proved HUR stability for heat equation on R^n by using Fourier transform method and result on its convolution. Further he has proved the HUR stability for heat equation of the type: $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad t > 0,\ 0 < x < l, a > 0$, with the initial condition $u(x,0) = \mu(x), \quad 0 \le x \le l$ and boundary conditions $u(0,t) = v_1(t), u_x(0,t) = v_2(t), t \ge 0$,

where $l \in \mathbb{R}^+, \mu(x) \in C(-\infty,\infty), v_1(t), v_2(t) \in C(-\infty,\infty)$ and $u(x,t) \in C_1^2(\mathbb{R} \times (0,\infty))$. The above stability result was proved by using Laplace transform.

In [41], authors have proved the HU stability of the hyperbolic partial differential equation of the type $\frac{\partial^2 u}{\partial x \partial y} = f(x,y,u(x,y), \frac{\partial u}{\partial x}(x,y), \frac{\partial u}{\partial y}(x,y))$ $, 0 \le x < a, 0 \le y < b$. They proved the following result.

Theorem 2.1.30 : [41]

One assumes that

(i) $a < \infty, b < \infty$; (ii) $f \in C([0,a] \times [0,b] \times \mathbb{B}^3, \mathbb{B})$;

(iii) $\exists\ L_f > 0$ such that $|f(x,y,z_1,z_2,z_3) - f(x,y,t_1,t_2,t_3)| \le L_f \max\{|z_i - t_i|, i = 1,2,3\}$, for all $x \in ([0,a], y \in ([0,b]$ and $z_1,z_2,z_3,t_1,t_2,t_3 \in \mathbb{B}$,where $\mathbb{B}$ is a real or complex Banach space.

Then,

(a) for $\phi \in C^1([0,a],\mathbb{B})$ and $\Psi \in C^1([0,b],\mathbb{B})$,the hyperbolic PDE

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) = f(x,y,u(x,y), \frac{\partial u}{\partial x}(x,y), \frac{\partial u}{\partial y}(x,y)), \tag{2.4}$$

has a unique solution, which satisfies

$u(x,0) = \phi(x), \forall x \in [0,a]$, $u(0,y) = \Psi(y), \forall y \in [0,b]$;

(b) Equation 6.12 is Hyers-Ulam stable.

They proved this by using the result involving integral inequalities. In the same paper authors have discussed the HUR stability of the equation 6.12.

In [6], the authors have proved the HU stability for poisson's problem with Dirichlet boundary conditions $\{ {}^{-\Delta u = f, \text{ in } \Omega}_{u=0, \text{ on } \partial\Omega}$, with respect to weak solutions in

$H_0^1(\Omega)$, where $f \in H^{-1}(\Omega), H : G_1 \longrightarrow G_2, G_1$ be a group and G_2 be a metric group, operator $A : H_0^1(\Omega) \longrightarrow H_0^1(\Omega)$, Ω be a bounded domain in R^d such that it's border $\partial\Omega$ is sufficiently smooth. They proved the HU stability using fixed point equation as $x = A(x)$, weakly picard operator and by the following result :

Theorem 2.1.31 :

Let (X,d) be a metric group. If $A : X \longrightarrow X$ is a ψ- weakly picard operator then the fixed point equation $x = A(x)$ is generalized HU stable.

Further, in the same paper, authors have proved the HU stability of nonlinear elliptic problem $\{ \begin{smallmatrix} -\Delta u(x) = f(x,y), in\Omega \\ u=0, \text{ on } \partial\Omega \end{smallmatrix}$, where $f : \Omega \times R \longrightarrow R$ and remaining symbols with their meaning as discussed above.

It may be mentioned that HU stability has application in Biology and Economics [2].

2.3 PRELIMINARIES

In this section, we shall state some basic theorems, without proofs, used during this work.

THEOREM 2.2.1 [22] :

Let X be a complex Banach space. Assume that $p,q : I \to C$ and $f : I \to X$ are continuous functions and $y_1 : I \to X$ is a nonzero twice continuously differentiable function which satisfies the differential equation $y_1''(x) + p(x)y_1'(x) + q(x)y_1(x) = 0$. If a twice continuously differentiable function $y : I \to X$ satisfies

$|y''(x) + p(x)y'(x) + q(x)y - f(x)| \leq \psi(x)$, for all $x \in I$, where $k = \frac{y}{y_1(a)} \in X$ and

$\psi : I \to (0,\infty)$, is a continuous function, then there exists a unique $x_0 \in X$ such that

$$\left\|y(x) - y_1(x)\left\{\int_a^x \left[e^{\left\{-\int_a^s\left(\frac{2y_1'(u)}{y_1(u)}+p(u)\right)du\right\}}\cdot \left(x_0 + \int_a^s \frac{f(v)}{y_1(v)}\cdot e^{\int_a^v\left(\frac{2y_1'(u)}{y_1(u)}+p(u)\right)du}dv\right)\right]ds + k\right\}\right\|$$

$$\leq ||y_1(x)|| \times \int_a^x \left[e^{-\Re\left(\int_a^s\left(\frac{2y_1'(u)}{y_1(u)}+p(u)\right)du\right)}\cdot \left|\int_s^b e^{\left\{\Re\left(\int_a^t\left(\frac{2y_1'(u)}{y_1(u)}+p(u)\right)du\right)\cdot\frac{\psi(t)}{||y_1(t)||}dt\right\}}\right|\right]ds.$$

THEOREM 2.2.2 [54] :

If $u(x,t) \in C_1^2(\mathbb{R}\times(0,\infty))$, then the initial-boundary value problem

$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}, t > 0, 0 < x < l,$

with initial condition $u(x,0) = \mu(x), \qquad 0 \leq x \leq l$

and the boundary conditions $u(0,t) = v_1(t)$, $u_x(0,t) = v_2(t) \qquad t \geq 0,$

where $\mu(x) \in C(-\infty,\infty)$, is stable in the sense of Hyers-Ulam-Rassias.

THEOREM 2.2.3 [18] **(Banach Contraction Principle)** :

Let (X,d) be a complete metric space and $T: X \to X$ be a contraction, that is, there exists $\alpha \in (0,1)$ such that $d(Tx,Ty) \leq \alpha d(x,y), \forall x,y \in X$. Then $\exists$ a unique $a \in X$ such that $Ta = a$. Moreover, $a = \lim_{n\to\infty} T^n x$ and $d(a,x) \leq \frac{1}{(1-\alpha)}d(x,Tx), \forall x \in X$.

THEOREM 2.2.4 [18] :

Let $c \in I, I = [a,b]$ with $a < b$, $p,q : I\times I \to \mathbb{R}$ be continuous functions with $p(x,t) \neq 0, \forall x,t \in I$, $\phi : I\times I \to (0,\infty)$ be a continuous function, $L: I\times I \to [1,\infty)$ be an integrable function and $f: I\times I\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Assume that there exists, $0 < \beta < 1$ such that

$$\int_c^x L(\tau,t)\phi(\tau,t)ds < \beta\phi(x,t);$$

$$h(x,c) = -\{p(x,c)y_x(x,c) - p_x(x,c)y(x,c) + q(x,c)y(x,c)\};$$

$$K(x,t,y(x,t)) = -\{p(x,t)\}^{-1}\{(p_x(x,t) - q(x,t))y(x,t) + h(x,c) - \int_c^t f(x,\tau,y(x,\tau))d\tau\}$$

and

$$|K(x,t,u(x,t)) - K(x,t,v(x,t))| \leq L(x,t)|u(x,t) - v(x,t)|,$$

$\forall c,x,t \in I$ and $h,y,u,v \in C(I\times I)$. Let $y : I\times I \to \mathbb{R}$ be a function such that

$$|p(x,t)y_{xt}(x,t) + q(x,t)y_t(x,t) + p_t(x,t)y_x(x,t) - p_x(x,t)y_t(x,t) - f(x,t,y(x,t))|$$
$$\leq \phi(x,t),$$

$\forall x,t \in I$ and $p_x(x,t) = q(x,t)$ holds.

Then there exists a unique solution $y_0 : I\times I \to \mathbb{R}$ of the differential equation

$$p(x,t)y_{xt}(x,t) + q(x,t)y_t(x,t) + p_t(x,t)y_x(x,t) - p_x(x,t)y_t(x,t) = f(x,t,u(x,t))$$

such that

$$|y(x,t) - y_0(x,t)| \leq \frac{\beta}{(1-\beta)}\phi(x,t), \ \ \forall x,t \in I.$$

THEOREM 2.2.5 [41] :

One assumes that

(i) $f \in C([0,\infty)\times[0,\infty\times\mathbb{B}^3,\mathbb{B});$

(ii) there exists $l_f \in C^1([0,\infty)\times[0,\infty,\mathbb{R}_+)$ such that

$|f(x,y,z_1,z_2,z_3) - f(x,y,t_1,t_2,t_3)| \leq l_f(x,y)max\{|z_i - t_i|, i = 1,2,3\},$

for all $x,y \in [0,\infty);$

(iii) there exist $\lambda_\phi^1, \lambda_\phi^2, \lambda_\phi^3 > 0$ such that

$\int_0^x \int_0^y \phi(s,t)dsdt \leq \lambda_\phi^1 \phi(x,y),\ \forall x,y \in [0,\infty);$

$\int_0^y \phi(s,t)dt \leq \lambda_\phi^2 \phi(x,y),\ \forall x,y \in [0,\infty);$

$\int_0^x ds \leq \lambda_\phi^3 \phi(x,y),\ \forall x,y \in [0,\infty);$

(iv) $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is increasing.

Then the hyperbolic partial differential equation

$\frac{\partial^2 u}{\partial x \partial y} = f(x,y,u(x,y),\frac{\partial u}{\partial x}(x,y),\frac{\partial u}{\partial y}(x,y)),\ 0 \leq x < a, 0 \leq y < b$ ($a = \infty$ and $b = \infty$),

is generalised Ulam-Hyers-Rassias stable.

Chapter 3

HYERS ULAM RASSIAS STABILITY OF THIRD ORDER LINEAR ORDINARY DIFFERENTIAL EQUATION

Some contents of this chapter is presented at the Conference [see CP1].

3.1 INTRODUCTION

In this chapter we study the Hyers Ulam Rassias stability of third order linear ordinary differential equation. As mentioned earlier generalized Hyers - Ulam stability of the linear differential equation of second order have been investigated in [22]. In fact, they proved the stability of the linear ordinary differential equation of the type

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x),$$

with the condition that there exists a nonzero twice differential function $y_1 : I \longrightarrow C$ such that $y_1'' + p(x)y_1' + q(x)y_1 = 0$, for all $x \in I = (a,b)$, where C is a complex Banach space.

We extend this idea to prove the stability of the third order non-homogeneous linear ordinary differential equation of the type

$$y'''(x) + p(x)y''(x) + q(x)y'(x) + r(x)y(x) = f(x), \tag{3.1}$$

where $y \in C^3[a,b], p,q,r,f \in C[a,b]$ and $-\infty < a < b < \infty$.

We prove the result by imposing certain integrability conditions on its coefficients. An example have been considered as an illustration.

First, we define the Hyers Ulam Rassias stability of the differential equation (3.1).

DEFINITION 3.1.1 : Let X be a normed space and let I be an open interval. We say that the differential equation (3.1) has the Hyers - Ulam - Rassias stable, if for any function $h : I \longrightarrow X$ satisfying the differential inequality

$$|h'''(x) + p(x)h''(x) + q(x)h'(x) + r(x)h(x) - f(x)| \leq \Psi(x), \text{for all } x \in I,$$

there exist a solution $g : I \longrightarrow X$ of (3.1) such that $|h(x) - g(x)| \leq \psi(x)$, for any $x \in I$, where $\Psi, \psi : I \longrightarrow (0, \infty)$ are continuous functions not depending on h and g explicitly.

3.2 MAIN RESULT

In this section we prove our main result of this chapter. We discuss the Hyers-Ulam-Rassias stability of (3.1). For the sake of convenience, all the integrals and derivatives will be viewed as existing.

Let $I = (a, b)$ be an arbitrary interval and $y_1 : I \longrightarrow C$ be a non- zero solution of corresponding homogeneous equation of (3.1), where

$$y_1'''(x) + p(x)y_1''(x) + q(x)y_1'(x) + r(x)y_1(x) = 0. \tag{3.2}$$

It may be noted that there exists a solution y_1 of (3.2) (may be complex valued) such that it does not vanish on I (see for example [12]). We have the following result.

THEOREM 3.2.1 : Let C be a Banach space. Assume that $p, q, r, f : I \longrightarrow C$ are continuous functions and $y_1 : I \longrightarrow C$ is a non-zero thrice continuously differential function which satisfies the differential inequality (3.2). If a thrice continuously differential function $y : I \longrightarrow C$ satisfies

$$|y'''(x) + p(x)y''(x) + q(x)y'(x) + r(x)y(x) - f(x)| \leq \Psi(x), \tag{3.3}$$

for all $x \in I$, where $\Psi : I \longrightarrow (0,\infty)$ is a continuous function, then there exist a unique $z(b) \in C$ such that

$$\Big|y(x) - y_1(x)\Big\{\int_a^x u_0(t)\Big[v_1'(a) + z(b)\int_a^t e^{-\int_a^s(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx} ds$$

$$+\int_a^t e^{-\int_a^s(\frac{2u_o'(x)}{u_0(x)}+p_1(x)dx)}\Big(\int_a^s\{\frac{f_1(l)}{u_0(l)}\}e^{\int_a^l(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx} dl\Big)ds\Big]dt + v(a)\Big\}\Big|$$

$$\leq |y_1(x)| \times \int_a^x |u_0(t)|\Big[\int_a^t e^{-\Re\Big(\int_a^s(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx\Big)}.$$

$$\Big|\int_s^b e^{\Re\Big(\int_a^t[\frac{2u_0'(x)}{u_0(x)}+p_1(x)]dx\Big)}\Psi_2(t)dt\Big|ds\Big]dt, \tag{3.4}$$

where $v(a) = \frac{y(a)}{y_1(a)} \in C$, $v_1'(a) = \frac{v'(a)}{u_0(a)}$, $\Psi_2(x) = \frac{\Psi_1(x)}{|u_0(x)|}$, $\Psi_1(x) = \frac{\Psi(x)}{|y_1(x)|}$, $\int_a^b e^{\Re\Big(\int_a^t[\frac{2u_0'(x)}{u_0(x)}+p_1(x)]dx\Big)}\Psi_2(t)dt < \infty$ and $u_0(x)$ is a twice differential function which is a solution of

$$u_0''(x) + p_1(x)u_0'(x) + q_1(x)u_0(x) = 0,$$

with $p_1(x) = \{\frac{3y_1'(x)}{y_1(x)} + p(x)\}$ and $q_1(x) = \{\frac{3y_1''(x)}{y_1(x)} + \frac{2p(x)y_1'(x)}{y_1(x)} + q(x)\}$.

Proof. : Assume that

$$v(x) = \frac{y(x)}{y_1(x)},\ \forall x \in I. \tag{3.5}$$

It follows from equation (3.2), (3.3) and (3.5) that

$$|(v(x)y_1(x))''' + p(x)(v(x)y_1(x))'' + q(x)(v(x)y_1(x))' + r(x)(v(x)y_1(x)) - f(x)|$$

$$= |(v'(x)y_1(x) + v(x)y_1'(x))'' + p(x)(v'(x)y_1(x) + v(x)y_1'(x))'$$

$$+ q(x)(v'(x)y_1(x)+v(x)y_1'(x))+r(x)(v(x)y_1(x))-f(x)|$$

$$=|(v''(x)y_1(x)+v'(x)y_1'(x)+v'(x)y_1'(x)+v(x)y_1''(x))'$$

$$+p(x)\left[v''(x)y_1(x)+v'(x)y_1'(x)+v'(x)y_1'(x)+v(x)y_1''(x)\right]$$

$$+q(x)[v'(x)y_1(x)+v(x)y_1'(x)]+r(x)(v(x)y_1(x))-f(x)|$$

$$=|\left[v'''(x)y_1(x)+v''(x)y_1'(x)+v''(x)y_1'(x)+v'(x)y_1''(x)+v''(x)y_1'(x)\right.$$

$$\left.+v'(x)y_1''(x)+v'(x)y_1''(x)+v(x)y_1'''(x)\right]+$$

$$p(x)[v''(x)y_1(x)+v'(x)y_1'(x)+v'(x)y_1'+v(x)y_1''(x)]$$

$$+ q(x)(v'(x)y_1(x)+v(x)y_1'(x))+r(x)v(x)y_1(x)-f(x)|$$

$$=|v'''(x)y_1(x)+v''(x)\{3y_1'(x)+p(x)y_1(x)\}$$

$$+ (v'(x))\{3y_1''(x)+2p(x)y_1'(x)+q(x)y_1(x)\}$$

$$+ v(x)\{y_1'''(x)+p(x)y_1''(x)+q(x)y_1'(x)+r(x)y_1(x)\}-f(x)|$$

$$=|v'''(x)y_1(x)+v''(x)\left\{3y_1'(x)+p(x)y_1(x)\right\}$$

$$+v'(x)\left[3y_1''(x)+2p(x)y_1'(x)+q(x)y_1(x)\right]-f(x)|$$

$$= |y_1(x)||v'''(x) + v''(x)\Big\{\frac{3y_1'(x)}{y_1(x)} + p(x)\Big\} + v'(x)\Big\{\frac{3y_1''(x)}{y_1(x)} + \frac{2p(x)y_1'(x)}{y_1(x)} + q(x)\Big\} - \frac{f(x)}{y_1(x)}|$$

$$\leq \Psi(x).$$

$$\Rightarrow |v'''(x) + v''(x)\Big\{\frac{3y_1'(x)}{y_1(x)} + p(x)\Big\} + v'(x)\Big\{\frac{3y_1''(x)}{y_1(x)} + \frac{2p(x)y_1'(x)}{y_1(x)} + q(x)\Big\} - \frac{f(x)}{y_1(x)}|$$

$$\leq \frac{\Psi(x)}{|y_1(x)|}.$$

Letting $\frac{\Psi(x)}{|y_1(x)|} = \Psi_1(x)$, we get

$$|v'''(x) + v''(x)\Big\{\frac{3y_1'(x)}{y_1(x)} + p(x)\Big\} + v'(x)\Big\{\frac{3y_1''(x)}{y_1(x)} + \frac{2p(x)y_1'(x)}{y_1(x)} + q(x)\Big\} - \frac{f(x)}{y_1(x)}| \leq \Psi_1(x). \tag{3.6}$$

Let $\{\frac{3y_1'(x)}{y_1(x)} + p(x)\} = p_1(x)$, $\{\frac{3y_1''(x)}{y_1(x)} + \frac{2p(x)y_1'(x)}{y_1(x)} + q(x)\} = q_1(x)$ and $f_1(x) = \frac{f(x)}{y_1(x)}$.

Then (3.6) becomes

$$|v'''(x) + p_1(x)v''(x) + q_1(x)v'(x) - f_1(x)| \leq \Psi_1(x). \tag{3.7}$$

Let $u_0(x)$ be a solution of

$$u_0''(x) + p_1(x)u_0'(x) + q_1(x)u_0(x) = 0. \tag{3.8}$$

Let

$$\frac{v'(x)}{u_0(x)} = v_1'(x), \tag{3.9}$$

and let $z(s) = e^{\int_a^s(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx} v_1''(s) - \int_a^s\{\frac{f_1(t)}{u_0(t)} e^{\int_a^t(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx}\}dt.$

Then

$$|z(s)-z(l)| = \Big|\int_l^s \frac{d}{dt}[e^{\int_a^t(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx} v_1''(t) - \int_a^t \frac{f_1(l)}{u_0(l)}(e^{\int_a^l(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx})dl]dt\Big|.$$

$$|z(s)-z(l)| = \Big|\int_l^s e^{\int_a^t(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx}\Big[v_1'''(t) + (\frac{2u_o'(x)}{u_0(x)} + p_1(x))v_1''(t) - \frac{f_1(t)}{u_0(t)}\Big]dt\Big|. \tag{3.10}$$

Then using the equations (3.7) and (3.9), we get
$\Big|u_O(x)\Big[v_1'''(x) + \Big(\frac{2u_O'(x)}{u_0(x)} + p_1(x)\Big)v_1''(x) - \frac{f_1(x)}{u_0(x)}\Big]\Big| \le \Psi_1(x).$

That is
$\Big|v_1'''(x) + \Big(\frac{2u_O'(x)}{u_0(x)} + p_1(x)\Big)v_1''(x) - \frac{f_1(x)}{u_0(x)}\Big| \le \frac{\Psi_1(x)}{|u_O(x)|}.$

Letting $\frac{\Psi_1(x)}{|u_O(x)|} = \Psi_2(x)$, we get

$$\Big|v_1'''(x) + \Big(\frac{2u_O'(x)}{u_0(x)} + p_1(x)\Big)v_1''(x) - \frac{f_1(x)}{u_0(x)}\Big| \le \Psi_2(x). \tag{3.11}$$

Then from (3.10), the integrability condition $\int_a^b e^{\Re(\int_a^t[\frac{2u_O'(x)}{u_0(x)}+p_1(x)]dx)}(\Psi_2(t))dt < \infty$, implies

$$|z(s)-z(l)| \le \Big|\int_l^s e^{\Re\Big(\int_a^t[\frac{2u_0'(x)}{u_0(x)}+p_1(x)]dx\Big)}(\Psi_2(t))dt\Big|. \tag{3.12}$$

which implies that $\{z(s)\}_{s\varepsilon I}$ is a Cauchy net. So

$$\lim_{s\to b} z(s) \equiv z(b)$$

exists, where $z(b) \in C$.

For any $x \in I$, consider

$$\Bigg|y(x)-y_1(x)\Bigg\{\int_a^x u_0(t)\Bigg[v_1'(a)+z(b)\int_a^t e^{-\int_a^s(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx}ds + \int_a^t e^{-\int_a^s(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx}\Big(\int_a^s \frac{f_1(l)}{u_0(l)}e^{\int_a^l(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx}dl\Big)ds\Bigg]dt+v(a)\Bigg\}\Bigg|$$

$$=\Bigg|v(x)y_1(x)-y_1(x)\Bigg\{\int_a^x u_0(t)\Bigg[v_1'(a)+z(b)\int_a^t e^{-\int_a^s(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx}ds + \int_a^t e^{-\int_a^s(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx}\Big(\int_a^s \frac{f_1(l)}{u_0(l)}e^{\int_a^l(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx}dl\Big)ds\Bigg]dt+v(a)\Bigg\}\Bigg|$$

$$=\Bigg|y_1(x)\Bigg\{v(x)-\int_a^x u_0(t)\Bigg[v_1'(a)+z(b)\int_a^t e^{-\int_a^s(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx}ds + \int_a^t e^{-\int_a^s(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx}\Big(\int_a^s \frac{f_1(l)}{u_0(l)}e^{\int_a^l(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx}dl\Big)ds\Bigg]dt+v(a)\Bigg\} - v(a)\Bigg|$$

$$=\Bigg|y_1(x)\Bigg\{\int_a^x v'(t)dt-\int_a^x u_0(t)\Bigg[v_1'(a)+z(b)\int_a^t e^{-\int_a^s(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx}ds + \int_a^t e^{-\int_a^s(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx}\Big(\int_a^s \frac{f_1(l)}{u_0(l)}e^{\int_a^l(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx}dl\Big)ds\Bigg]dt\Bigg\}\Bigg|$$

By using equation (3.9) we get,

$$=\Bigg|y_1(x)\Bigg\{\int_a^x u_0(t)v_1'(t)dt-\int_a^x u_0(t)\Bigg[v_1'(a)+z(b)\int_a^t e^{-\int_a^s(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx}ds + \int_a^t e^{-\int_a^s(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx}\Big(\int_a^s \frac{f_1(l)}{u_0(l)}e^{\int_a^l(\frac{2u_o'(x)}{u_0(x)}+p_1(x))dx}dl\Big)ds\Bigg]dt\Bigg\}\Bigg|$$

$$= \left| y_1(x) \left\{ \int_a^x u_0(t) \left[v_1'(t) - \left(v_1'(a) + z(b) \int_a^t e^{-\int_a^s (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dx} ds \right.\right.\right.\right.$$
$$\left.\left.\left.\left. + \int_a^t e^{-\int_a^s (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dx} \left(\int_a^s \frac{f_1(l)}{u_0(l)} e^{\int_a^l (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dx} dl \right) ds \right) \right] dt \right\} \right|$$

$$= \left| y_1(x) \left\{ \int_a^x u_0(t) \left[\int_a^t v_1''(s)ds - z(b) \int_a^t e^{-\int_a^s (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dx} ds \right.\right.\right.$$
$$\left.\left.\left. - \int_a^t e^{-\int_a^s (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dx} \left(\int_a^s \frac{f_1(l)}{u_0(l)} e^{\int_a^l (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dx} dl \right) ds \right] dt \right\} \right|$$

$$= \left| y_1(x) \left\{ \int_a^x u_0(t) \left[\int_a^t e^{-\int_a^s (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dx} \left(e^{\int_a^s (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dx} v_1''(s) \right.\right.\right.\right.$$
$$\left.\left.\left.\left. - z(b) - \int_a^s \frac{f_1(l)}{u_0(l)} e^{\int_a^l (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dx} dl \right) ds \right] dt \right\} \right|$$

$$= \left| y_1(x) \left\{ \int_a^x u_0(t) \left[\int_a^t e^{-\int_a^s (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dx} \{z(s) - z(b)\} ds \right] dt \right\} \right|$$

$$\leq |y_1(x)| \left| \int_a^x u_0(t) \left[\int_a^t e^{-\Re\left(\int_a^s (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dx \right)} \{z(s) - z(l)\} ds \right] dt \right| +$$
$$|y_1(x)| \left| \int_a^x u_0(t) \left[\int_a^t e^{-\Re\left(\int_a^s (\frac{2u_o'(x)}{u_0(x)} + p_1(x)dx \right)} \{z(l) - z(b)\} ds \right] dt \right|$$

$$\leq |y_1(x)| \times \int_a^x |u_0(t)| \left[\int_a^t e^{-\Re\left(\int_a^s (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dx \right)} |z(s) - z(l)| ds \right] dt +$$
$$|y_1(x)| \times \int_a^x |u_0(t)| \left[\int_a^t e^{-\Re\left(\int_a^s (\frac{2u_o'(x)}{u_0(x)} + p_1(x)dx \right)} |z(l) - z(b)| ds \right] dt$$

By using equation (3.12), we have

$$\leq |y_1(x)| \int_a^x |u_0(t)| \Bigg[\int_a^t e^{-\Re\left(\int_a^s (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dx \right)} .$$

$$\left| \int_l^s e^{\Re\left(\int_a^t [\frac{2u_0'(x)}{u_0(x)} + p_1(x)]dx \right)} \Psi_2(t)dt \right| ds \Bigg] dt$$

$$+ \ |y_1(x)| \times \int_a^x |u_0(t)| \Big[\int_a^t e^{-\Re\left(\int_a^s (\frac{2u_o'(x)}{u_0(x)} + p_1(x)dx \right)} |z(l) - z(b)| ds \Big] dt$$

$$\to |y_1(x)| \times \int_a^x |u_0(t)| \Bigg[\int_a^t e^{-\Re\left(\int_a^s (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dx \right)} .$$

$$\left| \int_s^b e^{\Re\left(\int_a^t [\frac{2u_0'(x)}{u_0(x)} + p_1(x)]dx \right)} \Psi_2(t)dt \right| ds \Bigg] dt,$$

as $l \to b$.

