A STUDY OF BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

A THESIS SUBMITTED IN PARTIAL FULFILLMENT FOR THE

DEGREE OF

DOCTOR OF PHILOSOPHY

IN **MATHEMATICS**

IN THE SCHOOL OF PHYSICAL AND APPLIED SCIENCES

GOA UNIVERSITY



By

HERAMB BALKRISHNA AIYA

SCHOOL OF PHYSICAL AND APPLIED SCIENCES

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FEBRUARY 2025

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DECLARATION

I, Mr. Heramb Balkrishna Aiya, hereby declare that this thesis represents work which has been carried out by me and that it has not been submitted, either in part or full, to any other University or Institution for the award of any research degree.

Place: Taleigao Plateau.

Date : 04/02/2025 **Heramb B. Aiya**

CERTIFICATE

We hereby certify that the above declaration of the candidate, Mr. Heramb B. Aiya is true and the work was carried out under our supervision.

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Chapter 1

INTRODUCTION

The theory of differential equations with deviating arguments has history of atleast 200 years. It mainly originated because of problems in number theory and geometry. Differential equations with deviating argument (or delay differential equations) were introduced to create more realistic models since many processes depend on past history or on future data.

The study of boundary value problems is important because of their applications in various fields such as mechanical vibration of elastic rods, heat flow, wave distribution, electromagnetic fields, fluid flow etc. Differential equations with deviating argument feature when dealing with modeling physical and biological situations. These equations are also used to study population dynamics, epidemic spread, control theory, distributed networks and neural networks. In economics, delay differential equations are used to study market fluctuations, consumer behavior, taxation effect etc. This makes study of boundary value problems for differential equations with deviating argument an important topic to investigate.

1.1 Research Objectives

The most natural question which arises in solving a mathematical problem is whether its solution exists or not. Many physical situations can be modelled into boundary value problems of differential equations with deviating arguments. These problems are mainly non-linear. So the primary thing is to look at different techniques of solving these problems analytically or numerically. In the absence of method of obtaining a solution, it is often seen that, theoretically one establishes the existence of a solution to the problem. This research work mainly deals with proving results on existence of solution for various boundary value problems of first and second order ordinary differential equations with deviating arguments by using different techniques.

Our main objectives of this research work are as follows:

1. To establish results on existence of solution for boundary value problems of first order differential equations with deviating arguments using Leray-Schauder alternative, Schauder's fixed point theorem, Banach fixed point theorem and Fixed point theorem on partially ordered metric spaces.

2. To establish results on existence of solution between lower and upper solutions for boundary value problems of first order differential equations with deviating arguments.

3. To develope Monotone Iterative Method, Quasilinearization Method, Extended Quasilinearization Method, Generalized Quasilinearization Method and Extended-Generalized Quasilinearization Method for boundary value problems of first order differential equations with deviating arguments.

4. To establish results on existence of solution for boundary value problems of second order differential equations with deviating arguments using Schauder's fixed point theorem and Banach fixed point theorem.

5. To establish results on existence of solution between lower and upper solutions for boundary value problems of second order differential equations with deviating arguments.

6. To develope Monotone Iterative Method, Quasilinearization Method, Extended Quasilinearization Method, Generalized Quasilinearization Method and Extended-Generalized Quasilinearization Method for boundary value problems of second order differential equations with deviating arguments.

1.2 Notations

The following notations will be used throughout this work.

1. $T > 0$, $\lambda \geq 0$ and $0 < r \leq T$.

2. $I = [0,\ T]$.

3. $\theta : I \to I$ is a continuous function.

4. B is the set of all functions defined and continuous on $[-r,\ T]$ and which are continuously differentiable on I.

5. $X = \{y \in B : y(t) = y(0),\ \forall\ t \in [-r,\ 0]\}$.

6. $Y = \{y \in B : y(t) = 0,\ \forall\ t \in [-r,\ 0]\}$.

7. $Z = C[I,\ \mathbb{R}]$.

8. $W = C\,'[I,\ \mathbb{R}]$.

9. $E = \{y \in B : y(t) = a,\ \forall\, t \in [-r,\ 0]\}$.

10. $U = \{y \in W : y(0) = a\}$.

11. $V = \{y \in W : y(0) = 0\}$.

12. $||z|| = \sup\limits_{t \in I} |z(t)|,\ \forall\, z \in Z$.

13. $||w||_0 = \sup\limits_{t \in I} |w(t)| + \sup\limits_{t \in I} |w'(t)|,\ \forall\, w \in W$.

14. $||x||_\infty = \sup\limits_{t \in [-r,\, T]} |x(t)| + \sup\limits_{t \in I} |x'(t)|,\ \forall\, x \in B$.

15. $d_\infty(w,\ z) = ||w - z||_\infty,\ \forall\, w,\ z \in E$.

16. $d_0(u,\ v) = ||u - v||_0,\ \forall\, u,\ v \in U$.

17. $||(y,\ \xi)||_\star = ||y||_\infty + |\xi|,\ \forall\, (y,\ \xi) \in Y \times \mathbb{R}$.

18. $||(y,\ \xi_1,\ \xi_2)||_{\star\star} = ||y||_\infty + |\xi_1| + |\xi_2|,\ \forall\, (y,\ \xi_1,\ \xi_2) \in Y \times \mathbb{R}^2$.

19. $||(v,\ \xi)||_\$ = ||v||_0 + |\xi|,\ \forall\, (v,\ \xi) \in V \times \mathbb{R}$.

20. $||(v,\ \xi_1,\ \xi_2)||_{\$\$} = ||v||_0 + |\xi_1| + |\xi_2|,\ \forall\, (v,\ \xi_1,\ \xi_2) \in V \times \mathbb{R}^2$.

21. For a linear operator $L : X \to Y \times \mathbb{R}$, $||L||_1 = \sup\limits_{||x||_\infty = 1} ||Lx||_\star$.

22. $D_i f(x_1,\ x_2, ... x_n) = \frac{\partial f(x_1,\, x_2, ... x_n)}{\partial x_i},\ i = 1,\ 2, ... n$.

23. $D_{i,\,j} f(x_1,\ x_2, ... x_n) = \frac{\partial^2 f(x_1,\, x_2, ... x_n)}{\partial x_i\, \partial x_j},\ i,\ j = 1,\ 2, ... n$.

Note that $(B,\ ||.||_\infty)$, $(X,\ ||.||_\infty)$, $(Y,\ ||.||_\infty)$, $(Z,\ ||.||)$, $(W,\ ||.||_0)$, $(V,\ ||.||_0)$, $(Y \times \mathbb{R},\ ||.||_\star)$ and $(Y \times \mathbb{R}^2,\ ||.||_{\star\star})$, $(V \times \mathbb{R},\ ||.||_\$)$ and $(V \times \mathbb{R}^2,\ ||.||_{\$\$})$ are Banach Spaces. Also $(U,\ d_0)$ and $(E,\ d_\infty)$ are complete metric spaces.

1.3 Abbreviations

Following abbreviations are being used consistently in this work.

1. ODE : Ordinary Differential Equation.
2. PDE : Partial Differential Equation.
3. IVP : Initial Value Problem.
4. BVP : Boundary Value Problem.
5. DEDA: Differential Equation with Deviating Argument.
6. DDE: Delay Differential Equation.
7. DECD: Differential Equation with Constant Delay.
8. DEVDA: Differential Equation with Variable Deviating Argument.
9. PBVP: Periodic Boundary Value Problem.
10. APBVP: Anti-Periodic Boundary Value Problem.
11. LBVP: Boundary Value Problem with Linear Condition.
12. NLBVP: Boundary Value Problem with Non-Linear Condition.
13. MIM: Monotone Iterative Method.
14. QM: Quasilinearization Method.
15. EQM: Extended Quasilinearization Method.
16. GQM: Generalized Quasilinearization Method.
17. EGQM: Exended-Generalized Quasilinearization Method.

1.4 Organization of the Thesis

We divide the thesis into 10 chapters.

Chapter 1 gives introduction to history and applications to boundary value problems of differential equations with deviating arguments. We also give main objectives of the thesis together with notations and abbreviations used throughout the thesis. Further, we highlight chapter wise description of the research work done.

Chapter 2 deals with literature survey for initial and boundary value problems of ordinary differential equations and differential equations with deviating arguments.

In Chapter 3, we investigate the following BVPs of first order DECD.

Let $a,\ b \in \mathbb{R}$ such that $a \neq 0$ and $b \neq 0$. Let $T > 0$, $\lambda \geq 0$, $I = [0,\ T]$, $0 < r \leq T$, $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Then the PBVP of first order DECD under investigation is given by

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = y(T). \end{aligned}\right\} \tag{3.1}$$

The APBVP of first order DECD under investigation is

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = -y(T). \end{aligned}\right\} \tag{3.2}$$

The LBVP of first order DECD under investigation is

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &ay(0) + by(T) = 0. \end{aligned}\right\} \tag{3.3}$$

The NLBVP of first order DECD under investigation is

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &g(y(0),\ y(T)) = 0. \end{aligned}\right\} \tag{3.4}$$

We also investigate the NLBVP of first order DECD given by

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = g(y(0),\ y(T)). \end{aligned}\right\} \tag{3.5}$$

We first prove results on existence of solution for problems (3.1), (3.2) and (3.3) by using Leray-Schauder alternative. Then using Schauder's fixed point theorem, we prove existence of solution for problem (3.5) by assuming f and g to be bounded continuous functions. Further, we prove a result on existence of solution between lower and upper solutions for PBVP (3.1) and results on existence of solution between weakly coupled lower and upper solutions for APBVP (3.2), LBVP (3.3) and NLBVP (3.4). We move on to prove results on existence of solution between coupled lower and upper solutions for problems (3.1) to (3.4).

In Chapter 4, we investigate the following BVPs of general first order DECD and DEVDA.

Let $a,\ b \in \mathbb{R}$ such that $a \neq 0$ and $b \neq 0$. Let $T > 0$, $\lambda \geq 0$, $I = [0,\ T]$, $0 < r \leq T$, $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, $\theta : I \to I$ be continuous and $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Then the PBVP of general first order DECD under investigation is given by

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = y(T). \end{aligned}\right\} \tag{4.1}$$

The APBVP of general first order DECD under investigation is

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = -y(T). \end{aligned}\right\} \tag{4.2}$$

The LBVP of general first order DECD under investigation is

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &ay(0) + by(T) = 0. \end{aligned}\right\} \tag{4.3}$$

The NLBVP of general first order DECD under investigation is

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &g(y(0),\ y(T)) = 0. \end{aligned}\right\} \tag{4.4}$$

We also investigate the NLBVP of general first order DECD given by

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = g(y(0),\ y(T)). \end{aligned}\right\} \tag{4.5}$$

The PBVP of first order DEVDA under investigation is given by

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(\theta(t))),\ t \in I, \\ &y(0) = y(T). \end{aligned}\right\} \tag{4.6}$$

The APBVP of first order DEVDA under investigation is

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(\theta(t))),\ t \in I, \\ &y(0) = -y(T). \end{aligned}\right\} \tag{4.7}$$

The LBVP of first order DEVDA under investigation is

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(\theta(t))),\ t \in I, \\ &ay(0) + by(T) = 0. \end{aligned}\right\} \tag{4.8}$$

The NLBVP of first order DEVDA under investigation is

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(\theta(t))),\ t \in I, \\ &g(y(0),\ y(T)) = 0. \end{aligned}\right\} \tag{4.9}$$

We first apply Leray-Schauder alternative to prove results on existence of solution for problems (4.1), (4.2), (4.3), (4.6), (4.7) and (4.8). Then, we use Schauder's fixed point theorem to prove existence of solution theorem for problem (4.5) by assuming f and g to be bounded continuous functions. Also, we apply Schauder's fixed point theorem to prove existence of solution for the

PBVP of DEVDA (4.6) by assuming f to be bounded continuous function. Further, we prove existence of solution between lower and upper solutions for the PBVPs (4.1) and (4.6). Results on existence of solution between weakly coupled lower and upper solutions for BVPs (4.2), (4.3), (4.4), (4.7), (4.8) and (4.9) are established. We move on to prove existence of solution between coupled lower and upper solutions for problems (4.1), (4.2), (4.3), (4.4), (4.6), (4.7), (4.8) and (4.9). Finally, we apply Banach fixed point theorem and Fixed point theorem on partially ordered metric space to prove existence of solution for the PBVP of DEVDA (4.6).

In Chapter 5, we develope monotone iterative method and quasilinearization method to establish the existence of solutions for the BVPs (3.1), (3.3) and (3.4).

In Chapter 6, we develope monotone iterative method and quasilinearization method to establish the existence of solutions for the BVPs of DECD (4.1), (4.3), (4.4). We also develope these methods to establish the existence of solutions for the BVPs of DEVDA (4.6), (4.8) and (4.9).

Chapter 7 deals with the investigation of the following boundary value problems of general first order DECD and first order DEVDA for developing extended quasilinearization method and extended-generalized quasilinearization method.

Let $a,\ b \in \mathbb{R}$ such that $a \neq 0$ and $b \neq 0$. Let $T > 0$, $\lambda \geq 0$, $I = [0,\ T]$, $0 < r \leq T$, $f,\ h : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, $\theta : I \to I$ be continuous and $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Then the PBVP of general first order DECD under investigation is given by

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(t-r)) + h(t,\ y(t),\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = y(T). \end{aligned}\right\} \tag{7.1}$$

The LBVP of general first order DECD under investigation is

$$\left.\begin{aligned} &y'(t)+\lambda y(t)=f(t,\ y(t),\ y(t-r))+h(t,\ y(t),\ y(t-r)),\ t\in I,\\ &y(t)=y(0),\ t\in[-r,\ 0],\\ &ay(0)+by(T)=0. \end{aligned}\right\} \tag{7.2}$$

The NLBVP of general first order DECD under investigation is

$$\left.\begin{aligned} &y'(t)+\lambda y(t)=f(t,\ y(t),\ y(t-r))+h(t,\ y(t),\ y(t-r)),\ t\in I,\\ &y(t)=y(0),\ t\in[-r,\ 0],\\ &g(y(0),\ y(T))=0. \end{aligned}\right\} \tag{7.3}$$

The PBVP of first order DEVDA under investigation is given by

$$\left.\begin{aligned} &y'(t)+\lambda y(t)=f(t,\ y(t),\ y(\theta(t)))+h(t,\ y(t),\ y(\theta(t))),\ t\in I,\\ &y(0)=y(T). \end{aligned}\right\} \tag{7.4}$$

The LBVP of first order DEVDA under investigation is

$$\left.\begin{aligned} &y'(t)+\lambda y(t)=f(t,\ y(t),\ y(\theta(t)))+h(t,\ y(t),\ y(\theta(t))),\ t\in I,\\ &ay(0)+by(T)=0. \end{aligned}\right\} \tag{7.5}$$

The NLBVP of first order DEVDA under investigation is

$$\left.\begin{aligned} &y'(t)+\lambda y(t)=f(t,\ y(t),\ y(\theta(t)))+h(t,\ y(t),\ y(\theta(t))),\ t\in I,\\ &g(y(0),\ y(T))=0. \end{aligned}\right\} \tag{7.6}$$

We also develope generalized quasilinearization method to establish the existence of solutions for the BVPs (4.1), (4.3), (4.4), (4.6), (4.8) and (4.9).

Chapter 8 deals with the study of existence of solution for various BVPs of second order DECD and second order DEVDA.

Let $a,\ b,\ \gamma_1,\ \gamma_2 \in \mathbb{R}$, $T > 0$, $\lambda \geq 0$, $I = [0,\ T]$, $0 < r \leq T$ and $\theta : I \to I$ be continuous. Then the BVPs for second order DECD under investigation are

$$\left.\begin{aligned} y''(t) &= f(t,\ y(t),\ y(t-r),\ y'(t)),\ t \in I, \\ y(t) &= a,\ t \in [-r,\ 0], \\ y(T) + y'(T) &= b. \end{aligned}\right\} \tag{8.1}$$

$$\left.\begin{aligned} y''(t) + \gamma_1 y'(t) + \gamma_2 y(t) \ &= f(t,\ y(t),\ y(t-r),\ y'(t)),\ t \in I, \\ y(t) &= a,\ t \in [-r,\ 0], \\ y(T) + y'(T) &= b. \end{aligned}\right\} \tag{8.2}$$

$$\left.\begin{aligned} y''(t) &= f(t,\ y(t),\ y(t-r),\ y'(t)),\ t \in I, \\ y(t) &= 0,\ t \in [-r,\ 0], \\ y(T) + y'(T) &= b. \end{aligned}\right\} \tag{8.3}$$

$$\left.\begin{aligned} -y''(t) + \lambda y(t) &= f(t,\ y(t),\ y(t-r)),\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T), \\ y'(0) &= y'(T). \end{aligned}\right\} \tag{8.4}$$

$$\left.\begin{aligned} -y''(t) &= f(t,\ y(t),\ y(t-r)),\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T), \\ y'(0) &= y'(T). \end{aligned}\right\} \tag{8.5}$$

$$\left.\begin{aligned} -y''(t) + \lambda y(t) &= f(t,\ y(t),\ y(t-r)),\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T), \\ y'(0) &= 0. \end{aligned}\right\} \tag{8.6}$$

$$\left.\begin{aligned} -y''(t) &= f(t,\ y(t),\ y(t-r)),\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T), \\ y'(0) &= 0. \end{aligned}\right\} \tag{8.7}$$

The BVPs for second order DEVDA under investigation are

$$\left.\begin{aligned} y''(t) &= f(t,\ y(t),\ y(\theta(t)),\ y'(t)),\ t \in I, \\ y(0) &= a, \\ y(T) + y'(T) &= b. \end{aligned}\right\} \tag{8.8}$$

$$\left.\begin{aligned} y''(t) + \gamma_1 y'(t) + \gamma_2 y(t) &= f(t,\ y(t),\ y(\theta(t)),\ y'(t)),\ t \in I, \\ y(0) &= a, \\ y(T) + y'(T) &= b. \end{aligned}\right\} \tag{8.9}$$

$$\left.\begin{aligned} y''(t) &= f(t,\ y(t),\ y(\theta(t)),\ y'(t)),\ t \in I, \\ y(0) &= 0, \\ y(T) + y'(T) &= b. \end{aligned}\right\} \tag{8.10}$$

$$\left.\begin{aligned} -y''(t) + \lambda y(t) &= f(t,\ y(t),\ y(\theta(t))),\ t \in I, \\ y(0) &= y(T), \\ y'(0) &= y'(T). \end{aligned}\right\} \tag{8.11}$$

$$\left.\begin{aligned} -y''(t) &= f(t,\ y(t),\ y(\theta(t))),\ t \in I, \\ y(0) &= y(T), \\ y'(0) &= y'(T). \end{aligned}\right\} \tag{8.12}$$

$$\left.\begin{aligned} -y''(t) + \lambda y(t) &= f(t,\ y(t),\ y(\theta(t))),\ t \in I, \\ y(0) &= y(T), \\ y'(0) &= 0. \end{aligned}\right\} \tag{8.13}$$

$$\left.\begin{aligned} -y''(t) &= f(t,\ y(t),\ y(\theta(t))),\ t \in I, \\ y(0) &= y(T), \\ y'(0) &= 0. \end{aligned}\right\} \tag{8.14}$$

We first apply Banach fixed point theorem to establish exitence of solution for the BVPs (8.1), (8.2), (8.8) and (8.9). We then prove existence of solution for the BVPs (8.3) and (8.10) using Schauder's fixed point theorem. Further,

we establish results on existence of solution between lower and upper solutions for BVPs (8.4), (8.5), (8.6), (8.7), (8.11), (8.12), (8.13) and (8.14).

In Chapter 9, we develope Monotone Iterative Method and Quasilinearization Method to prove the existence of solution for the BVPs (8.7) and (8.14).

Chapter 10 deals with the investigation of the following boundary value problems of second order DECD and second order DEVDA for developing extended quasilinearization method and extended-generalized quasilinearization method.

Let $T > 0,\ \lambda \geq 0,\ I = [0,\ T],\ 0 < r \leq T,\ f,\ h : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, $\theta : I \to I$ be continuous. Then the BVP of second order DECD under investigation is given by

$$\left.\begin{aligned} -y''(t) &= f(t,\ y(t),\ y(t-r)) + h(t,\ y(t),\ y(t-r)),\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T), \\ y'(0) &= 0. \end{aligned}\right\} \tag{10.1}$$

The BVP of second order DEVDA under investigation is given by

$$\left.\begin{aligned} -y''(t) &= f(t,\ y(t),\ y(\theta(t))) + h(t,\ y(t),\ y(\theta(t))),\ t \in I, \\ y(0) &= y(T), \\ y'(0) &= 0. \end{aligned}\right\} \tag{10.2}$$

We also develope generalized quasilinearization method for BVPs (8.7) and (8.14).

It is to be noted that in existing literature the function f is taken to be continuous and Lipchitz or continuous and decreasing in second coordinate for establishing existence of a solution for initial value problems as well as boundary value problems. In the present work, we take f to be continuous and non-decreasing in the second coordi-

nate or second and third co-ordinate, as the case may be to prove our most of the results. However, while using method of coupled lower and upper solutions, we assume f to be non increasing in second and third arguments.

At the end we summarize the research work done throughout the study and also mention the topics for future research work. We highlight the papers presented in national and international conferences, papers published in journals and papers communicated to various journals for publication. We wind up our thesis with complete bibliography.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

The chapter presents, in brief, the literature review on various boundary value problems of first and second order ordinary differential equation and differential equations with deviating arguments. Also theorems such as Leray-Schauder alternative, Schauder's fixed point theorem, Ascoli lemma, Fixed point theorem on partially ordered metric spaces and Banach fixed point theorem are stated which will be used time and again in proving many theorems on existence of solutions in our work.

2.2 Literature Review

As mentioned in the begining, the study of DDEs has a history of atleast 200 years[21]. A systematic treatment to the theory of DEDA started in the year 1949[54]. They first appeared as solution of PDEs[54]. Later DEDAs were studied in problems of geometry and were applied in automatic control theory[54].

A theory on differential equations and BVPs is covered in [12] and [13]. A thorough approach to differential equations and DEDA, is provided in [14]. Theorems on existence of solutions for BVPs and IVPs are proved using mainly fixed point theorems and one may refer [28] and [30] for various fixed point theorems used in the literature. One may refer [9], [20], [24], [26], [32], [40], [41], [47] and [52] for literature on existence of solutions for BVPs of first order DEDA. Monotone Iterative Method for IVPs and BVPs of first order DEDA is well estabished in [11], [23], [29], [31], [35], [44], [50] and [53]. The literature on quasilinearization method for IVPs and BVPs of first order DEDA can be found in [1], [2], [6], [7], [8], [15], [33], [34], [45], [46], [48], [49] and [51]. Existence of solution literature on BVPs of second order DEDA can be cited in [4], [10], [16], [19], [22], [25], [27], [32], [42] and [43]. For monotone iterative method and quasilinearization method for BVPs of second order DEDA, one may cite [5], [23], [31], [34], [36], [37], [38] and [39].

In 2003, Daniel Franco and others[18] published a paper in which a theorem on existence of solution between coupled lower and upper solution for NLBVP of first order ODE is proved.

In 2003, Daniel Franco and others[17] published a paper in which a theorem on existence of solution using Leray-Schauder alternative is proved.

In 2014, Bachar and Khamsi[3] applied fixed point theorem on partially ordered metric spaces to prove theorem on existence of solution for PBVP of first order DDE.

Monotone iterative techniques for discontinuous nonlinear differential equations are studied in detail by Ladde and others [31]. Lakshmikantham and Vatsala [34] have provided minute details on generalized quasilinearization for nonlinear problems.

In 2002, D. Franco and D. O'Regan [16] proved a theorem on existence of solution for second order BVP with nonlinear conditions.

2.3 Preliminaries

We now state some fixed point theorems and few other theorems required to prove theorems on existence of solutions for BVPs of first and second order DEDA.