Now, we prove the uniqueness of $z(b)$. Assume that $z(b_1), z(b_2) \in C$ also satisfies the inequality (3.4) in place of $z(b)$. Then we have,

$$|y_1(x)| \left| \int_a^x u_0(t) \left[\{z(b_2) - z(b_1)\} \int_a^t e^{-\int_a^s (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dx} ds \right] dt \right|$$

$$\leq 2|y_1(x)| \int_a^x |u_0(t)| \Bigg[\int_a^t e^{-\Re\left(\int_a^s (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dx \right)} .$$

$$\left| \int_s^b e^{\Re\left(\int_a^t [\frac{2u_0'(x)}{u_0(x)} + p_1(x)]dx \right)} \Psi_2(t)dt \right| ds \Bigg] dt.$$

$$\Rightarrow |z(b_2) - z(b_1)| \left| \int_a^x u_0(t) \left(\int_a^t e^{-\int_a^s (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dx} ds \right) dt \right|$$

$$\leq 2 \int_a^x |u_0(t)| \Bigg[\int_a^t e^{-\Re\left(\int_a^s (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dx \right)} .$$

$$\left| \int_s^b e^{\Re\left(\int_a^t [\frac{2u_0'(x)}{u_0(x)} + p_1(x)]dx \right)} \Psi_2(t)dt \right| ds \Bigg] dt.$$

$\Rightarrow |z(b_2) - z(b_1)|$

$$\leq \frac{2\int_a^x |u_0(t)| \Big[\int_a^t e^{-\Re\left(\int_a^s (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dx\right)} \Big|\int_s^b e^{\Re\left(\int_a^t [\frac{2u_0'(x)}{u_0(x)} + p_1(x)]dx\right)} \Psi_2(t)dt\Big| ds\Big] dt}{\Big|\int_a^x u_0(t)\Big(\int_a^t e^{-\int_a^s (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dx} ds\Big) dt\Big|}$$

It follows from the integrability hypothesis that

$$\Big|\int_s^b e^{\Re\left(\int_a^t [\frac{2u_0'(x)}{u_0(x)} + p_1(x)]dx\right)} \Psi_2(t)dt\Big| \to 0 \text{ as } s \to b.$$

This implies that $z(b_1) = z(b_2)$.

Hence, every third order linear differential equation has the Hyers - Ulam - Rassias stability with the condition that there exists a solution of corresponding homogeneous equation. □

Remark 3.2.2 It follows from Theorem (3.2.1) that

$$\tilde{y}(x) = y_1(x)\Big\{\int_a^x u_0(t)\Big[c_1 + c_2\int_a^t e^{-\int_a^s (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dxds}.$$
$$+ \int_a^t e^{-\int_a^s (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dx}\Big(\int_a^s \frac{f_1(l)}{u_0(l)} e^{\int_a^l (\frac{2u_o'(x)}{u_0(x)} + p_1(x))dx} dl\Big) ds\Big] dt + c_3\Big\} \tag{3.13}$$

is the general solution of the differential equation (3.1), where c_1, c_2, c_3 are arbitrary constants.

3.3 AN ILLUSTRATION

We now give an example in support of the result obtained in previous section.

Example 3.3.1 Consider the differential equation

$$y'''(x) - \lambda y(x) = f(x) \tag{3.14}$$

where $\lambda \in$ C. Let $I = (a,b)$ be an open interval and $a, b \in \mathbb{R}$ be arbitrarily given with $a < b$. Let $f : I \longrightarrow C$, $\psi : I \longrightarrow (0,\infty)$ be continuous functions. Assume that $y : I \longrightarrow C$ is a thrice continuously differentiable function satisfying the differential inequality

$$|y'''(x) - \lambda y(x) - f(x)| \leq \psi(x) \tag{3.15}$$

for all $x \in I$. Let $\lambda^{\frac{1}{3}}$ = a + *i* b and (3a - $\sqrt{3}$) <0.

We know that $y_1(x) = e^{-kx}$ is a solution of corresponding homogeneous equation (3.14), where k = $\lambda^{\frac{1}{3}}$.

It follows from **Theorem** 3.2.1 that there exist a solution $u_0 = e^{(\frac{3k}{2}+\frac{\sqrt{3}ki}{2})x}$ of $u_0''(x) + \frac{3y_1'}{y_1}u_0'(x) + \frac{3y_1''}{y_1}u_0(x) = 0$.

Again it follows from **Theorem** 3.2.1, **Remark** 3.2.2 and (3.15) that there exist a solution $\tilde{y} : I \longrightarrow C$ of (3.14) such that

$$\tilde{y}(x) = e^{-kx}\left\{\int_a^x e^{(\frac{3k}{2}+\frac{\sqrt{3}ki}{2})t}\left[c_1 + c_2\Big(\frac{1-e^{\sqrt{3}ki(a-t)}}{\sqrt{3}ki}\Big) + \int_a^t \left\{e^{-\sqrt{3}ki(s-a)}\Big(\int_a^s f(l)e^{(\sqrt{3}i-1)\frac{kl}{2}-\sqrt{3}kia}dl\Big)\right\}ds\right]dt + c_3\right\}, \tag{3.16}$$

for all $x \in I$ with the integrability condition $\int_a^b \{e^{\Re(\int_a^t \sqrt{3}kidx)}\psi_2(t)\}dt < \infty$ where $\psi_2(t) = \frac{\psi_1(t)}{|u_o|}, \psi_1(t) = \frac{\psi(t)}{|y_1|}$ and

$$\begin{aligned}
&\Bigg| y(x) - e^{-kx}\Bigg\{ \int_a^x e^{(\frac{3k}{2}+\frac{\sqrt{3}ki}{2})t}\Bigg[v_1'(a) + z(b)\Big(\frac{1-e^{\sqrt{3}ki(a-t)}}{\sqrt{3}ki}\Big) \\
&\qquad + \textstyle\int_a^t \Big\{ e^{-\sqrt{3}ki(s-a)}\Big(\int_a^s f(l) e^{(\sqrt{3}i-1)\frac{kl}{2}-\sqrt{3}kia} dl\Big)\Big\} ds\Bigg] dt + v(a)\Bigg\}\Bigg| \\
&\le |e^{-kx}| \times \int_a^x |e^{(\frac{3k}{2}+\frac{\sqrt{3}ki}{2})t}| \Big[\int_a^t e^{-\Re(\sqrt{3}ki(s-a))}. \\
&\qquad \Big| \textstyle\int_s^b e^{\Re(\sqrt{3}ki(t-a))}\Big(\frac{\Psi(t)}{|e^{-kt}||e^{(\frac{3k}{2}+\frac{\sqrt{3}ki}{2})t}|}\Big) dt\Big| ds\Big] dt.
\end{aligned}$$

Chapter 4

HYERS-ULAM-RASSIAS STABILITY OF LINEAR HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATIONS

Some contents of this chapter is published [see PP1 and PP3].

Some contents of this chapter is presented at the Conference [see CP2].

4.1 INTRODUCTION

In this chapter, we establish the Hyers-Ulam-Rassias (HUR) stability for linear homogeneous partial differential equations. We use the Laplace transform method to prove our results.

First , we establish the HUR stability of first order linear homogeneous partial differential equation, of the form

$$\frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x}, \qquad t > 0, 0 < x < l, a > 0 \tag{4.1}$$

with the initial condition

$$u(x,0) = \mu(x), \qquad 0 \leq x \leq l \tag{4.2}$$

and boundary condition

$$u(0,t) = v_0(t), \qquad t > 0, \tag{4.3}$$

where $l \in \mathbb{R}$, $\mu(x) \in C[0,l]$, $v_0(t) \in C(-\infty,\infty)$ and $u(x,t) \in C_1^1((0,l)\times(0,\infty))$.

Then, we establish the HUR stability of third order linear homogeneous partial differential equation, of the type

$$\frac{\partial u}{\partial t} = a^3\frac{\partial^3 u}{\partial x^3}, \qquad t > 0, 0 < x < l, a > 0 \tag{4.4}$$

with the initial condition

$$u(x,0) = \mu(x), \qquad 0 \leq x \leq l \tag{4.5}$$

and boundary conditions

$$u(0,t) = v_0(t),\ u_x(0,t) = v_1(t),\ u_{xx}(0,t) = v_2(t), t \geq 0, \tag{4.6}$$

where $l \in \mathbb{R}$, $\mu(x) \in C([0,l]$, $v_0(t), v_1(t), v_2(t) \in C(-\infty,\infty)$ and $u(x,t) \in C_1^3((0,l) \times (0,\infty))$.

Further , we extend the result and prove the HUR stability of the n^{th} order linear homogeneous partial differential equation, of the form

$$\frac{\partial u}{\partial t} = a^n \frac{\partial^n u}{\partial x^n}, \qquad t > 0, 0 < x < l, a > 0 \tag{4.7}$$

with initial condition

$$u(x,0) = \mu(x), \qquad 0 \le x \le l \tag{4.8}$$

and the boundary conditions

$$u(0,t) = v_0(t), u_x(0,t) = v_1(t), u_{xx}(0,t) = v_2(t), \cdots\cdots u_{xx\cdots x}(0,t) = v_{n-1}(t) \tag{4.9}$$

where $a, l \in \mathbb{R}$, $\mu(x) \in C[0,l]$, $v_0(t), v_1(t), v_2(t), v_3(t), \cdots\cdots, v_{n-1}(t) \in C(-\infty,\infty)$ and $u(x,t) \in C_1^n((0,l) \times (0,\infty))$.

First, we define HUR stability of the equation (4.1).

Definition 4.1 : We will say that the equation (4.1) is HUR stable with respect to $\phi(x,t) > 0$, if $\exists\ \psi(x,t) > 0$ such that for each $\varepsilon > 0$ and for each solution $w(x,t) \in C_1^1((0,l) \times (0,\infty))$ of the inequality

$$|\frac{\partial u}{\partial t} - a\frac{\partial u}{\partial x}| \le \varepsilon \phi(x,t) \tag{4.10}$$

with conditions (4.2) and (4.3), $\exists$ a solution $u(x,t) \in C_1^1((0,l) \times (0,\infty))$ of the equation (4.1) such that

$$|w(x,t) - u(x,t)| \le \varepsilon \psi(x,t) \tag{4.11}$$

$\forall (x,t) \in ((0,l) \times (0,\infty)), \phi(x,t) \in C((0,l) \times (0,\infty))$ and $\psi(x,t) \in C((0,l) \times (0,\infty))$.

Similarly, we can define the HUR Stability of Initial - Boundary value problems (I-BVPs) (4.4) – (4.6) and (4.7) - (4.9) .

Definition 4.2 : [58] For each function $f:(0,\infty)\to\mathbb{F}\,(\mathbb{R}\ \text{ or }\ \mathbb{C})$ of exponential order, the Laplace transform of $f(t)$ is defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt.$$

There exists a unique number $-\infty \leq \sigma < \infty$ such that this integral converges if $\Re(s) > \sigma$ and diverges if $\Re(s) < \sigma$. The number σ is called abscissa of convergence and is denoted by σ_f.

Definition 4.3 : [58] Let $f(t)$ be a continuous function whose Laplace transform $F(s)$ has the abscissa of convergence σ_f. Then the inverse Laplace transform is given by

$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\alpha+iy)t} F(\alpha+iy)dy$, for any real $\alpha > \sigma_f$.

4.2 HUR STABILITY OF (4.1)

In this section we prove the HUR stability of first order linear partial differential equation (4.1). We obtain the results by using the idea in [54].

Theorem 4.2.1 : If $w(x,t)\in C_1^1((0,l)\times(0,\infty))$ is an approximate solution of the I-BVP (4.1) - (4.3), then I-BVP (4.1) - (4.3) is HUR stable.

Proof : Given $\varepsilon > 0$. Suppose $w(x,t)$ be an approximate solution of the I-BVP (4.1) - (4.3). We have to show that, there exists an exact solution $u(x,t) \in C_1^1((0,l)\times(0,\infty))$ of the equation (4.1) such that $|w(x,t)-u(x,t)| \leq \varepsilon\psi(x,t)$, where $\psi(x,t)\in C((0,l)\times(0,\infty))$.

From the definition of HUR stability, we have

$$|\frac{\partial w}{\partial t} - a\frac{\partial w}{\partial x}| \leq \varepsilon\alpha(t-\frac{l}{a}).$$

$$\Rightarrow -\varepsilon\alpha(t-\frac{l}{a}) \leq \frac{\partial w}{\partial t} - a\frac{\partial w}{\partial x} \leq \varepsilon\alpha(t-\frac{l}{a}), \tag{4.12}$$

where $\alpha(t-c) = 0$, for $t \leq c$ and $\alpha(t-c) = x(t-c)$, for $t \geq c, c \geq 0$.

Taking Laplace transform of equation (4.12), we get

$-\varepsilon\mathcal{L}\{\alpha(t-\frac{l}{a})\} \leq \mathcal{L}\{\frac{\partial w}{\partial t} - a\frac{\partial w}{\partial x}\} \leq \varepsilon\mathcal{L}\{\alpha(t-\frac{l}{a})\}.$

$\Rightarrow |\mathcal{L}\{w_t - aw_x\}| \leq \varepsilon\mathcal{L}\{\alpha(t-\frac{l}{a})\}$. Hence

$$|\mathcal{L}\{w_t\} - a\mathcal{L}\{w_x\}| \leq \varepsilon\mathcal{L}\{\alpha(t-\frac{l}{a})\}. \tag{4.13}$$

Also since $w(0,t) = v_0(t)$, we get $\mathcal{L}\{w(0,t)\} = \mathcal{L}\{v_0(t)\} = W(0,p) = V_0(p)$.

Assuming the operation of differentiation w. r. t. x is interchangeable with integration w. r. t. t in Laplace transform, we get

$$\mathcal{L}\{\frac{\partial w}{\partial x}\} = \frac{dW}{dx}(x,p) \tag{4.14}$$

and

$$\mathcal{L}\{\frac{\partial w}{\partial t}\} = pW(x,p) - w(x,0). \tag{4.15}$$

From equations, (4.13),(4.14) and (4.15), we get

$$|pW(x,p) - w(x,0) - a\frac{dW}{dx}(x,p)| \leq \varepsilon\mathcal{L}\{\alpha(t-\frac{l}{a})\}.$$

$$\Rightarrow |-a\{\frac{dW}{dx}(x,p) - \frac{pW(x,p)}{a} + \frac{\mu(x)}{a}\}| \leq \varepsilon\mathcal{L}\{\alpha(t-\frac{l}{a})\}.$$

$$\Rightarrow |a\{\frac{dW}{dx}(x,p) - \frac{pW(x,p)}{a} + \frac{\mu(x)}{a}\}| \leq \varepsilon\mathcal{L}\{\alpha(t-\frac{l}{a})\}.$$

$$\Rightarrow |\frac{dW}{dx}(x,p) - \frac{pW(x,p)}{a} + \frac{\mu(x)}{a}| \leq \frac{\varepsilon}{a}\mathcal{L}\{\alpha(t-\frac{l}{a})\}.$$

$$\Rightarrow |\frac{dW}{dx}(x,p) - \frac{pW(x,p)}{a} + \frac{\mu(x)}{a}| \leq \frac{\varepsilon}{a} \times \frac{x}{p^2}e^{-\frac{pl}{a}}.$$

Hence

$$-\frac{\varepsilon}{a} \times \frac{x}{p^2}e^{-\frac{pl}{a}} \leq \frac{dW}{dx}(x,p) - \frac{pW(x,p)}{a} + \frac{\mu(x)}{a} \leq \frac{\varepsilon}{a} \times \frac{x}{p^2}e^{-\frac{pl}{a}}. \tag{4.16}$$

Integrating the equation (4.16) from 0 to x, we get

$$-\frac{\varepsilon x^2}{2ap^2}e^{-\frac{pl}{a}} \leq W(x,p) - W(0,p) - \frac{p}{a}\int_0^x W(s,p)ds + \frac{1}{a}\int_0^x \mu(s)ds \leq \frac{\varepsilon x^2}{2ap^2}e^{-\frac{pl}{a}}, \tag{4.17}$$

where

$$W(0,p) = \mathcal{L}\{w(0,t)\}. \tag{4.18}$$

It is easily verified that the function $U(x,p) = \mathcal{L}\{u(x,t)\}$, which is given by

$U(x,p) = W(0,p) + \frac{p}{a}\int_0^x U(s,p)ds - \frac{1}{a}\int_0^x \mu(s)ds$ has to satisfy the equation

$\frac{dW}{dx}(x,p) - \frac{pW(x,p)}{a} + \frac{\mu(x)}{a} = 0$ with the boundary condition (4.18).

Next, consider the difference

$\Delta = |W(x,p) - U(x,p)|.$

$= |W(x,p) - W(0,p) - \frac{p}{a}\int_0^x U(s,p)ds + \frac{1}{a}\int_0^x \mu(s)ds|.$

$= |W(x,p) - W(0,p) - \frac{p}{a}\int_0^x W(s,p)ds + \frac{1}{a}\int_0^x \mu(s)ds + \frac{p}{a}\int_0^x W(s,p)ds - \frac{P}{a}\int_0^x U(s,p)ds|.$

$\leq |W(x,p) - W(0,p) - \frac{p}{a}\int_0^x W(s,p)ds + \frac{1}{a}\int_0^x \mu(s)ds| + \frac{p}{a}\int_0^x |W(s,p) - U(s,p)|ds.$

$\leq \frac{\varepsilon x^2}{2ap^2}e^{-\frac{pl}{a}} + \frac{p}{a}\int_0^x |W(s,p) - U(s,p)|ds,$ (by equation (4.17)).

$\leq \frac{\varepsilon lx}{2ap^2}e^{-\frac{pl}{a}} + \frac{p}{a}\int_0^x |W(s,p) - U(s,p)|ds.$

By using Grownwall inequality, we get

$|W(x,p) - U(x,p)| \leq \frac{\varepsilon lx}{2ap^2}e^{-\frac{pl}{a}} \times e^{\int_0^x \frac{p}{a}ds}.$

$\Rightarrow |W(x,p) - U(x,p)| \leq \frac{\varepsilon lx}{2ap^2}e^{-\frac{pl}{a}} \times e^{\frac{px}{a}}.$

$\Rightarrow |W(x,p) - U(x,p)| \leq \frac{\varepsilon lx}{2ap^2}e^{-\frac{pl}{a}} \times e^{\frac{pl}{a}}.$

$\Rightarrow |W(x,p) - U(x,p)| \leq \frac{\varepsilon lx}{2ap^2}.$

$\Rightarrow |W(x,p) - U(x,p)| \leq \frac{\varepsilon l}{2a} \times \frac{x}{p^2}.$

$\Rightarrow |W(x,p) - U(x,p)| \leq \frac{\varepsilon l}{2a} \times \mathcal{L}\{\alpha(t)\}.$

$\Rightarrow -\frac{\varepsilon l}{2a} \times \mathcal{L}\{\alpha(t)\} \leq W(x,p) - U(x,p) \leq \frac{\varepsilon l}{2a} \times \mathcal{L}\{\alpha(t)\}.$

$\Rightarrow -\frac{\varepsilon l}{2a} \times \mathcal{L}\{\alpha(t)\} \leq \mathcal{L}\{w(x,t) - u(x,t)\} \leq \frac{\varepsilon l}{2a} \times \mathcal{L}\{\alpha(t)\}.$

Taking inverse Laplace transform, we get,

$-\frac{\varepsilon l}{2a} \times \alpha(t) \leq w(x,t) - u(x,t) \leq \frac{\varepsilon l}{2a} \times \alpha(t).$

$\Rightarrow |w(x,t) - u(x,t)| \leq \frac{\varepsilon l}{2a} \times \alpha(t).$

Consequently, we get

$$\max_{o \leq x \leq l} |w(x,t) - u(x,t)| \leq \frac{\varepsilon l}{2a} \times \alpha(t).$$

Hence the I-BVP (4.1) - (4.3) is HUR stable.

4.3 HUR STABILITY of (4.4)

In this section we discuss the HUR stability of third order linear partial differential equation (4.4).

Theorem 4.3.1 : If $w(x,t) \in C_1^3((0,l) \times (0,\infty))$ be an approximate solution of the I-BVP (4.4) - (4.6), then I-BVP (4.4) - (4.6) is HUR stable.

Proof : Given $\varepsilon > 0$. Suppose $w(x,t)$ be an approximate solution of the I-BVP (4.4) - (4.6). We have to show that there exists an exact solution $u(x,t) \in C_1^3((0,l) \times (0,\infty))$ of the equation (4.4) such that $|w(x,t) - u(x,t)| \leq \varepsilon \psi(x,t)$, where $\psi(x,t) \in C((0,l) \times (0,\infty))$.

From the definition of HUR stability we have

$$|\frac{\partial w}{\partial t}-a^3\frac{\partial^3 w}{\partial x^3}|\leq \varepsilon\alpha(t-\frac{l^3}{a^3}).$$

$$\Rightarrow -\varepsilon\alpha(t-\frac{l^3}{a^3})\leq \frac{\partial w}{\partial t}-a^3\frac{\partial^3 w}{\partial x^3}\leq \varepsilon\alpha(t-\frac{l^3}{a^3}), \tag{4.19}$$

where $\alpha(t-c)=0$, for $t\leq c$ and $\alpha(t-c)=x(t-c)$, for $t\geq c, c\geq 0$.

Taking Laplace transform of the equation (4.19), we get

$$-\varepsilon\mathcal{L}\{\alpha(t-\frac{l^3}{a^3})\}\leq \mathcal{L}\{\frac{\partial w}{\partial t}-a^3\frac{\partial^3 w}{\partial x^3}\}\leq \varepsilon\mathcal{L}\{\alpha(t-\frac{l^3}{a^3})\}.$$
$$\Rightarrow |\mathcal{L}\{\frac{\partial w}{\partial t}-a^3\frac{\partial^3 w}{\partial x^3}\}|\leq \varepsilon\mathcal{L}\{\alpha(t-\frac{l^3}{a^3})\}.$$

$$\Rightarrow |\mathcal{L}\left\{\frac{\partial w}{\partial t}\right\}-a^3\mathcal{L}\left\{\frac{\partial^3 w}{\partial x^3}\right\}|\leq \varepsilon\mathcal{L}\{\alpha(t-\frac{l^3}{a^3})\}. \tag{4.20}$$

Also since $w(x,t)$ satisfies boundary conditions (4.6) , we get

$\mathcal{L}\{w(0,t)\}=\mathcal{L}\{v_0(t)\}=W(0,p)=V_0(p), \mathcal{L}\{w_x(0,t)\}=\mathcal{L}\{v_1(t)\}=W_x(0,p)=V_1(p)$ and $\mathcal{L}\{w_{xx}(0,t)\}=\mathcal{L}\{v_2(t)\}=W_{xx}(0,p)=V_2(p)$.