Theorem 2.3.1 (Leray-Schauder Alternative [17]). *Let C be a complete convex subset of a locally convex Hausdorff linear topological space H and O an open subset of C with $p \in O$. In addition, let $A : \bar{O} \to C$ be a continuous and compact map. Then either A has a fixed point in $\bar{O}$ or there is $u \in \partial O$ and $\mu \in (0, 1)$ with $u = \mu Au + (1 - \mu)p$.*

Theorem 2.3.2 (Schauder's Fixed Point Theorem [28]). *Let H be a normed linear space, $S : H \to H$ be a continuous and compact map such that $S(H)$ is bounded. Then S has a fixed point.*

Theorem 2.3.3 (Contraction Mapping Theorem [30]). *Let $(C,\ d)$ be a complete metric space and $S : C \to C$ be a contraction mapping. Then S has a fixed point in C.*

Theorem 2.3.4 (Fixed Point Theorem on Partially Ordered Metric Spaces [3]). *Let $(C,\ d)$ be a complete metric space with partial order $\leq$ and $J : C \to C$ be an order preserving operator. Assume that $\exists\ k \in [0,\ 1)$ such that $d(Jx,\ Jy) \leq kd(x,\ y),\ \forall\ x,\ y \in C$ satisfying $x \leq y$. If $\exists\ x_0 \in C$ such that $x_0 \leq Jx_0$ or $Jx_0 \leq x_0$, then J has a unique fixed point in C.*

Lemma 2.3.5 (Arzela-Ascoli lemma [13]). *Let $\{f_n\}$ be an equicontinuous and uniformly bounded sequence of functions on a closed and bounded interval J. Then $\{f_n\}$ contains a subsequence $\{f_{n_k}\}$ which is uniformly convergent on J.*

Lemma 2.3.6. *[18] Let $L : W \to V \times \mathbb{R}$ be an operator defined by $L[y(t)] = (y(t) - y(0) + \lambda \int_0^t y(s)ds,\ y(0))$. Then*

1. *$L : W \to V \times \mathbb{R}$ is a continuous linear operator.*

2. *$L^{-1} : V \times \mathbb{R} \to W$ exists and it is defined as $L^{-1}(u(t), \omega) = \omega e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} u'(s)ds$.*

3. *$L^{-1} : V \times \mathbb{R} \to W$ is a continuous linear operator.*

Lemma 2.3.7. *[16] Let $L : W \to V \times \mathbb{R}^2$ be an operator defined by $L[y(t)] = (-y'(t) + y'(0) + \lambda \int_0^t y(s)ds,\ y(0),\ y(T))$. Then*

1. *$L : W \to V \times \mathbb{R}^2$ is a continuous and bijective linear operator.*

2. *$L^{-1} : V \times \mathbb{R}^2 \to W$ is a continuous linear operator.*

Chapter 3

FIRST ORDER DIFFERENTIAL EQUATIONS WITH CONSTANT DELAY

Some content of this chapter is published [see P1 and P2].

Some content of this chapter is presented at the Conference [see C1, C2 and C6].

3.1 Introduction

In this chapter, we investigate BVPs of first order DECD for existence of solution using Leray - Schauder alternative and Schauder's fixed point theorem. We employ the method of lower and upper solutions to these BVPs to prove some results on existence of solution between lower and upper solutions.

Let $a,\ b \in \mathbb{R}$ such that $a \neq 0$ and $b \neq 0$. Let $T > 0$, $\lambda \geq 0$, $I = [0,\ T]$, $0 < r \leq T$, $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Then the PBVP for first order DECD under investigation is given by

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ y(t-r)),\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T). \end{aligned}\right\} \tag{3.1}$$

The APBVP for first order DECD under investigation is

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ y(t-r)),\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= -y(T). \end{aligned}\right\} \tag{3.2}$$

The LBVP for first order DECD under investigation is

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ y(t-r)),\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ ay(0) + by(T) &= 0. \end{aligned}\right\} \tag{3.3}$$

The NLBVP for first order DECD under investigation is

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ y(t-r)),\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ g(y(0),\ y(T)) &= 0. \end{aligned}\right\} \tag{3.4}$$

We also investigate BVP for first order DECD given by

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = g(y(0),\ y(T)). \end{aligned}\right\} \tag{3.5}$$

We shall need the following Lemmas to prove theorems on existence of solution for the above BVPs.

Lemma 3.1.1. *Let $L : X \to Y \times \mathbb{R}$ be an operator defined by*

$$L[y(t)] = (u(t),\ y(0)),\ \textit{where}\ u(t) = \begin{cases} 0, & -r \le t \le 0, \\ y(t) - y(0) + \lambda \int\limits_0^t y(s)ds, & 0 \le t \le T. \end{cases}$$

Then

1. *$L : X \to Y \times \mathbb{R}$ is a continuous linear operator.*

2. *$L^{-1} : Y \times \mathbb{R} \to X$ exists and it is defined by*

$$L^{-1}(u(t),\ \omega) = \begin{cases} \omega, & -r \le t \le 0, \\ \omega e^{-\lambda t} + \int\limits_0^t e^{-\lambda(t-s)} u'(s)ds, & 0 \le t \le T. \end{cases}$$

3. *$L^{-1} : Y \times \mathbb{R} \to X$ is continuous linear operator.*

Proof. **(1)** Claim 1.1:- $L : X \to Y \times \mathbb{R}$ is a linear operator.

Let $y_1,\ y_2 \in X$ and $c_1,\ c_2 \in \mathbb{R}$ and $t \in [-r,\ T]$.

CASE-1.1:- Let $t \in [-r,\ 0]$.

$$\begin{aligned} \text{Then } (c_1y_1 + c_2y_2)(t) &= c_1y_1(t) + c_2y_2(t) \\ &= c_1y_1(0) + c_2y_2(0) \\ &= (c_1y_1 + c_2y_2)(0). \end{aligned}$$

Thus $L[(c_1y_1 + c_2y_2)(t)] = [0,\ (c_1y_1 + c_2y_2)(0)]$

$$= c_1[0,\ y_1(0)] + c_2[0,\ y_2(0)]$$

$$= c_1L[y_1(t)] + c_2L[y_2(t)].$$

CASE-1.2:- Let $t \in I$.

Let $L[y_1(t)] = (u_1(t),\ y_1(0)) = (L_1[y_1(t)],\ L_2[y_1(t)])$ and

$L[y_2(t)] = (u_2(t),\ y_2(0)) = (L_1[y_2(t)],\ L_2[y_2(t)])$,

where $L_1[y_1(t)] = u_1(t) = y_1(t) - y_1(0) + \lambda \int_0^t y_1(s)ds$,

$L_1[y_2(t)] = u_2(t) = y_2(t) - y_2(0) + \lambda \int_0^t y_2(s)ds$,

$L_2[y_1(t)] = y_1(0)$ and $L_2[y_2(t)] = y_2(0)$.

Therefore $c_1L_1[y_1(t)] = c_1u_1(t) = c_1y_1(t) - c_1y_1(0) + c_1\lambda \int_0^t y_1(s)ds$ and

$c_2L_1[y_2(t)] = c_2u_2(t) = c_2y_2(t) - c_2y_2(0) + c_2\lambda \int_0^t y_2(s)ds$.

So we have

$$c_1L_1[y_1(t)] + c_2L_1[y_2(t)] = c_1y_1(t) - c_1y_1(0) + c_1\lambda \int_0^t y_1(s)ds$$

$$+ c_2y_2(t) - c_2y_2(0) + c_2\lambda \int_0^t y_2(s)ds$$

$$= (c_1y_1 + c_2y_2)(t) - (c_1y_1 + c_2y_2)(0)$$

$$+ \lambda \int_0^t (c_1y_1 + c_2y_2)(s)ds$$

$$= L_1[(c_1y_1 + c_2y_2)(t)].$$

Also $c_1L_2[y_1(t)] + c_2L_2[y_2(t)] = c_1y_1(0) + c_2y_2(0)$

$$= (c_1y_1 + c_2y_2)(0)$$

$$= L_2[(c_1y_1 + c_2y_2)(t)].$$

Thus

$$\begin{aligned} c_1L[y_1(t)] + c_2L[y_2(t)] &= (c_1L_1[y_1(t)],\ c_1L_2[y_1(t)]) + (c_2L_1[y_2(t)],\ c_2L_2[y_2(t)]) \\ &= (c_1L_1[y_1(t)] + c_2L_1[y_2(t)],\ c_1L_2[y_1(t)] + c_2L_2[y_2(t)]) \\ &= (L_1[(c_1y_1 + c_2y_2)(t)],\ L_2[(c_1y_1 + c_2y_2)(t)]) \\ &= L[(c_1y_1 + c_2y_2)(t)]. \end{aligned}$$

Hence from CASE-1.1 and CASE-1.2, $L : X \to Y \times \mathbb{R}$ is a linear operator.

Claim 1.2:- $L : X \to Y \times \mathbb{R}$ is continuous.

Let $y \in X$ be arbitrary. Then $y(t) = y(0),\ \forall\ t \in [-r,\ 0]$.

Also $y \in C[-r,\ T]$ and $y \in C'(I)$.

Let $L[y(t)] = (u(t),\ y(0))$, where $u(t) = \begin{cases} 0, & -r \le t \le 0, \\ y(t) - y(0) + \lambda \int\limits_0^t y(s)ds, & 0 \le t \le T. \end{cases}$

Therefore $|y(0)| \le \sup\limits_{t \in [-r,\ T]} |y(t)| + \sup\limits_{t \in I} |y'(t)| = \ ||y||_\infty$.
Since $u(t) = 0,\ \forall\ t \in [-r,\ 0]$, $|u(t)| \le ||y||_\infty,\ \forall\ t \in [-r,\ 0]$.

Let $0 \le t \le T$. Then

$$\begin{aligned} |u(t)| &\le |y(t)| + |y(0)| + |\lambda| \int\limits_0^t |y(s)|ds \\ &\le ||y||_\infty + ||y||_\infty + |\lambda|\ T\ ||y||_\infty \\ &= (2 + |\lambda|\ T)\ ||y||_\infty. \end{aligned}$$

Thus $\sup\limits_{t \in [-r,\ T]} |u(t)| \le (2 + |\lambda|\ T)\ ||y||_\infty$.

Also, for $0 \le t \le T$, we have $u'(t) = y'(t) + \lambda y(t)$.

This implies

$$\begin{aligned}|u'(t)| &\leq |y'(t)| + |\lambda|\ |y(t)| \\ &\leq \sup_{t\in I}|y'(t)| + |\lambda| \sup_{t\in[-r,\ T]}|y(t)| \\ &\leq ||y||_\infty + |\lambda|\ ||y||_\infty \\ &= (1+|\lambda|)\ ||y||_\infty.\end{aligned}$$

So we have, $\sup_{t\in I}|u'(t)| \leq (1+|\lambda|)\ ||y||_\infty$.
Therefore $||u||_\infty \leq (3+|\lambda|+|\lambda|T)\ ||y||_\infty$. Thus

$$\begin{aligned}||Ly||_\star &= ||(u,\ y(0))||_\star \\ &= ||u||_\infty + |y(0)| \\ &\leq (3+|\lambda|+|\lambda|T)\ ||y||_\infty + ||y||_\infty \\ &= (4+|\lambda|+|\lambda|T)\ ||y||_\infty.\end{aligned}$$

Hence $L : X \to Y \times \mathbb{R}$ is a bounded linear operator and hence continuous.

(2) Claim 2.1:- $L : X \to Y \times \mathbb{R}$ is injective.

Let $L[y(t)] = (0,\ 0)$. Then $(y(t) - y(0) + \lambda \int_0^t y(s)ds,\ y(0)) = (0,\ 0)$.

This implies $y(t) - y(0) + \lambda \int_0^t y(s)ds = 0$ and $y(0) = 0$. Therefore

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = 0,\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = 0. \end{aligned}\right\}$$

Thus $y(t) = y(0)e^{-\lambda t} = 0,\ \forall\ t \in [-r,\ T]$.

Therefore $L[y(t)] = (0,\ 0)$ implies that $y(t) = 0$ on $[-r,\ T]$.

Hence $L : X \to Y \times \mathbb{R}$ is injective.

Claim 2.2:- $L : X \to Y \times \mathbb{R}$ is surjective.

Let $(u,\ \omega) \in Y \times \mathbb{R}$. Then $u(t) = 0,\ \forall\ t \in [-r,\ 0]$.

We wish to find $y \in X$ such that $L[y(t)] = (u(t),\ \omega)$.

i.e $(v(t),\ y(0)) = (u(t),\ \omega)$, where

$$v(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ y(t) - y(0) + \lambda \int\limits_0^t y(s)ds, & 0 \leq t \leq T. \end{cases}$$

i.e $v(t) = u(t)$ and $y(0) = \omega$.

Let such $y \in X$ exists. Then $y(t) = y(0),\ t \in [-r,\ 0]$ and
$u(t) = y(t) - y(0) + \lambda \int\limits_0^t y(s)ds,\ t \in I$. Therefore

$$\left.\begin{array}{l} y'(t) + \lambda y(t) = u'(t),\ t \in I, \\ y(t) = y(0),\ t \in [-r,\ 0], \\ y(0) = \omega. \end{array}\right\}$$

This implies $y(t) = \begin{cases} \omega, & -r \leq t \leq 0, \\ \omega e^{-\lambda t} + \int\limits_0^t e^{-\lambda(t-s)} u'(s)ds, & 0 \leq t \leq T. \end{cases}$

So to each $(u,\ \omega) \in Y \times \mathbb{R},\ \exists\ y \in X$ such that $L[y(t)] = (u(t),\ \omega)$.

Therefore $L : X \to Y \times \mathbb{R}$ is surjective.

Thus from Claim-2.1 and Claim-2.2, $L : X \to Y \times \mathbb{R}$ is bijective.

Hence $L^{-1} : Y \times \mathbb{R} \to X$ exists and it is defined by

$$L^{-1}(u(t),\ \omega) = \begin{cases} \omega, & -r \leq t \leq 0, \\ \omega e^{-\lambda t} + \int\limits_0^t e^{-\lambda(t-s)} u'(s)ds, & 0 \leq t \leq T. \end{cases}$$

(3) Since $L : X \to Y \times \mathbb{R}$ is bounded linear operator and $(X,\ ||.||_\infty)$ and $(Y \times \mathbb{R},\ ||.||_\star)$ are Banach spaces, $L^{-1} : Y \times \mathbb{R} \to X$ is bounded linear operator and hence continuous by bounded inverse theorem. $\square$

Lemma 3.1.2. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and $N : X \to Y \times \mathbb{R}$ be defined by $N[y(t)] = (v(t),\ y(T))$,*

$$\text{where } v(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ \int\limits_0^t f(s,\ y(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

Then $N : X \to Y \times \mathbb{R}$ is continuous and compact.

Proof. Claim 1:- $N : X \to Y \times \mathbb{R}$ is continuous.

Let $y \in X$ be arbitrary and $\{y_n\} \subseteq X$ such that $y_n \to y$ in $(X,\ ||.||_\infty)$.

Therefore $||y_n - y||_\infty \to 0$ as $n \to \infty$. This implies $y_n \to y$ uniformly on $[-r,\ T]$ and $y_n' \to y'$ uniformly on I.

Since $y_n(t) = y_n(0),\ \forall\ t \in [-r,\ 0]$ and $\forall\ t \in \mathbb{N}$, we have $y(t) = y(0)$,

$\forall\ t \in [-r,\ 0]$ as $y_n \to y$ pointwise on $[-r,\ 0]$.

Let $N[y_n(t)] = (v_n(t),\ y_n(T))$ and $N[y(t)] = (v(t),\ y(T))$, where

$$v_n(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ \int\limits_0^t f(s,\ y_n(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

and

$$v(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ \int\limits_0^t f(s,\ y(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

We show that $Ny_n \to Ny$ in $(Y \times \mathbb{R},\ ||.||_\star)$.

Since $y_n \to y$ uniformly on $[-r,\ T]$, $y_n(T) \to y(T)$ in $\mathbb{R}$.

Let $\epsilon > 0$ be arbitrary and $t \in [-r,\ 0]$. Then $|v_n(t) - v(t)| = 0 < \epsilon,\ \forall\ n \in \mathbb{N}$ and $\forall\ t \in [-r,\ 0]$.

Let $0 \leq t \leq T$.

Since $y_n \to y$ uniformly on $[-r,\ T]$, we have $y_n(s-r) \to y(s-r)$ uniformly on I. Therefore $f(s,\ y_n(s-r)) \to f(s,\ y(s-r))$ uniformly on I, as $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous.

So $\exists\ n_0 \in \mathbb{N}$ such that $|f(s,\ y_n(s-r)) - f(s,\ y(s-r))| < \frac{\epsilon}{T}$, $\forall\ n \geq n_0$ and $\forall\ s \in I$. Therefore

$$\begin{aligned} |v_n(t) - v(t)| &\leq \int_0^t |f(s,\ y_n(s-r)) - f(s,\ y(s-r))|ds \\ &\leq \int_0^T |f(s,\ y_n(s-r)) - f(s,\ y(s-r))|ds \\ &< \frac{\epsilon}{T}\int_0^T ds = \epsilon. \end{aligned}$$

So we have $|v_n(t) - v(t)| < \epsilon$, $\forall\ n \geq n_0$ and $\forall\ t \in I$. Thus $v_n \to v$ uniformly on $[-r,\ T]$. Since $v_n'(t) = f(t,\ y_n(t-r))$ and $v'(t) = f(t,\ y(t-r))$, we get $|v_n'(t) - v'(t)| = |f(t,\ y_n(t-r)) - f(t,\ y(t-r))| < \frac{\epsilon}{T}$, $\forall\ n \geq n_0$ and $\forall\ t \in I$. This implies $v_n' \to v'$ uniformly on I.

Therefore $\sup\limits_{t\in[-r,\ T]} |v_n(t) - v(t)| \to 0$ as $n \to \infty$ and $\sup\limits_{t\in I} |v_n'(t) - v'(t)| \to 0$ as $n \to \infty$.

So we have $||v_n - v||_\infty \to 0$ as $n \to \infty$. This implies

$$\begin{aligned} ||Ny_n - Ny||_\star &= ||(v_n,\ y_n(T)) - (v,\ y(T))||_\star \\ &= ||v_n - v||_\infty + |y_n(T) - y(T)|. \end{aligned}$$

Thus $||Ny_n - Ny||_\star \to 0$ as $n \to \infty$. Hence $N : X \to Y \times \mathbb{R}$ is continuous.

Claim 2:- $N : X \to Y \times \mathbb{R}$ is compact.

Let $\{y_n\}$ be a bounded sequence in $(X,\ ||.||_\infty)$.

Therefore $\exists\ K > 0$ such that $||y_n||_\infty \leq K,\ \forall\ n \in \mathbb{N}$. Let

$$Ny_n = (v_n,\ y_n(T)), \forall\ n \in \mathbb{N}, \text{ where } v_n(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ \int_0^t f(s,\ y_n(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

Since $||y_n||_\infty \leq K,\ \forall\ n \in \mathbb{N}$, we have $\sup\limits_{t\in[-r,\ T]} |y_n(t)| \leq K,\ \forall\ n \in \mathbb{N}$ and $\sup\limits_{t\in I} |y_n'(t)| \leq K,\ \forall\ n \in \mathbb{N}$. Note that $|v_n(t)| = 0,\ \forall\ t \in [-r,\ 0]$. Also $\exists\ \eta > 0$

such that $|f(t,\ y_n(t-r))| \leq \eta,\ \forall\ t \in I$ and $\forall\ n \in \mathbb{N}$.

Let $0 \leq t \leq T$. Then $|v_n(t)| \leq \int\limits_0^t |f(s,\ y_n(s-r))|ds \leq \eta \int\limits_0^T ds = \eta T,\ \forall\ t \in I$ and $\forall\ n \in \mathbb{N}$. This implies $\sup\limits_{t\in[-r,\ T]} |v_n(t)| \leq \eta T,\ \forall\ n \in \mathbb{N}$.

Since $v_n'(t) = f(t,\ y_n(t-r)),\ \forall\ n \in \mathbb{N}$, we have $|v_n'(t)| \leq \eta,\ \forall\ t \in I$ and $\forall\ n \in \mathbb{N}$. Therefore $\sup\limits_{t\in I} |v_n'(t)| \leq \eta,\ \forall\ n \in \mathbb{N}$.

So we get $||v_n||_\infty \leq \eta T + \eta = (\eta+1)T,\ \forall\ n \in \mathbb{N}$. Thus $\{v_n\}$ is bounded on $(Y,\ ||.||_\infty)$. Hence $\{v_n\}$ is uniformly bounded on $[-r,\ T]$ and $\{v_n'\}$ is uniformly bounded on I.

Let $t_0 \in [-r,\ T]$ be arbitrary and $\epsilon > 0$ be arbitrary.

<u>CASE-1</u>: Let $t_0 = -r$.

Choose $\delta = \min\left\{\frac{r}{2},\ \epsilon\right\}$. Then

$|v_n(t) - v_n(-r)| = 0 < \epsilon,\ \forall\ t$ such that $-r < t < -r + \delta$.

Therefore $\{v_n\}$ is equicontinuous at $-r$.

<u>CASE-2</u>: Let $t_0 = 0$.

Choose $\delta = \min\left\{\frac{r}{2},\ \frac{\epsilon}{\eta}\right\}$ and t be such that $|t| < \delta$.

If $t \leq t_0$, then $|v_n(t) - v_n(t_0)| = 0 < \epsilon$.

Let $t \geq t_0$. Then

$$|v_n(t) - v_n(t_0)| = |v_n(t)| \leq \int\limits_0^t f(s,\ y_n(s-r))ds$$
$$\leq t\eta < \delta\eta \leq \eta\frac{\epsilon}{\eta} = \epsilon.$$

This implies $|v_n(t) - v_n(t_0)| < \epsilon,\ \forall\ t$ such that $|t| < \delta$.

Therefore $\{v_n\}$ is equicontinuous at 0.

<u>CASE-3</u>: Let $t_0 = T$.

Choose $\delta = \min\left\{\frac{T}{2},\ \frac{\epsilon}{\eta}\right\}$ and let $T - \delta < t \leq T$.

$$\text{Then } |v_n(t) - v_n(t_0)| = |\int_0^t f(s,\ y_n(s-r))ds - \int_0^T f(s,\ y_n(s-r))ds|$$
$$= |\int_t^T f(s,\ y_n(s-r))ds|$$
$$\leq (T-t)\eta < \delta\eta \leq \eta\frac{\epsilon}{\eta} = \epsilon.$$

This implies $|v_n(t) - v_n(t_0)| < \epsilon,\ \forall\ t$ such that $T - \delta < t \leq T$.

Therefore $\{v_n\}$ is equicontinuous at T.

CASE-4: Let $-r < t_0 < 0$.

Choose $\delta = \min\left\{t_0 + \frac{r}{2},\ \frac{-t_0}{2}\right\}$. Then

$|v_n(t) - v_n(t_0)| = 0 < \epsilon,\ \forall\ t$ such that $|t - t_0| < \delta$.

Therefore $\{v_n\}$ is equicontinuous at t_0, where $-r < t_0 < 0$.

CASE-5: Let $0 < t_0 < T$.

Choose $\delta = \min\left\{\frac{t_0}{2},\ \frac{T-t_0}{2},\ \frac{\epsilon}{\eta}\right\}$ and let t be such that $|t - t_0| < \delta$. Then

$$|v_n(t) - v_n(t_0)| = |\int_0^t f(s,\ y_n(s-r))ds - \int_0^{t_0} f(s,\ y_n(s-r))ds|$$
$$= |\int_{t_0}^t f(s,\ y_n(s-r))ds|$$
$$\leq |\int_{t_0}^t |f(s,\ y_n(s-r))|ds| \leq \eta|t - t_0| < \delta\eta \leq \eta\frac{\epsilon}{\eta} = \epsilon.$$

This implies $|v_n(t) - v_n(t_0)| < \epsilon,\ \forall\ t$ such that $|t - t_0| < \delta$.

Therefore $\{v_n\}$ is equicontinuous at t_0, where $0 < t_0 < T$.