Interchanging of operations within Laplace transform, we get

$$\mathcal{L}\left\{\frac{\partial^3 w}{\partial x^3}\right\}=\frac{d^3W}{dx^3}(x,p) \tag{4.21}$$

and

$$\mathcal{L}\left\{\frac{\partial w}{\partial t}\right\}=pW(x,p)-w(x,0) \tag{4.22}$$

Equations (4.20), (4.21) and (4.22) give

$$|pW(x,p)-w(x,0)-a^3\frac{d^3W}{dx^3}(x,p)|\leq \varepsilon\mathcal{L}\{\alpha(t-\frac{l^3}{a^3})\}.$$

$$\Rightarrow |-a^3\{\frac{d^3W}{dx^3}(x,p)-\frac{pW(x,p)}{a^3}+\frac{\mu(x)}{a^3}\}|\leq \varepsilon\mathcal{L}\{\alpha(t-\frac{l^3}{a^3})\}.$$

$$\Rightarrow |a^3\{\frac{d^3W}{dx^3}(x,p)-\frac{pW(x,p)}{a^3}+\frac{\mu(x)}{a^3}\}|\leq \varepsilon\mathcal{L}\{\alpha(t-\frac{l^3}{a^3})\}.$$

$$\Rightarrow |\frac{d^3W}{dx^3}(x,p)-\frac{pW(x,p)}{a^3}+\frac{\mu(x)}{a^3}|\leq \frac{\varepsilon}{a^3}\mathcal{L}\{\alpha(t-\frac{l^3}{a^3})\}.$$

$$\Rightarrow \left|\frac{d^3W}{dx^3}(x,p)-\frac{pW(x,p)}{a^3}+\frac{\mu(x)}{a^3}\right| \le \frac{\varepsilon}{a^3}\times\frac{x}{p^2}e^{-\frac{pl^3}{a^3}}.$$

$$-\frac{\varepsilon x}{a^3p^2}\times e^{-\frac{pl^3}{a^3}} \le \frac{d^3W}{dx^3}(x,p)-\frac{pW(x,p)}{a^3}+\frac{\mu(x)}{a^3} \le \frac{\varepsilon x}{a^3p^2}\times e^{-\frac{pl^3}{a^3}}. \tag{4.23}$$

Integrating the inequality (4.23) thrice, from 0 to x we get

$$-\frac{\varepsilon x^4}{24p^2a^3}e^{-\frac{pl^3}{a^3}} \le W(x,p)-W(0,p)-\frac{dW}{dx}(0,p)x-\frac{d^2W}{dx^2}(0,p)\frac{x^2}{2}$$
$$-\frac{p}{2a^3}\int_0^x W(s,p)(x-s)^2ds+\frac{1}{2a^3}\int_0^x \mu(s)(x-s)^2ds \le \frac{\varepsilon x^4}{24p^2a^3}e^{-\frac{pl^3}{a^3}} \tag{4.24}$$

where

$$W(0,p)=V_0(p), W_x(0,p)=V_1(p), W_{xx}(0,p)=V_2(p). \tag{4.25}$$

It is easily verified that the function $U(x,p)=\mathcal{L}\{u(x,t)\}$ which is given by

$$U(x,p)= V_0(p)+V_1(p)x+V_2(p)\frac{x^2}{2}$$
$$+\frac{p}{2a^3}\int_0^x U(s,p)(x-s)^2ds-\frac{1}{2a^3}\int_0^x \mu(s)(x-s)^2ds$$

has to satisfy the equation

$$\frac{d^3W}{dx^3}(x,p)-\frac{pW(x,p)}{a^3}+\frac{\mu(x)}{a^3}=0$$

with the boundary condition (4.25).

Next, consider the difference ,

$\Delta=|W(x,p)-U(x,p)|$.

$$=|W(x,p)-V_0(p)-V_1(p)x-V_2(p)\frac{x^2}{2}-\frac{p}{2a^3}\int_0^x U(s,p)(x-s)^2ds+\frac{1}{2a^3}\int_0^x \mu(s)(x-s)^2ds|.$$

$$=|W(x,p)-V_0(p)-V_1(p)x-V_2(p)\frac{x^2}{2}-\frac{p}{2a^3}\int_0^x W(s,p)(x-s)^2ds+\frac{1}{2a^3}\int_0^x \mu(s)(x-s)^2ds$$
$$+\frac{p}{2a^3}\int_0^x W(s,p)(x-s)^2ds-\frac{p}{2a^3}\int_0^x U(s,p)(x-s)^2ds|.$$

$$\leq |W(x,p)-V_0(p)-V_1(p)x-V_2(p)\frac{x^2}{2}-\frac{p}{2a^3}\int_0^x W(s,p)(x-s)^2ds+\frac{1}{2a^3}\int_0^x \mu(s)(x-s)^2ds|$$
$$+\frac{p}{2a^3}\int_0^x |W(s,p)-U(s,p)|(x-s)^2ds.$$

$$\leq \frac{\varepsilon x^4}{24p^2a^3}e^{-\frac{pl^3}{a^3}}+\frac{p}{2a^3}\int_0^x |W(s,p)-U(s,p)|(x-s)^2ds \text{ (by equation (4.24))} .$$

$$\leq \frac{\varepsilon l^3 x}{24p^2a^3}e^{-\frac{pl^3}{a^3}}+\frac{p}{2a^3}\int_0^x |W(s,p)-U(s,p)|(x-s)^2ds.$$

By using Grownwall inequality, we get

$$|W(x,p)-U(x,p)| \leq \frac{\varepsilon l^3 x}{24p^2a^3}e^{-\frac{pl^3}{a^3}}\times e^{\frac{p}{2a^3}\int_0^x (x-s)^2ds}.$$

$$\Rightarrow |W(x,p)-U(x,p)| \leq \frac{\varepsilon l^3 x}{24p^2a^3}e^{-\frac{pl^3}{a^3}}\times e^{\frac{px^3}{6a^3}} .$$

$$\Rightarrow |W(x,p)-U(x,p)| \leq \frac{\varepsilon l^3 x}{24p^2a^3}e^{-\frac{pl^3}{a^3}}\times e^{\frac{pl^3}{6a^3}} .$$

$$\Rightarrow |W(x,p)-U(x,p)| \leq \frac{\varepsilon l^3 x}{24p^2a^3}e^{-\frac{5pl^3}{6a^3}} .$$

$$\Rightarrow |W(x,p)-U(x,p)| \leq \frac{\varepsilon l^3}{24a^3}\times\frac{x}{p^2}e^{-\frac{5pl^3}{6a^3}}.$$

$$\Rightarrow |W(x,p)-U(x,p)| \leq \frac{\varepsilon l^3}{24\,a^3}\times \mathcal{L}\{\alpha(t-\frac{5l^3}{6a^3})\}.$$

$$\Rightarrow -\frac{\varepsilon l^3}{24\,a^3}\times \mathcal{L}\{\alpha(t-\frac{5l^3}{6a^3})\} \leq W(x,p)-U(x,p) \leq \frac{\varepsilon l^3}{24\,a^3}\times \mathcal{L}\{\alpha(t-\frac{5l^3}{6a^3})\}.$$

$$\Rightarrow -\frac{\varepsilon l^3}{24\,a^3}\times \mathcal{L}\{\alpha(t-\frac{5l^3}{6a^3})\} \leq \mathcal{L}\{w(x,t)-u(x,t)\} \leq \frac{\varepsilon l^3}{24\,a^3}\times \mathcal{L}\{\alpha(t-\frac{5l^3}{6a^3})\}.$$

Taking inverse Laplace transform, we get,

$$-\frac{\varepsilon l^3}{24\,a^3}\times \{\alpha(t-\frac{5l^3}{6a^3})\} \leq w(x,t)-u(x,t) \leq \frac{\varepsilon l^3}{24\,a^3}\times \{\alpha(t-\frac{5l^3}{6a^3})\} .$$

$$\Rightarrow |w(x,t)-u(x,t)| \leq \frac{\varepsilon l^3}{24\,a^3}\times \{\alpha(t-\frac{5l^3}{6a^3})\}.$$

Consequently, we have

$$\max_{o\leq x\leq l}|w(x,t)-u(x,t)| \leq \frac{\varepsilon l^3}{24\,a^3}\{\alpha(t-\frac{5l^3}{6a^3})\}.$$

Hence the I-BVP (4.4) – (4.6) is HUR stable.

4.4 HUR STABILITY OF (4.7)

In this section we shall extend the results of previous sections and prove the HUR stability of nth order linear homogeneous partial differential equation (4.7).

Theorem 4.4.1 : If $w(x,t) \in C_1^n((0,l)\times(0,\infty))$ is an approximate solution of the I-BVP (4.7) - (4.9), then I-BVP (4.7) - (4.9) is HUR stable.

Proof : Let $\varepsilon > 0$ be given. Let $w(x,t)$ be an approximate solution of the I-BVP (4.7) - (4.9). We shall show that there is an exact solution $u(x,t) \in C_1^n((0,l)\times(0,\infty))$ of the equation (4.7) such that $|w(x,t)-u(x,t)| \leq \varepsilon\psi(x,t)$, where $\psi(x,t) \in C((0,l)\times(0,\infty))$.

Using the definition of HUR stability, we get

$$\left|\frac{\partial w}{\partial t} - a^n\frac{\partial^n w}{\partial x^n}\right| \leq \varepsilon\alpha(t-\frac{l^n}{a^n}).$$

This gives

$$-\varepsilon\alpha(t-\frac{l^n}{a^n}) \leq \frac{\partial w}{\partial t} - a^n\frac{\partial^n w}{\partial x^n} \leq \varepsilon\alpha(t-\frac{l^n}{a^n}), \tag{4.26}$$

where $\alpha(t-c)=0$, for $t\leq c$ and $\alpha(t-c)=x(t-c)$, for $t\geq c, c\geq 0$.

Taking Laplace transform of the equation (4.26), we get

$$-\varepsilon\mathcal{L}\{\alpha(t-\frac{l^n}{a^n})\} \leq \mathcal{L}\{\frac{\partial w}{\partial t} - a^n\frac{\partial^n w}{\partial x^n}\} \leq \varepsilon\mathcal{L}\{\alpha(t-\frac{l^n}{a^n})\},$$

and hence

$$\left|\mathcal{L}\{\frac{\partial w}{\partial t} - a^n\frac{\partial^n w}{\partial x^n}\}\right| \leq \varepsilon\mathcal{L}\{\alpha(t-\frac{l^n}{a^n})\}.$$

This gives,

$$\left|\mathcal{L}\left\{\frac{\partial w}{\partial t}\right\} - a^n \mathcal{L}\left\{\frac{\partial^n w}{\partial x^n}\right\}\right| \le \varepsilon \mathcal{L}\left\{\alpha(t - \frac{l^n}{a^n})\right\}. \tag{4.27}$$

Further, since $w(x,t)$ satisfies boundary conditions (4.9), we get

$\mathcal{L}\{w(0,t)\} = \mathcal{L}\{v_0(t)\} = W(0,p) = V_0(p)$,

$\mathcal{L}\{w_x(0,t)\} = \mathcal{L}\{v_1(t)\} = W_x(0,p) = V_1(p)$,

$\mathcal{L}\{w_{xx}(0,t)\} = \mathcal{L}\{v_2(t)\} = W_{xx}(0,p) = V_2(p), \cdots\cdots,$

$\cdots\cdots, \mathcal{L}\{w_{xx\cdots\cdots x}(0,t)\} = \mathcal{L}\{v_{n-1}(t)\} = W_{xx\cdots\cdots x}(0,p) = V_{n-1}(p).$

As $\mathcal{L}\left\{\frac{\partial^n w}{\partial x^n}\right\} = \frac{d^n W}{dx^n}(x,p)$, $\mathcal{L}\left\{\frac{\partial w}{\partial t}\right\} = pW(x,p) - w(x,0)$ and $\mathcal{L}\left\{\alpha(t - \frac{l^n}{a^n})\right\} = \frac{x}{p^2} e^{-p\frac{l^n}{a^n}}$, with (4.27), we get

$$\left| -a^n \{\frac{d^n W}{dx^n}(x,p) - \frac{pW(x,p)}{a^n} + \frac{\mu(x)}{a^n}\}\right| \le \varepsilon \frac{x}{p^2} e^{-p\frac{l^n}{a^n}}.$$

$$\Rightarrow |a^n \{\frac{d^n W}{dx^n}(x,p) - \frac{pW(x,p)}{a^n} + \frac{\mu(x)}{a^n}\}| \le \varepsilon \frac{x}{p^2} e^{-p\frac{l^n}{a^n}}.$$

$$\Rightarrow \left|\frac{d^n W}{dx^n}(x,p) - \frac{pW(x,p)}{a^n} + \frac{\mu(x)}{a^n}\right| \le \frac{\varepsilon}{a^n} \frac{x}{p^2} e^{-p\frac{l^n}{a^n}}.$$

Hence we get

$$-\frac{\varepsilon}{a^n} \frac{x}{p^2} e^{-p\frac{l^n}{a^n}} \le \frac{d^n W}{dx^n}(x,p) - \frac{pW(x,p)}{a^n} + \frac{\mu(x)}{a^n} \le \frac{\varepsilon}{a^n} \frac{x}{p^2} e^{-p\frac{l^n}{a^n}}. \tag{4.28}$$

Integrating the inequality (4.28), n times from 0 to x, we get

$$-\frac{\varepsilon x^{n+1}}{(n+1)!p^2 a^n} e^{-p\frac{l^n}{a^n}} \le W(x,p) - W(0,p) - \frac{dW(0,p)}{dx} x - \frac{d^2 W}{dx^2}(0,p)\frac{x^2}{2!}$$
$$-\frac{d^3 W}{dx^3}(0,p)\frac{x^3}{3!}\cdots\cdots - \frac{d^{n-1}W}{dx^{n-1}}(0,p)\frac{x^{n-1}}{n-1} - \frac{p}{(n-1)!a^n}\int_0^x W(s,p)(x-s)^{n-1}ds$$
$$+\frac{1}{(n-1)!a^n}\int_0^x \mu(s)(x-s)^{n-1}ds \le \frac{\varepsilon x^{n+1}}{(n+1)!p^2 a^n} e^{-p\frac{l^n}{a^n}},$$

i. e.

$$-\frac{\varepsilon x^{n+1}}{(n+1)!p^2 a^n} e^{-p\frac{l^n}{a^n}} \le W(x,p) - V_0(p) - V_1(p)x - V_2(p)\frac{x^2}{2!}$$
$$-V_3(p)\frac{x^3}{3!}\cdots\cdots - V_{n-1}(p)\frac{x^{n-1}}{n-1} - \frac{p}{(n-1)!a^n}\int_0^x W(s,p)(x-s)^{n-1}ds$$

$$+\frac{1}{(n-1)!a^n}\int_0^x \mu(s)(x-s)^{n-1}ds \le \frac{\varepsilon x^{n+1}}{(n+1)!p^2a^n}e^{-p\frac{l^n}{a^n}}. \tag{4.29}$$

It is easily verified that the function $U(x,p) = \mathcal{L}\{u(x,t)\}$ which is given by

$$U(x,p) = V_0(p) + V_1(p)x + V_2(p)\frac{x^2}{2!} + \cdots\cdots\cdots + V_{n-1}(p)\frac{x^{n-1}}{(n-1)!}$$
$$+\frac{p}{(n-1)!a^n}\int_0^x U(s,p)(x-s)^{n-1}ds - \frac{1}{(n-1)!a^n}\int_0^x \mu(s)(x-s)^{n-1}ds$$

has to satisfy the equation $\frac{d^nW}{dx^n}(x,p) - \frac{pW}{a^n}(x,p) + \frac{\mu(x)}{a^n} = 0$, with the boundary conditions

$$W(0,p) = V_0(p), W_x(0,p) = V_1(p), \cdots\cdots, W_{xx\cdots x}(0,p) = V_{n-1}(p). \tag{4.30}$$

Next consider, the difference,

$$\Delta = |W(x,p) - U(x,p)|.$$
$$= \Big| W(x,p) - V_0(p) - V_1(p)x - V_2(p)\frac{x^2}{2!} - \cdots\cdots\cdots - V_{n-1}(p)\frac{x^{n-1}}{(n-1)!}$$
$$-\frac{p}{(n-1)!a^n}\int_0^x U(s,p)(x-s)^{n-1}ds + \frac{1}{(n-1)!a^n}\int_0^x \mu(s)(x-s)^{n-1}ds\Big|.$$

$$= \Big| W(x,p) - V_0(p) - V_1(p)x - V_2(p)\frac{x^2}{2!} - \cdots\cdots\cdots - V_{n-1}(p)\frac{x^{n-1}}{(n-1)!}$$
$$-\frac{p}{(n-1)!a^n}\int_0^x W(s,p)(x-s)^{n-1}ds + \frac{1}{(n-1)!a^n}\int_0^x \mu(s)(x-s)^{n-1}ds$$
$$+\frac{p}{(n-1)!a^n}\int_0^x W(s,p)(x-s)^{n-1}ds - \frac{p}{(n-1)!a^n}\int_0^x U(s,p)(x-s)^{n-1}ds\Big|.$$

$$\le \Big| W(x,p) - V_0(p) - V_1(p)x - V_2(p)\frac{x^2}{2!} - \cdots\cdots\cdots - V_{n-1}(p)\frac{x^{n-1}}{(n-1)!}$$
$$-\frac{p}{(n-1)!a^n}\int_0^x W(s,p)(x-s)^{n-1}ds + \frac{1}{(n-1)!a^n}\int_0^x \mu(s)(x-s)^{n-1}ds\Big|$$
$$+\frac{p}{(n-1)!a^n}\int_0^x |W(s,p) - U(s,p)|(x-s)^{n-1}ds.$$

$$\le \frac{\varepsilon x^{n+1}}{(n+1)!p^2a^n}e^{-\frac{pl^n}{a^n}} + \frac{p}{(n-1)!a^n}\int_0^x |W(s,p) - U(s,p)|(x-s)^{n-1}ds,$$

(by equation (4.29)).

$$\le \frac{\varepsilon l^n}{(n+1)!a^n}\frac{x}{p^2}e^{-\frac{pl^n}{a^n}} + \frac{p}{(n-1)!a^n}\int_0^x |W(s,p) - U(s,p)|(x-s)^{n-1}ds.$$

By using Grownwall inequality we get

$$|W(x,p)-U(x,p)| \leq \frac{\varepsilon l^n}{(n+1)!a^n}\frac{x}{p^2}e^{-\frac{pl^n}{a^n}} \ e^{\int_0^x \frac{p}{(n-1)!a^n}(x-s)^{n-1}ds}.$$

$$\Rightarrow |W(x,p)-U(x,p)| \leq \frac{\varepsilon l^n}{(n+1)!a^n}\frac{x}{p^2}e^{-\frac{pl^n}{a^n}} \ e^{\frac{pl^n}{n!a^n}}.$$

$$\Rightarrow |W(x,p)-U(x,p)| \leq \frac{\varepsilon l^n}{(n+1)!a^n}\frac{x}{p^2} \ e^{-\frac{p(n!-1)l^n}{a^n n!}}.$$

$$\Rightarrow |W(x,p)-U(x,p)| \leq \frac{\varepsilon l^n}{(n+1)!a^n} \ \mathcal{L}\{\alpha(t-\frac{(n!-1)l^n}{n!a^n})\}.$$

$$\Rightarrow -\frac{\varepsilon l^n}{(n+1)!a^n} \ \mathcal{L}\{\alpha(t-\frac{(n!-1)l^n}{n!a^n})\} \leq W(x,p)-U(x,p) \leq \frac{\varepsilon l^n}{(n+1)!a^n} \ \mathcal{L}\{\alpha(t-\frac{(n!-1)l^n}{n!a^n})\}.$$

$$\Rightarrow -\frac{\varepsilon l^n}{(n+1)!a^n}\mathcal{L}\{\alpha(t-\frac{(n!-1)l^n}{n!a^n})\} \leq \mathcal{L}\{w(x,t)-u(x,t)\} \leq \frac{\varepsilon l^n}{(n+1)!a^n}\mathcal{L}\{\alpha(t-\frac{(n!-1)l^n}{n!a^n})\}.$$

Taking inverse Laplace transform, we get,

$$\Rightarrow -\frac{\varepsilon l^n}{(n+1)!a^n}\left\{\alpha(t-\frac{(n!-1)l^n}{n!a^n})\right\} \leq w(x,t)-u(x,t) \leq \frac{\varepsilon l^n}{(n+1)!a^n}\left\{\alpha(t-\frac{(n!-1)l^n}{n!a^n})\right\}.$$

$$\Rightarrow |w(x,t)-u(x,t)| \leq \frac{\varepsilon l^n}{(n+1)!a^n}\left\{\alpha(t-\frac{(n!-1)l^n}{n!a^n})\right\}.$$

Consequently, we have

$$\max_{o\leq x\leq l}|w(x,t)-u(x,t)| \leq \ \frac{\varepsilon l^n}{(n+1)!a^n}\left\{\alpha(t-\frac{(n!-1)l^n}{n!a^n})\right\}.$$

Hence the I-BVP (4.7) - (4.9) is HUR stable.

Remark 4.4.1. We have established the HUR stability for first, third and n^{th} order linear homogeneous partial differential equations (4.1), (4.4) and (4.7) respectively by employing Laplace transform method.

Chapter 5

HUR STABILITY OF LINEAR NON-HOMOGENEOUS PDE

Some contents of this chapter is presented at the Conference [see CP3].

5.1 INTRODUCTION

In this chapter, we study the HUR stability of the second and third order linear non-homogeneous partial differential equation. We prove the results by employing Banach contraction principle.

For this purpose, we consider second order linear non-homogeneous partial differential equation of the type

$$\begin{aligned} r(x,t)u_{tt}(x,t) + p(x,t)u_{xt}(x,t) + q(x,t)u_t(x,t) + p_t(x,t)u_x(x,t) \\ - p_x(x,t)u_t(x,t) = g(x,t,u(x,t)). \end{aligned} \tag{5.1}$$

Here $p,q,r : J \times J \to \mathbb{R}$ are differentiable functions at least once w. r. t. both the arguments and $r(x,t) \neq 0, \forall x,t \in J, J = [a,b]$ is a closed interval and $g : J \times J \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Also, we consider third order linear non-homogeneous partial differential equation of the type

$$\begin{aligned} s(x,t)u_{ttt}(x,t) + r(x,t)u_{tt}(x,t) + p(x,t)u_{xt}(x,t) + q(x,t)u_t(x,t) + p_t(x,t)u_x(x,t) \\ - p_x(x,t)u_t(x,t) = g(x,t,u(x,t)). \end{aligned} \tag{5.2}$$

Here $s,p,q,r : J \times J \to \mathbb{R}$ are differentiable functions at least once w. r. t. both the arguments and $s(x,t) \neq 0, \forall x,t \in J, J = [a,b]$ is a closed interval and $g : J \times J \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

We have proved the HUR stability of the above two equations. First, we need following definition.

Definition 5.1.1 : A function $u : J \times J \to \mathbb{R}$ is called a solution of the equation (5.1) (or (5.2)) if $u \in C^2(J \times J)$ (or $u \in C^3(J \times J)$) and satisfies the equation (5.1) (or (5.2)).

We now define the HUR stability for (5.1) and (5.2).

Definition 5.1.2 : The equation (5.1) is said to be HUR stable if the following holds:

Let $\phi : J\times J \to (0,\infty)$ be a continuous function. Then there exists a continuous function $\Psi : J\times J \to (0,\infty)$, which depends on ϕ such that whenever $u : J\times J \to \mathbb{R}$ is a continuous function with

$$\begin{aligned} |r(x,t)u_{tt}(x,t)+p(x,t)u_{xt}(x,t)+q(x,t)u_t(x,t)+p_t(x,t)u_x(x,t) \\ -p_x(x,t)u_t(x,t)-g(x,t,u(x,t))| \leq \phi(x,t), \end{aligned} \tag{5.3}$$

there exists a solution $u_0 : J\times J \to \mathbb{R}$ of (5.1) such that

$$|u(x,t)-u_0(x,t)| \leq \Psi(x,t),\ \ \forall (x,t) \in J\times J.$$

Definition 5.1.3: The equation (5.2) is said to be HUR stable if the following holds:

Let $\phi : J\times J \to (0,\infty)$ be a continuous function. Then there exists a continuous function $\Psi : J\times J \to (0,\infty)$, which depends on ϕ such that whenever $u : J\times J \to \mathbb{R}$ is a continuous function with

$$\begin{aligned} |s(x,t)u_{ttt}(x,t)+r(x,t)u_{tt}(x,t)+p(x,t)u_{xt}(x,t)+q(x,t)u_t(x,t) \\ +p_t(x,t)u_x(x,t)-\ p_x(x,t)u_t(x,t)-g(x,t,u(x,t))| \leq \phi(x,t), \end{aligned} \tag{5.4}$$

there exists a solution $u_0 : J\times J \to \mathbb{R}$ of (5.2) such that

$$|u(x,t)-u_0(x,t)| \leq \Psi(x,t),\ \ \forall (x,t) \in J\times J.$$

We need the following result.

Theorem 5.1.1:(**Banach Contraction Principle**) :

Let (Y,d) be a complete metric space, then each contraction map $T : Y \to Y$ has a unique fixed point, that is, there exists $b \in Y$ such that $Tb = b$.

Moreover, $d(b,w) \leq \frac{1}{(1-\alpha)} d(w,Tw), \forall w \in Y$ and $0 \leq \alpha < 1$.

We shall follow the approach of Gordji et al. [18] to establish our results.

5.2 HUR STABILITY OF (5.1)

In this section we prove the HUR stability of the second order linear non-homogeneous PDE (5.1). We have the following result.

Theorem 5.2.1: Let $c \in J$. Let p, q, r, g be as in (5.1) with following additional conditions:

(i) $|r(x,t)| \geq 1, \forall x,t \in J$.

(ii) $\phi : J \times J \to (0,\infty)$ be a continuous function and $M : J \times J \to [1,\infty)$ be an integrable function.

(iii) Assume that there exists $\alpha, 0 < \alpha < 1$ such that

$$\int_c^t M(x,s)\phi(x,s)ds \leq \alpha\phi(x,t). \tag{5.5}$$

Let

$$h(x,c) = -\Big\{r(x,c)u_t(x,c) + p(x,c)u_x(x,c) - p_x(x,c)u(x,c) + q(x,c)u(x,c)\Big\} \tag{5.6}$$

and

$$\begin{aligned} K(x,t,u(x,t)) = -\{|r(x,t)|\}^{-1} \times \Big\{p(x,t)u_x(x,t) + h(x,c) \\ - \int_c^t f(x,\tau,u(x,\tau))d\tau - \int_c^t u_t(x,\tau)r_t(x,\tau)d\tau\Big\}. \end{aligned} \tag{5.7}$$

Suppose that the following holds :

C1 : $|K(x,t,l(x,t)) - K(x,t,m(x,t))| \leq M(x,t)|l(x,t) - m(x,t)|, \ \forall\, c,x,t \in J$ and

$h,u,l,m \in C(J \times J)$.

C2 : $u : J \times J \to \mathbb{R}$ be a function satisfying the inequality (5.3) .

C3 : $p_x(x,t) = q(x,t), \forall (x,t) \in J \times J$.

Then there exists a unique solution $u_0 : J \times J \to \mathbb{R}$ of the equation (5.1) of the form

$$u_0(x,t) = u(x,c) + \int_c^t K(x,s,u_0(x,s))ds$$

such that

$$|u(x,t) - u_0(x,t)| \leq \frac{\alpha}{(1-\alpha)}\phi(x,t), \forall x,t \in J.$$

Proof : Consider

$$\begin{aligned}
&|r(x,t)u_{tt}(x,t) + p(x,t)u_{xt}(x,t) + q(x,t)u_t(x,t) + p_t(x,t)u_x(x,t) \\
&\quad - p_x(x,t)u_t(x,t) - g(x,t,u(x,t))| \\
&= |\{r(x,t)u_t(x,t) + p(x,t)u_x(x,t) - p_x(x,t)u(x,t) + q(x,t)u(x,t)\}_t \\
&\quad - g(x,t,u(x,t)) - r_t(x,t)u_t(x,t)|
\end{aligned}$$

From the inequality (5.3), we get

$$\begin{aligned}
&|\left\{r(x,t)u_t(x,t) + p(x,t)u_x(x,t) - p_x(x,t)u(x,t) + q(x,t)u(x,t)\right\}_t \\
&\quad - g(x,t,u(x,t)) - r_t(x,t)u_t(x,t)| \leq \phi(x,t).
\end{aligned}$$

Integrating from c to t, we get,

$$\begin{aligned}
&|r(x,t)u_t(x,t) + p(x,t)u_x(x,t) - p_x(x,t)u(x,t) + q(x,t)u(x,t) \\
&\qquad - \left\{r(x,c)u_t(x,c) + p(x,c)u_x(x,c) - p_x(x,c)u(x,c) + q(x,c)u(x,c)\right\} \\
&\qquad - \textstyle\int_c^t g(x,\tau,u(x,\tau))d\tau - \int_c^t u_\tau(x,\tau)r_\tau(x,\tau)d\tau| \leq \int_c^t \phi(x,\tau)d\tau.
\end{aligned}$$

Using condition C3, we get,

$$\begin{aligned}
&\Big| r(x,t)u_t(x,t) + p(x,t)u_x(x,t) + h(x,c) - \textstyle\int_c^t g(x,\tau,u(x,\tau))d\tau \\
&\qquad - \textstyle\int_c^t u_\tau(x,\tau)r_\tau(x,\tau)\Big| \leq \int_c^t \phi(x,\tau)d\tau,
\end{aligned}$$

where $h(x,c)$ is given by equation (5.6).

$$\begin{aligned}
&\Rightarrow |r(x,t)|\Big|u_t(x,t) + |r(x,t)|^{-1}\Big[p(x,t)u_x(x,t) + h(x,c) - \textstyle\int_c^t g(x,\tau,u(x,\tau))d\tau \\
&\qquad - \textstyle\int_c^t u_\tau(x,\tau)r_\tau(x,\tau)\Big]\Big| \leq \int_c^t \phi(x,\tau)d\tau.
\end{aligned}$$

$$\Rightarrow |r(x,t)||u_t(x,t) - K(x,t,u(x,t))| \leq \textstyle\int_c^t \phi(x,\tau)d\tau,$$

where $K(x,t,u(x,t))$ is given by equation (5.7).

$\Rightarrow |u_t(x,t)-K(x,t,u(x,t))| \leq |r(x,t)|^{-1}\int_c^t \phi(x,\tau)d\tau.$

$\Rightarrow |u_t(x,t)-K(x,t,u(x,t))| \leq \int_c^t \phi(x,\tau)d\tau, \qquad (\because |r(x,t)| \geq 1).$

Since $M : J\times J \to [1,\infty)$ is an integrable function, we have

$|u_t(x,t)-K(x,t,u(x,t))| \leq \int_c^t M(x,\tau)\phi(x,\tau)d\tau.$

Using inequality (5.5), we have

$\Rightarrow |u_t(x,t)-K(x,t,u(x,t))| \leq \int_c^t M(x,\tau)\phi(x,\tau)d\tau \leq \alpha\phi(x,t).$

$\Rightarrow |u_t(x,t)-K(x,t,u(x,t))| \leq \alpha\phi(x,t).$

$\Rightarrow |u_t(x,t)-K(x,t,u(x,t))| \leq \phi(x,t), \qquad (\because 0<\alpha<1).$

$$\Rightarrow -\phi(x,t) \leq u_t(x,t)-K(x,t,u(x,t)) \leq \phi(x,t). \tag{5.8}$$

$\Rightarrow u_t(x,t)-K(x,t,u(x,t)) \leq \phi(x,t).$

Integrating from c to t, we get

$$\int_c^t \{u_\tau(x,\tau)-K(x,\tau,u(x,\tau))\}d\tau \leq \int_c^t \phi(x,\tau)d\tau$$

$$\therefore u(x,t)-\left\{u(x,c)+\int_c^t K(x,\tau,u(x,\tau))d\tau\right\} \leq \int_c^t \phi(x,\tau)d\tau.$$

Since $M : J\times J \to [1,\infty)$ is an integrable function, we have

$$u(x,t)-\left\{u(x,c)+\int_c^t K(x,\tau,u(x,\tau))d\tau\right\} \leq \int_c^t M(x,\tau)\phi(x,\tau)d\tau.$$

By using inequality (5.5), we get

$$\begin{aligned} u(x,t)-\left\{u(x,c)+\int_c^t K(x,\tau,u(x,\tau))d\tau\right\} &\leq \int_c^t M(x,\tau)\phi(x,\tau)d\tau \\ &\leq \alpha\phi(x,t). \end{aligned}$$

Thus

$$u(x,t) - \left\{ u(x,c) + \int_c^t K(x,\tau,u(x,\tau))d\tau \right\} \leq \alpha\phi(x,t) \tag{5.9}$$

In a similar way, from the left inequality of (5.8), we obtain

$$-\left[u(x,t) - \left\{ u(x,c) + \int_c^t K(x,\tau,u(x,\tau))d\tau \right\} \right] \leq \alpha\phi(x,t) \tag{5.10}$$

From the inequalities (5.9) and (5.10), we get

$$\left| u(x,t) - \left\{ u(x,c) + \int_c^t K(x,\tau,u(x,\tau))d\tau \right\} \right| \leq \alpha\phi(x,t). \tag{5.11}$$

Let Y be the set of all continuously differentiable functions $l : J \times J \to \mathbb{R}$. We define a metric d and an operator T on Y as follows : For $l, m \in Y$

$$d(l,m) = \sup_{x,t \in J} \frac{|l(x,t) - m(x,t)|}{\phi(x,t)}$$

and the operator

$$(Tm)(x,t) = u(x,c) + \int_c^t K(x,\tau,m(x,\tau))d\tau. \tag{5.12}$$

Consider

$$d(Tl,Tm) = \sup_{x,t\in J}\left\{\frac{(Tl)(x,t)-(Tm)(x,t)}{\phi(x,t)}\right\}$$

$$= \sup_{x,t\in J}\left\{\frac{\int_c^t K(x,\tau,l(x,\tau))d\tau - \int_c^t K(x,\tau,m(x,\tau))d\tau}{\phi(x,t)}\right\}$$

$$\leq \sup_{x,t\in J}\left\{\frac{\int_c^t |K(x,\tau,l(x,\tau)) - K(x,\tau,m(x,\tau))|d\tau}{\phi(x,t)}\right\}.$$

By using condition C1, we get

$$d(Tl,Tm) \leq \sup_{x,t\in J}\left\{\frac{\int_c^t \{M(x,\tau)|l(x,\tau)-m(x,\tau)|\}d\tau}{\phi(x,t)}\right\}$$

$$= \sup_{x,t\in J}\left\{\frac{\int_c^t \{M(x,\tau)\phi(x,\tau)\times\frac{|l(x,\tau)-m(x,\tau)|}{\phi(x,\tau)}\}d\tau}{\phi(x,t)}\right\}$$

$$\leq \ \sup_{x,t\in J}\left\{\frac{\int_c^t\{M(x,\tau)\phi(x,\tau)\times\sup_{x,\tau\in J}\frac{|l(x,\tau)-m(x,\tau)|}{\phi(x,\tau)}\}d\tau}{\phi(x,t)}\right\}$$

$$= \ d(l,m)\times\sup_{x,t\in J}\left\{\frac{\int_c^t\{M(x,\tau)\phi(x,\tau)\}d\tau}{\phi(x,t)}\right\}$$

By using inequality (5.5), we get

$d(Tl,Tm)\leq \alpha d(l,m)$.

Hence by Banach contraction principle, there exists a unique $u_0\in X$ such that

$Tu_0 = u_0$, that is

$u(x,c)+\int_c^t K(x,\tau,u_0(x,\tau))d\tau = u_0(x,t)$ (by using equation (5.12))

and

$$d(u_0,u)\leq \frac{1}{(1-\alpha)}d(u,Tu). \tag{5.13}$$

Now by inequality (5.11), we get

$$|u(x,t)-(Tu)(x,t)|\leq \alpha\phi(x,t).$$

$$\Rightarrow \frac{|u(x,t)-(Tu)(x,t)|}{\phi(x,t)}\leq \alpha.$$

$$\Rightarrow sup_{x,t\in J}\frac{|u(x,t)-(Tu)(x,t)|}{\phi(x,t)}\leq \alpha.$$

Thus

$$d(u,Tu)\leq \alpha \tag{5.14}$$

Again

$$d(u_0,u) = \sup_{x,t\in J}\frac{|u_0(x,t)-u(x,t)|}{\phi(x,t)}.$$

From equation (5.13), we get

$$d(u_0,u) \le \frac{1}{(1-\alpha)} d(u,Tu).$$

$$\sup_{x,t\in J} \frac{|u_0(x,t)-u(x,t)|}{\phi(x,t)} \le \frac{1}{(1-\alpha)} d(u,Tu).$$

$$\frac{|u_0(x,t)-u(x,t)|}{\phi(x,t)} \le \sup_{x,t\in J} \frac{|u_0(x,t)-u(x,t)|}{\phi(x,t)} \le \frac{1}{(1-\alpha)} d(u,Tu).$$

$$\frac{|u_0(x,t)-u(x,t)|}{\phi(x,t)} \le \frac{1}{(1-\alpha)} d(u,Tu).$$

From equation (5.14), we get

$$\frac{|u_0(x,t)-u(x,t)|}{\phi(x,t)} \le \frac{1}{(1-\alpha)} \alpha.$$

$$|u(x,t)-u_0(x,t)| \le \frac{\alpha}{(1-\alpha)} \phi(x,t), \ \forall x,t \in J.$$

Consequently, the equation (5.1) is HUR stable.

5.3 HUR STABILITY OF (5.2)

In this section we prove the HUR stability of third order linear non-homogeneous PDE (5.2). We prove the following result.

Theorem 6.3.1: Let $c \in J$. Let p, q, r, s, g be as in (5.2) with additional conditions:

(i) $|s(x,t)| \ge 1, \forall x,t \in J.$

(ii) $\phi : J \times J \to (0,\infty)$ be a continuous function and $M : J \times J \to [1,\infty)$ be an integrable function.

(iii) Assume that there exists $\alpha, 0 < \alpha < 1$ such that

$$\int_c^t M(x,s)\phi(x,s)ds \le \alpha\phi(x,t) \tag{5.15}$$

and

$$\int_c^t \int_c^y \{M(x,z)\phi(x,z)\}dzdy \le \alpha\phi(x,t). \tag{5.16}$$

Let

$$h(x,c) = -\Big\{s(x,c)u_{tt}(x,c)+r(x,c)u_t(x,c)+p(x,c)u_x(x,c)-p_x(x,c)u(x,c)+q(x,c)u(x,c)\Big\} \tag{5.17}$$

and

$$K(x,t,u(x,t)) = -\{|s(x,t)|\}^{-1}\times\Big\{r(x,t)u_t(x,t)+p(x,t)u_x(x,t)+h(x,c) - \int_c^t g(x,\tau,u(x,\tau))d\tau - \int_c^t u_\tau(x,\tau)r_\tau(x,\tau)d\tau - \int_c^t s_\tau(x,\tau)u_{\tau\tau}(x,\tau)d\tau\Big\}. \tag{5.18}$$

Suppose that the following holds :

C1 : $|K(x,t,l(x,t))-K(x,t,m(x,t))| \leq M(x,t)|l(x,t)-m(x,t)|, \quad \forall\, c,x,t \in J$ and $h,u,l,m \in C(J\times J)$.

C2 : $u : J\times J \to \mathbb{R}$ be a function satisfying the inequality (5.4).

C3 : $p_x(x,t) = q(x,t), \forall (x,t) \in J\times J$.

Then there exists a unique solution $u_0 : J\times J \to \mathbb{R}$ of the equation (5.2) of the form

$$u_0(x,t) = u(x,c) + \textstyle\int_c^t \int_c^y K(x,z,u_0(x,z))dzdy$$

such that

$$|u(x,t)-u_0(x,t)| \leq \frac{\alpha}{(1-\alpha)}\phi(x,t), \forall x,t \in J.$$

Proof : Consider

$$|s(x,t)u_{ttt}(x,t)+r(x,t)u_{tt}(x,t)+p(x,t)u_{xt}(x,t)+q(x,t)u_t(x,t)+p_t(x,t)u_x(x,t) - p_x(x,t)u_t(x,t)-g(x,t,u(x,t))|$$

$$= \Big|\Big\{s(x,t)u_{tt}(x,t)+r(x,t)u_t(x,t)+p(x,t)u_x(x,t)-p_x(x,t)u(x,t)+q(x,t)u(x,t)\Big\}_t - g(x,t,u(x,t))-r_t(x,t)u_t(x,t)-s_t(x,t)u_{tt}(x,t)\Big|.$$

From the inequality (5.4), we get

$$\Big|\big\{s(x,t)u_{tt}(x,t)+r(x,t)u_t(x,t)+p(x,t)u_x(x,t)-p_x(x,t)u(x,t)+q(x,t)u(x,t)\big\}_t$$
$$-g(x,t,u(x,t))-r_t(x,t)u_t(x,t)-s_t(x,t)u_{tt}(x,t)\Big|\leq\phi(x,t).$$

Integrating from c to t, we get

$$\Big|s(x,t)u_{tt}(x,t)+r(x,t)u_t(x,t)+p(x,t)u_x(x,t)-p_x(x,t)u(x,t)+q(x,t)u(x,t)$$
$$-\big\{s(x,c)u_{tt}(x,c)+r(x,c)u_t(x,c)+p(x,c)u_x(x,c)-p_x(x,c)u(x,c)+q(x,c)u(x,c)\big\}$$
$$-\int_c^t g(x,\tau,u(x,\tau))d\tau-\int_c^t u_\tau(x,\tau)r_\tau(x,\tau)d\tau-\int_c^t s_\tau(x,\tau)u_{\tau\tau}(x,\tau)d\tau\Big|\leq\int_c^t\phi(x,\tau)d\tau.$$

Using condition C3, we get

$$\Big|s(x,t)u_{tt}(x,t)+r(x,t)u_t(x,t)+p(x,t)u_x(x,t)+h(x,c)-\int_c^t g(x,\tau,u(x,\tau))d\tau$$
$$-\int_c^t u_\tau(x,\tau)r_\tau(x,\tau)d\tau-\int_c^t s_\tau(x,\tau)u_{\tau\tau}(x,\tau)d\tau\Big|\leq\int_c^t\phi(x,\tau)d\tau,$$

where $h(x,c)$ is given by equation (5.17).

$$\Rightarrow|s(x,t)|\Big|u_{tt}(x,t)+|s(x,t)|^{-1}\Big\{r(x,t)u_t(x,t)+p(x,t)u_x(x,t)+h(x,c)$$
$$-\int_c^t g(x,\tau,u(x,\tau))d\tau-\int_c^t u_\tau(x,\tau)r_\tau(x,\tau)d\tau-\int_c^t s_\tau(x,\tau)u_{\tau\tau}(x,\tau)d\tau\Big\}\Big|$$
$$\leq\int_c^t\phi(x,\tau)d\tau.$$

$$\Rightarrow|s(x,t)||u_{tt}(x,t)-K(x,t,u(x,t))|\leq\int_c^t\phi(x,\tau)d\tau,$$

where $K(x,t,u(x,t))$ is given by equation (5.18).

$$\Rightarrow|u_{tt}(x,t)-K(x,t,u(x,t))|\leq|s(x,t)|^{-1}\int_c^t\phi(x,\tau)d\tau.$$

$$\Rightarrow|u_{tt}(x,t)-K(x,t,u(x,t))|\leq\int_c^t\phi(x,\tau)d\tau,\qquad(\because|s(x,t)|\geq1).$$

Since $M:J\times J\to[1,\infty)$ is an integrable function, we have

$$|u_{tt}(x,t)-K(x,t,u(x,t))|\leq\int_c^t M(x,\tau)\phi(x,\tau)d\tau.$$

Using inequality (5.15), we have

$$|u_{tt}(x,t)-K(x,t,u(x,t))|\leq\int_c^t M(x,\tau)\phi(x,\tau)d\tau\leq\alpha\phi(x,t).$$

$\Rightarrow |u_{tt}(x,t) - K(x,t,u(x,t))| \leq \alpha\phi(x,t).$

$\Rightarrow |u_{tt}(x,t) - K(x,t,u(x,t))| \leq \phi(x,t), \qquad (\because 0 < \alpha < 1).$

$$\Rightarrow -\phi(x,t) \leq u_{tt}(x,t) - K(x,t,u(x,t)) \leq \phi(x,t). \tag{5.19}$$

$\Rightarrow u_{tt}(x,t) - K(x,t,u(x,t)) \leq \phi(x,t).$

Integrating from c to t, we get

$$\int_c^t \{u_{\tau\tau}(x,\tau) - K(x,\tau,u(x,\tau))\}d\tau \leq \int_c^t \phi(x,\tau)d\tau.$$

$\therefore u_t(x,t) - u_t(x,c) - \int_c^t K(x,\tau,u(x,\tau))d\tau \leq \int_c^t \phi(x,\tau)d\tau.$

Since $M : J \times J \to [1,\infty)$ is an integrable function we have,

$$u_t(x,t) - \left\{u_t(x,c) + \int_c^t K(x,\tau,u(x,\tau))d\tau\right\} \leq \int_c^t M(x,\tau)\phi(x,\tau)d\tau.$$

By using inequality (5.15), we get

$$u_t(x,t) - \left\{u_t(x,c) + \int_c^t K(x,\tau,u(x,\tau))d\tau\right\} \leq \int_c^t M(x,\tau)\phi(x,\tau)d\tau$$

$$\leq \alpha\phi(x,t).$$

$\Rightarrow u_t(x,t) - u_t(x,c) - \int_c^t K(x,\tau,u(x,\tau))d\tau \leq \alpha\phi(x,t).$

$$\Rightarrow u_t(x,t) - u_t(x,c) - \int_c^t K(x,\tau,u(x,\tau))d\tau \leq \phi(x,t), \quad (\because 0 < \alpha < 1). \tag{5.20}$$

Again, integrating from c to t, we get

$u(x,t) - u(x,c) - \{u(x,c) - u(x,c)\} - \int_c^t \int_c^y K(x,z,u(x,z))dzdy \leq \int_c^t \phi(x,\tau)d\tau.$

$\Rightarrow u(x,t) - u(x,c) - \int_c^t \int_c^y K(x,z,u(x,z))dzdy \leq \int_c^t \phi(x,\tau)d\tau.$

Since $M : J \times J \to [1,\infty)$ is an integrable function, we have

$\Rightarrow u(x,t) - \left\{u(x,c) + \int_c^t \int_c^y K(x,z,u(x,z))dzdy\right\} \leq \int_c^t M(x,\tau)\phi(x,\tau)d\tau.$

Again, by using inequality (5.15), we get

$$\Rightarrow u(x,t)-\left\{u(x,c)+\int_c^t\int_c^y K(x,z,u(x,z))dzdy\right\} \leq \int_c^t M(x,\tau)\phi(x,\tau)d\tau.$$
$$\leq \alpha\phi(x,t).$$

Thus

$$u(x,t)-\left\{u(x,c)+\int_c^t\int_c^y K(x,z,u(x,z))dzdy\right\} \leq \alpha\phi(x,t). \quad (5.21)$$

In a similar way, from the left inequality of (5.19), we obtain

$$-\left[u(x,t)-\left\{u(x,c)+\int_c^t\int_c^y K(x,z,u(x,z))dzdy\right\}\right] \leq \alpha\phi(x,t). \quad (5.22)$$

From inequalities (5.21) and (5.22), we get

$$\left|u(x,t)-\left\{u(x,c)+\int_c^t\int_c^y K(x,z,u(x,z))dzdy\right\}\right| \leq \alpha\phi(x,t). \quad (5.23)$$

Let Y be the set of all continuously differentiable functions $l : J\times J \to \mathbb{R}$. We define a metric d and an operator T on Y as follows :

For $l,m \in Y$

$$d(l,m) = \sup_{x,t\in J} \frac{|l(x,t)-m(x,t)|}{\phi(x,t)}$$

and the operator

$$(Tm)(x,t) = u(x,c)+\int_c^t\int_c^y K(x,z,m(x,z))dzdy. \quad (5.24)$$

Consider

$$d(Tl,Tm) = \sup_{x,t\in J}\left\{\frac{(Tl)(x,t)-(Tm)(x,t)}{\phi(x,t)}\right\}$$

$$= \sup_{x,t\in J}\left\{\frac{\int_c^t\int_c^y K(x,z,l(x,z))dzdy-\int_c^t\int_c^y K(x,z,m(x,z))dzdy}{\phi(x,t)}\right\}$$

$$\leq \sup_{x,t\in J}\left\{\frac{\int_c^t\int_c^y |K(x,z,l(x,z))-K(x,z,m(x,z))|dzdy}{\phi(x,t)}\right\}$$

By using condition C1, we get,

$$\text{d(Tl,Tm)} \leq \sup_{x,t\in J}\left\{\frac{\int_c^t\int_c^y\{M(x,z)|l(x,z)-m(x,z)|\}dzdy}{\phi(x,t)}\right\}$$

$$= \sup_{x,t\in J}\left\{\frac{\int_c^t\int_c^y\{M(x,z)\phi(x,z)\times\frac{|l(x,z)-m(x,z)|}{\phi(x,z)}\}dzdy}{\phi(x,t)}\right\}$$

$$\leq \sup_{x,t\in J}\left\{\frac{\int_c^t\int_c^y\{M(x,z)\phi(x,z)\times\sup_{x,z\in J}\frac{|l(x,z)-m(x,z)|}{\phi(x,z)}\}dzdy}{\phi(x,t)}\right\}$$

$$= d(l,m)\times\sup_{x,t\in J}\left\{\frac{\int_c^t\int_c^y\{M(x,z)\phi(x,z)\}dzdy}{\phi(x,t)}\right\}$$

By using inequality (5.16), we get

$$d(Tl,Tm) \leq \alpha d(l,m).$$

Hence by using Banach contraction principle, there exists a unique $u_0 \in X$ such that $Tu_0 = u_0$, that is

$$u(x,c)+\int_c^t\int_c^y K(x,z,u_0(x,z))dzdy = u_0(x,t) \qquad \text{(by using equation (5.24))}$$

and

$$d(u_0,u) \leq \frac{1}{(1-\alpha)}d(u,Tu). \tag{5.25}$$

Now by inequality (5.23), we get

$$|u(x,t)-(Tu)(x,t)| \leq \alpha\phi(x,t).$$

$$\Rightarrow \frac{|u(x,t)-(Tu)(x,t)|}{\phi(x,t)} \leq \alpha.$$

$$\Rightarrow \sup_{x,t\in J}\frac{|u(x,t)-(Tu)(x,t)|}{\phi(x,t)} \leq \alpha.$$

Thus

$$d(u,Tu) \leq \alpha \tag{5.26}$$

Again

$$d(u_0,u) = \sup_{x,t\in J}\frac{|u_0(x,t)-u(x,t)|}{\phi(x,t)}.$$

From equation (5.25), we get

$$d(u_0,u) \leq \frac{1}{(1-\alpha)}d(u,Tu).$$

$$\sup_{x,t\in J}\frac{|u_0(x,t)-u(x,t)|}{\phi(x,t)} \leq \frac{1}{(1-\alpha)}d(u,Tu).$$

$$\frac{|u_0(x,t)-u(x,t)|}{\phi(x,t)} \leq sup_{x,t\in J}\frac{|u_0(x,t)-u(x,t)|}{\phi(x,t)} \leq \frac{1}{(1-\alpha)}d(u,Tu).$$

$$\frac{|u_0(x,t)-u(x,t)|}{\phi(x,t)} \leq \frac{1}{(1-\alpha)}d(u,Tu).$$

From equation (5.26) we get,

$$\frac{|u_0(x,t)-u(x,t)|}{\phi(x,t)} \leq \frac{1}{(1-\alpha)}\alpha.$$

$$|u(x,t)-u_0(x,t)| \leq \frac{\alpha}{(1-\alpha)}\phi(x,t), \ \forall x,t \in J.$$

Hence the third order PDE (5.2) is HUR stable.

Chapter 6

HYERS - ULAM STABILITY OF NON-LINEAR ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

Some contents of this chapter is published [see PP2].

6.1 INTRODUCTION

In this chapter, we study the HU stability of the first and second order non-linear ordinary and partial differential equations. We employ the well known Banach contraction principle to establish our results. Also, we prove the HU stability of the second order non-linear ordinary and partial differential equations by using Grownwall type inequality and some integral inequalities .

We consider the following first order partial differential equation

$$u_x(x,t)+K(x,u(x,t))=0, \tag{6.1}$$

where $K: J\times\mathbb{R}\to\mathbb{R}$ is a continuous function, $u(x,t)\in C^1(J\times J), J=[a,b]$ be a closed interval and the second order partial differential equation

$$u_{xx}(x,t)+F(x,u)u_x(x,t)+H(x,u)=0, \tag{6.2}$$

where $F,H: J\times\mathbb{R}\to\mathbb{R}$ are continuous functions and $u(x,t)\in C^2(J\times J)$.

First, we define HU stability for these two equations.