Hence, from CASE-1 to CASE-5, we conclude that $\{v_n\}$ is equicontinuous on $[-r,\ T]$.

Now by using Arzela-Ascoli's lemma we get a subsequence $\{v_{(n,m)}\}$ of $\{v_n\}$ which is uniformly convergent to some function v on $[-r,\ T]$. Let $\{y_{(n,m)}\}$

be the corresponding subsequence of $\{y_n\}$. Since $||y_n||_\infty \leq K,\ \forall\ n \in \mathbb{N}$, we have $||y_{(n,m)}||_\infty \leq K,\ \forall\ m \in \mathbb{N}$. This implies $\{y_{(n,m)}\}$ is uniformly bounded on $[-r,\ T]$ and $\{y'_{(n,m)}\}$ is uniformly bounded on I. Therefore $\{v_{(n,m)}\}$ is uniformly bounded on $[-r,\ T]$ and $\{v'_{(n,m)}\}$ is uniformly bounded on I. Since $v'_{(n,m)}(t) = f(t,\ y_{(n,m)}(t-r)),\ \forall\ t \in I$ and as $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous, we have $\{v'_{(n,m)}\}$ equicontinuous on I.

Therefore by Arzela-Ascoli's lemma $\{v'_{(n,m)}\}$ has a subsequence $\{v'_{(n,m,q)}\}$ converging uniformly to u on I. Since $\{v_{(n,m)}\}$ is a subsequence of $\{v_n\}$ and $\{v_{(n,m,q)}\}$ is a subsequence of $\{v_{(n,m)}\}$, we have $\{v_{(n,m,q)}\}$ is a subsequence of $\{v_n\}$. Now $v_{(n,m)} \to v$ uniformly on $[-r,\ T]$ implies $v_{(n,m,q)} \to v$ uniformly on $[-r,\ T]$. Therefore $v_{(n,m,q)} \to v$ uniformly on I. Since $v'_{(n,m,q)} \to u$ uniformly on I, we have $u = v'$ on I. So we have $v_{(n,m,q)} \to v$ uniformly on $[-r,\ T]$ as $q \to \infty$ and $v'_{(n,m,q)} \to v'$ uniformly on I as $q \to \infty$. Thus $||v_{(n,m,q)} - v||_\infty \to 0$ as $q \to \infty$. Let $\{y_{(n,m,q)}\}$ be the corresponding subsequence of $\{y_n\}$. Now $||y_n||_\infty \leq K,\ \forall\ n \in \mathbb{N}$ implies $||y_{(n,m,q)}||_\infty \leq K,\ \forall\ q \in \mathbb{N}$.

This implies $|y_{(n,m,q)}(t)| \leq K,\ \forall\ q \in \mathbb{N}$ and $\forall\ t \in [-r,\ T]$.

So we have $|y_{(n,m,q)}(T)| \leq K,\ \forall\ q \in \mathbb{N}$.

Therefore by Bolzano-Weierstrass's theorem, $\{y_{(n,m,q)}(T)\}$ has a convergent subsequence $\{y_l(T)\}$ converging to $\gamma \in \mathbb{R}$ as $l \to \infty$. Let $\{v_l\}$ be the corresponding subsequence of $\{v_{(n,m,q)}\}$ in $(Y,\ ||.||_\infty)$. Since $v_{(n,m,q)} \to v$ in $(Y,\ ||.||_\infty)$ as $q \to \infty$, we have $v_l \to v$ in $(Y,\ ||.||_\infty)$ as $l \to \infty$.

$$\begin{aligned} \text{So we have } ||Ny_l - (v,\gamma)||_\star &= ||(v_l,\ y_l(T)) - (v,\ \gamma)||_\star \\ &= ||v_l - v||_\infty + |y_l(T) - \gamma| \to 0 \text{ as } l \to \infty. \end{aligned}$$

Therefore $Ny_l \to (v,\ \gamma)$ in $(Y \times \mathbb{R},\ ||.||_\star)$ as $l \to \infty$.

Hence $N : X \to Y \times \mathbb{R}$ is a compact operator. $\square$

Lemma 3.1.3. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and $N : X \to Y \times \mathbb{R}$ be defined by $N[y(t)] = (v(t),\ -y(T))$,*

$$\text{where } v(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ \int_0^t f(s,\ y(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

Then $N : X \to Y \times \mathbb{R}$ is continuous and compact.

Proof. Proof is similar to the proof of the Lemma 3.1.2 and hence omitted. □

Lemma 3.1.4. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and $N : X \to Y \times \mathbb{R}$ be defined by $N[y(t)] = (v(t),\ -\frac{b}{a}y(T))$,*

$$\text{where } v(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ \int_0^t f(s,\ y(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

Then $N : X \to Y \times \mathbb{R}$ is continuous and compact.

Proof. Proof is similar to the proof of the Lemma 3.1.2 and hence omitted. □

Lemma 3.1.5. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and*
$p : I \times \mathbb{R} \to \mathbb{R}$ be continuous. Let $N : X \to Y \times \mathbb{R}$ be defined by
$N[y(t)] = (v(t),\ p(T,\ y(T)))$, where

$$v(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ \int_0^t f(s,\ y(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

Then $N : X \to Y \times \mathbb{R}$ is continuous and compact.

Proof. Proof is similar to the proof of the Lemma 3.1.2 and hence omitted. □

Lemma 3.1.6. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and $h : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Let $N : X \to Y \times \mathbb{R}$ be defined by $N[y(t)] = (v(t),\ h(y(0),\ y(T)))$, where*

$$v(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ \int_0^t f(s,\ y(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

Then $N : X \to Y \times \mathbb{R}$ is continuous and compact.

Proof. Proof is similar to the proof of the Lemma 3.1.2 and hence omitted. □

3.2 Existence Theorems using Leray - Schauder Alternative

In this section, we estabish some results on existence of solution to the BVPs using Leray - Schauder alternative.

Definition 3.2.1. *A function $u : [-r,\ T] \to \mathbb{R}$ is said to be a solution of PBVP* (3.1) *if*

1. u is continuous on $[-r,\ T]$ and continuously differentiable on I.

2. $u'(t) + \lambda u(t) = f(t,\ u(t-r)),\ t \in I$.

3. $u(t) = u(0),\ t \in [-r,\ 0]$.

4. $u(0) = u(T)$.

Similarly we can define solution for other given BVPs of first order DECD.

Theorem 3.2.1. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous. If $\exists\ M > 0$ independent of μ with $||y||_\infty \neq M$ for any solution y of*

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = \mu f(t,\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = \mu y(T), \end{aligned}\right\} \tag{3.6}$$

for each $\mu \in (0,\ 1)$, then the PBVP (3.1) has atleast one solution in X.

Proof. Let $L : X \to Y \times \mathbb{R}$ be defined by $L[y(t)] = (u(t),\ y(0))$, where

$$u(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ y(t) - y(0) + \lambda \int\limits_0^t y(s)ds, & 0 \leq t \leq T. \end{cases}$$

By Lemma 3.1.1, $L : X \to Y \times \mathbb{R}$ is linear, bijective and continuous.

Also $L^{-1} : Y \times \mathbb{R} \to X$ exists and is given by

$$L^{-1}(u(t),\ \gamma) = \begin{cases} \gamma, & -r \leq t \leq 0, \\ \gamma e^{-\lambda t} + \int\limits_0^t e^{-\lambda(t-s)} u'(s)ds, & 0 \leq t \leq T. \end{cases}$$

Further $L^{-1} : Y \times \mathbb{R} \to X$ is bounded and hence continuous.

Let $N : X \to Y \times \mathbb{R}$ be defined by $N[y(t)] = (v(t),\ y(T))$, where

$$v(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ \int\limits_0^t f(s,\ y(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

Then by Lemma 3.1.2, $N : X \to Y \times \mathbb{R}$ is continuous and compact.

Let $S = L^{-1}N : X \to X$. Then

$$S[y(t)] = \begin{cases} y(T), & -r \leq t \leq 0, \\ y(T)e^{-\lambda t} + \int\limits_0^t e^{-\lambda(t-s)} f(s,\ y(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

A fixed point of S is a solution of the PBVP (3.1).

Let $\{y_n\}$ be a bounded sequence in $(X,\ ||.||_\infty)$.

Since $N : X \to Y \times \mathbb{R}$ is a compact operator, we can find a convergent subsequence $\{Ny_{n_l}\}$ of $\{Ny_n\}$ as $l \to \infty$. Also it is given that

$L^{-1} : Y \times \mathbb{R} \to X$ is continuous linear operator. Therefore $\{L^{-1}[Ny_{n_l}]\}$ is convergent in $(X,\ ||.||_\infty)$ as $l \to \infty$. This implies $\{Sy_{n_l}\}$ is convergent in $(X,\ ||.||_\infty)$ as $l \to \infty$. Thus $S : X \to X$ is a compact operator. Since composition of continuous operators is continuous, $S : X \to X$ is a continuous operator.

Let $U = \{y \in X : ||y||_\infty < M\}$. Then U is open in $(X,\ ||.||_\infty)$.

By Leray-Schauder alternative either S has a fixed point in $\bar{U}$ or

$\exists\ \mu \in (0,\ 1)$ and $y \in \partial U$ such that $y = \mu Sy$.

Let if possible S has no fixed point in $\bar{U}$. Then $\exists\ y \in \partial U$ and $\mu \in (0,\ 1)$ such that $y = \mu Sy$. Therefore y is a solution of BVP (3.6).

Since $y \in \partial U$, we have $||y||_\infty = M$, which is a contradiction.

Therefore S has fixed point in $\bar{U}$. This implies $\exists\ y \in \bar{U}$ such that $y = Sy$.

Hence y is a solution of the PBVP (3.1). □

Theorem 3.2.2. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous. If $\exists\ M > 0$ independent of μ with $||y||_\infty \neq M$ for any solution y of*

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = \mu f(t,\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = -\mu y(T), \end{aligned}\right\} \tag{3.7}$$

for each $\mu \in (0,\ 1)$, then the APBVP (3.2) has atleast one solution in X.

Proof. Let $L : X \to Y \times \mathbb{R}$ be defined as in the proof of the Theorem 3.2.1.

By Lemma 3.1.1, $L : X \to Y \times \mathbb{R}$ is linear, bijective and continuous.

Also $L^{-1} : Y \times \mathbb{R} \to X$ exists and is given by

$$L^{-1}(u(t),\ \gamma) = \begin{cases} \gamma, & -r \leq t \leq 0, \\ \gamma e^{-\lambda t} + \int\limits_0^t e^{-\lambda(t-s)} u'(s) ds, & 0 \leq t \leq T. \end{cases}$$

Further $L^{-1} : Y \times \mathbb{R} \to X$ is bounded and hence continuous.

Let $N : X \to Y \times \mathbb{R}$ be defined by $N[y(t)] = (v(t),\ -y(T))$, where

$$v(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ \int\limits_0^t f(s,\ y(s-r)) ds, & 0 \leq t \leq T. \end{cases}$$

Then by Lemma 3.1.3, $N : X \to Y \times \mathbb{R}$ is continuous and compact.

Let $S = L^{-1}N : X \to X$. Then

$$S[y(t)] = \begin{cases} -y(T), & -r \leq t \leq 0, \\ -y(T)e^{-\lambda t} + \int\limits_0^t e^{-\lambda(t-s)} f(s,\ y(s-r)) ds, & 0 \leq t \leq T. \end{cases}$$

Also S is compact and continuous.

The remaining part of the proof is similar to the proof of the Theorem 3.2.1.

Hence there exists atleast one solution in X to the APBVP (3.2). □

Theorem 3.2.3. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous. If $\exists\ M > 0$ independent of μ with $||y||_\infty \neq M$ for any solution y of*

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = \mu f(t,\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = -\mu \frac{b}{a} y(T), \end{aligned}\right\} \tag{3.8}$$

for each $\mu \in (0,\ 1)$, then the LBVP (3.3) has atleast one solution in X.

Proof. Let $L : X \to Y \times \mathbb{R}$ be defined as in the proof of the Theorem 3.2.1.

By Lemma 3.1.1, $L : X \to Y \times \mathbb{R}$ is linear, bijective and continuous.

Also $L^{-1} : Y \times \mathbb{R} \to X$ exists and is given by

$$L^{-1}(u(t),\ \gamma) = \begin{cases} \gamma, & -r \leq t \leq 0, \\ \gamma e^{-\lambda t} + \int\limits_0^t e^{-\lambda(t-s)} u'(s) ds, & 0 \leq t \leq T. \end{cases}$$

Further $L^{-1} : Y \times \mathbb{R} \to X$ is bounded and hence continuous.

Let $N : X \to Y \times \mathbb{R}$ be defined by $N[y(t)] = (v(t),\ -\frac{b}{a} y(T))$, where

$$v(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ \int\limits_0^t f(s,\ y(s-r)) ds, & 0 \leq t \leq T. \end{cases}$$

Then by Lemma 3.1.4, $N : X \to Y \times \mathbb{R}$ is continuous and compact.

Let $S = L^{-1}N : X \to X$. Then

$$S[y(t)] = \begin{cases} -\frac{b}{a} y(T), & -r \leq t \leq 0, \\ -\frac{b}{a} y(T) e^{-\lambda t} + \int\limits_0^t e^{-\lambda(t-s)} f(s,\ y(s-r)) ds, & 0 \leq t \leq T. \end{cases}$$

Also S is compact and continuous.

The remaining part of the proof is similar to the proof of the Theorem 3.2.1.

Hence there exists atleast one solution in X to the APBVP (3.3). □

3.3 Existence Theorem using Schauder's Fixed Point Theorem

In this section, we first prove a result on existence of solution for BVP (3.5), by assuming f and g to be continuous and bounded, using Schauder's fixed point theorem.

Definition 3.3.1. *A function $u : [-r,\ T] \to \mathbb{R}$ is said to be a solution of BVP (3.5) if 1. u is continuous on $[-r,\ T]$ and continuously differentiable on I.*

2. $u'(t) = f(t,\ u(t-r)),\ t \in I$.

3. $u(t) = u(0),\ t \in [-r,\ 0]$.

4. $u(0) = g(u(0),\ u(T))$.

Theorem 3.3.1. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be both bounded and continuous functions. Then the BVP (3.5) has atleast one solution $y \in X$.*

Proof. As $g : \mathbb{R}^2 \to \mathbb{R}$ is bounded, $\exists\ K_1 > 0$ such that $|g(x,\ y)| \leq K_1$, $\forall\ (x,\ y) \in \mathbb{R}^2$.

Also $f : I \times \mathbb{R} \to \mathbb{R}$ is bounded implies that $\exists\ K_2 > 0$ such that $|f(t,\ x)| \leq K_2,\ \forall\ (t,\ x) \in I \times \mathbb{R}$.

Let $A : X \to X$ be defined by

$$A[y(t)] = \begin{cases} g(y(0),\ y(T)), & -r \leq t \leq 0, \\ g(y(0),\ y(T)) + \int\limits_0^t f(s,\ y(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

A fixed point of $A : X \to X$ is a solution of the BVP (3.5).

Claim 1:- $A : X \to X$ is continuous.

Let $y \in X$ be arbitrary and $\{y_n\} \subseteq X$ such that $y_n \to y$ in $(X,\ ||.||_\infty)$.

Then $||y_n - y||_\infty \to 0$ as $n \to \infty$. This implies $y_n \to y$ uniformly on $[-r,\ T]$ and $y_n' \to y'$ uniformly on I. Since $y_n(t) = y_n(0),\ \forall\ t \in [-r,\ 0]$ and $\forall\ t \in \mathbb{N}$, we have $y(t) = y(0),\ \forall\ t \in [-r,\ 0]$ as $y_n \to y$ pointwise on $[-r,\ 0]$.

Let $A[y_n(t)] = v_n(t)$ and $A[y(t)] = v(t)$, where

$$v_n(t) = \begin{cases} g(y_n(0),\ y_n(T)), & -r \leq t \leq 0, \\ g(y_n(0),\ y_n(T)) + \int\limits_0^t f(s,\ y_n(s-r))ds, & 0 \leq t \leq T, \end{cases}$$

and

$$v(t) = \begin{cases} g(y(0),\ y(T)), & -r \le t \le 0, \\ g(y(0),\ y(T)) + \int_0^t f(s,\ y(s-r))ds, & 0 \le t \le T. \end{cases}$$

We show that $Ay_n \to Ay$ in $(X,\ ||.||_\infty)$.

Let $\epsilon > 0$ be arbitrary.

Since $y_n \to y$ uniformly on $[-r,\ T]$, we have $y_n(0) \to y(0)$ and $y_n(T) \to y(T)$ in $\mathbb{R}$. This implies $g(y_n(0),\ y_n(T)) \to g(y(T),\ y(0))$ in $\mathbb{R}$, as g is continuous. Therefore $\exists\ n_1 \in \mathbb{N}$ such that $|g(y_n(0),\ y_n(T)) - g(y(T),\ y(0))| < \frac{\epsilon}{2},\ \forall\ n \ge n_1$.

Let $-r \le t \le 0$ be arbitrary.

Then $|v_n(t) - v(t)| = |g(y_n(0),\ y_n(T)) - g(y(T),\ y(0))| < \dfrac{\epsilon}{2},\ \forall\ n \ge n_1$.

Let $0 \le t \le T$.

Since $y_n \to y$ uniformly on $[-r,\ T]$, $y_n(s-r) \to y(s-r)$ uniformly on I.

This implies $f(s,\ y_n(s-r)) \to f(s,\ y(s-r))$ uniformly on I as $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous. Therefore $\exists\ n_2 \in \mathbb{N}$ such that

$|f(s,\ y_n(s-r)) - f(s,\ y(s-r))| < \frac{\epsilon}{2T},\ \forall\ n \ge n_2$ and $\forall\ s \in I$.

Let $n_0 = \max\{n_1,\ n_2\}$.

Then $|f(s,\ y_n(s-r)) - f(s,\ y(s-r))| < \frac{\epsilon}{2T},\ \forall\ n \ge n_0$ and $\forall\ s \in I$ and $|g(y_n(0),\ y_n(T)) - g(y(T),\ y(0))| < \frac{\epsilon}{2},\ \forall\ n \ge n_0$. This implies

$$\begin{aligned} |v_n(t) - v(t)| &\le |g(y_n(0),\ y_n(T)) - g(y(0),\ y(T))| \\ &\quad + \int_0^t |f(s,\ y_n(s-r)) - f(s,\ y(s-r))|ds \\ &< \frac{\epsilon}{2} + \int_0^T |f(s,\ y_n(s-r)) - f(s,\ y(s-r))|ds \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2T}\int_0^T ds = \epsilon. \end{aligned}$$

Therefore $|v_n(t) - v(t)| < \epsilon,\ \forall\ n \ge n_0$ and $\forall\ t \in I$.

Also we have $|v_n(t) - v(t)| < \epsilon,\ \forall\ n \geq n_0$ and $\forall\ t \in [-r,\ 0]$. This implies $|v_n(t) - v(t)| < \epsilon,\ \forall\ n \geq n_0$ and $\forall\ t \in [-r,\ T]$. Thus $v_n \to v$ uniformly on $[-r,\ T]$. For $t \in I$, we have $v_n'(t) = f(t,\ y_n(t-r))$ and $v'(t) = f(t,\ y(t-r))$. So we get

$$|v_n'(t) - v'(t)| = |f(t,\ y_n(t-r)) - f(t,\ y(t-r))| < \frac{\epsilon}{2T},\ \forall\ n \geq n_0 \text{ and } \forall\ t \in I.$$

This implies $v_n' \to v'$ uniformly on I.

Therefore $\sup\limits_{t\in[-r,\ T]} |v_n(t) - v(t)| \to 0$ and $\sup\limits_{t\in I} |v_n'(t) - v'(t)| \to 0$ as $n \to \infty$.
Thus $||v_n - v||_\infty \to 0$ as $n \to \infty$. This implies $||Ay_n - Ay||_\infty \to 0$ as $n \to \infty$.

So we have $Ay_n \to Ay$ in $(X,\ ||.||_\infty)$.

Hence $A : X \to X$ is continuous.

Claim 2:- $A : X \to X$ is compact.

Let $\{y_n\}$ be a bounded sequence in $(X,\ ||.||_\infty)$. Then $\exists\ K > 0$ such that $||y_n||_\infty \leq K,\ \forall\ n \in \mathbb{N}$.

Let $A[y_n(t)] = v_n(t)$,

$$\text{where } v_n(t) = \begin{cases} g(y_n(0),\ y_n(T)), & -r \leq t \leq 0, \\ g(y_n(0),\ y_n(T)) + \int\limits_0^t f(s,\ y_n(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

Now $||y_n||_\infty \leq K,\ \forall\ n \in \mathbb{N}$ implies $\sup\limits_{t\in[-r,\ T]} |y_n(t)| \leq K,\ \forall\ n \in \mathbb{N}$ and $\sup\limits_{t\in I} |y_n'(t)| \leq K,\ \forall\ n \in \mathbb{N}$.
Let $M = \max\{K_1,\ TK_2\}$ and $-r \leq t \leq 0$. Then

$|v_n(t)| = |g(y_n(0),\ y_n(T))| \leq K_1 \leq M \leq 2M,\ \forall\ n \in \mathbb{N}$ and $\forall\ t \in [-r,\ 0]$.

Let $0 \leq t \leq T$.

$$\text{Therefore } |v_n(t)| = |g(y_n(0),\ y_n(T)) + \int_0^t f(s,\ y_n(s-r))ds|$$

$$\leq |g(y_n(0),\ y_n(T))| + |\int_0^t f(s,\ y_n(s-r))ds|$$

$$\leq K_1 + \int_0^t |f(s,\ y_n(s-r))|ds$$

$$\leq K_1 + \int_0^T |f(s,\ y_n(s-r))|ds$$

$$\leq K_1 + K_2T \leq 2M,\ \forall\ t \in I \text{ and } \forall\ n \in \mathbb{N}.$$

This implies $\sup_{t\in[-r,\ T]} |v_n(t)| \leq 2M,\ \forall\ n \in \mathbb{N}$.
Now $v_n'(t) = f(t,\ y_n(t-r)),\ \forall\ n \in \mathbb{N}$. So we have $|v_n'(t)| \leq K_2,\ \forall\ t \in I$ and $\forall\ n \in \mathbb{N}$. Therefore $\sup_{t\in I} |v_n'(t)| \leq K_2,\ \forall\ n \in \mathbb{N}$. This gives $||v_n||_\infty \leq K_2$, $\forall\ n \in \mathbb{N}$. Therefore $\{v_n\}$ is bounded on $(X,\ ||.||_\infty)$. Thus $\{v_n\}$ is uniformly bounded on $[-r,\ T]$ and $\{v_n'\}$ is uniformly bounded on I.

Let $t_0 \in [-r,\ T]$ be arbitrary and $\epsilon > 0$ be arbitrary.

CASE-1: Let $t_0 = -r$. Choose $\delta = \min\left\{\frac{r}{2},\ \epsilon\right\}$.

$$\text{Then } |v_n(t) - v_n(-r)| = |g(y_n(0),\ y_n(T)) - g(y_n(0),\ y_n(T))|$$

$$= 0 < \epsilon,\ \forall\ t \text{ such that } -r < t < -r + \delta.$$

This implies $\{v_n\}$ is equicontinuous at $-r$.

CASE-2: Let $t_0 = 0$.

Choose $\delta = \min\left\{\frac{r}{2},\ \frac{\epsilon}{K_2}\right\}$ and $t \in (-\delta,\ \delta)$ be arbitrary.

$$\text{If } t \leq t_0, \text{ then } |v_n(t) - v_n(t_0)| = |g(y_n(0),\ y_n(T)) - g(y_n(0),\ y_n(T))|$$

$$= 0 < \epsilon.$$

Let $t \geq t_0$. Then

$$\begin{aligned}
|v_n(t) - v_n(t_0)| &= |g(y_n(0),\ y_n(T)) \\
&\quad + \int_0^t f(s,\ y_n(s-r))ds - g(y_n(0),\ y_n(T))| \\
&= |\int_0^t f(s,\ y_n(s-r))ds| \\
&\leq \int_0^t |f(s,\ y_n(s-r))|ds \leq tK_2 < \delta K_2 \leq K_2 \frac{\epsilon}{K_2} = \epsilon.
\end{aligned}$$

So we have $|v_n(t) - v_n(t_0)| < \epsilon,\ \forall\ t$ such that $|t| < \delta$.