Definition 6.1.1 : The equation (6.1) is said to be HU stable if the following holds:

Let $\varepsilon\geq 0$. Assume that, for any function $u(x,t)\in C^1$ satisfying the differential inequality

$$|u_x(x,t)+K(x,u(x,t))|\leq\varepsilon,\ \forall x,t\in J, \tag{6.3}$$

there exists a solution $u_0(x,t)\in C^1$ of equation (6.1) and $M(\varepsilon)>0$ such that

$$|u(x,t)-u_0(x,t)|\leq M(\varepsilon),\ \forall (x,t)\in J\times J.$$

Next, we define HU stability for equation (6.2).

Definition 6.1.2 : The equation (6.2) is said to be HU stable if the following holds:

Let $\varepsilon \geq 0$. For any function $u(x,t) \in C^2$ satisfying the differential inequality

$$|u_{xx}(x,t) + F(x,u)u_x(x,t) + H(x,u)| \leq \varepsilon, \; \forall \, x,t \in J, \tag{6.4}$$

there exists a solution $u_0(x,t) \in C^2$ of equation (6.2) and $M(\varepsilon) > 0$ such that

$$|u(x,t) - u_0(x,t)| \leq M(\varepsilon), \; \forall (x,t) \in J \times J.$$

Further we have proved, the HU stability for the second order non-linear ordinary and partial differential equations of the form:

$$u_{xx}(x,t) = f(x,t,u(x,t),u_x(x,t)), \;\; 0 \leq x \leq a, 0 \leq t \leq b, \tag{6.5}$$

and

$$u_{xt}(x,t) = f(x,t,u(x,t),u_x(x,t)) \;\; 0 \leq x \leq a, 0 \leq t \leq b, \tag{6.6}$$

where $a,b \in (0,\infty), f \in C([0,a] \times [0,b] \times \mathbb{B}^2, \mathbb{B})$ and $(\mathbb{B}, ||.||)$ be a real or complex Banach space.

Now we define HU stability of the equations (6.5) and (6.6).

Definition 6.1.3 : Equation (6.5) is HU stable if $\exists$ real constants $c_f^1, c_f^2 > 0$ such that for any $\varepsilon > 0$ and for any solution $v(x,t)$ of the inequality

$$||v_{xx}(x,t) - f(x,t,v(x,t),v_x(x,t))|| \leq \varepsilon, \;\; \forall \, x \in [0,a], \forall \, t \in [0,b], \tag{6.7}$$

$\exists$ a solution $u(x,t)$ of (6.5) with $||v(x,t) - u(x,t)|| \leq \varepsilon \, c_f^1$ and

$||v_x(x,t) - u_x(x,t)|| \leq \varepsilon \, c_f^2, \;\; \forall \, x \in [0,a], \forall \, t \in [0,b].$

Remark 6.1.4 : A function $v(x,t)$ is a solution to the inequality (6.7) iff $\exists$ a continuous function $g(x,t)$ which depends on $v(x,t)$ such that

i) $||g(x,t)|| \leq \varepsilon$,

ii) $\forall\, x \in [0,a], \forall\, t \in [0,b], v_{xx}(x,t) = f(x,t,v(x,t),v_x(x,t)) + g(x,t)$.

Definition 6.1.5 : Equation (6.6) is HU stable if $\exists$ real constants c_f^3, $c_f^4 > 0$ such that for any $\varepsilon > 0$ and for any solution $v(x,t)$ of the inequality

$$||v_{xt}(x,t) - f(x,t,v(x,t),v_x(x,t))|| \leq \varepsilon, \;\; \forall\, x \in [0,a], \;\; \forall\, t \in [0,b], \qquad (6.8)$$

$\exists$ a solution $u(x,t)$ of (6.6) with

$||v(x,t) - u(x,t)|| \leq \varepsilon\, c_f^3$ and

$||v_x(x,t) - u_x(x,t)|| \leq \varepsilon\, c_f^4, \;\; \forall\, x \in [0,a], \forall\, t \in [0,b]$.

Remark 6.1.6 : A function $v(x,t)$ is a solution to the inequality (6.8) iff $\exists$ a continuous function $g(x,t)$ which depends on $v(x,t)$ such that

i) $||g(x,t)|| \leq \varepsilon$,

ii) $\forall\, x \in [0,a], \forall\, t \in [0,b], v_{xt}(x,t) = f(x,t,v(x,t),v_x(x,t)) + g(x,t)$.

In proving main results, we need the following results.

Theorem 6.1.1 *(Banach Contraction Principle)* [18] :
Let (X,d) be a complete metric space and $T : X \to X$ be a contraction, that is, there exists $\alpha \in (0,1)$ such that $d(Tx,Ty) \leq \alpha d(x,y), \forall x,y \in X$. Then $\exists$ a unique $a \in X$ such that $Ta = a$. Moreover, $a = \lim_{n \to \infty} T^n x$ and $d(a,x) \leq \frac{1}{(1-\alpha)} d(x,Tx), \forall x \in X$.

Lemma 6.1.2 : [50] One assumes that

i) $u, v, h \in C(\mathbb{R}_+^n, \mathbb{R}_+)$,

ii) for any $t \geq t_0$, $u(t) \leq h(t) + \int_{t_0}^{t} v(s)u(s)ds$.

iii) $h(t)$ is positive and increasing.

Then $u(t) \leq h(t) \times \exp\{\int_{t_0}^{t} v(r)dr\}$, for any $t \geq t_0$.

6.2 HU STABILITY OF (6.1) and (6.2)

In this section, we prove the HU stability of (6.1) and (6.2). We prove these results by using Banach contraction principle. First, we have the following result for (6.1).

Theorem 6.2.1 : Let $x_0 \in J$ and $K : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$|K(x,v(x,t)) - K(x,w(x,t))| \leq \lambda |v(x,t) - w(x,t)|, \forall x,t \in J, \tag{6.9}$$

where $\lambda > 0, \lambda \in R$ and $v(x,t), w(x,t) \in C^1$. Let

$$M_1 = \sup_{x \in J} \left| \int_{x_0}^{x} ds \right|, \tag{6.10}$$

with $0 < \lambda M_1 < 1$. Let $u(x,t) \in C^1$ satisfy

$$|u_x(x,t) + K(x,u(x,t))| \leq \varepsilon, \forall x,t \in J, \tag{6.11}$$

then there exists a unique function $u_0(x,t) \in C^1$, such that

$$\frac{\partial u_0}{\partial x}(x,t) + K(x,u_0(x,t)) = 0 \quad \text{and} \quad |u(x,t) - u_0(x,t)| \leq \frac{M_1}{1-\lambda M_1}\varepsilon.$$

Proof. Consider the differential equation

$$u_x(x,t) + K(x,u(x,t)) = 0, \ \forall x,t \in J. \tag{6.12}$$

We define a metric d and an operator P on C^1, respectively by

$$d(\zeta,\eta) = \sup_{x,t \in J} \left| \zeta(x,t) - \eta(x,t) \right| \text{ and}$$

$$(P\zeta)(x,t) = u(x_0,t) - \int_{x_0}^{x} K(s,\zeta(s,t))ds, \forall \zeta \in C^1. \tag{6.13}$$

Consider,

$$d(P\zeta,P\eta) = \sup_{x,t \in J} \left| (P\zeta)(x,t) - (P\eta)(x,t) \right|$$

$$
\begin{aligned}
&= \sup_{x,t\in J}\left|-\int_{x_0}^{x} K(s,\zeta(s,t))ds+\int_{x_0}^{x} K(s,\eta(s,t))ds\right| \\
&= \sup_{x,t\in J}\left|\int_{x_0}^{x} K(s,\zeta(s,t))ds-\int_{x_0}^{x} K(s,\eta(s,t))ds\right| \\
&\leq \sup_{x,t\in J}\left|\int_{x_0}^{x} |K(s,\zeta(s,t))-K(s,\eta(s,t))|ds\right| \\
&\leq \sup_{x,t\in J}\left|\int_{x_0}^{x} \lambda|\zeta(s,t))-\eta(s,t)|ds\right| \quad \text{(by equation (6.9))} \\
&\leq \lambda \sup_{x,t\in J}\left|\int_{x_0}^{x} \sup_{s,t\in J}|\zeta(s,t))-\eta(s,t)|ds\right| \\
&\leq \lambda \sup_{x,t\in J}\left[d(\zeta,\eta)\times\left|\int_{x_0}^{x} ds\right|\right] \\
&\leq \lambda M_1 d(\zeta,\eta) \quad \text{(by equation (6.10))}.
\end{aligned}
$$

Then by using Banach contraction principle, there exists a unique $u_0(x,t) \in C^1$ such that $Pu_0(x,t)=u_0(x,t)$. Thus $u_0(x,t)$ satisfy $u(x_0,t)-\int_{x_0}^{x} K(s,u_0(s,t))ds = u_0(x,t)$ and

$$
d(u_0,u) \leq \frac{1}{1-\lambda M_1} d(u,Pu). \tag{6.14}
$$

Now by inequality (6.11) we get,

$$
-\varepsilon \leq \frac{\partial u}{\partial x}(x,t)+K(x,u(x,t)) \leq \varepsilon, \forall x,t \in J.
$$

Integrating from x_0 to x, we get

$$
-\varepsilon \int_{x_0}^{x} ds \leq \int_{x_0}^{x}\{\frac{\partial u}{\partial s}(s,t)+K(s,u(s,t))\}ds \leq \varepsilon \int_{x_0}^{x} ds,
$$

$$
\Rightarrow -\varepsilon \int_{x_0}^{x} ds \leq u(x,t)-u(x_0,t)+\int_{x_0}^{x} K(s,u(s,t))ds \leq \varepsilon \int_{x_0}^{x} ds.
$$

$$
\begin{aligned}
\Rightarrow -\varepsilon \sup_{x\in J}\int_{x_0}^{x} ds \leq -\varepsilon \int_{x_0}^{x} ds \leq u(x,t)-u(x_0,t)+\int_{x_0}^{x} K(s,u(s,t))ds \\
\leq \varepsilon \int_{x_0}^{x} ds \leq \sup_{x\in J}\int_{x_0}^{x} ds.
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow -\varepsilon \sup_{x\in J}\left|\int_{x_0}^{x} ds\right| \leq -\varepsilon \sup_{x\in J}\int_{x_0}^{x} ds \leq -\varepsilon \int_{x_0}^{x} ds \leq u(x,t)-u(x_0,t)+\int_{x_0}^{x} K(s,u(s,t))ds \\
\leq \varepsilon \int_{x_0}^{x} ds \leq \sup_{x\in J}\int_{x_0}^{x} ds \leq \varepsilon \sup_{x\in J}\left|\int_{x_0}^{x} ds\right|.
\end{aligned}
$$

$$\Rightarrow -\varepsilon M_1 \leq u(x,t) - u(x_0,t) + \int_x^{x_0} K(s,u(s,t))ds \leq \varepsilon M_1$$

$$\Rightarrow |u(x,t) - u(x_0,t) + \int_x^{x_0} K(s,u(s,t))ds| \leq \varepsilon M_1.$$

$$\Rightarrow |u(x,t) - (Pu)(x,t)| \leq \varepsilon M_1.$$

$$\Rightarrow \sup_{x,t\in J} |u(x,t) - (Pu)(x,t)| \leq \varepsilon M_1.$$

$$\Rightarrow \ d(u,Pu) \leq \varepsilon M_1.$$

Using this inequality and equation (6.14), we get

$$\begin{aligned}|u(x,t) - u_0(x,t)| &= |u_0(x,t) - u(x,t)| \\ &\leq \sup_{x,t\in J} \Big|u_0(x,t) - u(x,t)\Big| \\ &= d(u_0(x,t), u(x,t)). \\ &\leq \frac{1}{1-\lambda M_1} d(u,Pu). \\ &\leq \frac{M_1}{1-\lambda M_1}\varepsilon = M(\varepsilon)\ .\end{aligned}$$

Hence the result. □

Next, we prove the HU stability of (6.2).

Theorem 6.2.2 : Let $x_0 \in J$ and $F,H : J\times\mathbb{R} \to \mathbb{R}$ be a continuous functions such that

$$|F(x,v(x,t))v_x(x,t) - F(x,w(x,t))w_x(x,t)| \leq \lambda_1|v(x,t) - w(x,t)|, \tag{6.15}$$

and

$$|H(x,v(x,t)) - H(x,w(x,t))| \leq \lambda_2|v(x,t) - w(x,t)|,\ \forall x,t \in J, \tag{6.16}$$

where $\lambda_1,\lambda_2 > 0, \lambda_1,\lambda_2 \in \mathbb{R}$ and $v(x,t), w(x,t) \in C^2(J\times J)$.

Let

$$M_2 = \sup_{x,y\in J} \Big| \int_{x_0}^{x} \int_{x_0}^{y} dsdy \Big|, \tag{6.17}$$

with $0 < \{\lambda_1 + \lambda_2\}M_2 < 1$. If $u(x,t) \in C^2(J \times J)$ satisfy

$$|u_{xx}(x,t) + F(x,u)u_x(x,t) + H(x,u)| \le \varepsilon, \ \forall x,t \in J, \tag{6.18}$$

then there exists, a unique function, $u_0(x,t) \in C^2(J \times J)$ such that

$$\frac{\partial^2 u_0}{\partial x^2}(x,t) + F(x,u_0(x,t))\frac{\partial u_0}{\partial x}(x,t) + H(x,u_0(x,t)) = 0$$

and

$$|u(x,t) - u_0(x,t)| \le \frac{M_2}{1 - \{\lambda_1 + \lambda_2\}M_2}\varepsilon.$$

Proof. Consider the differential equation

$$\frac{\partial^2 u}{\partial x^2}(x,t) + F(x,u(x,t))\frac{\partial u}{\partial x}(x,t) + H(x,u(x,t)) = 0, \forall x,t \in J. \tag{6.19}$$

We define a metric d and an operator P on $C^2(J \times J)$, respectively by

$$d(\zeta,\eta) = \sup_{x,t \in J} |\zeta(x,t) - \eta(x,t)| \quad \text{and}$$

$$(P\zeta)(x,t) = u(x_0,t) - \int_{x_0}^{x}\int_{x_0}^{y} F(s,\zeta(s,t))\zeta_s(s,t)dsdy - \int_{x_0}^{x}\int_{x_0}^{y} H(s,\zeta(s,t))dsdy, \tag{6.20}$$

$\forall \ \zeta \in C^2(J \times J)$.

Consider

$$\begin{aligned} d(P\zeta,P\eta) &= \sup_{x,t\in J}\Big|(P\zeta)(x,t) - (P\eta)(x,t)\Big| \\ &= \sup_{x,t\in J}\Big| - \textstyle\int_{x_0}^{x}\int_{x_0}^{y} F(s,\zeta(s,t))\zeta_s(s,t)dsdy - \int_{x_0}^{x}\int_{x_0}^{y} H(s,\zeta(s,t))dsdy \\ &\qquad + \textstyle\int_{x_0}^{x}\int_{x_0}^{y} F(s,\eta(s,t))\eta_s(s,t)dsdy + \int_{x_0}^{x}\int_{x_0}^{y} H(s,\eta(s,t))dsdy\Big| \\ &= \sup_{x,t\in J}\Big| \textstyle\int_{x_0}^{x}\int_{x_0}^{y} F(s,\zeta(s,t))\zeta_s(s,t)dsdy + \int_{x_0}^{x}\int_{x_0}^{y} H(s,\zeta(s,t))dsdy \\ &\qquad - \textstyle\int_{x_0}^{x}\int_{x_0}^{y} F(s,\eta(s,t))\eta_s(s,t)dsdy - \int_{x_0}^{x}\int_{x_0}^{y} H(s,\eta(s,t))dsdy\Big| \end{aligned}$$

$$= \sup_{x,t\in J}\Big| \int_{x_0}^{x}\int_{x_0}^{y} F(s,\zeta(s,t))\zeta_s(s,t)dsdy - \int_{x_0}^{x}\int_{x_0}^{y} F(s,\eta(s,t))\eta_s(s,t)dsdy$$

$$+ \int_{x_0}^{x}\int_{x_0}^{y} H(s,\zeta(s,t))dsdy - \int_{x_0}^{x}\int_{x_0}^{y} H(s,\eta(s,t))dsdy\Big|$$

$$\leq \sup_{x,t\in J}\Big| \int_{x_0}^{x}\int_{x_0}^{y} F(s,\zeta(s,t))\zeta_s(s,t)dsdy - \int_{x_0}^{x}\int_{x_0}^{y} F(s,\eta(s,t))\eta_s(s,t)dsdy\Big|$$

$$+ \sup_{x,t\in J}\Big| \int_{x_0}^{x}\int_{x_0}^{y} H(s,\zeta(s,t))dsdy - \int_{x_0}^{x}\int_{x_0}^{y} H(s,\eta(s,t))dsdy\Big|$$

$$\leq \sup_{x,t\in J}\Big| \int_{x_0}^{x}\int_{x_0}^{y} |F(s,\zeta(s,t))\zeta_s(s,t) - F(s,\eta(s,t))\eta_s(s,t)|dsdy\Big|$$

$$+ \sup_{x,t\in J}\Big| \int_{x_0}^{x}\int_{x_0}^{y} |H(s,\zeta(s,t)) - H(s,\eta(s,t))|dsdy\Big|$$

$$\leq \sup_{x,t\in J}\Big| \int_{x_0}^{x}\int_{x_0}^{y} \lambda_1|\zeta(s,t)-\eta(s,t)|dsdy\Big|$$

$$+ \sup_{x,t\in J}\Big| \int_{x_0}^{x}\int_{x_0}^{y} \lambda_2|\zeta(s,t)-\eta(s,t)|dsdy\Big|$$

(by equation (6.15) and (6.16))

$$\leq \lambda_1 \sup_{x,t\in J}\Big| \int_{x_0}^{x}\int_{x_0}^{y} \sup_{s,t\in J}|\zeta(s,t)-\eta(s,t)|\, dsdy\Big| +$$

$$\lambda_2 \sup_{x,t\in J}\Big| \int_{x_0}^{x}\int_{x_0}^{y} \sup_{s,t\in J}|\zeta(s,t)-\eta(s,t)|\, dsdy\Big|$$

$$\leq \lambda_1 \sup_{x,t\in J}\Big| \int_{x_0}^{x}\int_{x_0}^{y} d(\zeta,\eta)dsdy\Big| + \lambda_2 \sup_{x,t\in J}\Big| \int_{x_0}^{x}\int_{x_0}^{y} d(\zeta,\eta)dsdy\Big|$$

$$\leq \{\lambda_1+\lambda_2\} \sup_{x,t\in J}\Big[d(\zeta,\eta) \Big| \int_{x_0}^{x}\int_{x_0}^{y} dsdy \Big| \Big]$$

$$\leq \{\lambda_1+\lambda_2\} d(\zeta,\eta) M_2 \qquad \text{(by equation (6.17))}$$

$$\leq \{\lambda_1+\lambda_2\} M_2 \times d(\zeta,\eta).$$

Hence using Banach contraction principle, there exists a unique, $u_0(x,t) \in C^2(J\times J)$ such that $Pu_0(x,t) = u_0(x,t)$. Thus $u_0(x,t)$ satisfy

$$u(x_0,t)-\int_{x_0}^{x}\int_{x_0}^{y}F(s,u_0(s,t))u_s(s,t))dsdy-\int_{x_0}^{x}\int_{x_0}^{y}H(s,u_0(s,t))dsdy=u_0(x,t)$$

and

$$d(u_0,u)\leq\frac{1}{1-(\lambda_1+\lambda_2)M_2}d(u,Pu). \tag{6.21}$$

Now by inequality (6.18), we get

$$-\varepsilon\leq\frac{\partial^2 u}{\partial x^2}(x,t)+F(x,u)\frac{\partial u}{\partial x}(x,t)+H(x,u)\leq\varepsilon,\quad \forall x,t\in J.$$

Integrating from x_0 to x, we derive

$$-\varepsilon\int_{x_0}^{x}ds\leq\frac{\partial u}{\partial x}(x,t)-\frac{\partial u}{\partial x}(x_0,t)+\int_{x_0}^{x}F(s,u(s,t))u_s(s,t)ds+\int_{x_0}^{x}H(s,u(s,t))ds\leq\varepsilon\int_{x_0}^{x}ds.$$

Again integrating from x_0 to x, we obtain

$$-\varepsilon\int_{x_0}^{x}\int_{x_0}^{y}dsdy\leq u(x,t)-u(x_0,t)-[u(x_0,t)-u(x_0,t)]+\int_{x_0}^{x}\int_{x_0}^{y}F(s,u(s,t))u_s(s,t)dsdy$$
$$+\int_{x_0}^{x}\int_{x_0}^{y}H(s,u(s,t))dsdy\leq\varepsilon\int_{x_0}^{x}\int_{x_0}^{y}dsdy.$$

By using the equation (6.20), we get

$$-\varepsilon\int_{x_0}^{x}\int_{x_0}^{y}dsdy\leq u(x,t)-(Pu)(x,t)\leq\varepsilon\int_{x_0}^{x}\int_{x_0}^{y}dsdy.$$

$$\Rightarrow -\varepsilon\sup_{x,y\in J}\int_{x_0}^{x}\int_{x_0}^{y}dsdy\leq-\varepsilon\int_{x_0}^{x}\int_{x_0}^{y}dsdy\leq u(x,t)-(Pu)(x,t)\leq\varepsilon\int_{x_0}^{x}\int_{x_0}^{y}dsdy$$
$$\leq\varepsilon\sup_{x,y\in J}\int_{x_0}^{x}\int_{x_0}^{y}dsdy$$

$$\Rightarrow -\varepsilon\sup_{x,y\in J}\left|\int_{x_0}^{x}\int_{x_0}^{y}dsdy\right|\leq-\varepsilon\sup_{x,y\in J}\int_{x_0}^{x}\int_{x_0}^{y}dsdy\leq-\varepsilon\int_{x_0}^{x}\int_{x_0}^{y}dsdy\leq u(x,t)-(Pu)(x,t)$$
$$\leq\varepsilon\int_{x_0}^{x}\int_{x_0}^{y}dsdy\leq\varepsilon\sup_{x,y\in J}\int_{x_0}^{x}\int_{x_0}^{y}dsdy\leq\varepsilon\sup_{x,y\in J}\left|\int_{x_0}^{x}\int_{x_0}^{y}dsdy\right|.$$

$$\Rightarrow -\varepsilon\sup_{x,y\in J}\left|\int_{x_0}^{x}\int_{x_0}^{y}dsdy\right|\leq u(x,t)-(Pu)(x,t)\leq\varepsilon\sup_{x,y\in J}\left|\int_{x_0}^{x}\int_{x_0}^{y}dsdy\right|.$$

$$\Rightarrow -\varepsilon M_2\leq u(x,t)-(Pu)(x,t)\leq\varepsilon M_2$$

$$\Rightarrow |u(x,t)-(Pu)(x,t)|\leq\varepsilon M_2.$$

$$\Rightarrow \sup_{x,t\in J}\left|u(x,t)-(Pu)(x,t)\right|\leq\varepsilon M_2.$$

$\Rightarrow\ d(u,Pu) \leq \varepsilon M_2$, which with equation (6.21) yields

$$
\begin{aligned}
|u(x,t) - u_0(x,t)| &= |u_0(x,t) - u(x,t)| \\
&\leq \sup_{x,t\in J} |u_0(x,t) - u(x,t)| \\
&= d(u_0(x,t), u(x,t)). \\
&\leq \frac{1}{1-\{\lambda_1+\lambda_2\}M_2} d(u,Pu). \\
&\leq \frac{M_2}{1-\{\lambda_1+\lambda_2\}M_2}\varepsilon = M(\varepsilon)
\end{aligned}
$$

Hence the result. □

Thus, we have proved the HU stability of first and second order partial differential equations (6.1) and (6.2) respectively by employing Banach contraction principle.

6.3 INTEGRAL INEQUALITIES

In this section, we have developed some integral inequalities, which will be used to prove results in the next section. First, we prove the following.

Theorem 6.3.1 : If $v(x,t)$ is a solution to the inequality (6.7) then (v, v_x) satisfies the following integral inequality system.

$$||v(x,t) - v(0,t) - \int_0^x \int_0^y f(s,t,v(s,t),v_s(s,t))dsdy|| \leq \frac{\varepsilon x^2}{2},$$

$$||v_x(x,t) - v_x(0,t) - \int_0^x f(s,t,v(s,t),v_s(s,t))ds|| \leq \varepsilon x.$$

Proof. Let $v(x,t)$ be a solution to the inequality

$$||v_{xx}(x,t) - f(x,t,v(x,t),v_x(x,t))|| \leq \varepsilon.$$

Integrating w.r.t. x, we get

$$\int_0^x ||v_{ss}(s,t) - f(s,t,v(s,t),v_s(s,t))||ds \leq \int_0^x \varepsilon ds.$$

Since

$||\int_0^x \{v_{ss}(s,t) - f(s,t,v(s,t),v_s(s,t))\}ds|| \leq \int_0^x ||v_{ss}(s,t) - f(s,t,v(s,t),v_s(s,t))||ds,$

we get

$||\int_0^x \{v_{ss}(s,t) - f(s,t,v(s,t),v_s(s,t))\}ds|| \leq \int_0^x \varepsilon ds.$

$\Rightarrow ||v_x(x,t) - v_x(0,t) - \int_0^x f(s,t,v(s,t),v_s(s,t))ds|| \leq \varepsilon x.$

Integrating w.r.t. x, we get

$\int_0^x ||v_y(y,t) - v_y(0,t) - \int_0^y f(s,t,v(s,t),v_s(s,t))ds||dy \leq \int_0^x \varepsilon s ds.$

Since $||\int_0^x \{v_y(y,t) - v_y(0,t) - \int_0^y f(s,t,v(s,t),v_s(s,t))ds\}dy||$

$$\leq \int_0^x ||v_y(y,t) - v_y(0,t) - \int_0^y f(s,t,v(s,t),v_s(s,t))ds||dy,$$

we get

$||\int_0^x \{v_y(y,t) - v_y(0,t) - \int_0^y f(s,t,v(s,t),v_s(s,t))ds\}dy|| \leq \int_0^x \varepsilon s ds.$

$$\Rightarrow ||v(x,t) - v(0,t) - \{v(0,t) - v(0,t)\} - \int_0^x \int_0^y f(s,t,v(s,t),v_s(s,t))dsdy|| \leq \frac{\varepsilon x^2}{2}.$$

$\Rightarrow ||v(x,t) - v(0,t) - \int_0^x \int_0^y f(s,t,v(s,t),v_s(s,t))dsdy|| \leq \frac{\varepsilon x^2}{2}.$ □

Next, we prove the result related to the inequality (6.8).