This implies $\{v_n\}$ is equicontinuous at 0.

CASE-3: Let $t_0 = T$.

Choose $\delta = \min\left\{\frac{T}{2},\ \frac{\epsilon}{K_2}\right\}$ and let $T - \delta < t \leq T$. Then

$$\begin{aligned}
|v_n(t) - v_n(t_0)| &= |\int_0^t f(s,\ y_n(s-r))ds - \int_0^T f(s,\ y_n(s-r))ds| \\
&= |\int_T^t f(s,\ y_n(s-r))ds| \\
&= |-\int_t^T f(s,\ y_n(s-r))ds| \\
&= |\int_t^T f(s,\ y_n(s-r))ds| \\
&\leq \int_t^T |f(s,\ y_n(s-r))|ds \leq (T-t)K_2 < \delta K_2 \\
&\leq K_2 \frac{\epsilon}{K_2} = \epsilon.
\end{aligned}$$

Therefore $|v_n(t) - v_n(t_0)| < \epsilon,\ \forall\ t$ such that $T - \delta < t \leq T$.

This implies $\{v_n\}$ is equicontinuous at T.

CASE-4: Let $-r < t_0 < 0$. Choose $\delta = \min\left\{t_0 + \frac{r}{2},\ \frac{-t_0}{2}\right\}$.

$$\begin{aligned}\text{Then } |v_n(t) - v_n(t_0)| &= |g(y_n(0),\ y_n(T)) - g(y_n(0),\ y_n(T))| \\ &= 0 < \epsilon,\ \forall\ t \text{ such that } |t - t_0| < \delta.\end{aligned}$$

This implies $\{v_n\}$ is equicontinuous at t_0, where $-r < t_0 < 0$.

CASE-5: Let $0 < t_0 < T$.

Choose $\delta = \min\left\{\frac{t_0}{2},\ \frac{T-t_0}{2},\ \frac{\epsilon}{K_2}\right\}$ and let t be such that $|t - t_0| < \delta$. Then

$$\begin{aligned}|v_n(t) - v_n(t_0)| &= |\int_0^t f(s,\ y_n(s-r))ds - \int_0^{t_0} f(s,\ y_n(s-r))ds| \\ &= |\int_{t_0}^t f(s,\ y_n(s-r))ds| \\ &\leq \int_{t_0}^t |f(s,\ y_n(s-r))|ds \\ &\leq |\int_{t_0}^t |f(s,\ y_n(s-r))|ds| \\ &\leq |t - t_0|K_2 < \delta K_2 \leq K_2\frac{\epsilon}{K_2} = \epsilon.\end{aligned}$$

So we get $|v_n(t) - v_n(t_0)| < \epsilon,\ \forall\ t$ such that $|t - t_0| < \delta$.

Thus $\{v_n\}$ is equicontinuous at t_0, where $0 < t_0 < T$.

Hence from CASE-1 to CASE-5, we conclude that $\{v_n\}$ is equicontinuous on $[-r,\ T]$.

By Arzela-Ascoli's lemma, $\{v_n\}$ has a subsequence $\{v_{(n,m)}\}$ converging uniformly to v on $[-r,\ T]$. Let $\{y_{(n,m)}\}$ be the corresponding subsequence of $\{y_n\}$. Since $||y_n||_\infty \leq K,\ \forall\ n \in \mathbb{N}$, we have $||y_{(n,m)}||_\infty \leq K,\ \forall\ m \in \mathbb{N}$. This implies $\{y_{(n,m)}\}$ is uniformly bounded on $[-r,\ T]$ and $\{y'_{(n,m)}\}$ is uniformly bounded on I. Therefore $\{v_{(n,m)}\}$ is uniformly bounded on $[-r,\ T]$ and $\{v'_{(n,m)}\}$ is uniformly bounded on I. Since $v'_{(n,m)}(t) = f(t,\ y_{(n,m)}(t-r)),\ \forall\ t \in I$

and $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous, we have $\{v'_{(n,m)}\}$ equicontinuous on I.

Therefore by Arzela-Ascoli's lemma $\{v'_{(n,m)}\}$ has a subsequence $\{v'_{(n,m,q)}\}$ converging uniformly to u on I.

Since $\{v_{(n,m)}\}$ is a subsequence of $\{v_n\}$ and $\{v_{(n,m,q)}\}$ is a subsequence of $\{v_{(n,m)}\}$, $\{v_{(n,m,q)}\}$ is a subsequence of $\{v_n\}$. Also as $v_{(n,m)} \to v$ uniformly on $[-r,\ T]$, we have $v_{(n,m,q)} \to v$ uniformly on $[-r,\ T]$. This implies $v_{(n,m,q)} \to v$ uniformly on I. As $v'_{(n,m,q)} \to u$ uniformly on I, we have $u = v'$ on I. Therefore $v_{(n,m,q)} \to v$ uniformly on $[-r,\ T]$ as $q \to \infty$ and $v'_{(n,m,q)} \to v'$ uniformly on I as $q \to \infty$. Thus $||v_{(n,m,q)} - v||_\infty \to 0$ as $q \to \infty$. Further as v is continuous on $[-r,\ T]$ and differentiable on I, we have $v \in X$ as $(X,\ ||.||_\infty)$ is a Banach space.

Hence $A : X \to X$ is a compact operator.

Claim 3:- $A(X)$ is bounded.

Let $v \in A(X)$ be arbitrary. Then $\exists\ y \in X$ such that $v(t) = A[y(t)]$.

$$\text{This implies } v(t) = \begin{cases} g(y(0),\ y(T)), & -r \le t \le 0, \\ g(y(0),\ y(T)) + \int\limits_0^t f(s,\ y(s-r))ds, & 0 \le t \le T. \end{cases}$$

Let $-r \le t \le 0$. Then $|v(t)| = |g(y(0),\ y(T))| \le K_1 \le M \le 2M$.

Let $0 \le t \le T$. Then

$$\begin{aligned} |v(t)| &= |g(y(0),\ y(T)) + \int_0^t f(s,\ y(s-r))ds| \\ &\le |g(y(0),\ y(T))| + |\int_0^t f(s,\ y(s-r))ds| \\ &\le K_1 + \int_0^t |f(s,\ y(s-r))|ds \le K_1 + K_2 \int_0^T ds \le 2M. \end{aligned}$$

This implies $\sup\limits_{t \in [-r,\ T]} |v(t)| \le 2M$. As $v'(t) = f(t,\ y(t-r)),\ \forall\ t \in I$, we have $|v'(t)| \le K_2,\ \forall\ t \in I$. So we get $\sup\limits_{t \in I} |v'(t)| \le K_2$. Thus $||v||_\infty \le 2M + K_2$.

Therefore $A(X)$ is bounded in $(X,\ ||.||_\infty)$. Now by Schauder's fixed point theorem, $A : X \to X$ has a fixed point $y \in X$.

Hence y is a solution of the BVP (3.5). □

3.4 Method of Lower and Upper Solutions

In this section, we first give definitions of lower solution, upper solution, strict lower solution, strict upper solution and solution for first order DECD

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ y(t-r)),\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \end{aligned}\right\} \tag{3.9}$$

to prove that upper solution exceeds lower solution under certain conditions on a function $f : I \times \mathbb{R} \to \mathbb{R}$, lower and upper solution.

Definition 3.4.1. *A function $\alpha \in B$ is called a lower solution of first order DECD* (3.9) *if 1.* $\alpha'(t) + \lambda\alpha(t) \le f(t,\ \alpha(t-r)),\ t \in I.$

2. $\alpha(t) \le \alpha(0),\ t \in [-r,\ 0].$

A function $\beta \in B$ is called an upper solution of first order DECD (3.9) *if above inequalities are reversed.*

Definition 3.4.2. *A function $\alpha \in B$ is called a strict lower solution of first order DECD* (3.9) *if 1.* $\alpha'(t) + \lambda\alpha(t) < f(t,\ \alpha(t-r)),\ t \in I.$

2. $\alpha(t) \le \alpha(0),\ t \in [-r,\ 0].$

Definition 3.4.3. *A function $\beta \in B$ is called a strict upper solution of first order DECD* (3.9) *if 1.* $\beta'(t) + \lambda\beta(t) > f(t,\ \beta(t-r)),\ t \in I.$

2. $\beta(t) \ge \beta(0),\ t \in [-r,\ 0].$

Definition 3.4.4. *A function $u \in B$ is called a solution of first order DECD* (3.9) *if 1.* $u'(t) + \lambda u(t) = f(t,\ u(t-r)),\ t \in I.$

2. $u(t) = u(0),\ t \in [-r,\ 0].$

Theorem 3.4.1. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(t,\ u) \leq f(t,\ v),\ \forall\ (t,\ u), (t,\ v) \in I \times \mathbb{R}$ such that $u \leq v$. Let $\alpha,\ \beta \in X$ be a strict lower and upper solutions respectively of first order DECD* (3.9)*. Then $\alpha(0) \leq \beta(0)$ implies $\alpha(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$ and $\alpha(t) < \beta(t),\ \forall\ t \in (0,\ T]$.*

Proof. As $\alpha \in X$ is strict lower solution of first order DECD (3.9), we have

1. $\alpha'(t) + \lambda\alpha(t) < f(t,\ \alpha(t-r)),\ t \in I$,

2. $\alpha(t) = \alpha(0),\ t \in [-r,\ 0]$.

Since $\beta \in X$ is an upper solution of first order DECD (3.9), we have

3. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \beta(t-r)),\ t \in I$,

4. $\beta(t) = \beta(0),\ t \in [-r,\ 0]$.

It is given that $\alpha(0) \leq \beta(0)$. Also $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$.

Therefore $\alpha(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$.

We next prove that $\alpha(t) < \beta(t),\ \forall\ t \in (0,\ T]$.

Let if possible $\exists\ t_1 \in (0,\ T]$ such that $\alpha(t) < \beta(t),\ \forall\ t \in (0,\ t_1)$ and $\alpha(t_1) = \beta(t_1)$. Therefore $\alpha(t_1 - h) < \beta(t_1 - h)$ for sufficiently small $h > 0$.

This implies $\frac{\beta(t_1)-\beta(t_1-h)}{h} < \frac{\alpha(t_1)-\alpha(t_1-h)}{h}$. Thus $\beta'(t_1) \leq \alpha'(t_1)$.

So we have $\beta'(t_1) + \lambda\beta(t_1) \leq \alpha'(t_1) + \lambda\alpha(t_1)$. Therefore

$f(t_1,\ \beta(t_1 - r)) \leq \beta'(t_1) + \lambda\beta(t_1) \leq \alpha'(t_1) + \lambda\alpha(t_1) < f(t_1,\ \alpha(t_1 - r))$.

This gives $f(t_1,\ \beta(t_1 - r)) < f(t_1,\ \alpha(t_1 - r))$.

CASE-1: Let $0 < t_1 \leq r$.

Therefore $-r < t_1 - r \leq 0$.

So we have $\alpha(t_1 - r) \leq \beta(t_1 - r)$.

As f is non-decreasing in the second argument, we have

$f(t_1,\ \alpha(t_1 - r)) \leq f(t_1,\ \beta(t_1 - r))$, which is a contradiction.

CASE-2: Let $r < t_1 \leq T$.

This gives $0 < t_1 - r < t_1$.

So we have $\alpha(t_1 - r) < \beta(t_1 - r)$.

As f is non-decreasing in the second argument, we have

$f(t_1,\ \alpha(t_1 - r)) \leq f(t_1,\ \beta(t_1 - r))$, which is a contradiction.

Thus $\alpha(t) < \beta(t),\ \forall\, t \in (0, T]$.

Hence $\alpha(t) \leq \beta(t),\ \forall\, t \in [-r,\ 0]$ and $\alpha(t) < \beta(t),\ \forall\, t \in (0,\ T]$. □

Theorem 3.4.2. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(t,\ u) \leq f(t,\ v),\ \forall\ (t,\ u), (t,\ v) \in I \times \mathbb{R}$ such that $u \leq v$. Let $\alpha,\ \beta \in X$ be a lower and strict upper solutions respectively of first order DECD* (3.9)*. Then $\alpha(0) \leq \beta(0)$ implies $\alpha(t) \leq \beta(t),\ \forall\, t \in [-r,\ 0]$ and $\alpha(t) < \beta(t),\ \forall\, t \in (0,\ T]$.*

Proof. Similar to the proof of the Theorem 3.4.1. □

Next we define lower and upper solution for the PBVP (3.1) to prove result on existence of solution between them using Schauder's fixed point theorem. We also define strict lower and upper solution for the PBVP to prove result on existence of solution between the two.

Definition 3.4.5. *A function $\alpha \in B$ is called a lower solution of PBVP* (3.1) *if 1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t - r)),\ t \in I$.*

2. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$.

3. $\alpha(0) \leq \alpha(T)$.

Definition 3.4.6. *A function $\beta \in B$ is called an upper solution of PBVP* (3.1) *if 1. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \beta(t - r)),\ t \in I$.*

2. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0]$.

3. $\beta(0) \geq \beta(T)$.

Definition 3.4.7. *A function $\alpha \in B$ is called a strict lower solution of PBVP* (3.1) *if 1. $\alpha'(t) + \lambda\alpha(t) < f(t,\ \alpha(t - r)),\ t \in I$.*

2. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$.

3. $\alpha(0) \leq \alpha(T)$.

Definition 3.4.8. *A function $\beta \in B$ is called a strict upper solution of PBVP* (3.1) *if* 1. $\beta'(t) + \lambda\beta(t) > f(t,\ \beta(t-r)),\ t \in I.$

2. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$

3. $\beta(0) \geq \beta(T).$

We have the following existence theorem.

Theorem 3.4.3. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy* $f(t,\ x) \leq f(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in I \times \mathbb{R}$ *such that $x \leq y$. Let $\alpha,\ \beta \in X$ be lower and upper solutions of PBVP* (3.1) *such that* $\alpha(t) \leq \beta(t),$ $\forall\ t \in [-r,\ T]$. *Then the PBVP* (3.1) *has atleast one solution $y \in X$ such that* $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T].$

Proof. Since $\alpha,\ \beta \in X$ are lower and upper solutions of PBVP (3.1), we have

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t-r)),\ t \in I,$
2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \beta(t-r)),\ t \in I,$
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0],$
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0],$
5. $\alpha(0) \leq \alpha(T),$
6. $\beta(0) \geq \beta(T).$

As $\alpha,\ \beta \in X$, we get $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0].$

Let $p : I \times \mathbb{R} \to \mathbb{R}$ be defined by $p(t,\ x) = \max\{\alpha(t),\ \min\{x,\ \beta(t)\}\},$

$\forall\ (t,\ x) \in I \times \mathbb{R}$. Since $\alpha(t) \leq \max\{\alpha(t),\ \{x,\ \beta(t)\}\}$, we have

$\alpha(t) \leq p(t,\ x),\ \forall\ (t,\ x) \in I \times \mathbb{R}$. Also $\alpha(t) \leq \beta(t),\ \forall\ t \in I$ as

$\alpha(t) \leq \beta(t),\ \forall\ t \in [-r,\ T].$

Now $\min\{x,\ \beta(t)\} \leq \beta(t),\ \forall\ (t,\ x) \in I \times \mathbb{R}$ implies that

$\max\{\alpha(t),\ \min\{x,\ \beta(t)\}\} \leq \beta(t),\ \forall\ (t,\ x) \in I \times \mathbb{R}.$

So we have $p(t,\ x) \leq \beta(t),\ \forall\ (t,\ x) \in I \times \mathbb{R}.$

Therefore $\alpha(t) \leq p(t,\ x) \leq \beta(t),\ \forall\ (t,\ x) \in I \times \mathbb{R}.$

Hence $p : I \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded.

Let $F^\star : I \times \mathbb{R} \to \mathbb{R}$ be defined by

$$F^\star(t,\ y) = \begin{cases} f(t,\ \alpha(t-r)), & y \le \alpha(t-r), \\ f(t,\ y), & \alpha(t-r) \le y \le \beta(t-r), \\ f(t,\ \beta(t-r)), & y \ge \beta(t-r). \end{cases}$$

Then $F^\star : I \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded.

Let $L : X \to Y \times \mathbb{R}$ be defined by $L[y(t)] = (u(t),\ y(0))$, where

$$u(t) = \begin{cases} 0, & -r \le t \le 0, \\ y(t) - y(0) + \lambda \int\limits_0^t y(s)ds, & 0 \le t \le T. \end{cases}$$

By Lemma 3.1.1, $L : X \to Y \times \mathbb{R}$ is linear, bijective and continuous.

Also $L^{-1} : Y \times \mathbb{R} \to X$ exists and is given by

$$L^{-1}(u(t),\ \gamma) = \begin{cases} \gamma, & -r \le t \le 0, \\ \gamma e^{-\lambda t} + \int\limits_0^t e^{-\lambda(t-s)} u'(s)ds, & 0 \le t \le T. \end{cases}$$

Further $L^{-1} : Y \times \mathbb{R} \to X$ is bounded and hence continuous.

Let $N : X \to Y \times \mathbb{R}$ be defined as $N[y(t)] = (v(t),\ p(T,\ y(T)))$,

$$\text{where } v(t) = \begin{cases} 0, & -r \le t \le 0, \\ \int\limits_0^t F^\star(s,\ y(s-r))ds, & 0 \le t \le T. \end{cases}$$

Then $N : X \to Y \times \mathbb{R}$ is compact and continuous.

Let $S = L^{-1}N : X \to X$. Then

$$S[y(t)] = \begin{cases} p(T,\ y(T)), & -r \le t \le 0, \\ p(T,\ y(T))e^{-\lambda t} + \int\limits_0^t e^{-\lambda(t-s)} F^\star(s,\ y(s-r))ds, & 0 \le t \le T. \end{cases}$$

A fixed point of $S : X \to X$ is a solution of

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = F^*(t,\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = p(T,\ y(T)). \end{aligned}\right\} \tag{3.10}$$

Since $L^{-1} : Y \times \mathbb{R} \to X$ is continuous linear operator and $N : X \to Y \times \mathbb{R}$ is compact, $S : X \to X$ is a compact operator. Because composition of continuous operators is continuous, we have $S : X \to X$ to be continuous. Further p and $F^\star$ are bounded implies that $S(X)$ is a bounded subset of $(X,\ ||.||_\infty)$. Therefore by Schauder's fixed point theorem, $S : X \to X$ has a fixed point $y \in X$. Thus y is a solution of the BVP (3.10).

We show that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$.

Since $\alpha(0) \leq \alpha(T) \leq p(T,\ y(T)) \leq \beta(T) \leq \beta(0)$, we have

$\alpha(0) \leq p(T,\ y(T)) \leq \beta(0)$. This implies $\alpha(0) \leq y(0) \leq \beta(0)$.

Therefore $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$.

Hence it is enough to show that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in (0,\ T]$.

Let $\epsilon > 0$ be arbitrary.

Define $\alpha_\epsilon(t) = \alpha(t) - \epsilon(t + 2r),\ \forall\ t \in [-r,\ T]$ and

$$\beta_\epsilon(t) = \beta(t) + \epsilon(t + 2r),\ \forall\ t \in [-r,\ T].$$

Therefore $\alpha_\epsilon(t) < \alpha(t),\ \forall\ t \in [-r,\ T]$ and $\beta_\epsilon(t) > \beta(t),\ \forall\ t \in [-r,\ T]$.

Also $\alpha'_\epsilon(t) = \alpha'(t) - \epsilon,\ \forall\ t \in I$ and $\beta'_\epsilon(t) = \beta'(t) + \epsilon,\ \forall\ t \in I$.

This implies $\alpha'_\epsilon(t) < \alpha'(t),\ \forall\ t \in I$ and $\beta'_\epsilon(t) > \beta'(t),\ \forall\ t \in I$.

Since $\alpha(0) \leq y(0) \leq \beta(0)$, we have $\alpha_\epsilon(0) < \alpha(0) \leq y(0) \leq \beta(0) < \beta_\epsilon(0)$.

So we get $\alpha_\epsilon(0) < y(0) < \beta_\epsilon(0)$.

Therefore we prove that $\alpha_\epsilon(t) < y(t) < \beta_\epsilon(t),\ \forall\ t \in (0,\ T]$.

Let if possible $\exists\ t_1\ \in (0,\ T]$ such that $\alpha_\epsilon(t) < y(t) < \beta_\epsilon(t)$,

$\forall\ t \in (0,\ t_1)$ and $y(t_1) = \beta_\epsilon(t_1)$. Therefore $y(t_1) > \beta(t_1)$.

Also $y(t_1 - h) < \beta_\epsilon(t_1 - h)$ for sufficiently small $h > 0$.

This implies $\frac{\beta_\epsilon(t_1)-\beta_\epsilon(t_1-h)}{h} < \frac{y(t_1)-y(t_1-h)}{h}$. Therefore $\beta'_\epsilon(t_1) \leq y'(t_1)$.

CASE-1: Let $0 < t_1 \leq r$.

So we have $-r < t_1 - r \leq 0$.

This implies $\alpha(t_1 - r) \leq y(t_1 - r) \leq \beta(t_1 - r)$.

$$\begin{aligned}\text{Therefore } y'(t_1) &= F^\star(t_1,\ y(t_1 - r)) - \lambda y(t_1)\\ &= f(t_1,\ y(t_1 - r)) - \lambda y(t_1)\\ &\leq f(t_1,\ \beta(t_1 - r)) - \lambda\beta(t_1)\\ &\leq \beta'(t_1) < \beta'_\epsilon(t_1), \text{ which is a contradiction.}\end{aligned}$$

CASE-2: Let $r < t_1 \leq T$.

So we have $0 < t_1 - r \leq t_1$.

This implies $\alpha_\epsilon(t_1 - r) < y(t_1 - r) < \beta_\epsilon(t_1 - r)$.

CASE-2.1: Let $y(t_1 - r) < \alpha(t_1 - r)$.

$$\begin{aligned}\text{Therefore } y'(t_1) &= F^\star(t_1,\ y(t_1 - r)) - \lambda y(t_1)\\ &= f(t_1,\ \alpha(t_1 - r)) - \lambda y(t_1)\\ &\leq f(t_1,\ \beta(t_1 - r)) - \lambda y(t_1)\\ &< f(t_1,\ \beta(t_1 - r)) - \lambda\beta(t_1)\\ &\leq \beta'(t_1) < \beta'_\epsilon(t_1), \text{ which is a contradiction.}\end{aligned}$$

CASE-2.2: Let $\alpha(t_1 - r) \leq y(t_1 - r) \leq \beta(t_1 - r)$.

$$\begin{aligned}\text{Therefore } y'(t_1) &= F^\star(t_1,\ y(t_1 - r)) - \lambda y(t_1)\\ &= f(t_1,\ y(t_1 - r)) - \lambda y(t_1)\\ &\leq f(t_1,\ \beta(t_1 - r)) - \lambda\beta(t_1)\\ &\leq \beta'(t_1) < \beta'_\epsilon(t_1), \text{ which is a contradiction.}\end{aligned}$$

CASE-2.3: Let $y(t_1 - r) > \beta(t_1 - r)$.

Therefore $y'(t_1) = F^\star(t_1,\ y(t_1 - r)) - \lambda y(t_1)$

$= f(t_1,\ \beta(t_1 - r)) - \lambda y(t_1)$

$< f(t_1,\ \beta(t_1 - r)) - \lambda\beta(t_1)$

$\leq \beta'(t_1) < \beta'_\epsilon(t_1)$, which is a contradiction.