Theorem 6.3.2 : : If $v(x,t)$ is a solution to the inequality (6.8), then (v, v_x, v_t) satisfies the following integral inequalities.

$||v(x,t) - v(x,0) - v(0,t) + v(0,0) - \int_0^t \int_0^x f(s,z,v(s,z),v_s(s,z))dsdz|| \leq \varepsilon xt,$

$||v_t(x,t) - v_t(0,t) - \int_0^x f(s,t,v(s,t),v_s(s,t))ds|| \leq \varepsilon x,$

$||v_x(x,t) - v_x(x,0) - \int_0^t f(x,z,v(x,z),v_x(x,z))dz|| \leq \varepsilon t.$

Proof. : Let $v(x,t)$ be a solution to the inequality

$$||v_{xt}(x,t) - f(x,t,v(x,t),v_x(x,t))|| \leq \varepsilon, \tag{6.22}$$

Integrating w. r. t. x, we get

$$\int_0^x ||v_{st}(s,t) - f(s,t,v(s,t),v_s(s,t))||ds \leq \int_0^x \varepsilon ds.$$

Since

$$|| \int_0^x \{v_{st}(s,t) - f(s,t,v(s,t),v_s(s,t))\}ds|| \leq \int_0^x ||v_{st}(s,t) - f(s,t,v(s,t),v_s(s,t))||ds,$$

we get,

$$|| \int_0^x \{v_{st}(s,t) - f(s,t,v(s,t),v_s(s,t))\}ds|| \leq \int_0^x \varepsilon ds.$$

$$\Rightarrow ||v_t(x,t) - v_t(0,t) - \int_0^x f(s,t,v(s,t),v_s(s,t))ds|| \leq \varepsilon x.$$

Integrating w. r. t. t, we get

$$\int_0^t ||v_z(x,z) - v_z(0,z) - \int_0^x f(s,z,v(s,z),v_s(s,z))ds||dz \leq \int_0^t \varepsilon x dz.$$

Since

$$|| \int_0^t \{v_z(x,z) - v_z(0,z) - \int_0^x f(s,z,v(s,z),v_s(s,z))ds\}dz||$$

$$\leq \int_0^t ||v_z(x,z) - v_z(0,z) - \int_0^x f(s,z,v(s,z),v_s(s,z))ds||dz,$$

we get,

$$|| \int_0^t \{v_z(x,z) - v_z(0,z) - \int_0^x f(s,z,v(s,z),v_s(s,z))ds\}dz|| \leq \int_0^t \varepsilon x dz.$$

$$\Rightarrow ||v(x,t) - v(x,0) - v(0,t) + v(0,0) - \int_0^t \int_0^x f(s,z,v(s,z),v_s(s,z))dsdz|| \leq \varepsilon xt.$$

Integrating equation (6.22) w. r. t. t we get,

$$\int_0^t ||v_{xz}(x,z) - f(x,z,v(x,z),v_x(x,z))||dz \leq \int_0^t \varepsilon dz.$$

Since

$$|| \int_0^t \{v_{xz}(x,z) - f(x,z,v(x,z),v_x(x,z))\}dz|| \leq \int_0^t ||v_{xz}(x,z) - f(x,z,v(x,z),v_x(x,z))||dz,$$

we get,

$$|| \int_0^t \{v_{xz}(x,z) - f(x,z,v(x,z),v_x(x,z))\}dz|| \leq \int_0^t \varepsilon dz.$$

$$\Rightarrow ||v_x(x,t) - v_x(x,0) - \int_0^t f(x,z,v(x,z),v_x(x,z))dz|| \leq \varepsilon t.$$

which proves the required integral inequalities. □

6.4 HU STABILITY OF (6.5) and (6.6)

In this section, we prove the HU stability of second order non-linear ordinary and partial differential equations (6.5) and (6.6). We use integral inequalities established in the previous section. First, we have the result for the equation (6.5).

Theorem 6.4.1 : Assume that

i) $f \in C([0,a]\times[0,b]\times\mathbb{B}^2,\mathbb{B})$, where $a,b \in (0,\infty)$.

ii) There exists $L_f(x,t) \in C^1([0,a]\times[0,b],\mathbb{R}_+)$ such that $\int_0^a L_f(x,t)dx < \infty$ and $||f(x,t,z_1,z_2)-f(x,t,t_1,t_2)|| \leq L_f(x,t)\max_{i\in\{1,2\}}\{||z_i-t_i||\}, \forall\, x \in [0,a], \forall\, t \in [0,b]$ and $z_1,\, z_2,\, t_1,\, t_2 \in \mathbb{B}$.

Then (6.5) is HU stable.

Proof. Let $v(x,t)$ be a solution to the inequality

$||v_{xx}(x,t)-f(x,t,v(x,t),v_x(x,t))|| \leq \varepsilon, \forall\, x \in [0,a], \forall\, t \in [0,b]$.

Let $u(x,t)$ be the unique solution to the problem

$$\begin{aligned} u_{xx}(x,t) &= f(x,t,u(x,t),u_x(x,t)), \\ u(0,t) &= v(0,t), \forall\, t \in [0,b], \\ u(x,0) &= v(x,0), \forall\, x \in [0,a]. \end{aligned} \tag{6.23}$$

Then $(u,u_x(x,t))$ satisfies the following system.

$$\begin{aligned} u(x,t) &= v(0,t) + \int_0^x\int_0^y f(s,t,u(s,t),u_s(s,t))dsdy, \\ u_x(x,t) &= v_x(0,t) + \int_0^x f(s,t,u(s,t),u_s(s,t))ds. \end{aligned} \tag{6.24}$$

By using **Theorem 6.3.1**, it follows that

$$||v(x,t)-v(0,t)-\int_0^x\int_0^y f(s,t,v(s,t),v_s(s,t))dsdy|| \leq \frac{\varepsilon x^2}{2}, \tag{6.25}$$

and

$$||v_x(x,t) - v_x(0,t) - \int_0^x f(s,t,v(s,t),v_s(s,t))ds|| \le \varepsilon x. \tag{6.26}$$

Consider

$||v(x,t) - u(x,t)|| = ||v(x,t) - v(0,t) - \int_0^x \int_0^y f(s,t,u(s,t),u_s(s,t))dsdy||.$

$$= ||v(x,t) - v(0,t) - \int_0^x \int_0^y f(s,t,v(s,t),v_s(s,t))dsdy$$
$$+ \int_0^x \int_0^y f(s,t,v(s,t),v_s(s,t))dsdy$$
$$- \int_0^x \int_0^y f(s,t,u(s,t),u_s(s,t))dsdy||.$$

$$\le ||v(x,t) - v(0,t) - \int_0^x \int_0^y f(s,t,v(s,t),v_s(s,t))dsdy||$$
$$+|| \int_0^x \int_0^y f(s,t,v(s,t),v_s(s,t))dsdy - \int_0^x \int_0^y f(s,t,u(s,t),u_s(s,t))dsdy||.$$

$$\le ||v(x,t) - v(0,t) - \int_0^x \int_0^y f(s,t,v(s,t),v_s(s,t))dsdy||$$
$$+ \int_0^x \int_0^y ||f(s,t,v(s,t),v_s(s,t)) - f(s,t,u(s,t),u_s(s,t))||dsdy.$$

$$\le \frac{\varepsilon x^2}{2} + \int_0^x \int_0^y L_f(s,t) \times \max\{||v(s,t) - u(s,t)||, ||v_s(s,t) - u_s(s,t)||\}dsdy.$$

(by using hypothesis (ii) and equation (6.25))

$$\le \frac{\varepsilon x^2}{2} + \int_0^x dy \int_0^y \left(L_f(s,t) \times \max\{||v(s,t) - u(s,t)||, ||v_s(s,t) - u_s(s,t)||\}\right)ds.$$

$$= \frac{\varepsilon x^2}{2} +$$
$$\int_0^x (x-s)L_f(s,t) \times \max\{||v(s,t) - u(s,t)||, ||v_s(s,t) - u_s(s,t)||\}ds.$$

$$\le \varepsilon\left(x + \frac{x^2}{2}\right) +$$
$$\int_0^x a \times L_f(s,t) \times \max\{||v(s,t) - u(s,t)||, ||v_s(s,t) - u_s(s,t)||\}ds,$$

$$(\because\ (x-s) \le \max(x-s) \le (a-x) \le a).$$

$$\le \varepsilon\left(x + \frac{x^2}{2}\right) +$$
$$\int_0^x (1+a) \times L_f(s,t) \times \max\{||v(s,t) - u(s,t)||, ||v_s(s,t) - u_s(s,t)||\}ds.$$

$$||v(x,t)-u(x,t)|| \leq \varepsilon\left(x+\frac{x^2}{2}\right)+\int_0^x G(s,t)\times \max\{||v(s,t)-u(s,t)||,||v_s(s,t)-u_s(s,t)||\}ds, \tag{6.27}$$

where $G(s,t) = (1+a)\times L_f(s,t)$.

Again consider

$$||v_x(x,t)-u_x(x,t)|| = ||v_x(x,t)-v_x(0,t)-\int_0^x f(s,t,u(s,t),u_s(s,t))ds||.$$

$$= ||v_x(x,t)-v_x(0,t)-\int_0^x f(s,t,v(s,t),v_s(s,t))ds$$
$$+\int_0^x f(s,t,v(s,t),v_s(s,t))ds-\int_0^x f(s,t,u(s,t),u_s(s,t))ds||.$$

$$\leq ||v_x(x,t)-v_x(0,t)-\int_0^x f(s,t,v(s,t),v_s(s,t))ds||$$
$$+||\int_0^x f(s,t,v(s,t),v_s(s,t))ds-\int_0^x f(s,t,u(s,t),u_s(s,t))ds||.$$

$$\leq \varepsilon x+\int_0^x ||f(s,t,v(s,t),v_s(s,t))-f(s,t,u(s,t),u_s(s,t))||ds.$$

(by using Theorem 6.3.1)

By using hypothesis (ii), we get

$$||v_x(x,t)-u_x(x,t)|| \leq \varepsilon x+\int_0^x L_f(s,t)\times \max\{||v(s,t)-u(s,t)||,||v_s(s,t)-u_s(s,t)||\}ds.$$

$$\leq \varepsilon\left(x+\frac{x^2}{2}\right)+$$
$$\int_0^x (1+a)\times L_f(s,t)\times \max\{||v(s,t)-u(s,t)||,||v_s(s,t)-u_s(s,t)||\}ds.$$

$$||v_x(x,t)-u_x(x,t)|| \leq \varepsilon\left(x+\frac{x^2}{2}\right)+\int_0^x G(s,t)\times \max\{||v(s,t)-u(s,t)||,||v_s(s,t)-u_s(s,t)||\}ds, \tag{6.28}$$

where $G(s,t) = (1+a)\times L_f(s,t)$.

By using equations (6.27) and (6.28), we get

$$\max\{||v(x,t)-u(x,t)||,||v_x(x,t)-u_x(x,t)||\} \le \varepsilon\left(x+\frac{x^2}{2}\right)+$$
$$\int_0^x G(s,t)\times\max\{||v(s,t)-u(s,t)||,||v_s(s,t)-u_s(s,t)||ds.$$

By using Lemma 6.1.2, we get

$$\max\{||v(x,t)-u(x,t)||,||v_x(x,t)-u_x(x,t)||\} \le \varepsilon\left(x+\frac{x^2}{2}\right)\times\exp\left(\int_0^x G(s,t)ds\right).$$

From which it follows that, each of $||v(x,t)-u(x,t)||$ and $||v_x(x,t)-u_x(x,t)||$ is $\le \varepsilon\left(x+\frac{x^2}{2}\right)\times\exp\left(\int_0^x G(s,t)ds\right)$.

$$i.e.\ ||v(x,t)-u(x,t)|| \le \varepsilon\left(x+\frac{x^2}{2}\right)\times\exp\left(\int_0^x G(s,t)ds\right) \quad \text{and}$$
$$||v_x(x,t)-u_x(x,t)|| \le \varepsilon\left(x+\frac{x^2}{2}\right)\times\exp\left(\int_0^x G(s,t)ds\right).$$

$$\implies ||v(x,t)-u(x,t)|| \le \varepsilon\left(a+\frac{a^2}{2}\right)\times\exp\left(\int_0^a G(s,t)ds\right) \quad \text{and}$$
$$||v_x(x,t)-u_x(x,t)|| \le \varepsilon\left(a+\frac{a^2}{2}\right)\times\exp\left(\int_0^a G(s,t)ds\right).$$

$$\implies ||v(x,t)-u(x,t)|| \le \varepsilon\times c_f \quad \text{and} \quad ||v_x(x,t)-u_x(x,t)|| \le \varepsilon\times c_f,$$
$$\text{where } c_f=\left(a+\frac{a^2}{2}\right)\times\max_{t\in[0,b]}\exp\left(\int_0^a G(s,t)ds\right).$$

Hence the equation (6.5) is HU stable. □

Next, we prove result for the equation (6.6).

Theorem 6.4.2: Assume that

i) $f\in C([0,a]\times[0,b]\times\mathbb{B}^2,\mathbb{B})$, where $a,b\in(0,\infty)$.

ii) There exists $L_f(x,t)\in C^1([0,a]\times[0,b],\mathbb{R}_+)$ such that $\int_0^b L_f(x,t)dt<\infty$ and

$$||f(x,t,z_1,z_2)-f(x,t,t_1,t_2)|| \le L_f(x,t)\max_{i\in\{1,2\}}\{||z_i-t_i||\},\ \forall\, x\in[0,a],\ \forall\, t\in[0,b]$$

and $z_1,\ z_2,\ t_1,\ t_2\in\mathbb{B}$.

Then the equation (6.6) is HU stable.

Proof. : Let $v(x,t)$ be a solution to the inequality

$$||v_{xt}(x,t) - f(x,t,v(x,t),v_x(x,t))|| \leq \varepsilon, \ \forall x \in [0,a], \forall t \in [0,b].$$

Let $u(x,t)$ be the unique solution to the problem

$$\begin{aligned} u_{xt}(x,t) &= f(x,t,u(x,t),u_x(x,t)), \\ u(0,t) &= v(0,t), \forall t \in [0,b], \\ u(x,0) &= v(x,0), \forall x \in [0,a]. \end{aligned} \tag{6.29}$$

Then $(u, u_x(x,t), u_t(x,t))$ satisfies the following system.

$$\begin{aligned} u(x,t) &= v(x,0) + v(0,t) - v(0,0) + \int_0^t \int_0^x f(s,z,u(s,z),u_s(s,z))dsdz, \\ u_t(x,t) &= v_t(0,t) + \int_0^x f(s,t,u(s,t),u_s(s,t))ds, \\ u_x(x,t) &= v_x(x,0) + \int_0^t f(x,z,u(x,z),u_x(x,z))dz. \end{aligned} \tag{6.30}$$

By using Theorem 6.3.2, it follows that

$$\left|\left|v(x,t) - v(x,0) - v(0,t) + v(0,0) - \int_0^t \int_0^x f(s,z,v(s,z),v_s(s,z))dsdz\right|\right| \leq \varepsilon xt, \tag{6.31}$$

and

$$||v_x(x,t) - v_x(x,0) - \int_0^t f(x,z,v(x,z),v_x(x,z))dz|| \leq \varepsilon t. \tag{6.32}$$

Using equation (6.30), we get

$$\begin{aligned} ||v(x,t) - u(x,t)|| &= ||v(x,t) - v(x,0) - v(0,t) + v(0,0) \\ &\quad - \textstyle\int_0^t \int_0^x f(s,z,u(s,z),u_s(s,z))dsdz||. \\ &= ||v(x,t) - v(x,0) - v(0,t) + v(0,0) \\ &\quad - \textstyle\int_0^t \int_0^x f(s,z,v(s,z),v_s(s,z))dsdz \\ &\quad + \textstyle\int_0^t \int_0^x f(s,z,v(s,z),v_s(s,z))dsdz \\ &\quad - \textstyle\int_0^t \int_0^x f(s,z,u(s,z),u_s(s,z))dsdz||. \end{aligned}$$

$$\leq ||v(x,t) - v(x,0) - v(0,t) + v(0,0)$$
$$- \textstyle\int_0^t \int_0^x f(s,z,v(s,z),v_s(s,z))dsdz||$$
$$+|| \textstyle\int_0^t \int_0^x f(s,z,v(s,z),v_s(s,z))dsdz$$
$$- \textstyle\int_0^t \int_0^x f(s,z,u(s,z),u_s(s,z))dsdz||.$$

$$\leq ||v(x,t) - v(x,0) - v(0,t) + v(0,0)$$
$$- \textstyle\int_0^t \int_0^x f(s,z,v(s,z),v_s(s,z))dsdz||$$
$$+ \textstyle\int_0^t \int_0^x ||f(s,z,v(s,z),v_s(s,z)) - f(s,z,u(s,z),u_s(s,z))||dsdz.$$

$$\leq \varepsilon xt$$
$$+ \textstyle\int_0^t \int_0^x ||f(s,z,v(s,z),v_s(s,z)) - f(s,z,u(s,z),u_s(s,z))||dsdz.$$

(By using equation (6.31)).

$$= \varepsilon xt$$
$$+ \textstyle\int_0^t dz \cdot x\Big(||f(x_1,z,v(x_1,z),v_x(x_1,z)) - f(x_1,z,u(x_1,z),u_x(x_1,z))||\Big),$$

where $x_1 \in (0,x)$.

$$||v(x,t)-u(x,t)|| \leq \varepsilon xt + \int_0^t L_f(x_1,z) \times \max\Big\{||v(x_1,z)-u(x_1,z)||, ||v_x(x_1,z)-u_x(x_1,z)||\Big\} xdz, \tag{6.33}$$

(by hypothesis (ii)).

Again, by using equation (6.30), we get

$$||v_x(x,t) - u_x(x,t)|| = ||v_x(x,t) - v_x(x,0) - \textstyle\int_0^t f(x,z,u(x,z),u_x(x,z))dz||.$$

$$= ||v_x(x,t) - v_x(x,0) - \textstyle\int_0^t f(x,z,v(x,z),v_x(x,z))dz$$
$$+ \textstyle\int_0^t f(x,z,v(x,z),v_x(x,z))dz - \int_0^t f(x,z,u(x,z),u_x(x,z))dz||.$$

$$\leq ||v_x(x,t) - v_x(x,0) - \textstyle\int_0^t f(x,z,v(x,z),v_x(x,z))dz||$$
$$+|| \textstyle\int_0^t f(x,z,v(x,z),v_x(x,z))dz - \int_0^t f(x,z,u(x,z),u_x(x,z))dz||.$$

$$||v_x(x,t)-u_x(x,t)|| \leq ||v_x(x,t)-v_x(x,0)-\int_0^t f(x,z,v(x,z),v_x(x,z))dz|| + \int_0^t ||\{f(x,z,v(x,z),v_x(x,z))-f(x,z,u(x,z),u_x(x,z))\}dz||.$$

$$||v_x(x,t)-u_x(x,t)|| \leq \varepsilon t + \int_0^t \{L_f(x,z)\times \max\left[||v(x,z)-u(x,z)||, ||v_x(x,z)-u_x(x,z)||\right]\}dz. \tag{6.34}$$

(By using equation (6.32) and hypothesis (ii)).

Again, we have the following inequalities

(i) $\varepsilon xt \leq \varepsilon(1+a)b$ and $\varepsilon t \leq \varepsilon(1+a)b, \ \forall\, x \in [0,a], \ \forall\, t \in [0,b]$.

(ii) $L_f(x,z) \leq \max_{x\in[0,a]} \{L_f(x,z)\} = G(z), \ \forall\, z \in [0,b]$.

(iii) $||v(x,z)-u(x,z)|| \leq \max_{x\in[0,a]} \{||v(x,z)-u(x,z)||\} = h_1(z), \forall\, z \in [0,b]$.

(iv) $||v_x(x,z)-u_x(x,z)|| \leq \max_{y\in[0,a]} \{||v_x(y,z)-u_x(y,z)||\} = h_2(z), \forall\, z \in [0,b]$.

Using the above inequalities (i)-(iv) and the equation (6.33), we get

$$||v(x,t)-u(x,t)|| \leq \varepsilon(1+a)b + (1+a)\int_0^t G(z)\times \max[h_1(z),h_2(z)]dz. \tag{6.35}$$

Again using the inequalities (i)-(iv) and the Equation (6.34), we get

$$||v_x(x,t)-u_x(x,t)|| \leq \varepsilon(1+a)b + (1+a)\int_0^t G(z)\times \max[h_1(z),h_2(z)]dz. \tag{6.36}$$

Using equation (6.35), we get,

$$h_1(t) = \max_{x\in[0,a]} ||v(x,t)-u(x,t)|| \leq \varepsilon(1+a)b + (1+a)\int_0^t G(z)\times \max[h_1(z),h_2(z)]dz. \tag{6.37}$$

Using equation (6.36), we get,

$$h_2(t) = \max_{y\in[0,a]} ||v_y(y,t)-u_y(y,t)|| \leq \varepsilon(1+a)b + (1+a)\int_0^t G(z)\times \max[h_1(z),h_2(z)]dz. \tag{6.38}$$

Using equations (6.37) and (6.38), we get

$$\max[h_1(t),h_2(t)] \le \varepsilon(1+a)b+(1+a)\int_0^t G(z)\times\max[h_1(z),h_2(z)]dz.$$

By using Lemma 6.1.2, we get

$$\max_{t\in[0,b]}[h_1(t),h_2(t)] \le \varepsilon b(1+a)^2\times\exp\left\{\int_0^b G(z)dz\right\}. \tag{6.39}$$

From equation (6.39), we get

$$||v(x,t)-u(x,t)|| \le \varepsilon b(1+a)^2\times\exp\left\{\int_0^b G(z)dz\right\}$$

and

$$||v_x(x,t)-u_x(x,t)|| \le \varepsilon b(1+a)^2\times\exp\left\{\int_0^b G(z)dz\right\}.$$

$$\Longrightarrow\ ||v(x,t)-u(x,t)|| \le \varepsilon\times c_f \quad\text{and}\quad ||v_x(x,t)-u_x(x,t)|| \le \varepsilon\times c_f,$$

where $c_f=b(1+a)^2\times\exp\left\{\int_0^b G(z)dz\right\}$.

Hence the equation (6.6) is HU stable. □

Thus, we have proved the HU stability of the second order non-linear ordinary and partial differential equations (6.5) and (6.6) respectively by using the integral inequalities.

Chapter 7

GENERALISED HUR STABILITY OF SECOND ORDER NON-LINEAR ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

7.1 INTRODUCTION

In this chapter, we have studied the generalised HUR stability for the second order non-linear ordinary and partial differential equations by using the results in [41].

We have obtained the generalised HUR stability for the second order non-linear ordinary differential equation of the type

$$u_{xx}(x,t) = f(x,t,u(x,t),u_x(x,t)), \tag{7.1}$$

where $f \in C([0,a]\times[0,b]\times\mathbb{B}^2,\mathbb{B})$ and $(\mathbb{B},||.||)$ is a real or complex Banach space.

Further we prove the generalised HUR stability for the second order non-linear partial differential equation of the type

$$u_{xt}(x,t) = f(x,t,u(x,t),u_x(x,t),u_t(x,t),u_{xx}(x,t)), \tag{7.2}$$

where $f \in C([0,a]\times[0,b]\times\mathbb{B}^4,\mathbb{B})$.

We have obtained the required integral inequalities to prove our main results.

First, we define generalised HUR stability of the differential equations (7.1) and (7.2).

Definition 7.1.1 : Equation (7.1) is generalised HUR stable if there exists real constants $c_{1(f,\phi)}, c_{2(f,\phi)} > 0$ such that for any continuous function $\phi : [0,a]\times[0,b] \to (0,\infty)$ and for any soluion $v(x,t)$ of the inequality

$$||v_{xx}(x,t) - f(x,t,v(x,t),v_x(x,t))|| \le \phi(x,t), \tag{7.3}$$

there exists a solution $u(x,t)$ of (7.1) with

$$||v(x,t)-u(x,t)|| \le c_{1(f,\phi)}\phi(x,t)$$

and $$||v_x(x,t)-u_x(x,t)|| \le c_{2(f,\phi)}\phi(x,t), \ \ \forall x \in [0,a],\ \forall t \in [0,b].$$

Remark 7.1.2 : A function $v(x,t)$ is a solution to the inequality (7.3) iff there exists a continuous function $g(x,t)$ which depends on $v(x,t)$ such that

i) $||g(x,t)|| \leq \phi(x,t)$,

ii) $v_{xx}(x,t) = f(x,t,v(x,t),v_x(x,t)) + g(x,t), \forall x \in [0,a],\ \forall\, t \in [0,b]$.