From the above discussed cases we conclude that

$\alpha_\epsilon(t) < y(t) < \beta_\epsilon(t),\ \forall\ t \in I$ and $\forall\ \epsilon > 0$. Therefore by taking $\epsilon \to 0$

we get, $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in I$. This gives $\alpha(t) \leq y(t) \leq \beta(t)$,

$\forall\ t \in [-r,\ T]$. Thus $F^\star(t,\ y(t - r)) = f(t,\ y(t - r)),\ \forall\ t \in I$ and

$p(T,\ y(T)) = y(T)$.

Hence y is a solution of (3.1) such that $\alpha(t) \leq y(t) \leq \beta(t)$,

$\forall\ t \in [-r,\ T]$. □

We shall illustrate theorem 3.4.3 with the following examples.

Example 3.4.1. Consider the following PBVP

$$\left.\begin{aligned} &y'(t) + y(t) = y(t-1),\ t \in [0,\ 2],\\ &y(t) = y(0),\ t \in [-1,\ 0],\\ &y(0) = y(2). \end{aligned}\right\}$$

Here $r = 1$, $T = 2$, $I = [0, 2]$ and $f(t, u) = u,\ \forall\ (t,\ u) \in I \times \mathbb{R}$.

Therefore $f(t,\ y(t - 1)) = y(t - 1)$.

Also f is non-decreasing in the second argument.

Let $\alpha : [-1,\ 2] \to \mathbb{R}$ be defined by $\alpha(t) = -1,\ \forall\ t \in [-1,\ 2]$.

Therefore $\alpha(0) = -1 = \alpha(2)$. Also $\alpha'(t) + \alpha(t) = -1 = \alpha(t - 1)$.

Let $\beta : [-1,\ 2] \to \mathbb{R}$ be defined by $\beta(t) = 1,\ \forall\ t \in [-1,\ 2]$.

Therefore $\beta(0) = 1 = \beta(2)$. Also $\beta'(t) + \beta(t) = 1 = \beta(t - 1)$.

Therefore we have 1. $\alpha'(t) + \alpha(t) = \alpha(t-1),\ t \in [0,\ 2]$.

2. $\beta'(t) + \beta(t) = \beta(t-1),\ t \in [0,\ 2]$.

3. $\alpha(t) = \alpha(0),\ t \in [-1,\ 0]$.

4. $\beta(t) = \beta(0),\ t \in [-1,\ 0]$.

5. $\alpha(t) \leq \beta(t),\ t \in [-1,\ 2]$.

6. $\alpha(0) = \alpha(2)$.

7. $\beta(0) = \beta(2)$.

So we can say that α and β are lower and upper solutions of the given PBVP such that $\alpha(t) \leq \beta(t),\ t \in [-1,\ 2]$.

Now let $y : [-1,\ 2] \to \mathbb{R}$ be defined by $y(t) = 0,\ \forall\ t \in [-1,\ 2]$.

Then y is a solution of the given PBVP such that $\alpha \leq y \leq \beta$ on $[-1,\ 2]$.

Example 3.4.2. Consider PBVP

$$\left.\begin{aligned} &y'(t) + y(t) = (y(t-1)+1)e^{-t},\ t \in [0,\ 1], \\ &y(t) = y(0),\ t \in [-1,\ 0], \\ &y(0) = y(1). \end{aligned}\right\}$$

Here $r = 1$, $T = 1$, $I = [0,\ 1]$ and $f(t,\ u) = (u+1)e^{-t},\ \forall\ (t,\ u) \in I \times \mathbb{R}$.

Therefore $f(t,\ y(t-1)) = (y(t-1)+1)e^{-t}$.

Also f is non-decreasing in the second argument.

Let $\alpha : [-1,\ 1] \to \mathbb{R}$ be defined by $\alpha(t) = \begin{cases} \frac{1-e^{-1}}{e-2}, & -1 \leq t \leq 0, \\ \frac{e^{-t}}{e-2} + \left(\frac{e-1}{e-2}\right) te^{-t} - \frac{e^{-1}}{e-2}, & 0 \leq t \leq 1. \end{cases}$

Therefore $\alpha(0) = \frac{1-e^{-1}}{e-2} = \alpha(1)$.

Let $0 \leq t \leq 1$. This implies $-1 \leq t-1 \leq 0$. Therefore $\alpha(t-1) = \alpha(0)$.

So we have $\alpha'(t) + \alpha(t) = \left(\frac{e-1}{e-2}\right) e^{-t} - \frac{e^{-1}}{e-2}$ and

$$(\alpha(t-1)+1)e^{-t} = \left(\tfrac{e-1}{e-2}\right) e^{-t} - \tfrac{e^{-1}e^{-t}}{e-2}.$$

Thus $\alpha'(t) + \alpha(t) \leq (\alpha(t-1)+1)e^{-t}$.

Let $\beta : [-1,\ 1] \to \mathbb{R}$ be defined by $\beta(t) = \begin{cases} \frac{1+e^{-1}}{e-2}, & -1 \leq t \leq 0, \\ \frac{e^{-t}}{e-2} + \left(\frac{e-1}{e-2}\right) te^{-t} + \frac{e^{-1}}{e-2}, & 0 \leq t \leq 1. \end{cases}$

Therefore $\beta(0) = \frac{1+e^{-1}}{e-2} = \beta(1)$.

Let $0 \leq t \leq 1$. This implies $-1 \leq t-1 \leq 0$. Therefore $\beta(t-1) = \beta(0)$.

So we have $\beta'(t) + \beta(t) = \left(\frac{e-1}{e-2}\right) e^{-t} + \frac{e^{-1}}{e-2}$ and

$$(\beta(t-1)+1)e^{-t} = \left(\frac{e-1}{e-2}\right) e^{-t} + \frac{e^{-1}e^{-t}}{e-2}.$$

Thus $\beta'(t) + \beta(t) \leq (\beta(t-1)+1)e^{-t}$.

Therefore we have 1. $\alpha'(t) + \alpha(t) \leq (\alpha(t-1)+1)e^{-t},\ t \in [0,\ 1]$.

2. $\beta'(t) + \beta(t) \geq (\beta(t-1)+1)e^{-t},\ t \in [0,\ 1]$.
3. $\alpha(t) = \alpha(0),\ t \in [-1,\ 0]$.
4. $\beta(t) = \beta(0),\ t \in [-1,\ 0]$.
5. $\alpha(t) \leq \beta(t),\ t \in [-1,\ 1]$.
6. $\alpha(0) = \alpha(1)$.
7. $\beta(0) = \beta(1)$.

So we can say that α and β are lower and upper solutions of the given PBVP such that $\alpha(t) \leq \beta(t),\ t \in [-1,\ 1]$.

Now let $y : [-1,\ 1] \to \mathbb{R}$ be defined by $y(t) = \begin{cases} \frac{1}{e-2}, & -1 \leq t \leq 0, \\ \frac{e^{-t}}{e-2} + \left(\frac{e-1}{e-2}\right) te^{-t}, & 0 \leq t \leq 1. \end{cases}$

Then y is a solution of the given PBVP such that $\alpha \leq y \leq \beta$ on $[-1,\ 1]$.

Next, we prove a result on existence of solution between strict lower and strict upper solution for the PBVP.

Theorem 3.4.4. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(t,\ x) \leq f(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in I \times \mathbb{R}$ such that $x \leq y$. Let $\alpha,\ \beta \in X$ be strict lower and upper solutions of PBVP (3.1) such that $\alpha(t) \leq \beta(t)$, $\forall\ t \in [-r,\ T]$. Then the PBVP (3.1) has atleast one solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$ and $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are strict lower and upper solutions of PBVP (3.1), we have 1. $\alpha'(t) + \lambda\alpha(t) < f(t,\ \alpha(t-r)),\ t \in I$,

2. $\beta'(t) + \lambda\beta(t) > f(t,\ \beta(t-r)),\ t \in I$,
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$,
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0]$,
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T]$,
6. $\alpha(0) \leq \alpha(T)$,
7. $\beta(0) \geq \beta(T)$.

As $\alpha,\ \beta \in X$, we get $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$. Defining $p,\ F^\star,\ L,\ N$ and S as in the proof of the Theorem 3.4.3 and using Schauder's fixed point theorem, we see that $S : X \to X$ has a fixed point $y \in X$. Therefore y is a solution of (3.10).

Proceeding as in the proof of the Theorem 3.4.3, we get $\alpha(0) \leq y(0) \leq \beta(0)$. Therefore $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$.

Next we show that $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.

Let if possible $\exists\ t_1\ \in (0,\ T]$ such that $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ t_1)$ and $y(t_1) = \beta(t_1)$. This implies that $y(t_1 - h) < \beta(t_1 - h)$ for sufficiently small $h > 0$. So we have $\frac{\beta(t_1)-\beta(t_1-h)}{h} < \frac{y(t_1)-y(t_1-h)}{h}$. Therefore $\beta'(t_1) \leq y'(t_1)$.

CASE-1: Let $0 < t_1 \leq r$. So we have $-r < t_1 - r \leq 0$.

This implies $\alpha(t_1 - r) \leq y(t_1 - r) \leq \beta(t_1 - r)$. Then

$$\begin{aligned} y'(t_1) &= F^\star(t_1,\ y(t_1 - r)) - \lambda y(t_1) \\ &= f(t_1,\ y(t_1 - r)) - \lambda y(t_1) \\ &\leq f(t_1,\ \beta(t_1 - r)) - \lambda\beta(t_1). \end{aligned}$$

Therefore $y'(t_1) < \beta'(t_1)$, which is a contradiction.

CASE-2: Let $r < t_1 \leq T$. So we have, $0 < t_1 - r \leq t_1$.

This implies, $\alpha(t_1 - r) < y(t_1 - r) < \beta(t_1 - r)$. Therefore

$$\begin{aligned} y'(t_1) &= F^{\star}(t_1,\ y(t_1 - r)) - \lambda y(t_1) \\ &= f(t_1,\ y(t_1 - r)) - \lambda y(t_1) \\ &\leq f(t_1,\ \beta(t_1 - r)) - \lambda \beta(t_1) \\ &< \beta'(t_1), \text{ which is a contradiction.} \end{aligned}$$

Thus from CASE-1 and CASE-2, we have $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.

Therefore $F^{\star}(t,\ y(t - r)) = f(t,\ y(t - r)),\ \forall\ t \in I$ and $p(T,\ y(T)) = y(T)$.

Hence y is a solution of the PBVP (3.1) such that $\alpha(t) \leq y(t) \leq \beta(t)$,

$\forall\ t \in [-r,\ 0]$ and $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$. □

3.5 Method of Weakly Coupled Lower and Upper Solutions

In this section, we define weakly coupled coupled lower and upper solutions for APBVP (3.2), LBVP (3.3) and NLBVP (3.4). Again we make use of Schauder's fixed point theorem to estabish theorems on existence of solution between weakly coupled lower and upper solutions for these problems. We further define strict weakly coupled lower and upper solutions APBVP, LBVP and NLBVP to establish results on existence of solution between the two.

Definition 3.5.1. *A pair of functions $\alpha,\ \beta \in B$ is called weakly coupled lower and upper solutions of APBVP* (3.2) *if*

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t-r)),\ t \in I.$
2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \beta(t-r)),\ t \in I.$
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T].$
6. $\alpha(0) \leq -\beta(T).$
7. $\beta(0) \geq -\alpha(T).$

Definition 3.5.2. *A pair of functions $\alpha,\ \beta \in B$ is called weakly coupled lower and upper solutions of LBVP* (3.3) *if*

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t-r)),\ t \in I.$
2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \beta(t-r)),\ t \in I.$
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T].$
6. $\max\{a\alpha(0) + b\alpha(T),\ a\alpha(0) + b\beta(T)\}$
   $\leq 0 \leq \min\{a\beta(0) + b\alpha(T),\ a\beta(0) + b\beta(T)\}.$

Definition 3.5.3. *A pair of functions $\alpha,\ \beta \in B$ is called weakly coupled lower and upper solutions of NLBVP* (3.4) *if*

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t-r)),\ t \in I.$
2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \beta(t-r)),\ t \in I.$
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T].$
6. $\max\{g(\alpha(0),\ \alpha(T)),\ g(\alpha(0),\ \beta(T))\} \leq 0 \leq \min\{g(\beta(0),\ \alpha(T)),\ g(\beta(0),\ \beta(T))\}.$

Definition 3.5.4. *A pair of weakly coupled lower and upper solutions $\alpha,\ \beta \in B$ of APBVP* (3.2)*, is called strict weakly coupled lower and upper solutions of APBVP* (3.2) *if 1.* $\alpha'(t) + \lambda\alpha(t) < f(t,\ \alpha(t-r)),\ t \in I.$

$$2.\ \beta'(t) + \lambda\beta(t) > f(t,\ \beta(t-r)),\ t \in I.$$

Similarly, we may define strict weakly coupled lower and upper solutions for LBVP (3.3) *and NLBVP* (3.4)*.*

We now prove a result on existence of solution between weakly coupled lower and upper solutions for APBVP (3.2), LBVP (3.3) and NLBVP (3.4).

Theorem 3.5.1. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(t,\ x) \leq f(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in I \times \mathbb{R}$ such that $x \leq y$. Let $\alpha,\ \beta \in X$ be weakly coupled lower and upper solutions of APBVP* (3.2) *Then the APBVP* (3.2) *has atleast one solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t)$, $\forall\ t \in [-r,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are weakly coupled lower and upper solutions of APBVP (3.2), we have 1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t-r)),\ t \in I.$

2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \beta(t-r)),\ t \in I.$
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T].$
6. $\alpha(0) \leq -\beta(T).$
7. $\beta(0) \geq -\alpha(T).$

As $\alpha,\ \beta \in X$, we get $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$. Let $p,\ F^\star$ and L be as defined in the proof of the Theorem 3.4.3.

Let $N : X \to Y \times \mathbb{R}$ be defined by $N[y(t)] = (v(t),\ -p(T,\ y(T)))$,

$$\text{where } v(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ \int\limits_0^t F^\star(s,\ y(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

Therefore $N : X \to Y \times \mathbb{R}$ is compact and continuous.

Let $S = L^{-1}N : X \to X$. Then

$$S[y(t)] = \begin{cases} -p(T,\ y(T)), & -r \le t \le 0, \\ -p(T,\ y(T))e^{-\lambda t} + \int\limits_0^t e^{-\lambda(t-s)}F^{\star}(s,\ y(s-r))ds;, & 0 \le t \le T. \end{cases}$$

By Schauder's fixed point theorem, $S : X \to X$ has a fixed point $y \in X$.

A fixed point of $S : X \to X$ is a solution of

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = F^{*}(t,\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = -p(T,\ y(T)). \end{aligned}\right\} \tag{3.11}$$

Therefore y is a solution of the BVP (3.11).

Since $-\beta(0) \le \alpha(T) \le p(T,\ y(T)) \le \beta(T) \le -\alpha(0)$, we have

$-\beta(0) \le p(T,\ y(T)) \le -\alpha(0)$. This implies $\alpha(0) \le -p(T,\ y(T)) \le \beta(0)$.

Therefore $\alpha(0) \le y(0) \le \beta(0)$. Thus $\alpha(t) \le y(t) \le \beta(t),\ \forall\ t \in [-r,\ 0]$.

The remaining part of the proof is similar to the proof of the Theorem 3.4.3. □

We illustrate theorem 3.5.1 with the following example.

Example 3.4.3. Consider the following APBVP

$$\left.\begin{aligned} &y'(t) + y(t) = y(t-1) + 1,\ t \in [0,\ 1], \\ &y(t) = y(0),\ t \in [-1,\ 0], \\ &y(0) = -y(1). \end{aligned}\right\}$$

Here $r = 1$, $T = 1$, $I = [0,\ 1]$ and $f(t,\ u) = u + 1,\ \forall\ (t,\ u) \in I \times \mathbb{R}$.

Therefore $f(t,\ y(t-1)) = y(t-1) + 1$.

Also f is non-decreasing in the second argument.

Let $\alpha : [-1,\ 1] \to \mathbb{R}$ be defined as $\alpha(t) = \begin{cases} \frac{e^{-1}}{2} - 1, & -1 \le t \le 0, \\ \frac{e^{-1}}{2} - e^{-t}, & 0 \le t \le 1. \end{cases}$

Therefore $\alpha(0) = \frac{e^{-1}}{2} - 1$ and $\alpha(1) = -\frac{e^{-1}}{2}$.

Let $0 \le t \le 1$. This implies $-1 \le t - 1 \le 0$.

Therefore $\alpha(t-1) = \alpha(0) = \frac{e^{-1}}{2} - 1$.

Thus $\alpha'(t) + \alpha(t) = \frac{e^{-1}}{2} = \alpha(t-1) + 1$.

Let $\beta : [-1, 1] \to \mathbb{R}$ be defined as $\beta(t) = \begin{cases} \frac{e^{-1}}{2}, & -1 \le t \le 0, \\ \frac{e^{-1}}{2} + 1 - e^{-t}, & 0 \le t \le 1. \end{cases}$

Therefore $\beta(0) = \frac{e^{-1}}{2}$ and $\beta(1) = \frac{e^{-1}}{2} - 1$.

Let $0 \le t \le 1$. This implies $-1 \le t - 1 \le 0$.

Therefore $\beta(t-1) = \beta(0) = \frac{e^{-1}}{2}$.

Thus $\beta'(t) + \beta(t) = \frac{e^{-1}}{2} + 1 = \beta(t-1) + 1$.

Therefore we have 1. $\alpha'(t) + \alpha(t) = \alpha(t-1) + 1,\ t \in [0,\ 1]$,

2. $\beta'(t) + \beta(t) = \beta(t-1) + 1,\ t \in [0,\ 1]$,
3. $\alpha(t) = \alpha(0),\ t \in [-1,\ 0]$,
4. $\beta(t) = \beta(0),\ t \in [-1,\ 0]$,
5. $\alpha(t) \le \beta(t),\ t \in [-1,\ 1]$,
6. $\alpha(0) = -\beta(1)$,
7. $\beta(0) = -\alpha(1)$.

So we can say that α and β are weakly coupled lower and upper solutions of the given APBVP.

Now let $y : [-1,\ 1] \to \mathbb{R}$ be defined as $y(t) = \begin{cases} \frac{e^{-1}-1}{2}, & -1 \le t \le 0, \\ \\ \frac{e^{-1}+1}{2} - e^{-t}, & 0 \le t \le 1. \end{cases}$

Then y is a solution of the given APBVP such that $\alpha \le y \le \beta$ on $[-1,\ 1]$.

Next, we prove the existence theorem for the LBVP between weakly coupled lower and upper solutions.

Theorem 3.5.2. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(t,\ x) \le f(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in I \times \mathbb{R}$ such that $x \le y$. Let $\alpha,\ \beta \in X$ be weakly coupled lower and upper solutions of LBVP (3.3). Then the LBVP (3.3) has atleast one solution $y \in X$ such that $\alpha(t) \le y(t) \le \beta(t),\ \forall\ t \in [-r,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are weakly coupled lower and upper solutions of LBVP (3.3), we have 1. $\alpha'(t) + \lambda\alpha(t) \le f(t,\ \alpha(t-r)),\ t \in I$.

2. $\beta'(t) + \lambda\beta(t) \ge f(t,\ \beta(t-r)),\ t \in I$.
3. $\alpha(t) \le \alpha(0),\ t \in [-r,\ 0]$.
4. $\beta(t) \ge \beta(0),\ t \in [-r,\ 0]$.
5. $\alpha(t) \le \beta(t),\ t \in [-r,\ T]$.
6. $\max\{a\alpha(0) + b\alpha(T),\ a\alpha(0) + b\beta(T)\} \le 0$
   $\le \min\{a\beta(0) + b\alpha(T),\ a\beta(0) + b\beta(T)\}$.

As $\alpha,\ \beta \in X$, we get $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$.
Let p, $F^\star$ and L be as defined in the proof of the Theorem 3.4.3.
Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $g(x,\ y) = ax + by,\ \forall\ (x,\ y) \in \mathbb{R}^2$.
Therefore $g : \mathbb{R}^2 \to \mathbb{R}$ is continuous and $g(y(0),\ y(T)) = ay(0) + by(T)$.
Let $g^\star : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$g^\star(x,\ y) = p(0,\ x) - g[p(0,\ x),\ p(T,\ y)],\ \forall\ (x,\ y) \in \mathbb{R}^2.$$
$$= p(0,\ x) - ap(0,\ x) - bp(T,\ y),\ \forall\ (x,\ y) \in \mathbb{R}^2.$$

Therefore $g^\star(x,\ y) = (1-a)p(0,\ x) - bp(T,\ y),\ \forall\ (x,\ y) \in \mathbb{R}^2$.

Since $\alpha(0) \leq p(0,\ x) \leq \beta(0),\ \forall\ x \in \mathbb{R}$, we have

$|p(0,\ x)| \leq K_1 = \max\{|\alpha(0)| + 1,\ |\beta(0)|\},\ \forall\ x \in \mathbb{R}$.

Also $\alpha(T) \leq p(T,\ y) \leq \beta(T),\ \forall\ y \in \mathbb{R}$ implies that

$|p(T,\ y)| \leq K_2 = \max\{|\alpha(T)| + 1,\ |\beta(T)|\},\ \forall\ y \in \mathbb{R}$.

Therefore $|g^\star(x,\ y)| \leq |1-a||p(0,\ x)| + |b||p(T,\ y)|,\ \forall\ (x,\ y) \in \mathbb{R}^2$.

$$\leq |1-a|K_1 + |b|K_2,\ \forall\ (x,\ y) \in \mathbb{R}^2.$$

Therefore $g^\star : \mathbb{R}^2 \to \mathbb{R}$ is bounded. Also $g^\star : \mathbb{R}^2 \to \mathbb{R}$ is continuous, as p and g are continuous.

Let $N : X \to Y \times \mathbb{R}$ be defined as $N[y(t)] = [v(t),\ g^\star(y(0),\ y(T))]$,

where $v(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ \int\limits_0^t F^\star(s,\ y(s-r))ds, & 0 \leq t \leq T. \end{cases}$

Then by Lemma 3.1.6, N is compact and continuous.

Let $S = L^{-1}N : X \to X$. Then

$$S[y(t)] = \begin{cases} g^\star(y(0), y(T)), & -r \leq t \leq 0, \\ g^\star(y(0), y(T))e^{-\lambda t} + \int\limits_0^t e^{-\lambda(t-s)}F^\star(s,\ y(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

A fixed point of $S : X \to X$ is a solution of

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= F^\star(t,\ y(t-r)),\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= g^\star(y(0),\ y(T)). \end{aligned}\right\} \tag{3.12}$$

Further $S : X \to X$ is compact and continuous. Therefore by Schauder's fixed point theorem, $S : X \to X$ has a fixed point $y \in X$.

Thus y is a solution of the BVP (3.12).

We first show that $\alpha(0) \leq y(0) \leq \beta(0)$.

Let if possible $y(0) > \beta(0)$.

CASE-I:- Let $b > 0$.

Therefore $\max\{a\alpha(0) + b\alpha(T),\ a\alpha(0) + b\beta(T)\} = a\alpha(0) + b\beta(T)$ and

$$\min\{a\beta(0) + b\alpha(T),\ a\beta(0) + b\beta(T)\} = a\beta(0) + b\alpha(T).$$

This implies that $a\alpha(0) + b\beta(T) \leq 0 \leq a\beta(0) + b\alpha(T)$.

So we get

$p(0,\ y(0)) = \max\{\alpha(0),\ \min\{y(0),\ \beta(0)\}\} = \max\{\alpha(0),\ \beta(0)\} = \beta(0)$.

Since $\alpha(T) \leq p(T,\ y(T))$, we have $b\alpha(T) \leq bp(T,\ y(T))$.

This gives $0 \leq a\beta(0) + b\alpha(T) \leq a\beta(0) + bp(T,\ y(T))$. Therefore

$$\begin{aligned} y(0) &= g^{\star}(y(0),\ y(T)) \\ &= p(0,\ y(0)) - ap(0,\ y(0)) - bp(T,\ y(T)) \\ &= \beta(0) - a\beta(0) - bp(T,\ y(T)) \\ &= \beta(0) - [a\beta(0) + bp(T,\ y(T))] \leq \beta(0). \end{aligned}$$

Thus $y(0) \leq \beta(0)$ is a contradiction as $y(0) > \beta(0)$.

CASE-II:- Let $b < 0$.

Therefore $\max\{a\alpha(0) + b\alpha(T),\ a\alpha(0) + b\beta(T)\} = a\alpha(0) + b\alpha(T)$ and

$$\min\{a\beta(0) + b\alpha(T),\ a\beta(0) + b\beta(T)\} = a\beta(0) + b\beta(T).$$

So we get $a\alpha(0) + b\alpha(T) \leq 0 \leq a\beta(0) + b\beta(T)$. This implies

$p(0,\ y(0)) = \max\{\alpha(0),\ \min\{y(0),\ \beta(0)\}\} = \max\{\alpha(0),\ \beta(0)\} = \beta(0)$.

Since $p(T,\ y(T)) \leq \beta(T)$, we have $b\beta(T) \leq bp(T,\ y(T))$.

This gives $0 \leq a\beta(0) + b\beta(T) \leq a\beta(0) + bp(T,\ y(T))$. Therefore

$$\begin{aligned} y(0) &= g^{\star}(y(0),\ y(T)) \\ &= p(0,\ y(0)) - ap(0,\ y(0)) - bp(T,\ y(T)) \\ &= \beta(0) - a\beta(0) - bp(T,\ y(T)). \end{aligned}$$

So we have $y(0) = \beta(0) - [a\beta(0) + bp(T,\ y(T))] \leq \beta(0)$.

Thus $y(0) \leq \beta(0)$ is a contradiction as $y(0) > \beta(0)$.

Therefore from CASE-I and CASE-II, we have $y(0) \leq \beta(0)$.

Similarly, it can be proved that $\alpha(0) \leq y(0)$. So we have $\alpha(0) \leq y(0) \leq \beta(0)$.

The remaining part of proof is similar to the proof of the Theorem 3.4.3. $\square$

Further, we prove the existence theorem for the NLBVP between weakly coupled lower and upper solutions.

Theorem 3.5.3. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy*
$f(t,\ x) \leq f(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in I \times \mathbb{R}$ *such that* $x \leq y$. *Let* $g : \mathbb{R}^2 \to \mathbb{R}$ *be continuous function which satisfies*

(1) $g(t,\ x) \leq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ *such that* $x \leq y$ *or*

(2) $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ *such that* $x \leq y$.

Let $\alpha,\ \beta \in X$ *be weakly coupled lower and upper solutions of NLBVP* (3.4). *Then the NLBVP* (3.4) *has atleast one solution* $y \in X$ *such that*
$\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$.

Proof. Without the loss of generality, let $g(t,\ x) \leq g(t,\ y)$,
$\forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$.

Since $\alpha,\ \beta \in X$ are weakly coupled lower and upper solutions of NLBVP (3.4), we have 1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t-r)),\ t \in I$,

2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \beta(t-r)),\ t \in I$,

3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$,

4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0]$,

5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T]$,

6. $\max\{g(\alpha(0),\ \alpha(T)),\ g(\alpha(0),\ \beta(T))\}$
$\leq 0 \leq \min\{g(\beta(0),\ \alpha(T)),\ g(\beta(0),\ \beta(T))\}$.

Since $\alpha,\ \beta \in X$, we get $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0)$,
$\forall\ t \in [-r,\ 0]$.

As $g(t,\ x) \leq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$,

we have $\max\{g(\alpha(0),\ \alpha(T)),\ g(\alpha(0),\ \beta(T))\} = g(\alpha(0),\ \beta(T))$ and

$$\min\{g(\beta(0),\ \alpha(T)),\ g(\beta(0),\ \beta(T))\} = g(\beta(0),\ \alpha(T)).$$

This implies $g(\alpha(0),\ \beta(T)) \leq 0 \leq g(\beta(0),\ \alpha(T))$.

Let p, $F^\star$ and L be as defined in the proof of the Theorem 3.4.3.

Let $g^\star : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$g^\star(x,\ y) = p(0,\ x) - g[p(0,\ x),\ p(T,\ y)],\ \forall\ (x,\ y) \in \mathbb{R}^2$.

Therefore $g^\star : \mathbb{R}^2 \to \mathbb{R}$ is continuous and bounded.

Let $N : X \to Y \times \mathbb{R}$ be defined by $N[y(t)] = [v(t),\ g^\star(y(0),\ y(T))]$,

$$\text{where } v(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ \int\limits_0^t F^\star(s,\ y(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

Then N is compact and continuous, by Lemma 3.1.6.

Let $S = L^{-1}N : X \to X$. Then

$$S[y(t)] = \begin{cases} g^\star(y(0), y(T)), & -r \leq t \leq 0, \\ g^\star(y(0), y(T))e^{-\lambda t} + \int\limits_0^t e^{-\lambda(t-s)} F^\star(s,\ y(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

A fixed point of $S : X \to X$ is a solution of

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= F^\star(t,\ y(t-r)),\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= g^\star(y(0),\ y(T)). \end{aligned}\right\} \tag{3.13}$$

By Schauder's fixed point theorem, $S : X \to X$ has a fixed point $y \in X$. Therefore y is a solution of the BVP (3.13).

We first prove that $\alpha(0) \leq y(0) \leq \beta(0)$.

Let if possible $y(0) > \beta(0)$. This implies

$p(0,\ y(0)) = \max\{\alpha(0),\ \min\{y(0),\ \beta(0)\}\} = \max\{\alpha(0),\ \beta(0)\} = \beta(0)$.

$$\begin{aligned}\text{Therefore } y(0) &= g^{\star}(y(0),\ y(T)) \\ &= p(0,\ y(0)) - g[p(0,\ y(0)),\ p(T,\ y(T))] \\ &= \beta(0) - g(\beta(0),\ \alpha(T)) \\ &\leq \beta(0).\end{aligned}$$

Thus $y(0) \leq \beta(0)$ is a contradiction to $y(0) > \beta(0)$.

Therefore $y(0) \leq \beta(0)$. Similarly, it can be proved that $\alpha(0) \leq y(0)$.

Hence $\alpha(0) \leq y(0) \leq \beta(0)$.

The remaining part of the proof is similar to the proof of the Theorem 3.4.3. □

Next, we prove a result on existence of solution theorem for strict weakly coupled lower and upper solution for APBVP (3.2), LBVP (3.3) and NLBVP (3.4).

Theorem 3.5.4. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(t,\ x) \leq f(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in I \times \mathbb{R}$ such that $x \leq y$. Let $\alpha,\ \beta \in X$ be strict weakly coupled lower and upper solutions of APBVP (3.2). Then the APBVP (3.2) has atleast one solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$ and $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are strict weakly coupled lower and upper solutions of APBVP (3.2), we have 1. $\alpha'(t) + \lambda\alpha(t) < f(t,\ \alpha(t-r)),\ t \in I$,

2. $\beta'(t) + \lambda\beta(t) > f(t,\ \beta(t-r)),\ t \in I$,
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$,
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0]$,
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T]$,
6. $\alpha(0) \leq -\beta(T)$,
7. $\beta(0) \geq -\alpha(T)$.

As $\alpha,\ \beta \in X$, we get $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$. Defining $p,\ F^{\star},\ L,\ N$ and S as in the proof of the Theorem 3.5.1 and using

Schauder's fixed point theorem, we see that $S : X \to X$ has a fixed point $y \in X$. This will imply that y is a solution of the BVP (3.11). Proceding as in proof of the Theorem 3.5.1, we obtain $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$. The remaining part of the proof is similar to the proof of the Theorem 3.4.4. □

Theorem 3.5.5. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(t,\ x) \leq f(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in I \times \mathbb{R}$ such that $x \leq y$. Let $\alpha,\ \beta \in X$ be strict weakly coupled lower and upper solutions of LBVP (3.3). Then the LBVP (3.3) has atleast one solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$ and $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are strict weakly coupled lower and upper solutions of LBVP (3.3), we have 1. $\alpha'(t) + \lambda\alpha(t) < f(t,\ \alpha(t-r)),\ t \in I$,

2. $\beta'(t) + \lambda\beta(t) > f(t, \beta(t-r)),\ t \in I$,
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$,
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0]$,
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T]$,
6. $\max\{a\alpha(0) + b\alpha(T),\ a\alpha(0) + b\beta(T)\} \leq 0$
   $\leq \min\{a\beta(0) + b\alpha(T),\ a\beta(0) + b\beta(T)\}$.

As $\alpha,\ \beta \in X$, we get $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$. Defining $p,\ F^\star,\ L,\ N$ and S as in the proof of the Theorem 3.5.2 and using Schauder's fixed point theorem, we see that $S : X \to X$ has a fixed point $y \in X$. This implies y is a solution of the BVP (3.12). Proceding as in the proof of the theorem 3.5.2, we obtain $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$. The remaining part of the proof is similar to the proof of the Theorem 3.4.4. □

Theorem 3.5.6. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(t,\ x) \leq f(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in I \times \mathbb{R}$ such that $x \leq y$. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous function which satisfies*

(1) $g(t,\ x) \leq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$ or

(2) $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$.

Let $\alpha,\ \beta \in X$ *be strict weakly coupled lower and upper solutions of NLBVP* (3.4). *Then the NLBVP* (3.4) *has atleast one solution* $y \in X$ *such that* $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$ *and* $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.

Proof. Without the loss of generality, let $g(t,\ x) \leq g(t,\ y)$,
$\forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$.
Since $\alpha,\ \beta \in X$ are strict weakly coupled lower and upper solutions of NLBVP (3.4), we have

1. $\alpha'(t) + \lambda\alpha(t) < f(t,\ \alpha(t-r)),\ t \in I$,
2. $\beta'(t) + \lambda\beta(t) > f(t,\ \beta(t-r)),\ t \in I$,
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$,
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0]$,
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T]$,
6. $\max\{g(\alpha(0),\ \alpha(T)),\ g(\alpha(0),\ \beta(T))\} \leq 0 \leq \min\{g(\beta(0),\ \alpha(T)),\ g(\beta(0),\ \beta(T))\}$.

As $\alpha,\ \beta \in X$, we get $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$.
Defining $p,\ F^\star,\ L,\ N$ and S as in the proof of the Theorem 3.5.3 and using Schauder's fixed point theorem, we see that $S : X \to X$ has a fixed point $y \in X$. This implies y is a solution of (3.13). Proceding as in the proof of the Theorem 3.5.3, we obtain $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$.
The remaining part of the proof is similar to the proof of the Theorem 3.4.4. □

3.6 Method of Coupled Lower and Upper Solutions

In this section, we define coupled lower and upper solutions for PBVP (3.1), APBVP (3.2), LBVP (3.3) and NLBVP (3.4) to estabish few results on existence of solution between coupled lower and upper solutions for these problems. We further define strict coupled lower and upper solutions for the above problems to establish results on existence of solution between the two.

Definition 3.6.1. *A pair of functions $\alpha,\ \beta \in B$ is called coupled lower and upper solutions of PBVP* (3.1) *if*

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \beta(t-r)),\ t \in I.$
2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \alpha(t-r)),\ t \in I.$
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T].$
6. $\alpha(0) \leq \alpha(T).$
7. $\beta(0) \geq \beta(T).$

Definition 3.6.2. *A pair of functions $\alpha,\ \beta \in B$ is called coupled lower and upper solutions of APBVP* (3.2) *if*

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \beta(t-r)),\ t \in I.$
2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \alpha(t-r)),\ t \in I.$
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T].$
6. $\alpha(0) \leq -\beta(T).$
7. $\beta(0) \geq -\alpha(T).$

Definition 3.6.3. *A pair of functions $\alpha,\ \beta \in B$ is called coupled lower and upper solutions of LBVP* (3.3) *if*

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \beta(t-r)),\ t \in I.$
2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \alpha(t-r)),\ t \in I.$
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T].$
6. $\max\{a\alpha(0) + b\alpha(T),\ a\alpha(0) + b\beta(T)\} \leq 0$
   $\leq \min\{a\beta(0) + b\alpha(T),\ a\beta(0) + b\beta(T)\}.$

Definition 3.6.4. *A pair of functions $\alpha,\ \beta \in B$ is called coupled lower and upper solutions of NLBVP* (3.4) *if*

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \beta(t-r)),\ t \in I.$

2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \alpha(t-r)),\ t \in I.$

3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$

4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$

5. $\alpha(t) \leq \beta(t),\ \ t \in [-r,\ T].$

6. $\max\{g(\alpha(0),\ \alpha(T)),\ g(\alpha(0),\ \beta(T))\} \leq 0 \leq \min\{g(\beta(0),\ \alpha(T)),\ g(\beta(0),\ \beta(T))\}.$

Definition 3.6.5. *A pair of coupled lower and upper solutions $\alpha,\ \beta \in B$ of PBVP* (3.1) *is called strict coupled lower and upper solutions of PBVP* (3.1) *if 1.* $\alpha'(t) + \lambda\alpha(t) < f(t,\ \beta(t-r)),\ t \in I.$

2. $\beta'(t) + \lambda\beta(t) > f(t,\ \alpha(t-r)),\ t \in I.$

Similarly we may define strict coupled lower and upper solutions for APBVP, LBVP and NLBVP of first order DECD.

We now prove results on existence of solution between coupled lower and upper solutions for PBVP, APBVP, LBVP and NLBVP of first order DECD.

Theorem 3.6.1. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy* $f(t,\ x) \geq f(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in I \times \mathbb{R}$ *such that $x \leq y$. Let $\alpha,\ \beta \in X$ be coupled lower and upper solutions of PBVP* (3.1)*. Then the PBVP* (3.1) *has atleast one solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are coupled lower and upper solutions of PBVP (3.1), we have 1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \beta(t-r)),\ t \in I,$

2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \alpha(t-r)),\ t \in I,$

3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0],$

4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0],$

5. $\alpha(0) \leq \alpha(T),$

6. $\beta(0) \geq \beta(T).$

Also $\alpha,\ \beta \in X$ implies $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and

$\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$.

Defining $p,\ F^\star,\ L,\ N$ and S as in the proof of the Theorem 3.4.3 and using Schauder's fixed point theorem, we see that $S : X \to X$ has a fixed point $y \in X$. This will imply that y is a solution of the BVP (3.10).

Proceeding as in the proof of the Theorem 3.4.3, we get $\alpha(0) \leq y(0) \leq \beta(0)$.

This implies $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$.

Therefore it is enough to show that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in (0,\ T]$.

Let $\epsilon > 0$ be arbitrary.

Let α_ϵ and β_ϵ be as defined in the proof of theorem 3.4.3.

So we get $\alpha_\epsilon(0) < y(0) < \beta_\epsilon(0)$. Therefore we prove that

$\alpha_\epsilon(t) < y(t) < \beta_\epsilon(t),\ \forall\ t \in (0,\ T]$.

Let if possible $\exists\ t_1\ \in (0,\ T]$ such that $\alpha_\epsilon(t) < y(t) < \beta_\epsilon(t)$,

$\forall\ t \in (0,\ t_1),\ y(t_1) = \beta_\epsilon(t_1)$ and $\beta'_\epsilon(t_1) \leq y'(t_1)$.

CASE-1: Let $0 < t_1 \leq r$. So we have $-r < t_1 - r \leq 0$.

This implies $\alpha(t_1 - r) \leq y(t_1 - r) \leq \beta(t_1 - r)$. Therefore

$$\begin{aligned} y'(t_1) &= F^\star(t_1,\ y(t_1 - r)) - \lambda y(t_1) \\ &= f(t_1,\ y(t_1 - r)) - \lambda y(t_1) \\ &\leq f(t_1,\ \alpha(t_1 - r)) - \lambda\beta(t_1) \\ &\leq \beta'(t_1) < \beta'_\epsilon(t_1), \text{ which is a contradiction.} \end{aligned}$$

CASE-2: Let $r < t_1 \leq T$. So we have $0 < t_1 - r \leq t_1$.

This implies $\alpha_\epsilon(t_1 - r) < y(t_1 - r) < \beta_\epsilon(t_1 - r)$.

CASE-2.1: Let $y(t_1 - r) < \alpha(t_1 - r)$. Then

$$\begin{aligned} y'(t_1) &= F^\star(t_1,\ y(t_1 - r)) - \lambda y(t_1) \\ &= f(t_1,\ \alpha(t_1 - r)) - \lambda y(t_1) \\ &< f(t_1,\ \alpha(t_1 - r)) - \lambda\beta(t_1). \end{aligned}$$

Therefore $y'(t_1) < \beta'_\epsilon(t_1) < \beta'(t_1)$, which is a contradiction

CASE-2.2: Let $\alpha(t_1 - r) \leq y(t_1 - r) \leq \beta(t_1 - r)$.

$$\begin{aligned}\text{Therefore } y'(t_1) &= F^\star(t_1,\ y(t_1 - r)) - \lambda y(t_1)\\ &= f(t_1,\ y(t_1 - r)) - \lambda y(t_1)\\ &\leq f(t_1,\ \alpha(t_1 - r)) - \lambda\beta(t_1)\\ &\leq \beta'(t_1) < \beta'_\epsilon(t_1), \text{ which is a contradiction.}\end{aligned}$$

CASE-2.3: Let $y(t_1 - r) > \beta(t_1 - r)$.

$$\begin{aligned}\text{Therefore } y'(t_1) &= F^\star(t_1,\ y(t_1 - r)) - \lambda y(t_1)\\ &= f(t_1,\ \beta(t_1 - r)) - \lambda y(t_1)\\ &\leq f(t_1,\ \alpha(t_1 - r)) - \lambda\beta(t_1)\\ &\leq \beta'(t_1) < \beta'_\epsilon(t_1), \text{ which is a contradiction.}\end{aligned}$$

Thus from the above discussed cases, we conclude that
$\alpha_\epsilon(t) < y(t) < \beta_\epsilon(t),\ \forall\ t \in I$ and $\forall \epsilon > 0$. Therefore by taking $\epsilon \to 0$, we get $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in I$. This implies $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$. So we have $F^\star(t,\ y(t - r)) = f(t,\ y(t - r)),\ \forall\ t \in I$ and $p(T,\ y(T)) = y(T)$. Hence y is a solution of the PBVP (3.1) such that
$\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$. □

Theorem 3.6.2. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(t,\ x) \geq f(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in I \times \mathbb{R}$ such that $x \leq y$. Let $\alpha,\ \beta \in X$ be coupled lower and upper solutions of APBVP* (3.2)*. Then the APBVP* (3.2) *has atleast one solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are coupled lower and upper solutions of APBVP (3.2), we have 1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \beta(t-r)),\ t \in I,$

2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \alpha(t-r)),\ t \in I,$
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0],$
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0],$
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T],$
6. $\alpha(0) \leq -\beta(T),$
7. $\beta(0) \geq -\alpha(T).$

As $\alpha,\ \beta \in X$, we get $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$. Defining $p,\ F^{\star},\ L,\ N$ and S as in the proof of the Theorem 3.5.1 and using Schauder's fixed point theorem, we see that $S : X \to X$ has a fixed point $y \in X$. Therefore y is a solution of the BVP (3.11). Proceding as in the proof of the Theorem 3.5.1, we obtain $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$.
The remaining part of the proof is similar to the proof of the Theorem 3.6.1. □

Theorem 3.6.3. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(t,\ x) \geq f(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in I \times \mathbb{R}$ such that $x \leq y$. Let $\alpha,\ \beta \in X$ be coupled lower and upper solutions of LBVP* (3.3)*. Then the LBVP* (3.3) *has atleast one solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are coupled lower and upper solutions of LBVP (3.3), we have 1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \beta(t-r)),\ t \in I,$

2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \alpha(t-r)),\ t \in I,$
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0],$
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0],$
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T],$
6. $\max\{a\alpha(0) + b\alpha(T),\ a\alpha(0) + b\beta(T)\} \leq 0$
   $\leq \min\{a\beta(0) + b\alpha(T),\ a\beta(0) + b\beta(T)\}.$

As $\alpha,\ \beta \in X$, we get $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$. Defining $p,\ F^{\star},\ L,\ N$ and S as in the proof of the Theorem 3.5.2 and using

Schauder's fixed point theorem, we see that $S : X \to X$ has a fixed point $y \in X$. This will imply that y is a solution of the BVP (3.12). Proceding as in the proof of the Theorem 3.5.2, we obtain $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$. The remaining part of the proof is similar to the proof the of Theorem 3.6.1. $\square$

Theorem 3.6.4. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(t,\ x) \geq f(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in I \times \mathbb{R}$ such that $x \leq y$. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous function which satisfies*

(1) $g(t,\ x) \leq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$ or

(2) $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$.

Let $\alpha,\ \beta \in X$ be coupled lower and upper solutions of NLBVP (3.4). Then the NLBVP (3.4) has atleast one solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t)$, $\forall\ t \in [-r,\ T]$.

Proof. Without the loss of generality, let $g(t,\ x) \leq g(t,\ y)$, $\forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$.

Since $\alpha,\ \beta \in X$ are coupled lower and upper solutions of NLBVP (3.4), we have

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \beta(t-r)),\ t \in I$,
2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \alpha(t-r)),\ t \in I$,
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$,
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0]$,
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T]$,
6. $\max\{g(\alpha(0),\ \alpha(T)),\ g(\alpha(0),\ \beta(T))\} \leq 0 \leq \min\{g(\beta(0),\ \alpha(T)),\ g(\beta(0),\ \beta(T))\}$.

Also $\alpha,\ \beta \in X$ implies $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$.

Defining $p,\ F^\star,\ L,\ N$ and S as in the proof of the Theorem 3.5.3 and using Schauder's fixed point theorem, we see that $S : X \to X$ has a fixed point $y \in X$. This will imply that y is a solution of the BVP (3.13).

Proceding as in the proof of the Theorem 3.5.3, we obtain

$\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$.

The remaining part of the proof is similar to the proof of the Theorem 3.6.1. $\square$

We further prove results on existence of solution between strict coupled lower and upper solutions for PBVP, APBVP, LBVP and NLBVP of first order DECD.

Theorem 3.6.5. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(t,\ x) \geq f(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in I \times \mathbb{R}$ such that $x \leq y$. Let $\alpha,\ \beta \in X$ be strict coupled lower and upper solutions of PBVP* (3.1)*. Then the PBVP* (3.1) *has atleast one solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$ and $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are strict coupled lower and upper solutions of PBVP (3.1), we have 1. $\alpha'(t) + \lambda\alpha(t) < f(t,\ \beta(t-r)),\ t \in I$,

2. $\beta'(t) + \lambda\beta(t) > f(t,\ \alpha(t-r)),\ t \in I$,
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$,
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0]$,
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T]$,
6. $\alpha(0) \leq \alpha(T)$,
7. $\beta(0) \geq \beta(T)$.

As $\alpha,\ \beta \in X$, we get $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$. Defining $p,\ F^\star,\ L,\ N$ and S as in the proof of the Theorem 3.4.3 and using Schauder's fixed point theorem, we see that $S : X \to X$ has a fixed point $y \in X$. This will imply that y is a solution of the BVP (3.10).

Proceeding as in the proof of the Theorem 3.4.3, we get $\alpha(0) \leq y(0) \leq \beta(0)$.

This implies $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$.

Therefore it is enough show that $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.

Let if possible $\exists\ t_1\ \in (0,\ T]$ such that $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ t_1)$ and $y(t_1) = \beta(t_1)$. This implies that $y(t_1 - h) < \beta(t_1 - h)$ for sufficiently small $h > 0$. Therefore $\frac{\beta(t_1)-\beta(t_1-h)}{h} < \frac{y(t_1)-y(t_1-h)}{h}$. Thus $\beta'(t_1) \leq y'(t_1)$.

CASE-1: Let $0 < t_1 \leq r$. So we get $-r < t_1 - r \leq 0$.