Definition 7.1.3 : Equation (7.2) is generalised HUR stable if there exists real constants $c_{1(f,\phi)}, c_{2(f,\phi)}, c_{3(f,\phi)}, \varepsilon > 0$ such that for any continuous function $\phi : [0,a]\times[0,b] \to (0,\infty)$ and for any soluion $v(x,t)$ of the inequality

$$||v_{xt}(x,t) - f(x,t,v(x,t),v_x(x,t),v_t(x,t),v_{xx}(x,t))|| \leq \varepsilon\phi(x,t), \tag{7.4}$$

there exists a solution $u(x,t)$ of (7.2) with

$$||v(x,t) - u(x,t)|| \leq c_{1(f,\phi)}\phi(x,t),$$

$$||v_x(x,t) - u_x(x,t)|| \leq c_{2(f,\phi)}\phi(x,t)$$

and
$$||v_t(x,t) - u_t(x,t)|| \leq c_{3(f,\phi)}\phi(x,t), \forall x \in [0,a],\ \forall t \in [0,b].$$

Remark 7.1.4: A function $v(x,t)$ is a solution to the inequality (7.4) iff there exists a continuous function $g(x,t)$ which depends on $v(x,t)$ such that

i) $||g(x,t)|| \leq \phi(x,t)$,

ii) $v_{xt}(x,t) = f(x,t,v(x,t),v_x(x,t),v_t(x,t),v_{xx}(x,t)) + g(x,t), \forall x \in [0,a],\ \forall t \in [0,b]$.

In proving main results following Gronwall type Lemmas are required.

Lemma 7.1.5 : [50] Assume that

i) $u,v,h \in C(\mathbb{R}^n_+, \mathbb{R}_+)$.

ii) For any $t \geq t_0$, $u(t) \leq h(t) + \int_{t_0}^t v(s)u(s)ds$.

iii) $h(t)$ is positive and increasing.

Then $u(t) \leq h(t)\ \exp\{\int_{t_0}^t v(r)dr\}$, for any $t \geq t_0$.

Lemma 7.1.6 : [50] Let $u(x,y), \eta(x,y)$ and $c(x,y)$ be nonnegative continuous functions defined on $x,y \in R_+$, and let $\eta(x,y)$ be nondecreasing function in each variable $x,y \in R_+$. If $u(x,y) \leq \eta(x,y) + \int_0^x \int_0^y c(s,t)u(s,t)dsdt$, for $x,y \in R_+$, then

$$u(x,y) \leq \eta(x,y) \times \exp\{\int_0^x \int_0^y c(s,t)dsdt\}, \quad \text{for } x,y \in R_+.$$

7.2 INTEGRAL INEQUALITIES

In this section, we have proved some results on inequalities, which are required in proving our main results. First, we prove a result for the inequality (7.3).

Theorem 7.2.1 : If $v(x,t)$ is a solution to the inequality (7.3), then (v, v_x) satisfies the following integral inequality system.

$$||v(x,t) - v(0,t) - \int_0^x \int_0^y f(s,t,v(s,t),v_s(s,t))dsdy|| \leq \int_0^x \int_0^z \phi(s,t)dsdz,$$

$$||v_x(x,t) - v_x(0,t) - \int_0^x f(s,t,v(s,t),v_s(s,t))ds|| \leq \int_0^x \phi(s,t)ds.$$

Proof. If $v(x,t)$ is a solution to the inequality

$$||v_{xx}(x,t) - f(x,t,v(x,t),v_x(x,t))|| \leq \phi(x,t),$$

then integrating w. r. t. x, we get

$$\int_0^x ||v_{ss}(s,t) - f(s,t,v(s,t),v_s(s,t))||ds \leq \int_0^x \phi(s,t)ds.$$

Since

$$|| \int_0^x \{v_{ss}(s,t) - f(s,t,v(s,t),v_s(s,t))\}ds|| \leq \int_0^x ||v_{ss}(s,t) - f(s,t,v(s,t),v_s(s,t))||ds,$$

we get

$$|| \int_0^x \{v_{ss}(s,t) - f(s,t,v(s,t),v_s(s,t))\}ds|| \leq \int_0^x \phi(s,t)ds.$$

$$\Rightarrow \quad ||v_x(x,t) - v_x(0,t) - \int_0^x f(s,t,v(s,t),v_s(s,t))ds|| \leq \int_0^x \phi(s,t)ds.$$

Integrating w. r. t. x, we get

$\int_0^x ||v_y(y,t) - v_y(0,t) - \int_0^y f(s,t,v(s,t),v_s(s,t))ds||dy \le \int_0^x \int_0^z \phi(s,t)dsdz.$

Since $|| \int_0^x \{v_y(y,t) - v_y(0,t) - \int_0^y f(s,t,v(s,t),v_s(s,t))ds\}dy||$

$$\le \int_0^x ||v_y(y,t) - v_y(0,t) - \int_0^y f(s,t,v(s,t),v_s(s,t))ds||dy,$$

we get

$$|| \int_0^x \{v_y(y,t) - v_y(0,t) - \int_0^y f(s,t,v(s,t),v_s(s,t))ds\}dy|| \le \int_0^x \int_0^z \phi(s,t)dsdz.$$

$$\Rightarrow ||v(x,t) - v(0,t) - \{v(0,t) - v(0,t)\} - \int_0^x \int_0^y f(s,t,v(s,t),v_s(s,t))dsdy|| \le \int_0^x \int_0^z \phi(s,t)dsdz.$$

$$\Rightarrow ||v(x,t) - v(0,t) - \int_0^x \int_0^y f(s,t,v(s,t),v_s(s,t))dsdy|| \le \int_0^x \int_0^z \phi(s,t)dsdz. \qquad \square$$

Similarly, we have the result for the inequality (7.4).

Theorem 7.2.2 : If $v(x,t)$ is a solution to the inequality (7.4), then v, v_t and v_x satisfy the integral inequalities

$$||v(x,t) - v(x,0) - v(0,t) + v(0,0) - \int_0^t \int_0^x f(s,z,v(s,z),v_s(s,z),v_z(s,z),v_{ss}(s,z))dsdz|| \le \int_0^t \int_0^x \phi(s,z)dsdz,$$

$$||v_t(x,t) - v_t(0,t) - \int_0^x f(s,t,v(s,t),v_s(s,t),v_t(s,t),v_{ss}(s,t))ds|| \le \int_0^x \phi(s,t)ds$$

and

$$||v_x(x,t) - v_x(x,0) - \int_0^t f(x,z,v(x,z),v_x(x,z),v_z(x,z),v_{xx}(x,z))dz|| \le \int_0^t \phi(x,z)dz$$

respectively.

Proof. If $v(x,t)$ is a solution to the inequality

$$||v_{xt}(x,t) - f(x,t,v(x,t),v_x(x,t),v_t(x,t),v_{xx}(x,t))|| \le \phi(x,t), \qquad (7.5)$$

then integrating w. r. t. x, we get

$$\int_0^x ||v_{st}(s,t) - f(s,t,v(s,t),v_s(s,t),v_t(s,t),v_{ss}(s,t))||ds \le \int_0^x \phi(s,t)ds.$$

Since

$$||\int_0^x \{v_{st}(s,t) - f(s,t,v(s,t),v_s(s,t),v_t(s,t),v_{ss}(s,t))\}ds||$$
$$\leq \int_0^x ||v_{st}(s,t) - f(s,t,v(s,t),v_s(s,t),v_t(s,t),v_{ss}(s,t))||ds,$$

we get

$$||\int_0^x \{v_{st}(s,t) - f(s,t,v(s,t),v_s(s,t),v_t(s,t),v_{ss}(s,t))\}ds|| \leq \int_0^x \phi(s,t)ds.$$

$$\Rightarrow ||v_t(x,t) - v_t(0,t) - \int_0^x f(s,t,v(s,t),v_s(s,t),v_t(s,t),v_{ss}(s,t))ds|| \leq \int_0^x \phi(s,t)ds.$$

Integrating w. r. t. t, we get

$$\int_0^t ||v_z(x,z) - v_z(0,z) - \int_0^x f(s,z,v(s,z),v_s(s,z),v_z(s,z),v_{ss}(s,z))ds||dz$$
$$\leq \int_0^t \int_0^x \phi(s,z)dsdz.$$

Since

$$||\int_0^t \{v_z(x,z) - v_z(0,z) - \int_0^x f(s,z,v(s,z),v_s(s,z),v_z(s,z),v_{ss}(s,z))ds\}dz||$$
$$\leq \int_0^t ||v_z(x,z) - v_z(0,z) - \int_0^x f(s,z,v(s,z),v_s(s,z),v_z(s,z),v_{ss}(s,z))ds||dz,$$

we get

$$||\int_0^t \{v_z(x,z) - v_z(0,z) - \int_0^x f(s,z,v(s,z),v_s(s,z),v_z(s,z),v_{ss}(s,z))ds\}dz||$$
$$\leq \int_0^t \int_0^x \phi(s,z)dsdz.$$

$$\Rightarrow ||v(x,t) - v(x,0) - v(0,t) + v(0,0)$$
$$- \int_0^t \int_0^x f(s,z,v(s,z),v_s(s,z),v_z(s,z),v_{ss}(s,z))dsdz|| \leq \int_0^t \int_0^x \phi(s,z)dsdz.$$

Integrating equation (7.5) w. r. t. t, we get

$$\int_0^t ||v_{xz}(x,z) - f(x,z,v(x,z),v_x(x,z),v_z(x,z),v_{xx}(x,z))||dz \leq \int_0^t \phi(x,z)dz.$$

Since

$$||\int_0^t \{v_{xz}(x,z) - f(x,z,v(x,z),v_x(x,z),v_z(x,z),v_{xx}(x,z))\}dz||$$
$$\leq \int_0^t ||v_{xz}(x,z) - f(x,z,v(x,z),v_x(x,z),v_z(x,z),v_{xx}(x,z))||dz,$$

we have

$|| \int_0^t \{v_{xz}(x,z) - f(x,z,v(x,z),v_x(x,z),v_z(x,z),v_{xx}(x,z))\}dz|| \leq \int_0^t \phi(x,z)dz.$

$\Rightarrow ||v_x(x,t) - v_x(x,0) - \int_0^t f(x,z,v(x,z),v_x(x,z),v_z(x,z),v_{xx}(x,z))dz|| \leq \int_0^t \phi(x,z)dz.$

□

7.3 GENERALISED HUR STABILITY OF (7.1)

In this section we prove generalised HUR stability for the second order non-linear ordinary differential equation (7.1). We have the following result.

Theorem 3.1 : Assume that

i) $f \in C([0,a] \times [0,b] \times \mathbb{B}^2, \mathbb{B})$.

ii) There exists $L_f(x,t) \in C^1([0,a] \times [0,b], \mathbb{R}_+)$ such that $\int_0^a L_f(x,t)dx < \infty$ and $||f(x,t,z_1,z_2) - f(x,t,t_1,t_2)|| \leq L_f(x,t) \max_{i \in \{1,2\}} \{||z_i - t_i||\}, \forall\, x \in [0,a], \forall\, t \in [0,b]$ and $z_1,\, z_2,\, t_1,\, t_2 \in \mathbb{B}$.

iii) $\phi : [0,a] \times [0,b] \to (0,\infty)$ is an increasing function.

iv) There exist constants $\lambda_\phi^1, \lambda_\phi^2 > 0$ such that

$$\int_0^x \int_0^z \phi(s,t)dsdz \leq \lambda_\phi^1 \phi(x,t), \quad \int_0^x \phi(s,t)ds \leq \lambda_\phi^2 \phi(x,t), \forall x \in [0,a], \forall t \in [0,b].$$

Then (7.1) is generalised HUR stable.

Proof. : Let $v(x,t)$ be a solution to the inequality

$||v_{xx}(x,t) - f(x,t,v(x,t),v_x(x,t))|| \leq \phi(x,t),\ \forall x \in [0,a], \forall t \in [0,b].$

Let $u(x,t)$ be the unique solution to the problem

$$\begin{aligned} u_{xx}(x,t) &= f(x,t,u(x,t),u_x(x,t)), \\ u(0,t) &= v(0,t), \forall t \in [0,b], \\ u(x,0) &= v(x,0), \forall x \in [0,a]. \end{aligned} \tag{7.6}$$

If $u(x,t)$ is a solution to (7.6), then $(u,u_x(x,t))$ satisfies the following system

$$\begin{aligned} u(x,t) &= v(0,t)+\int_0^x\int_0^z f(s,t,u(s,t),u_s(s,t))dsdz,\\ u_x(x,t) &= v_x(0,t)+\int_0^x f(s,t,u(s,t),u_s(s,t))ds. \end{aligned} \tag{7.7}$$

Then by using **Theorem 7.2.1** and hypothesis (iv), it follows that

$||v(x,t)-v(0,t)-\int_0^x\int_0^y f(s,t,v(s,t),v_s(s,t))dsdy|| \le \int_0^x\int_0^z \phi(s,t))dsdz,$

i.e.

$$||v(x,t)-v(0,t)-\int_0^x\int_0^y f(s,t,v(s,t),v_s(s,t))dsdy|| \le \lambda_\phi^1\phi(x,t) \tag{7.8}$$

and

$||v_x(x,t)-v_x(0,t)-\int_0^x f(s,t,v(s,t),v_s(s,t))ds|| \le \int_0^x \phi(s,t)ds$, i.e.

$$||v_x(x,t)-v_x(0,t)-\int_0^x f(s,t,v(s,t),v_s(s,t))ds|| \le \lambda_\phi^2\phi(x,t). \tag{7.9}$$

Consider

$$\begin{aligned} ||v(x,t)-u(x,t)|| &= ||v(x,t)-v(0,t)-\int_0^x\int_0^z f(s,t,u(s,t),u_s(s,t))dsdz||.\\ &= ||v(x,t)-v(0,t)-\int_0^x\int_0^z f(s,t,v(s,t),v_s(s,t))dsdz\\ &\quad +\int_0^x\int_0^z f(s,t,v(s,t),v_s(s,t))dsdz\\ &\quad -\int_0^x\int_0^z f(s,t,u(s,t),u_s(s,t))dsdz||.\\ &\le ||v(x,t)-v(0,t)-\int_0^x\int_0^z f(s,t,v(s,t),v_s(s,t))dsdz||\\ &\quad +||\int_0^x\int_0^z f(s,t,v(s,t),v_s(s,t))dsdz-\int_0^x\int_0^z f(s,t,u(s,t),u_s(s,t))dsdz||.\\ &\le ||v(x,t)-v(0,t)-\int_0^x\int_0^z f(s,t,v(s,t),v_s(s,t))dsdz||\\ &\quad +\int_0^x\int_0^z ||f(s,t,v(s,t),v_s(s,t))-f(s,t,u(s,t),u_s(s,t))||dsdz. \end{aligned}$$

$$||v(x,t)-u(x,t)|| \leq \lambda_\phi^1 \phi(x,t)$$
$$+\int_0^x \int_0^z L_f(s,t) \times \max\{||v(s,t)-u(s,t)||, ||v_s(s,t)-u_s(s,t)||\} ds dz.$$

(by using hypothesis (ii) and equation (7.8))

$$\leq \lambda_\phi^1 \phi(x,t)$$
$$+\int_0^x dz \int_0^z L_f(s,t) \times \max\{||v(s,t)-u(s,t)||, ||v_s(s,t)-u_s(s,t)||\} ds.$$

$$\leq \lambda_\phi^1 \phi(x,t)$$
$$+\int_0^x (x-s) L_f(s,t) \times \max\{||v(s,t)-u(s,t)||, ||v_s(s,t)-u_s(s,t)||\} ds.$$

$$\leq \lambda_\phi^1 \phi(x,t)$$
$$+\int_0^x a L_f(s,t) \times \max\{||v(s,t)-u(s,t)||, ||v_s(s,t)-u_s(s,t)||\} ds,$$

$$(\because x-s \leq \max(x-s) \leq (a-x) \leq a).$$

$$\leq \lambda_\phi^1 \phi(x,t)$$
$$+\int_0^x (1+a) L_f(s,t) \times \max\{||v(s,t)-u(s,t)||, ||v_s(s,t)-u_s(s,t)||\} ds,$$

$$||v(x,t)-u(x,t)|| \leq \lambda_\phi^1 \phi(x,t) + \int_0^x G(s,t) \times \max\{||v(s,t)-u(s,t)||, ||v_s(s,t)-u_s(s,t)||\} ds, \tag{7.10}$$

where G(s,t) = (1+a)$L_f(s,t)$.

Again consider

$$||v_x(x,t)-u_x(x,t)|| = ||v_x(x,t)-v_x(0,t)-\int_0^x f(s,t,u(s,t),u_s(s,t)) ds||.$$

$$= ||v_x(x,t)-v_x(0,t)-\int_0^x f(s,t,v(s,t),v_s(s,t)) ds$$
$$+\int_0^x f(s,t,v(s,t),v_s(s,t)) ds - \int_0^x f(s,t,u(s,t),u_s(s,t)) ds||.$$

$$\leq ||v_x(x,t)-v_x(0,t)-\int_0^x f(s,t,v(s,t),v_s(s,t)) ds||$$
$$+||\int_0^x f(s,t,v(s,t),v_s(s,t)) ds - \int_0^x f(s,t,u(s,t),u_s(s,t)) ds||.$$

$$\leq \int_0^x \phi(s,t)ds$$

$$+\int_0^x ||f(s,t,v(s,t),v_s(s,t)) - f(s,t,u(s,t),u_s(s,t))||ds.$$

(by using Theorem (7.2.1))

By using hypothesis (ii) and (iv), we get

$$||v_x(x,t) - u_x(x,t)|| \leq \lambda_\phi^2 \phi(x,t)$$

$$+\int_0^x L_f(s,t) \times max\{||v(s,t) - u(s,t)||, ||v_s(s,t) - u_s(s,t)||\}ds.$$

$$\leq \lambda_\phi^2 \phi(x,t)$$

$$+\int_0^x (1+a)L_f(s,t) \times max\{||v(s,t) - u(s,t)||, ||v_s(s,t) - u_s(s,t)||\}ds,$$

$$||v_x(x,t) - u_x(x,t)|| \leq \lambda_\phi^2 \phi(x,t) + \int_0^x G(s,t) \times \max\{||v(s,t) - u(s,t)||, ||v_s(s,t) - u_s(s,t)||\}ds, \tag{7.11}$$

where G(s,t) = (1+a)L$_f(s,t)$.

Let $\lambda_\phi^3(x,t) = \max\{\lambda_\phi^1(x,t), \lambda_\phi^2(x,t)\}$.

Then equation (7.10), yields

$$||v(x,t) - u(x,t)|| \leq \lambda_\phi^3 \phi(x,t) + \int_0^x G(s,t) \times \max\{||v(s,t) - u(s,t)||, ||v_s(s,t) - u_s(s,t)||\}ds$$

and equation (7.11), yields

$$||v_x(x,t) - u_x(x,t)|| \leq \lambda_\phi^3 \phi(x,t) + \int_0^x G(s,t) \times \max\{||v(s,t) - u(s,t)||, ||v_s(s,t) - u_s(s,t)||\}ds.$$

From these two yields

$$\max\{||v(x,t) - u(x,t)||, ||v_x(x,t) - u_x(x,t)||\} \leq \lambda_\phi^3 \phi(x,t) +$$

$$\int_0^x G(s,t) \times \max\{||v(s,t) - u(s,t)||, ||v_s(s,t) - u_s(s,t)||\}ds.$$

By using Lemma (7.1.5), we get

$$\max\{||v(x,t) - u(x,t)||, ||v_x(x,t) - u_x(x,t)||\} \leq \lambda_\phi^3 \phi(x,t) \times \exp\{\int_0^x G(s,t)ds\}.$$

From which it follows that

i. e. $||v(x,t)-u(x,t)|| \leq \lambda_{\phi}^{3}\phi(x,t)\times\exp\{\int_0^x G(s,t)ds\}$ and

$$||v_x(x,t)-u_x(x,t)|| \leq \lambda_{\phi}^{3}\phi(x,t)\times\exp\{\int_0^x G(s,t)ds\}.$$

$\implies ||v(x,t)-u(x,t)|| \leq \lambda_{\phi}^{3}\phi(x,t)\times\exp\{\int_0^a G(s,t)ds\}$ and

$$||v_x(x,t)-u_x(x,t)|| \leq \lambda_{\phi}^{3}\phi(x,t)\times\exp\{\int_0^a G(s,t)ds\}.$$

$||v(x,t)-u(x,t)|| \leq \phi(x,t)\times c_{(f,\phi)}$ and $||v_x(x,t)-u_x(x,t)|| \leq \phi(x,t)\times c_{(f,\phi)}$,

where $c_{(f,\phi)} = \lambda_{\phi}^{3}\times\max_{0\leq t\leq b}\exp\{\int_0^a G(s,t)ds\}$.

Hence equation (7.1) is generalised HUR stable. □

7.4 GENERALISED HUR STABILITY OF (7.2).

In this section we prove generalised HUR stability for the second order non-linear partial differential equation (7.2). We have the following result.

Theorem 4.1 : Assume that

i) $f\in C([0,a]\times[0,b]\times\mathbb{B}^4,\mathbb{B})$.

ii) There exists $L_f(x,t)\in C^1([0,a]\times[0,b],\mathbb{R}_+)$ such that $L_f(x,t)$ is integrable and

$||f(x,t,z_1,z_2,z_3,z_4)-f(x,t,t_1,t_2,t_3,t_4|| \leq L_f(x,t)\min_{i\in\{1,2,3,4\}}\{||z_i-t_i||\}$,

$\forall\, x\in[0,a], \forall\, t\in[0,b]$ and $z_1,\ z_2,\ z_3,\ z_4,\ t_1,\ t_2,\ t_3,\ t_4\in\mathbb{B}$.

iii) $\phi:[0,a]\times[0,b]\to(0,\infty)$ is an increasing function in each variable.

iv) There exist constants $\lambda_{\phi}^{1},\lambda_{\phi}^{2},\lambda_{\phi}^{3}>0$ such that $\int_0^t\int_0^x\phi(s,z)dsdz\leq\lambda_{\phi}^{1}\phi(x,t)$,

$$\int_0^t\phi(x,z)dz\leq\lambda_{\phi}^{2}\phi(x,t),\ \int_0^x\phi(s,t)ds\leq\lambda_{\phi}^{3}\phi(x,t),\ \forall\, x\in[0,a],\forall\, t\in[0,b]$$

Then (7.2) is generalised HUR stable.

Proof. :

Let $v(x,t)$ be a solution to the inequality

$||v_{xt}(x,t) - f(x,t,v(x,t),v_x(x,t),v_t(x,t),v_{xx}(x,t))|| \leq \phi(x,t),\ \forall x \in [o,a], \forall t \in [o,b].$

Let $u(x,t)$ be the unique solution to the problem

$$\begin{aligned} u_{xt}(x,t) &= f(x,t,u(x,t),u_x(x,t),u_t(x,t),u_{xx}(x,t)), \\ u(0,t) &= v(0,t), \forall t \in [o,b], \\ u(x,0) &= v(x,0), \forall x \in [o,a]. \end{aligned} \tag{7.12}$$

If $u(x,t)$ is a solution to equation (7.12), then $(u, u_t(x,t), u_x(x,t))$ satisfies the following system.

$$\begin{aligned} u(x,t) &= v(x,0) + v(0,t) - v(0,0) \\ &\quad + \int_0^t \int_0^x f(s,z,u(s,z),u_s(s,z),u_z(s,z),u_{ss}(s,z))dsdz, \\ u_t(x,t) &= v_t(0,t) + \int_0^x f(s,t,u(s,t),u_s(s,t),u_t(s,t),u_{ss}(s,t))ds, \\ u_x(x,t) &= v_x(x,0) + \int_0^t f(x,z,u(x,z),u_x(x,z),u_z(x,z),u_{xx}(x,z))dz. \end{aligned} \tag{7.13}$$

Then by using **Theorem 7.2.2** and hypothesis (iv), it follows that

$$\left|\left|v(x,t) - v(x,0) - v(0,t) + v(0,0) - \int_0^t \int_0^x f(s,z,v(s,z),v_s(s,z),v_z(s,z),v_{ss}(s,z))dsdz\right|\right|$$
$$\leq \int_0^t \int_0^x \phi(s,z)dsdz.$$

That is

$$\begin{aligned} \Bigg|\Bigg| v(x,t) - v(x,0) - \int_0^t \int_0^x f(s,z,v(s,z),v_s(s,z),v_z(s,z),v_{ss}(s,z))dsdz \\ -v(0,t) + v(0,0)\Bigg|\Bigg| \leq \lambda_\phi^1 \phi(x,t). \end{aligned} \tag{7.14}$$

Also

$$||v_t(x,t) - v_t(0,t) - \int_0^x f(s,t,v(s,t),v_s(s,t),v_t(s,t),v_{ss}(s,t))ds|| \leq \int_0^x \phi(s,t)ds.$$

$$\Rightarrow ||v_t(x,t) - v_t(0,t) - \int_0^x f(s,t,v(s,t),v_s(s,t),v_t(s,t),v_{ss}(s,t))ds|| \leq \lambda_\phi^3 \phi(x,t), \tag{7.15}$$

(By using hypothesis (iv))

Next

$$||v_x(x,t)-v_x(x,0)-\int_0^t f(x,z,v(x,z),v_x(x,z),v_z(x,z),v_{xx}(x,z))dz|| \leq \int_0^t \phi(x,z)dz.$$

$$\Rightarrow ||v_x(x,t)-v_x(x,0)-\int_0^t f(x,z,v(x,z),v_x(x,z),v_z(x,z),v_{xx}(x,z))dz|| \leq \lambda_\phi^2 \phi(x,t). \tag{7.16}$$