This implies $\alpha(t_1 - r) \leq y(t_1 - r) \leq \beta(t_1 - r)$.

$$\begin{aligned}\text{Therefore } y'(t_1) &= F^\star(t_1,\ y(t_1 - r)) - \lambda y(t_1)\\ &= f(t_1,\ y(t_1 - r)) - \lambda y(t_1)\\ &\leq f(t_1,\ \alpha(t_1 - r)) - \lambda\beta(t_1)\\ &< \beta'(t_1), \text{ which is a contradiction.}\end{aligned}$$

CASE-2: Let $r < t_1 \leq T$. So we get $0 < t_1 - r \leq t_1$.

This implies $\alpha(t_1 - r) < y(t_1 - r) < \beta(t_1 - r)$.

$$\begin{aligned}\text{Therefore } y'(t_1) &= F^\star(t_1,\ y(t_1 - r)) - \lambda y(t_1)\\ &= f(t_1,\ y(t_1 - r)) - \lambda y(t_1)\\ &\leq f(t_1,\ \alpha(t_1 - r)) - \lambda\beta(t_1)\\ &< \beta'(t_1), \text{ which is a contradiction.}\end{aligned}$$

Thus from CASE-1 and CASE-2, we have $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.

Therefore $F^\star(t,\ y(t - r)) = f(t,\ y(t - r)),\ \forall\ t \in I$ and $p(T,\ y(T)) = y(T)$.

Hence y is a solution of the PBVP (3.1) such that $\alpha(t) \leq y(t) \leq \beta(t)$,

$\forall\ t \in [-r,\ 0]$ and $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$. □

Theorem 3.6.6. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(t,\ x) \geq f(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in I \times \mathbb{R}$ such that $x \leq y$. Let $\alpha,\ \beta \in X$ be strict coupled lower and upper solutions of APBVP* (3.2)*. Then the APBVP* (3.2) *has atleast one solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$ and $\alpha(t) < y(t) < \beta(t),\ forall\ t \in (0,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are strict coupled lower and upper solutions of (3.2), we have 1. $\alpha'(t) + \lambda\alpha(t) < f(t,\ \beta(t-r)),\ t \in I$,

2. $\beta'(t) + \lambda\beta(t) > f(t,\ \alpha(t-r)),\ t \in I$,

3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$,

4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0]$,

5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T]$,

6. $\alpha(0) \leq -\beta(T)$,

7. $\beta(0) \geq -\alpha(T)$.

As $\alpha,\ \beta \in X$, we get $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$. Defining $p,\ F^{\star},\ L,\ N$ and S as in the proof of the Theorem 3.5.1 and using Schauder's fixed point theorem, we see that $S : X \to X$ has a fixed point $y \in X$. This will imply that y is a solution of the BVP (3.11). Proceding as in the proof of the Theorem 3.5.1, we obtain $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$. The remaining part of the proof is similar to the proof of the Theorem 3.6.5. □

Theorem 3.6.7. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(t,\ x) \geq f(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in I \times \mathbb{R}$ such that $x \leq y$. Let $\alpha,\ \beta \in X$ be strict coupled lower and upper solutions of LBVP* (3.3)*. Then the LBVP* (3.3) *has atleast one solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$ and $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are strict coupled lower and upper solutions of LBVP (3.3), we have 1. $\alpha'(t) + \lambda\alpha(t) < f(t,\ \beta(t-r)),\ t \in I$,

2. $\beta'(t) + \lambda\beta(t) > f(t,\ \alpha(t-r)),\ t \in I$,

3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$,

4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0]$,

5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T]$,

6. $\max\{a\alpha(0) + b\alpha(T),\ a\alpha(0) + b\beta(T)\} \leq 0$
$\leq \min\{a\beta(0) + b\alpha(T),\ a\beta(0) + b\beta(T)\}$.

As $\alpha,\ \beta \in X$, we get $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$.

Defining $p,\ F^{\star},\ L,\ N$ and S as in the proof of the Theorem 3.5.2 and using Schauder's fixed point theorem, we see that $S : X \to X$ has a fixed point $y \in X$. Then y is a solution of the BVP (3.12). Proceeding as in the proof of the Theorem 3.5.2, we obtain $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$.

The remaining part of the proof is similar to the proof of the Theorem 3.6.5. □

Theorem 3.6.8. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy*

$f(t,\ x) \geq f(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in I \times \mathbb{R}$ *such that* $x \leq y$. *Let* $g : \mathbb{R}^2 \to \mathbb{R}$ *be continuous function which satisfies*

(1) $g(t,\ x) \leq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ *such that* $x \leq y$ *or*

(2) $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ *such that* $x \leq y$.

Let $\alpha,\ \beta \in X$ *be strict coupled lower and upper solutions of NLBVP* (3.4). *Then the NLBVP* (3.4) *has atleast one solution* $y \in X$ *such that*

$\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$ *and* $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.

Proof. Without the loss of generality, let $g(t,\ x) \leq g(t,\ y)$,

$\forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$.

Since $\alpha,\ \beta \in X$ are strict coupled lower and upper solutions of NLBVP (3.4), we have 1. $\alpha'(t) + \lambda\alpha(t) < f(t,\ \beta(t-r)),\ t \in I$,

2. $\beta'(t) + \lambda\beta(t) > f(t,\ \alpha(t-r)),\ t \in I$,

3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$,

4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0]$,

5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T]$,

6. $\max\{g(\alpha(0),\ \alpha(T)),\ g(\alpha(0),\ \beta(T))\}$

$\leq 0 \leq \min\{g(\beta(0),\ \alpha(T)),\ g(\beta(0),\ \beta(T))\}$.

Also $\alpha,\ \beta \in X$ implies $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and

$\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$.

Defining $p,\ F^{\star},\ L,\ N$ and S as in the proof of the Theorem 3.5.3 and using Schauder's fixed point theorem, we see that $S : X \to X$ has a fixed point $y \in X$. This will imply that y is a solution of the BVP (3.13). Proceeding as in

the proof of the Theorem 3.5.3, we obtain $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\, t \in [-r,\ 0]$. The remaining part of the proof is similar to the proof of the Theorem 3.6.5. $\square$

Chapter 4

FIRST ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

4.1 Introduction

In this chapter, we investigate BVPs of general first order DECD and first order DEVDA for existence of solution using Leray - Schauder alternative and Schauder's fixed point theorem. We employ method of lower and upper solution to prove some results on existence of solution between them for PBVP of general first order DECD and first order DEVDA. We also establish results on existence of solution between weakly coupled lower and upper solutions for APBVP, LBVP and NLBVP of general first order DECD and first order DEVDA. Further, we establish results on existence of solution between coupled lower and upper solutions for PBVP, APBVP, LBVP and NLBVP of general first order DECD and first order DEVDA. Apart from this, we use fixed point theorem on partially ordered metric spaces and Banach fixed point theorem to prove results on existence of solution for PBVP of first order DEVDA.

Let $a,\ b \in \mathbb{R}$ such that $a \neq 0$ and $b \neq 0$. Let $T > 0$, $\lambda \geq 0$, $I = [0,\ T]$, $0 < r \leq T$, $\theta : I \to I$ be continuous, $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Then the PBVP for general first order DECD under investigation is given by

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = y(T). \end{aligned}\right\} \tag{4.1}$$

The APBVP for general first order DECD under investigation is

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = -\ y(T). \end{aligned}\right\} \tag{4.2}$$

The LBVP for general first order DECD under investigation is,

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &ay(0) + by(T) = 0. \end{aligned}\right\} \tag{4.3}$$

The NLBVP for general first order DECD under investigation is

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &g(y(0),\ y(T)) = 0. \end{aligned}\right\} \tag{4.4}$$

We also investigate BVP for general first order DECD given by

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = g(y(0),\ y(T)). \end{aligned}\right\} \tag{4.5}$$

The PBVP for first order DEVDA under investigation is given by

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(\theta(t))),\ t \in I, \\ &y(0) = y(T). \end{aligned}\right\} \tag{4.6}$$

The APBVP for first order DEVDA under investigation is

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(\theta(t))),\ t \in I, \\ &y(0) = -\ y(T). \end{aligned}\right\} \tag{4.7}$$

The LBVP for first order DEVDA under investigation is

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(\theta(t))),\ t \in I, \\ &ay(0) + by(T) = 0. \end{aligned}\right\} \tag{4.8}$$

The NLBVP for first order DEVDA under investigation is

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(\theta(t))),\ t \in I, \\ &g(y(0),\ y(T)) = 0. \end{aligned}\right\} \tag{4.9}$$

We will need the following lemmas to prove existence of solution results for the above BVPs.

Lemma 4.1.1. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and $N : X \to Y \times \mathbb{R}$ be defined by $N[y(t)] = (v(t),\ y(T))$, where*

$$v(t) = \begin{cases} 0, & -r \le t \le 0, \\ \int\limits_0^t f(s,\ y(s),\ y(s-r))ds, & 0 \le t \le T. \end{cases}$$

Then $N : X \to Y \times \mathbb{R}$ is continuous and compact.

Proof. Similar to the proof of the Lemma 3.1.2. □

Lemma 4.1.2. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and $N : X \to Y \times \mathbb{R}$ be defined by $N[y(t)] = (v(t),\ -y(T))$, where*

$$v(t) = \begin{cases} 0, & -r \le t \le 0, \\ \int\limits_0^t f(s,\ y(s),\ y(s-r))ds, & 0 \le t \le T. \end{cases}$$

Then $N : X \to Y \times \mathbb{R}$ is continuous and compact.

Proof. Similar to the proof of the Lemma 3.1.2. □

Lemma 4.1.3. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and $N : X \to Y \times \mathbb{R}$ be defined by $N[y(t)] = (v(t),\ -\frac{b}{a}y(T))$, where*

$$v(t) = \begin{cases} 0, & -r \le t \le 0, \\ \int\limits_0^t f(s,\ y(s),\ y(s-r))ds, & 0 \le t \le T. \end{cases}$$

Then $N : X \to Y \times \mathbb{R}$ is continuous and compact.

Proof. Similar to the proof of the Lemma 3.1.2. □

Lemma 4.1.4. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and $p : I \times \mathbb{R} \to \mathbb{R}$ be continuous. Let $N : X \to Y \times \mathbb{R}$ be defined by $N[y(t)] = (v(t),\ p(T,\ y(T)))$,*

$$\text{where } v(t) = \begin{cases} 0, & -r \le t \le 0, \\ \int\limits_0^t f(s,\ y(s),\ y(s-r))ds, & 0 \le t \le T. \end{cases}$$

Then $N : X \to Y \times \mathbb{R}$ is continuous and compact.

Proof. Similar to the proof of the Lemma 3.1.2. □

Lemma 4.1.5. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and $h : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Let $N : X \to Y \times \mathbb{R}$ be defined by $N[y(t)] = (v(t),\ h(y(0),\ y(T)))$, where*

$$v(t) = \begin{cases} 0, & -r \le t \le 0, \\ \int\limits_0^t f(s,\ y(s),\ y(s-r))ds, & 0 \le t \le T. \end{cases}$$

Then $N : X \to Y \times \mathbb{R}$ is continuous and compact.

Proof. Similar to the proof of the Lemma 3.1.2. □

Lemma 4.1.6. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and $Q : W \to V \times \mathbb{R}$ be defined by $Q[y(t)] = (v(t),\ y(T))$, where $v(t) = \int\limits_0^t f(s,\ y(s),\ y(\theta(s)))ds$, $t \in I$. Then $Q : W \to V \times \mathbb{R}$ is continuous and compact.*

Proof. Similar to the proof of the Lemma 3.1.2. □

Lemma 4.1.7. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and $Q : W \to V \times \mathbb{R}$ be defined by $Q[y(t)] = (v(t),\ -y(T))$, where $v(t) = \int\limits_0^t f(s,\ y(s),\ y(\theta(s)))ds$, $t \in I$. Then $Q : W \to V \times \mathbb{R}$ is continuous and compact.*

Proof. Similar to the proof of the Lemma 3.1.2. □

Lemma 4.1.8. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and $Q : W \to V \times \mathbb{R}$ be defined by $Q[y(t)] = (v(t),\ -\frac{b}{a}y(T))$, where $v(t) = \int_0^t f(s,\ y(s),\ y(\theta(s)))ds$, $t \in I$. Then $Q : W \to V \times \mathbb{R}$ is continuous and compact.*

Proof. Similar to the proof of the Lemma 3.1.2. □

Lemma 4.1.9. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, $p : I \times \mathbb{R} \to \mathbb{R}$ be continuous and $Q : W \to V \times \mathbb{R}$ be defined by $Q[y(t)] = (v(t),\ p(T,\ y(T)))$, where $v(t) = \int_0^t f(s,\ y(s),\ y(\theta(s)))ds,\ t \in I$. Then $Q : W \to V \times \mathbb{R}$ is continuous and compact.*

Proof. Similar to the proof of the Lemma 3.1.2. □

Lemma 4.1.10. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, $h : \mathbb{R}^2 \to \mathbb{R}$ be continuous and $Q : W \to V \times \mathbb{R}$ be defined by $Q[y(t)] = (v(t),\ h(y(0),\ y(T)))$, where $v(t) = \int_0^t f(s,\ y(s),\ y(\theta(s)))ds,\ t \in I$. Then $Q : W \to V \times \mathbb{R}$ is continuous and compact.*

Proof. Similar to the proof of the Lemma 3.1.2. □

4.2 Existence Theorems using Leray - Schauder Alternative

In this section, we estabish results on existence of solution to above BVPs of using Leray - Schauder alternative. We start with definition of solution of PBVP (4.1).

Definition 4.2.1. *A function* $u : [-r,\ T] \to \mathbb{R}$ *is said to be a solution of PBVP* (4.1) *if*

1. u *is continuous on* $[-r,\ T]$ *and continuously differentiable on* I*.*

2. $u'(t) + \lambda u(t) = f(t,\ u(t),\ u(t-r)),\ t \in I$*.*

3. $u(t) = u(0),\ t \in [-r,\ 0]$*.*

4. $u(0) = u(T)$*.*

Similarly we can give definitions of solution for other given BVPs of general first order DECD.

Definition 4.2.2. *A function* $u : I \to \mathbb{R}$ *is said to be a solution of PBVP* (4.6) *if 1.* u *is continuously differentiable on* I*.*

2. $u'(t) + \lambda u(t) = f(t,\ u(t),\ u(\theta(t))),\ t \in I$*.*

3. $u(0) = u(T)$*.*

Similarly we can give definitions of solution for other given BVPs of first order DEVDA.

We now have the following existence results.

Theorem 4.2.1. *Let* $f : I \times \mathbb{R}^2 \to \mathbb{R}$ *be continuous. If* $\exists\ M > 0$ *independent of* μ *with* $||y||_\infty \neq M$ *for any solution* y *of*

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = \mu f(t,\ y(t),\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = \mu y(T), \end{aligned}\right\} \tag{4.10}$$

for each $\mu \in (0,\ 1)$*, then the PBVP* (4.1) *has atleast one solution in* X*.*

Proof. Similar to the proof of the Theorem 3.2.1. □

Theorem 4.2.2. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. If $\exists\ M > 0$ independent of μ with $||y||_\infty \neq M$ for any solution y of*

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = \mu f(t,\ y(t),\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = -\mu y(T), \end{aligned}\right\} \tag{4.11}$$

for each $\mu \in (0,\ 1)$, then the APBVP (4.2) has atleast one solution in X.

Proof. Similar to the proof of the Theorem 3.2.2. □

Theorem 4.2.3. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. If $\exists\ M > 0$ independent of μ with $||y||_\infty \neq M$ for any solution y of*

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = \mu f(t,\ y(t),\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = -\mu \frac{b}{a} y(T), \end{aligned}\right\} \tag{4.12}$$

for each $\mu \in (0,\ 1)$, then the LBVP (4.3) has atleast one solution in X.

Proof. Similar to the proof of the Theorem 3.2.3. □

Theorem 4.2.4. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. If $\exists\ M > 0$ independent of μ with $||y||_\infty \neq M$ for any solution y of*

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = \mu f(t,\ y(t),\ y(\theta(t))),\ t \in I, \\ &y(0) = \mu y(T), \end{aligned}\right\} \tag{4.13}$$

for each $\mu \in (0,\ 1)$, then the PBVP (4.6) has atleast one solution in X.

Proof. Similar to the proof of the Theorem 3.2.1. □

Theorem 4.2.5. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. If $\exists\, M > 0$ independent of μ with $||y||_\infty \neq M$ for any solution y of*

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = \mu f(t,\ y(t),\ y(\theta(t))),\ t \in I, \\ &y(0) = -\ \mu y(T), \end{aligned}\right\} \tag{4.14}$$

for each $\mu \in (0,\ 1)$, then the APBVP (4.7) has atleast one solution in X.

Proof. Similar to the proof of the Theorem 3.2.2. □

Theorem 4.2.6. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. If $\exists\, M > 0$ independent of μ with $||y||_\infty \neq M$ for any solution y of*

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = \mu f(t,\ y(t),\ y(\theta(t))),\ t \in I, \\ &y(0) = -\ \mu \frac{b}{a} y(T), \end{aligned}\right\} \tag{4.15}$$

for each $\mu \in (0,\ 1)$, then the LBVP (4.8) has atleast one solution in X.

Proof. Similar to the proof of the Theorem 3.2.3. □

4.3 Existence Theorems using Schauder's Fixed Point Theorem

In this section, we first prove the existence of solution for BVP (4.5), by assuming f and g both to be bounded and continuous. We make use of Schauder's fixed point theorem to obtain the result.

Definition 4.3.1. *A function* $u : [-r,\ T] \to \mathbb{R}$ *is said to be a solution of BVP* (4.5) *if*

1. u *is continuous on* $[-r,\ T]$ *and continuously differentiable on* I.

2. $u'(t) = f(t,\ u(t),\ u(t-r)),\ t \in I$.

3. $u(t) = u(0),\ t \in [-r,\ 0]$.

4. $u(0) = g(u(0),\ u(T))$.

Theorem 4.3.1. *Let* $f : I \times \mathbb{R}^2 \to \mathbb{R}$ *be bounded and continuous function,* $g : \mathbb{R}^2 \to \mathbb{R}$ *be bounded and continuous function. Then the BVP* (4.5) *has atleast one solution* $y \in X$.

Proof. Similar to the proof of the Theorem 3.3.1. □

Next we establish existence of solution result for PBVP of first order DE-VDA (4.6) using of Schauder's fixed point theorem by assuming $f : I \times \mathbb{R}^2 \to \mathbb{R}$ to be continuous and bounded. We break down the result into two lemmas.

Lemma 4.3.2. *Let* $\lambda > 0$ *and* $f : I \times \mathbb{R}^2 \to \mathbb{R}$ *be continuous. Then* $J : Z \to Z$ *defined by*
$J[y(t)] = \int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} f(s,\ y(s),\ y(\theta(s)))ds\ +\ \int_0^t e^{-\lambda(t-s)} f(s,\ y(s),\ y(\theta(s)))ds$
is continuous.

Proof. Let $y_n \to y$ in $(Z,\ ||.||)$.

Therefore $||y_n - y|| \to 0$ as $n \to \infty$. This implies $y_n \to y$ uniformly on I.

Let $Jy_n = v_n$ and $Jy = v$.

We claim that $v_n \to v$ uniformly on I.

Let $\epsilon > 0$ be arbitrary.

Since $y_n \to y$ uniformly on I, $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and $\theta : I \to I$ is continuous, we have $f(t,\ y_n(t),\ y_n(\theta(t))) \to f(t,\ y(t),\ y(\theta(t)))$
uniformly on I.

Therefore $\exists\ n_0 \in \mathbb{N}$ such that $|f(t,\ y_n(t),\ y_n(\theta(t))) - f(t,\ y(t),\ y(\theta(t)))| < \lambda\epsilon$, $\forall\ n \geq n_0$ and $\forall\ t \in I$. So we have

$$\begin{aligned}
|v_n(t) - v(t)| = |&\int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} f(s,\ y_n(s),\ y_n(\theta(s)))ds \\
&+ \int_0^t e^{-\lambda(t-s)} f(s,\ y_n(s),\ y_n(\theta(s)))ds \\
&- \int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} f(s,\ y(s),\ y(\theta(s)))ds \\
&- \int_0^t e^{-\lambda(t-s)} f(s,\ y(s),\ y(\theta(s)))ds| \\
\leq &\int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} |f(s,\ y_n(s),\ y_n(\theta(s))) \ -\ f(s,\ y(s),\ y(\theta(s)))|ds \\
&+ \int_0^t e^{-\lambda(t-s)} |f(s,\ y_n(s),\ y_n(\theta(s))) \ -\ f(s,\ y(s),\ y(\theta(s)))|ds \\
< &\lambda\epsilon \int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} ds \ +\ \lambda\epsilon \int_0^t e^{-\lambda(t-s)} ds = \epsilon.
\end{aligned}$$

Therefore $|v_n(t) - v(t)| < \epsilon,\ \forall\ n \geq n_0$ and $\forall\ t \in I$. Thus $v_n \to v$ uniformly on I. Hence $J : Z \to Z$ is a continuous operator. □

Lemma 4.3.3. *Let $\lambda > 0$ and $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and bounded. Let $J : Z \to Z$ defined by*
$J[y(t)] = \int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} f(s,\ y(s),\ y(\theta(s)))ds \ +\ \int_0^t e^{-\lambda(t-s)} f(s,\ y(s),\ y(\theta(s)))ds.$
Then $J(Z)$ is a bounded subset of $(Z,\ \|.\|)$ and $J : Z \to Z$ is compact.

Proof. Since $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is bounded, $\exists\ M > 0$ such that
$|f(t,\ u,\ v)| \leq M,\ \forall\ (t,\ u,\ v) \in I \times \mathbb{R}^2$.

Let $y \in Z$. Then

$$\begin{aligned}
|J[y(t)]| &= |\int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} f(s,\ y(s),\ y(\theta(s)))ds \\
&\quad + \int_0^t e^{-\lambda(t-s)} f(s,\ y(s),\ y(\theta(s)))ds| \\
&\leq \int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} |f(s,\ y(s),\ y(\theta(s)))|ds \\
&\quad + \int_0^t e^{-\lambda(t-s)} |f(s,\ y(s),\ y(\theta(s)))|ds \\
&\leq M \int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} ds \ + \ M \int_0^t e^{-\lambda(t-s)} ds = \frac{M}{\lambda},\ \forall\ t \in I.
\end{aligned}$$

This implies $\|Jy\| \leq \frac{M}{\lambda},\ \forall\ y \in Z$.

Therefore $J(Z)$ is bounded subset of $(Z,\ \|.\|)$.

Next we prove that $J : Z \to Z$ is a compact operator.

Let $\{y_n\}$ be a bounded sequence in $(Z,\ \|.\|)$. Denote $v_n = Jy_n$.

Since $J(Z)$ is a bounded subset of $(Z,\ \|.\|)$, $\{v_n\}$ is bounded sequence in $(Z,\ \|.\|)$. Therefore $\{v_n\}$ is uniformly bounded sequence on I.

We show that $\{v_n\}$ is equicontinuous on I uniformly.

Let $t_0 \in [-r,\ T]$ be arbitrary and $\epsilon > 0$ be arbitrary.

<u>CASE-1</u>: Let $t_0 = 0$. Consider

$$\begin{aligned}
|v_n(t) - v_n(t_0)| &= |v_n(t) - v_n(0)| \\
&\leq |\int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} f(s,\ y_n(s),\ y_n(\theta(s)))ds \\
&\quad + \int_0^t e^{-\lambda(t-s)} f(s,\ y_n(s),\ y_n(\theta(s)))ds \\
&\quad - \int_0^T \frac{e^{-\lambda(T-s)}}{1-e^{-\lambda T}} f(s,\ y_n(s),\ y_n(\theta(s)))ds|.
\end{aligned}$$

So we have

$$|v_n(t) - v_n(t_0)| \leq \int_0^T \frac{e^{-\lambda(T-s)}}{1-e^{-\lambda T}}(1-e^{-\lambda T})|f(s,\ y_n(s),\ y_n(\theta(s)))|ds$$
$$+ \int_0^t e^{-\lambda(t-s)}|f(s,\ y_n(s),\ y_n(\theta(s)))|ds$$
$$\leq \frac{e^{-\lambda T}(1-e^{-\lambda t})}{1-e^{-\lambda T}} M \int_0^T e^{\lambda s}ds \ +\ Me^{-\lambda t}\int_0^t e^{\lambda s}ds$$
$$= \frac{2M}{\lambda}(1-e^{-\lambda t}).$$

CASE-1.1: Let $0 < \frac{\lambda\epsilon}{2M} < 1$.