Using equation (7.13), we get

$$\begin{aligned}
||v(x,t)-u(x,t)|| &= ||v(x,t)-v(x,0)-v(0,t)+v(0,0) \\
&\quad -\int_0^t\int_0^x f(s,z,u(s,z),u_s(s,z),u_z(s,z),u_{ss}(s,z))dsdz||. \\
&= ||v(x,t)-v(x,0)-v(0,t)+v(0,0) \\
&\quad -\int_0^t\int_0^x f(s,z,v(s,z),v_s(s,z),v_z(s,z),v_{ss}(s,z))dsdz \\
&\quad +\int_0^t\int_0^x f(s,z,v(s,z),v_s(s,z),v_z(s,z),v_{ss}(s,z))dsdz \\
&\quad -\int_0^t\int_0^x f(s,z,u(s,z),u_s(s,z),u_z(s,z),u_{ss}(s,z))dsdz||. \\
&\leq ||v(x,t)-v(x,0)-v(0,t)+v(0,0) \\
&\quad -\int_0^t\int_0^x f(s,z,v(s,z),v_s(s,z),v_z(s,z),v_{ss}(s,z))dsdz|| \\
&\quad +||\int_0^t\int_0^x f(s,z,v(s,z),v_s(s,z),v_z(s,z),v_{ss}(s,z))dsdz \\
&\quad -\int_0^t\int_0^x f(s,z,u(s,z),u_s(s,z),u_z(s,z),u_{ss}(s,z))dsdz||. \\
&\leq ||v(x,t)-v(x,0)-v(0,t)+v(0,0) \\
&\qquad -\int_0^t\int_0^x f(s,z,v(s,z),v_s(s,z),v_z(s,z),v_{ss}(s,z))dsdz|| \\
&\quad +\int_0^t\int_0^x ||\{f(s,z,v(s,z),v_s(s,z),v_z(s,z),v_{ss}(s,z)) \\
&\qquad -f(s,z,u(s,z),u_s(s,z),u_z(s,z),u_{ss}(s,z))\}||dsdz. \\
&\leq \lambda_\phi^1 \phi(x,t) \\
&\quad +\int_0^t\int_0^x \Big\{L_f(s,z)\ \min\big[||v(s,z)-u(s,z)||, ||v_s(s,z)-u_s(s,z)||, \\
&\qquad ||v_z(s,z)-u_z(s,z)||, ||v_{ss}(s,z)-u_{ss}(s,z)||\big]\Big\}dsdz.
\end{aligned}$$

(by using hypothesis (ii) and equation (7.14))

$$||v(x,t)-u(x,t)|| \le \lambda_\phi^1 \phi(x,t) + \int_0^t \int_0^x \Big\{ L_f(s,z)\, ||v(s,z)-u(s,z)|| \Big\} dsdz.$$

By using Lemma (7.1.6), we get

$$||v(x,t)-u(x,t)|| \le \lambda_\phi^1 \phi(x,t) \times \exp\Big\{ \int_0^t \int_0^x L_f(s,z) dsdz \Big\}.$$

$$\le \lambda_\phi^1 \phi(x,t) \times \exp\Big\{ \int_0^b \int_0^a L_f(s,z) dsdz \Big\}.$$

$$\le c_{1(f,\phi)} \phi(x,t), \quad \text{where } c_{1(f,\phi)} = \lambda_\phi^1 \times \exp\Big\{ \int_0^b \int_0^a L_f(s,z) dsdz \Big\}.$$

Again, by using equation (7.13), we get

$$||v_x(x,t)-u_x(x,t)|| = ||v_x(x,t)-v_x(x,0) - \int_0^t f(x,z,u(x,z),u_x(x,z),u_z(x,z),u_{xx}(x,z))dz||.$$

i.e.

$$\begin{aligned}
||v_x(x,t)-u_x(x,t)|| = ||v_x(x,t) - v_x(x,0) \\
&- \int_0^t f(x,z,v(x,z),v_x(x,z),v_z(x,z),v_{xx}(x,z))dz \\
&+ \int_0^t f(x,z,v(x,z),v_x(x,z),v_z(x,z),v_{xx}(x,z))dz \\
&- \int_0^t f(x,z,u(x,z),u_x(x,z),u_z(x,z),u_{xx}(x,z))dz||.
\end{aligned}$$

$$\begin{aligned}
\le ||v_x(x,t) - v_x(x,0) \\
&- \int_0^t f(x,z,v(x,z),v_x(x,z),v_z(x,z),v_{xx}(x,z))dz|| \\
&+|| \int_0^t f(x,z,v(x,z),v_x(x,z),v_z(x,z),v_{xx}(x,z))dz \\
&- \int_0^t f(x,z,u(x,z),u_x(x,z),u_z(x,z),u_{xx}(x,z))dz||.
\end{aligned}$$

$$\begin{aligned}
\le ||v_x(x,t) - v_x(x,0) \\
&- \int_0^t f(x,z,v(x,z),v_x(x,z),v_z(x,z),v_{xx}(x,z))dz|| \\
&+ \int_0^t ||\{ f(x,z,v(x,z),v_x(x,z),v_z(x,z),v_{xx}(x,z)) \\
&- f(x,z,u(x,z),u_x(x,z),u_z(x,z),u_{xx}(x,z)) \Big\} dz||.
\end{aligned}$$

$$\Rightarrow ||v_x(x,t)-u_x(x,t)|| \leq \lambda_\phi^2 \phi(x,t) + \int_0^t \Big\{ L_f(x,z) \min\Big[||v(x,z)-u(x,z)||, ||v_x(x,z)-u_x(x,z)||, ||v_z(x,z)-u_z(x,z)||, ||v_{xx}(x,z)-u_{xx}(x,z)||\Big]\Big\} dz,$$

(By using equation (7.16) and hypothesis (ii)).

$$\Rightarrow ||v_x(x,t)-u_x(x,t)|| \leq \lambda_\phi^2 \phi(x,t) + \int_0^t \Big\{ L_f(x,z)\, ||v_x(x,z)-u_x(x,z)|| \Big\} dz, \ \forall\, t \in [0,b].$$

By using Lemma (7.1.5)(considering x fixed), we get

$$||v_x(x,t)-u_x(x,t)|| \leq \lambda_\phi^2 \phi(x,t) \times \exp\Big\{ \int_0^t L_f(x,z)dz \Big\}, \ \forall\, t \ \in [0,b].$$

$$\leq \lambda_\phi^2 \phi(x,t) \times \exp\Big\{ \int_0^b L_f(x,z)dz \Big\}.$$

$$\leq c_{2(f,\phi)} \phi(x,t), \quad \text{where } c_{2(f,\phi)} = \lambda_\phi^2 \times \max_{0\leq x \leq a} \exp\Big\{ \int_0^b L_f(x,z)dz \Big\}.$$

Next

$$||v_t(x,t)-u_t(x,t)|| = ||v_t(x,t) - v_t(0,t) - \int_0^x f(s,t,u(s,t),u_s(s,t),u_t(s,t),u_{ss}(s,t))ds||.$$

i.e.

$$||v_t(x,t)-u_t(x,t)|| = ||v_t(x,t) - v_t(0,t) - \int_0^x f(s,t,v(s,t),v_s(s,t),v_t(s,t),v_{ss}(s,t))ds + \int_0^x f(s,t,v(s,t),v_s(s,t),v_t(s,t),v_{ss}(s,t))ds - \int_0^x f(s,t,u(s,t),u_s(s,t),u_t(s,t),u_{ss}(s,t))ds||.$$

i.e.

$$||v_t(x,t)-u_t(x,t)|| \leq ||v_t(x,t)-v_t(0,t)-\int_0^x f(s,t,v(s,t),v_s(s,t),v_t(s,t),v_{ss}(s,t))ds|| + || \int_0^x f(s,t,v(s,t),v_s(s,t),v_t(s,t),v_{ss}(s,t))ds - \int_0^x f(s,t,u(s,t),u_s(s,t),u_t(s,t),u_{ss}(s,t))ds||.$$

$$\leq ||v_t(x,t)-v_t(0,t)-\int_0^x f(s,t,v(s,t),v_s(s,t),v_t(s,t),v_{ss}(s,t))ds||$$

$$+\int_0^x \{||f(s,t,v(s,t),v_s(s,t),v_t(s,t),v_{ss}(s,t))$$
$$-f(s,t,u(s,t),u_s(s,t),u_t(s,t),v_{ss}(s,t))||\}ds.$$

$$\Rightarrow ||v_t(x,t)-u_t(x,t)|| \le \lambda_\phi^3 \phi(x,t)$$
$$+\int_0^x \Big\{L_f(s,t)\ \min\Big[||v(s,t)-u(s,t)||, ||v_s(s,t)-u_s(s,t)||,$$
$$||v_t(s,t)-u_t(s,t)||, ||v_{ss}(s,t)-u_{ss}(s,t)||\Big]\Big\}ds,\ \forall\, x\in[0,a].$$

(By using equation (7.15) and hypothesis (ii))

$$\Rightarrow ||v_t(x,t)-u_t(x,t)|| \le \lambda_\phi^3 \phi(x,t) + \int_0^x \Big\{L_f(s,t)||v_t(s,t)-u_t(s,t)||\Big\}ds,\ \forall\, x\in[0,a].$$

By using Lemma (7.1.5)(considering t fixed), we get

$$||v_t(x,t)-u_t(x,t)|| \le \lambda_\phi^3 \phi(x,t)\times \exp\Big\{\int_0^x L_f(s,t)ds\Big\},\ \forall\, x\in[0,a].$$
$$\le \lambda_\phi^3 \phi(x,t)\times \exp\Big\{\int_0^a L_f(s,t)ds\Big\}.$$
$$\le c_{3(f,\phi)}\phi(x,t), \quad \text{where } c_{3(f,\phi)} = \lambda_\phi^3 \times \max_{0\le t\le b} \exp\Big\{\int_0^a L_f(s,t)ds\Big\}.$$

Thus we have real constants $c_{1(f,\phi)}, c_{2(f,\phi)}$ and $c_{3(f,\phi)}$ such that for any function $\phi(x,t)$ as in (iii) and for any soluion $v(x,t)$ of the inequality (7.4), the solution $u(x,t)$ of (7.2) satisfy the following :

$$||v(x,t)-u(x,t)|| \le c_{1(f,\phi)}\phi(x,t), \quad \forall\, x\in[0,a],\ \forall\, t\in[0,b],$$

$$||v_x(x,t)-u_x(x,t)|| \le c_{2(f,\phi)}\phi(x,t), \quad \forall\, t\in[0,b],$$

and $\ ||v_t(x,t)-u_t(x,t)|| \le c_{3(f,\phi)}\phi(x,t), \quad \forall\, x\in[0,a].$

Hence equation (7.2) is generalised HUR stable. □

Thus, we have proved the generalised HUR stability of second order nonlinear ordinary and partial differential equations (7.1) and (7.2) respectively by employing Gronwall type Lemmas.

Summary

In this thesis, as stated in the research objective, we have studied the HU and HUR stablility of different types of differential equations. It includes the stability of non-linear ordinary differential equations, linear partial differential equations and non-linear partial differential equations. We have investigated these stability results by using various approaches viz. Laplace transform method, Banach contraction principle and some integral inequalities, etc. The following are some of the salient features that characterize this thesis:

1. The first chapter deals with general introduction of the topic and the problems taken up for the research.

2. Chapter 2 deals with survey of the available literature on HU and HUR stability of different types of equations such as functional equations, difference equations and differential equations. It reflects the present status of the work done on HU and HUR stability of different types of equations.

3. Chapter 3 is devoted to the study of HUR stability of third order ordinary differential equation. In this chapter we have studied the HUR stability of third order ordinary differential equation. This HUR stability result is established by imposing certain integrability conditions on the coefficients

of the differential equation and by using the result in [22]. An example is provided in support of the result.

4. In chapter 4, we have established the HUR stability for first, third and n^{th} order linear homogeneous partial differential equations. These results are proved by employing Laplace transform method and using the idea in [54].

5. Chapter 5 focuses on HUR stability of linear non-homogeneous partial differential equations. Here we have proved the HUR stability of the second order partial differential equation . Further , we have established the HUR stability for the third order non-homogeneous partial differential equation. These results are proved by using Banach contraction principle and the results found in [18].

6. The Chapter 6 deals with the HU stability of non-linear ordinary and partial differential equations. We have proved the HU stability of first order non-linear partial differential equation and second order non-linear partial differential equation . These results are proved by employing Banach's contraction principle.

   Further, in this chapter we have established the HU stability of second order non-linear ordinary differential equation and second order non-linear partial differential equation .These results are established by employing Grownwall type inequality and some integral inequalities.

7. The Chapter 7 focuses on the generalised HUR stability. First we have discussed generalised HUR stability of second order non-linear ordinary differential equation. Then we have established the generalised HUR sta-

bility for the second order non-linear partial differential equation . These results are proved by employing Grownwall type inequality, some integral inequalities and using the result in [41].

Problems for further study

The following topics are suggested for further study:

There is an ample opportunity to study HU and HUR stability for delay differential equations in wide range. This can be studied in two categories of ordinary and partial differential equations. The applications of HU and HUR stability are not yet discussed fully.

Publications and Presentations

We conclude by mentioning the results reported in this thesis that are published in the journals and presented at the conferences.

(I) **Papers Published in Journals [UGC Care listed]**

(PP1) V. P. Sonalkar, A. N. Mohapatra and Y. S.Valaulikar, Hyers-Ulam-Rassias stability of linear partial differential equation,*J. of Appl. Sc. and Comp.*,**6**(3)(2019), 839- 846.

(PP2) V.P. Sonalkar, A. N. Mohapatra and Y. S. Valaulikar, Hyers-Ulam stability of first and second order partial differential equations, *Jnanabha*, **50**(2)(2020),38-43.

(PP3) V.P. Sonalkar, A. N. Mohapatra and Y. S. Valaulikar, Hyers-Ulam-Rassias stability of nth order linear partial differential equation, *The Mathematics student*, Vol. 91, Nos. 3-4, July- December (2022), 11-17.

(II) **Papers Communicated to Journals**

- **(C1)** V. P. Sonalkar, A. N. Mohapatra and Y. S. Valaulikar, Generalised Hyers-Ulam- Rassias stability of second order non-linear ordinary and partial differential equation.
- **(C2)** V. P. Sonalkar, A. N. Mohapatra and Y. S. Valaulikar, Hyers-Ulam-Rassias stability of second order partial differential equation.
- **(C3)** V. P. Sonalkar, A. N. Mohapatra and Y. S. Valaulikar, Hyers-Ulam-Rassias stability of third order partial differential equation.
- **(C4)** V. P. Sonalkar, A. N. Mohapatra and Y. S. Valaulikar, Hyers-Ulam stability of second order non-linear ordinary and partial differential equations.

(III) **Papers presented in the Conferences**

(CP1) V. P. Sonalkar and A. N. Mohapatra, *Hyers-Ulam-Rassais stability for third order ordinary differential equations.* Paper presented at National Conference on "Applied Mathematics : Numerical Analysis, Algebra and Computational Mathematics" at G. H. College, Haveri, on Jan. 31, 2015.

(CP2) V. P. Sonalkar, A. N. Mohapatra and Y. S.Valaulikar, *Hyers-Ulam-Rassais stability of linear partial differential equation.* Paper presented at **One Day International Conference** on "Modern Trends in Mathematics" at H. K. R. H. College, Uthamapalayam, Tamilnadu on Sept. 10, 2018.

(CP3) V. P. Sonalkar, A. N. Mohapatra and Y. S.Valaulikar, *Hyers-Ulam-Rassais stability of second order partial differential equation.* Paper presented in **One Day National Conference** on "Recent Trends in Science and Technology" at K. L. E. College of Science and Commerce, Kalamboli, Navi Mumbai on Feb. 07, 2020.

Bibliography

[1] M. R. Abdollahpour, A. G. Najati, H. Kim, *Hyers-Ulam stability of a differential equation of third order*, Int. J. of Math. Analysis, Vol. 6, No. 59, 2943-2948, 2012.

[2] E.Ahmed, A.M.A. El- Sayed, H. A. A. El-Saka, G. A. Ashry, *On application of Hyers-Ulam stability in Biology and Economics*, ArXiv 2010.

[3] Q. H.Alqifiary, *Some properties of second order differential equation* , Mathematica Moravila, Vol. 17, (1), 89 - 94, 2013.

[4] Q. H.Alqifiary, S.M. jung *On the Hyers-Ulam stability of differential equations of second order*, Abstract and Applied Analysis, Vol. 2014, 1-8, 2014.

[5] C. Alsina, R. Ger *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl., Vol. 2, 373-380, 1998.

[6] S.Andras , A. R. Meszaros *Ulam- Hyers stability of elliptic partial differential equations in sobolev spaces* , J. of Applied math. and computation, 229, 131-138,2014.

[7] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, Vol. 2, 64-66, 1950.

[8] C. Badea, *On the Hyers-Ulam stability of mappings: the direct method*, In 'Stability of Mappings of Hyers-Ulam Type' (edited by Th. M. Rassias and J. Tabor), Hadronic Press, Palm Harbor, Florida, pp. 7–13, 1994.

[9] J. Brzdek, D. Popa and B. Xu, *Hyers-Ulam stability for linear equations of higher orders*, Acta Mathematica Hungarica, Vol. 120, no. 1-2, pp.1-8, 2008.

[10] C. Buse, O. Saierli and A Tabassum, *Spectral characterizations for Hyers-Ulam stability*, Qualitative theory of differential equations, Vol. 30, 1-14, 2014.

[11] L. C Adariu and V. Radu, *Fixed points and the stability of Jensen's functional equation,* J. Inequal. Pure Appl. Math. 4, Issue 1, Article 4, 7 pages, 2003.

[12] R. Camporesi and Antonio J. Discala, A generalization of a theorem of Mammana, Collquium Mathematicum, Vol. 122, No. 2, p. 1-8, 2011.

[13] J. Chmieli'nski and S.M. Jung, *The stability of the Wigner equation on a restricted domain*, J. Math. Anal. Appl. 254, 309-320, 2001.

[14] D. S. Cimpean, Dorian Popa, *On the stability of linear differential equation of higher order with constant coefficients*, Appl. Math. and Computation, Vol. 217, issue 8, 4141-4146, 2010.

[15] S. Czerwik, *On the stability of the quadratic mapping in normed spaces,* Abh. Math. Sem. Hamburg Vol. 62, 59–64, 1992.

[16] P. Gavruta, *Hyers-Ulam stability of Hosszu's equation*, Functional Equations and Inequalities (edited by T. M. Rassias), Kluwer Academic, Dordrecht, Vol. 518, 105–110, 2000.

[17] P. Gavruta, S. Jung, Y. Li., *Hyers-Ulam stability for second order linear differential equations with boundary conditions*, Elec. J. of Diff. Eqtns, Vol. 2011, No. 80, p. 1-5, 2011.

[18] M. E. Gordji, Y. J.Cho, M. B. Ghaemi, B. Alizadeh, *Stability of second order partial differential equations*, J. of Inequalities and Applications, Vol. 2011:81, 1-10, 2011.

[19] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl., Acad. Science, USA 27, 222-224, 1941.

[20] D. H. Hyers and S. M. Ulam, *On approximate isometries*, Bull. Amer. Math. Soc. 51, 288–292, 1945.

[21] A.Javadian, *Approximately n - order linear differential equations* , Inter. Jour. nonlinear analysis and applications, 6 , No. 1, 135-139, 2015.

[22] A.Javadian, E. Sorouri, G. H. Kim, M. Eshaghi Gordji, *Generalized Hyers-Ulam stability of a second order linear differential equations* , Applied Mathematics, Vol. 2011, Article ID 813137, doi : 10.1155/2011/813137, 2011.

[23] S.M. Jung, *On the modified Hyers-Ulam-Rassias stability of the functional equation for gamma function*, Mathematica (Cluj), 39 (62), 235–239,1997.

[24] S.M. Jung, *Stability of the quadratic equation of Pexider type*, Abh. Math. Sem. Univ, Hemburg,70, 175-190,2000.

[25] S.M. Jung, *On the stability of a multiplicative functional equation*, J. Math. Anal. Appl. 254 , 247–261, 2001.

[26] S. M. Jung, *Hyers-Ulam stability of linear differential equations of first order*, Appl. Math. Lett., Vol. 17, 1135-1140, 2004.

[27] S. M. Jung , *Hyers-Ulam stability of linear differential equations of first order III,* J. Math. Anal. Appl., Vol. 311, 139-146, 2005.

[28] S. M. Jung, *Hyers-Ulam stability of linear differential equation of first order II*, Appl. Math. Lett., Vol. 19, 854-858, 2006.

[29] S. M. Jung , *Hyers-Ulam stability of linear differential equations of first order I*, Int. Jou. of Appl. Math. and Statistics, Vol. 7, 96-100, 2007.

[30] S. M. Jung, *Hyers-Ulam stability of linear partial differential equation of first order,* Applied Mathematics Letters, Vol. 22, 70-74, 2009.

[31] S.M. Jung, *Functional equation* $f(x) = pf(x-1) - qf(x-2)$ *and its Hyers-Ulam stability*, J. Inequal. Appl. 2009(2009), Article ID 181678, 10 pages, (doi:10.1155/2009/181678).

[32] S. M. Jung, *A fixed point approach to the stability of differential equations* $y' = f(x,y)$, Bull Malaya Sci. Soc. 33, 47- 56, 2010.

[33] S.M. Jung, *Hyers - Ulam - Rassias stability of functional equations in nonlinear analysis*, Springer optimization and it's Applications, Vol. 48, 2011.

[34] S. M.Jung, Min S., *On approximate Euler differential equations*, Abstr Ana 2009, (Article ID 537963), 8 pages, 2009.

[35] S. M. Jung and P. K. Sahoo, *Hyers-Ulam stability of the quadratic equation of Pexider type*, J. Korean Math. Soc., 38:645–665, 2001.

[36] S. M. Jung and P. K. Sahoo, *Hyers-Ulam-Rassias stability of a functional equation of Davison in rings*, Nonlin. Funct. Anal. and Appl., 11:891–896, 2006.

[37] G. H. Kim, Bing XU and W. Zhang, *Notes on stability of the generalized Gamma functional equation*, International Journal of Mathematics and Mathematical Sciences, 32:1, 57-63, 2002.

[38] G. H. Kim and Y. W. Lee, *The stability of the Beta functional equation*, Studia Univ. Babes-Bolyai Math. 40, 89–96, 2000.

[39] Y. Li, Y. Shen, *Hyers-Ulam stability of linear differential equations of second order*, Applied Mathematics Letters, Vol. 23, No.3, 306-309, 2010.

[40] L. Losonczi, *On the stability of Hosszu's functional equation*, Results in Mathematics, V. 29(3), 305–310,1996.

[41] N. Lungu, C. Craciun, *Ulam- Hyers - Rassias stability of a hyperbolic partial differential equation*, ISRN Mathematical analysis, Vol. 2012, Article ID 609754, 10 pages, doi :10.5402/2012/609754.

[42] N. Lungu and D. Popa, *Hyers-Ulam stability of first order differential equation* , J. of Math. Anal. and Applications, Vol. 385, 86 - 91, 2012.

[43] T. Miura, S. Miyajima, S. Takahasi, *A characterisation of Hyers-Ulam stability of first order linear differential operators*, J. Math. Anal. Appl., Vol. 286, 136–146, 2003.

[44] T. Miura, S. Miyajima, S. Takahasi, *Hyers-Ulam stability of first order linear*

differential operator with constant coefficients, Math. Nachr., Vol. 258, 90-96, 2003.

[45] A. N. Mohapatra, *Hyers-Ulam and Hyers - Ulam - Aoki - Rassias stability for ordinary differential equations*, Application and Applied Mathematics, 10(1), 149-161, 2015.

[46] R. Murali , A. P. Selvan, *On the generalized Hyers-Ulam stability of linear ordinary differential equations of higher order*, International Journal of Pure and Applied Mathematics 117(12), 317–326, 2017.

[47] R. Murali , A. P. Selvan, *Hyers-Ulam-Rassias stability for the linear ordinary differential equation of third order*, Kragujevac J. Math. 42(4), 579–590, 2018.

[48] M. Obloza, *Hyers stability of the linear differential equations*, Rocznik Nauk.-Dydakt. Prace Mat., Vol. 13, 259-270, 1993.

[49] M. Obloza , *Connection between Hyers and Lyapunov stability of the ordinary differential equations*, Rocznik Nauk.-Dydakt. Prace Mat., Vol. 14, 141–146, 1997.

[50] B. G. Pachpatte, *Inequalities for Differential and Integral equations*, Academic Press, San Diego,Calif, USA, 1998.

[51] D. Popa, I. Rosa , *Hyers-Ulam stability of the linear differential operator with nonconstant coefficients*, Applied Mathematics and Computation, Vol. 212, 1562–1568,2012.

[52] M. N. Qarawani,*Hyers-Ulam stability of linear and non linear differential equations of second order*, Inter. J. of Applied Mathematics Research, Vol. 1, No. 4, 422 - 432, 2012.

[53] M. N. Qarawani, *Hyers-Ulam stability of a generalized second order non linear differential equation*, Applied Mathematics, Vol. 3,1857 - 1861, 2012.

[54] M. N. Qarawani, *Hyers-Ulam stability for Heat equation*, Applied Mathematics, Vol.4, 1001 - 1008, 2013.

[55] Th. M. Rassias, *On the stability of linear mapping in Banach spaces* , proc. Amer. math. soc., 72, 297-300, 1978.

[56] Th. M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math., Vol. 62, 23-130, 2000.

[57] K. Ravi, R. Murali , A. P. Selvan, *Ulam stability of a general n^{th} order linear differential equation with constant coefficients*, Asian Journal of Mathematics and Computer Research, 11(1), 61–68, 2016.

[58] H. Rezaei, S. M. Jung, T. M. Rassias, Laplace ransform and Hyers-Ulam stability of linear differential equations , Journal of Mathematical Analysis and Applications, Vol. 403, 244 - 251, 2013.

[59] P. K. Sahoo and PI. Kannapan, *Introduction to Functional equations*, CRC Press , chapman and Hall book , Florida , 2011, ISBN 978 - 1 - 4398 - 4111 - 2.

[60] S. E. Takahasi, T. Miura , S. Miyajima, *Hyers-Ulam stability of Banach*

space valued linear differential equations $y' = \lambda y$, Bull. of Korean Math. Soc., Vol. 39, 309 - 315, 2002.

[61] S. Takahasi, H.Takagi, T. Miura, S. Miyajima, *The Hyers-Ulam stability constants of first order linear differential operators*, J. Math. Analy. and Applications , Vol. 296, No.- 2, 403-409, 2004.

[62] S. M. Ulam, *A collection of Mathematical problems*, Interscience publication, New York, 1960.