Let $\frac{2M}{\lambda}|1-e^{-\lambda t}| < \epsilon$. Then $|1-e^{-\lambda t}| < \frac{\lambda\epsilon}{2M}$.

This implies $-\frac{\lambda\epsilon}{2M} < 1-e^{-\lambda t} < \frac{\lambda\epsilon}{2M}$. So we have $1-\frac{\lambda\epsilon}{2M} < e^{-\lambda t} < 1+\frac{\lambda\epsilon}{2M}$.

Therefore $\log[1-\frac{\lambda\epsilon}{2M}] < -\lambda t < \log[1+\frac{\lambda\epsilon}{2M}]$.

Thus $-\frac{1}{\lambda}\log[1+\frac{\lambda\epsilon}{2M}] < t < -\frac{1}{\lambda}\log[1-\frac{\lambda\epsilon}{2M}]$.

Choose $\delta = \min\{\frac{T}{2}, -\frac{1}{\lambda}\log[1-\frac{\lambda\epsilon}{2M}]\}$.

Let $0 \leq t < \delta$.

Then $t < -\frac{1}{\lambda}\log[1-\frac{\lambda\epsilon}{2M}]$. This implies $\log[1-\frac{\lambda\epsilon}{2M}] < -\lambda t$.

So we have $1-\frac{\lambda\epsilon}{2M} < e^{-\lambda t}$. Therefore $1-e^{-\lambda t} < \frac{\lambda\epsilon}{2M}$. Thus $\frac{2M}{\lambda}(1-e^{-\lambda t}) < \epsilon$.

Hence $|v_n(t) - v_n(t_0)| < \epsilon$, $\forall\ t$ such that $0 \leq t < \delta$.

CASE-1.2: Let $\frac{\lambda\epsilon}{2M} \geq 1$. Choose $\delta = \min\{\frac{T}{2}, -\frac{1}{\lambda}\log(\frac{1}{2})\}$.

Let $0 \leq t < \delta$. Then $t < -\frac{1}{\lambda}\log(\frac{1}{2})$.

This implies $\log(\frac{1}{2}) < -\lambda t$. So we have $\frac{1}{2} < e^{-\lambda t}$. This gives $-e^{-\lambda t} < -\frac{1}{2}$.

Therefore $1-e^{-\lambda t} < \frac{1}{2} < \frac{\lambda\epsilon}{2M}$. Thus $\frac{2M}{\lambda}(1-e^{-\lambda t}) < \epsilon$.

Hence $|v_n(t) - v_n(t_0)| < \epsilon$, $\forall\ t$ such that $0 \leq t < \delta$.

From CASE 1.1 and CASE 1.2, we may conclude that $\{v_n\}$ is equicontinuous at $t_0 = 0$.

CASE-2: Let $t_0 = T$. Then for $t \leq T$, we have

$$
\begin{aligned}
|v_n(t) - v_n(t_0)| &= |v_n(t) - v_n(T)| \\
&= |\int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} f(s,\ y_n(s),\ y_n(\theta(s)))ds \\
&\quad + \int_0^t e^{-\lambda(t-s)} f(s,\ y_n(s),\ y_n(\theta(s)))ds \\
&\quad - \int_0^T \frac{e^{-\lambda(2T-s)}}{1-e^{-\lambda T}} f(s, y_n(s), y_n(\theta(s)))ds \\
&\quad - \int_0^T e^{-\lambda(T-s)} f(s,\ y_n(s),\ y_n(\theta(s)))ds| \\
&= |\int_0^T [\frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} - \frac{e^{-\lambda(2T-s)}}{1-e^{-\lambda T}}] f(s, y_n(s), y_n(\theta(s)))ds \\
&\quad + \int_0^t e^{-\lambda(t-s)} f(s,\ y_n(s),\ y_n(\theta(s)))ds \\
&\quad - \int_0^t e^{-\lambda(T-s)} f(s,\ y_n(s),\ y_n(\theta(s)))ds \\
&\quad - \int_t^T e^{-\lambda(T-s)} f(s,\ y_n(s),\ y_n(\theta(s)))ds| \\
&\leq M \int_0^T \frac{e^{-\lambda(T-s)}}{1-e^{-\lambda T}} |e^{-\lambda t} - e^{-\lambda T}|ds \ +\ M \int_0^t e^{\lambda s} |e^{-\lambda t} - e^{-\lambda T}|ds \\
&\quad + \ M \int_t^T e^{-\lambda(T-s)} ds \\
&\leq \frac{M}{\lambda} \frac{e^{-\lambda T}}{1-e^{-\lambda T}} (e^{-\lambda t} - e^{-\lambda T})(e^{\lambda T} - 1) \\
&\quad + \ \frac{M}{\lambda}(e^{-\lambda t} - e^{-\lambda T})(e^{\lambda t} - 1) \ +\ \frac{M}{\lambda} e^{-\lambda T}(e^{\lambda T} - e^{\lambda t}) \\
&= \frac{M}{\lambda}(e^{-\lambda t} - e^{-\lambda T}) \ +\ \frac{M}{\lambda}(1 - e^{-\lambda(T-t)}) \ -\ \frac{M}{\lambda}(e^{-\lambda t} - e^{-\lambda T}) \\
&\quad + \ \frac{M}{\lambda}(1 - e^{-\lambda(T-t)}) \\
&= \frac{2M}{\lambda}(1 - e^{-\lambda(T-t)}).
\end{aligned}
$$

CASE-2.1: Let $0 < \frac{\lambda\epsilon}{2M} < 1$. Choose $\delta = \min\{\frac{T}{2}, -\frac{1}{\lambda}\log[1 - \frac{\lambda\epsilon}{2M}]\}$.

Let $T - \delta \leq t < T$. Then $T - t < -\frac{1}{\lambda}\log[1 - \frac{\lambda\epsilon}{2M}]$.

This implies $\log[1 - \frac{\lambda\epsilon}{2M}] < -\lambda(T - t)$. So we have $1 - \frac{\lambda\epsilon}{2M} < e^{-\lambda(T-t)}$.

Therefore $1 - e^{-\lambda(T-t)} < \frac{\lambda\epsilon}{2M}$. Thus $\frac{2M}{\lambda}(1 - e^{-\lambda(T-t)}) < \epsilon$.

Hence $|v_n(t) - v_n(t_0)| < \epsilon$, $\forall\ t$ such that $T - \delta < t \leq T$.

CASE-2.2: Let $\frac{\lambda\epsilon}{2M} \geq 1$. Choose $\delta = \min\{\frac{T}{2}, -\frac{1}{\lambda}\log(\frac{1}{2})\}$.

Let $T - \delta \leq t < T$. Then $T - t < -\frac{1}{\lambda}\log(\frac{1}{2})$. This implies $\log(\frac{1}{2}) < -\lambda(T - t)$.

So we have $\frac{1}{2} < e^{-\lambda(T-t)}$. This gives $-e^{-\lambda(T-t)} < -\frac{1}{2}$.

Therefore $1 - e^{-\lambda(T-t)} < \frac{1}{2} < \frac{\lambda\epsilon}{2M}$. Thus $\frac{2M}{\lambda}(1 - e^{-\lambda(T-t)}) < \epsilon$.

Hence $|v_n(t) - v_n(t_0)| < \epsilon$, $\forall\ t$ such that $T - \delta < t \leq T$.

From CASE 2.1 and CASE 2.2, we may conclude that $\{v_n\}$ is equicontinuous at $t_0 = T$.

CASE-3: Let $0 < t_0 < T$. Let $t \geq t_0$. Then

$$\begin{aligned}
|v_n(t) - v_n(t_0)| = |&\int_0^T \frac{e^{-\lambda(T-s+t)}}{1 - e^{-\lambda T}} f(s,\ y_n(s),\ y_n(\theta(s)))ds \\
&+ \int_0^t e^{-\lambda(t-s)} f(s,\ y_n(s),\ y_n(\theta(s)))ds \\
&- \int_0^T \frac{e^{-\lambda(T-s+t_0)}}{1 - e^{-\lambda T}} f(s, y_n(s), y_n(\theta(s)))ds \\
&- \int_0^{t_0} e^{-\lambda(t_0-s)} f(s,\ y_n(s),\ y_n(\theta(s)))ds|.
\end{aligned}$$

So we have

$$
\begin{aligned}
|v_n(t)-v_n(t_0)| &= |\int_0^T [\frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} - \frac{e^{-\lambda(T-s+t_0)}}{1-e^{-\lambda T}}] f(s,\ y_n(s),\ y_n(\theta(s)))ds. \\
&\quad + \int_0^{t_0} e^{-\lambda(t-s)} f(s,\ y_n(s),\ y_n(\theta(s)))ds \\
&\quad + \int_{t_0}^{t} e^{-\lambda(t-s)} f(s,\ y_n(s),\ y_n(\theta(s)))ds \\
&\quad - \int_0^{t_0} e^{-\lambda(t_0-s)} f(s,\ y_n(s),\ y_n(\theta(s)))ds| \\
&\le \int_0^T \frac{e^{-\lambda(T-s)}}{1-e^{-\lambda T}} |e^{-\lambda t} - e^{-\lambda t_0}||f(s,\ y_n(s),\ y_n(\theta(s)))|ds \\
&\quad + \int_0^{t_0} e^{\lambda s} |e^{-\lambda t} - e^{-\lambda t_0}||f(s,\ y_n(s),\ y_n(\theta(s)))|ds \\
&\quad + \int_{t_0}^{t} e^{-\lambda(t-s)} |f(s,\ y_n(s),\ y_n(\theta(s)))|ds \\
&\le M \frac{e^{-\lambda T}}{1-e^{-\lambda T}} |e^{-\lambda t} - e^{-\lambda t_0}| \int_0^T e^{\lambda s} ds \ +\ M|e^{-\lambda t} - e^{-\lambda t_0}| \int_0^{t_0} e^{\lambda s} ds \\
&\quad + \ M e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s} ds \\
&\le \frac{M}{\lambda} \frac{e^{-\lambda T}}{1-e^{-\lambda T}} |e^{-\lambda t} - e^{-\lambda t_0}|(e^{\lambda T} - 1) \\
&\quad + \ \frac{M}{\lambda} |e^{-\lambda t} - e^{-\lambda t_0}|(e^{\lambda t_0} - 1) \ +\ \frac{M}{\lambda} e^{-\lambda t}(e^{\lambda t} - e^{\lambda t_0}) \\
&= \frac{M}{\lambda} |e^{-\lambda t} - e^{-\lambda t_0}| \ +\ \frac{M}{\lambda}(1 - e^{-\lambda(t-t_0)}) \ -\ \frac{M}{\lambda} |e^{-\lambda t} - e^{-\lambda t_0}| \\
&\quad + \ \frac{M}{\lambda}(1 - e^{-\lambda(t-t_0)}) \\
&= \frac{2M}{\lambda}(1 - e^{-\lambda(t-t_0)}) = \frac{2M}{\lambda}(1 - e^{-\lambda|t-t_0|}).
\end{aligned}
$$

Let $t \leq t_0$. Then

$$
\begin{aligned}
|v_n(t) - v_n(t_0)| &= |\int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} f(s,\ y_n(s),\ y_n(\theta(s)))ds \\
&\quad + \int_0^t e^{-\lambda(t-s)} f(s,\ y_n(s),\ y_n(\theta(s)))ds \\
&\quad - \int_0^T \frac{e^{-\lambda(T-s+t_0)}}{1-e^{-\lambda T}} f(s,\ y_n(s),\ y_n(\theta(s)))ds \\
&\quad - \int_0^{t_0} e^{-\lambda(t_0-s)} f(s,\ y_n(s),\ y_n(\theta(s)))ds| \\
&= |\int_0^T [\frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} - \frac{e^{-\lambda(T-s+t_0)}}{1-e^{-\lambda T}}] f(s,\ y_n(s),\ y_n(\theta(s)))ds \\
&\quad + \int_0^t e^{-\lambda(t-s)} f(s,\ y_n(s),\ y_n(\theta(s)))ds \\
&\quad - \int_0^t e^{-\lambda(t_0-s)} f(s,\ y_n(s),\ y_n(\theta(s))) \\
&\quad - \int_t^{t_0} e^{-\lambda(t_0-s)} f(s, y_n(s), y_n(\theta(s)))ds| \\
&\leq \int_0^T \frac{e^{-\lambda(T-s)}}{1-e^{-\lambda T}} |e^{-\lambda t} - e^{-\lambda t_0}||f(s,\ y_n(s),\ y_n(\theta(s)))|ds \\
&\quad + \int_0^t e^{\lambda s} |e^{-\lambda t} - e^{-\lambda t_0}||f(s,\ y_n(s),\ y_n(\theta(s)))|ds \\
&\quad + \int_t^{t_0} e^{-\lambda(t_0-s)} |f(s,\ y_n(s),\ y_n(\theta(s)))|ds \\
&\leq M \frac{e^{-\lambda T}}{1-e^{-\lambda T}} |e^{-\lambda t} - e^{-\lambda t_0}| \int_0^T e^{\lambda s} ds \\
&\quad + M|e^{-\lambda t} - e^{-\lambda t_0}| \int_0^t e^{\lambda s} ds \ + \ M e^{-\lambda t_0} \int_t^{t_0} e^{\lambda s} ds.
\end{aligned}
$$

So we have

$$\begin{aligned}|v_n(t)-v_n(t_0)| &\le \frac{M}{\lambda}\frac{e^{-\lambda T}}{1-e^{-\lambda T}}|e^{-\lambda t}-e^{-\lambda t_0}|(e^{\lambda T}-1)\\ &\quad + \frac{M}{\lambda}|e^{-\lambda t}-e^{-\lambda t_0}|(e^{\lambda t}-1) + \frac{M}{\lambda}e^{-\lambda t_0}(e^{\lambda t_0}-e^{\lambda t})\\ &= \frac{M}{\lambda}|e^{-\lambda t}-e^{-\lambda t_0}| + \frac{M}{\lambda}(1-e^{-\lambda(t_0-t)}) - \frac{M}{\lambda}|e^{-\lambda t}-e^{-\lambda t_0}|\\ &\quad + \frac{M}{\lambda}(1-e^{-\lambda(t_0-t)})\\ &= \frac{2M}{\lambda}(1-e^{-\lambda(t_0-t)}) = \frac{2M}{\lambda}(1-e^{-\lambda|t-t_0|}).\end{aligned}$$

CASE-3.1: Let $0 < \frac{\lambda\epsilon}{2M} < 1$.

Choose $\delta = \min\{\frac{T}{2}, -\frac{1}{\lambda}\log[1-\frac{\lambda\epsilon}{2M}]\}$.

Let $|t-t_0| < \delta$.

Then $|t-t_0| < -\frac{1}{\lambda}\log[1-\frac{\lambda\epsilon}{2M}]$. This implies $\log[1-\frac{\lambda\epsilon}{2M}] < -\lambda|t-t_0|$.

So we have $1-\frac{\lambda\epsilon}{2M} < e^{-\lambda|t-t_0|}$. Therefore $1-e^{-\lambda|t-t_0|} < \frac{\lambda\epsilon}{2M}$.

Thus $\frac{2M}{\lambda}(1-e^{-\lambda|t-t_0|}) < \epsilon$.

Hence $|v_n(t)-v_n(t_0)| < \epsilon$, $\forall\ t$ such that $|t-t_0| < \delta$.

CASE-3.2: Let $\frac{\lambda\epsilon}{2M} \ge 1$.

Choose $\delta = \min\{\frac{T}{2}, -\frac{1}{\lambda}\log(\frac{1}{2})\}$.

Let $|t-t_0| < \delta$.

Then $|t-t_0| < -\frac{1}{\lambda}\log(\frac{1}{2})$. This implies $\log(\frac{1}{2}) < -\lambda|t-t_0|$.

So we have $\frac{1}{2} < e^{-\lambda|t-t_0|}$. This gives $-e^{-\lambda|t-t_0|} < -\frac{1}{2}$.

Therefore $1-e^{-\lambda|t-t_0|} < \frac{1}{2} < \frac{\lambda\epsilon}{2M}$. Thus $\frac{2M}{\lambda}(1-e^{-\lambda|t-t_0|}) < \epsilon$.

Hence $|v_n(t)-v_n(t_0)| < \epsilon$, $\forall\ t$ such that $|t-t_0| < \delta$.

From CASE 3.1 and CASE 3.2 we may conclude that $\{v_n\}$ is equicontinuous at t_0, where $0 < t_0 < T$.

Hence from CASE-1 to CASE-3 we conclude that $\{v_n\}$ is equicontinuous on I uniformly.

Now using Arzela-Ascoli lemma, $\{v_n\}$ has a subsequence $\{v_{n_k}\}$ converging uniformly to v on I. Therefore $v_{n_k} \to v$ on $(Z,\ ||.||)$. Thus $\{Jy_n\}$ has a

subsequence $\{Jy_{n_k}\}$ converging in $(Z,\ ||.||)$.

Hence $J : Z \to Z$ is a compact operator. □

Theorem 4.3.4. *Let $\lambda > 0$ and $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and bounded. Then PBVP (4.6) has atleast one solution in $(Z,\ ||.||)$.*

Proof. Let $J : Z \to Z$ be defined by

$J[y(t)] = \int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} f(s,\ y(s),\ y(\theta(s)))ds\ +\ \int_0^t e^{-\lambda(t-s)} f(s,\ y(s),\ y(\theta(s)))ds.$

A fixed point of J is a solution of (4.6). Since $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $J : Z \to Z$ is continuous by Lemma 4.3.2.

Also $J(Z)$ is bounded and $J : Z \to Z$ is compact operator as $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and bounded (Using Lemma 4.3.3).

Therefore by Schauder's fixed point theorem $J : Z \to Z$ has a fixed point $y \in Z$. Thus y is a solution of (4.6). □

4.4 Method of Lower and Upper Solutions

In this section, we establish results showing existence of solution lying between lower solution and upper solution. We start with definitions of lower solution, upper solution, strict lower solution, strict upper solution and solution for general first order DECD

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ y(t),\ y(t-r)),\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \end{aligned}\right\} \tag{4.16}$$

and

$$\left.\begin{aligned} y'(t) + \lambda y(t) = f(t,\ y(t),\ y(\theta(t))).\ t \in I, \end{aligned}\right\} \tag{4.17}$$

We also prove that upper solution exceeds lower solution under certain conditions on a function $f : I \times \mathbb{R}^2 \to \mathbb{R}$, lower and upper solution.

Definition 4.4.1. *A function $u \in B$ is called a solution of general first order DECD* (4.16) *if*

1. $u'(t) + \lambda u(t) = f(t,\ u(t),\ u(t-r)),\ t \in I.$

2. $u(t) = u(0),\ t \in [-r,\ 0].$

Definition 4.4.2. *A function $u \in W$ is called a solution of general first order DECD* (4.16) *if* $u'(t) + \lambda u(t) = f(t,\ u(t),\ u(\theta(t))),\ t \in I.$

Definition 4.4.3. *A function $\alpha \in B$ is called a lower solution of general first order DECD* (4.16) *if*

1. $\alpha'(t) + \lambda \alpha(t) \leq f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I.$

2. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$

A function $\beta \in B$ is called an upper solution of general first order DECD (4.16) *if the above inequalities are reversed.*

Definition 4.4.4. *A function $\alpha \in W$ is called a lower solution of first order DEVDA* (4.17) *if* $\alpha'(t) + \lambda \alpha(t) \leq f(t,\ \alpha(t),\ \alpha(\theta(t))),\ t \in I.$

A function $\beta \in W$ is called an upper solution of first order DEVDA (4.17) *if the above inequalities are reversed.*

Definition 4.4.5. *A function $\alpha \in B$ is called a strict lower solution of general first order DECD* (4.16) *if*

1. $\alpha'(t) + \lambda \alpha(t) < f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I.$

2. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$

Definition 4.4.6. *A function $\alpha \in W$ is called a strict lower solution of first order DEVDA* (4.17) *if* $\alpha'(t) + \lambda \alpha(t) < f(t,\ \alpha(t),\ \alpha(\theta(t))),\ t \in I.$

Definition 4.4.7. *A function $\beta \in B$ is called a strict upper solution of general first order DECD* (4.16) *if*

1. $\beta'(t) + \lambda \beta(t) > f(t,\ \beta(t),\ \beta(t-r)),\ t \in I.$

2. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$

Definition 4.4.8. *A function $\beta \in B$ is called a strict upper solution of first order DEVDA* (4.17) *if* $\beta'(t) + \lambda\beta(t) > f(t,\ \beta(t),\ \beta(\theta(t))),\ t \in I$.

Theorem 4.4.1. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy* $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ *such that* $u_1 \leq u_2$ *and* $v_1 \leq v_2$. *Let $\alpha \in X$ be a strict lower solution of general first order DECD* (4.16). *Let $\beta \in X$ be an upper solution of general first order DECD* (4.16). *Then $\alpha(0) \leq \beta(0)$ implies $\alpha(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$ and $\alpha(t) < \beta(t),\ \forall\ t \in (0,\ T]$.*

Proof. Similar to the proof of the Theorem 3.4.1. □

Theorem 4.4.2. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy* $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ *such that* $u_1 \leq u_2$ *and* $v_1 \leq v_2$. *Let $\alpha \in X$ be a lower solution of general first order DECD* (4.16). *Let $\beta \in X$ be a strict upper solution of general first order DECD* (4.16). *Then $\alpha(0) \leq \beta(0)$ implies $\alpha(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$ and $\alpha(t) < \beta(t),\ \forall\ t \in (0,\ T]$.*

Proof. Similar to the proof of the Theorem 3.4.1. □

Theorem 4.4.3. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy* $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ *such that* $u_1 \leq u_2$ *and* $v_1 \leq v_2$. *Let $\alpha \in W$ be a strict lower solution of first order DEVDA* (4.17). *Let $\beta \in W$ be an upper solution of first order DEVDA* (4.17). *Then $\alpha(0) \leq \beta(0)$ implies $\alpha(t) < \beta(t),\ \forall\ t \in (0,\ T]$.*

Proof. Similar to the proof of the Theorem 3.4.1. □

Theorem 4.4.4. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy* $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ *such that* $u_1 \leq u_2$ *and* $v_1 \leq v_2$. *Let $\alpha \in W$ be a lower solution of first order DEVDA* (4.17). *Let $\beta \in W$ be a strict upper solution of first order DEVDA* (4.17). *Then $\alpha(0) \leq \beta(0)$ implies $\alpha(t) < \beta(t),\ \forall\ t \in (0,\ T]$.*

Proof. Similar to the proof of the Theorem 3.4.1. □

We now define lower and upper solution for PBVPs of general first order DECD (4.1) and first order DEVDA (4.6). We use Schauder's fixed point theorem to establish results on existence of solution between lower and upper solutions for PBVP by assuming f to be non-decreasing in second and third argument. Also we define strict lower and upper solutions for the PBVPs to establish results on existence of solution between the two.

Definition 4.4.9. *A function $\alpha \in B$ is called a lower solution of PBVP* (4.1) *if 1.* $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I$.

2. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$.

3. $\alpha(0) \leq \alpha(T)$.

Similarly, a function $\beta \in B$ is called an upper solution of PBVP (4.1) *if the above inequalities are reversed by replacing α with β.*

Definition 4.4.10. *A function $\alpha \in W$ is called a lower solution of PBVP* (4.6) *if 1.* $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t),\ \alpha(\theta(t))),\ t \in I$.

2. $\alpha(0) \leq \alpha(T)$.

Similarly, a function $\beta \in W$ is called an upper solution of PBVP (4.6) *if the above inequalities are reversed by replacing α with β.*

Definition 4.4.11. *A function $\alpha \in B$ is called a strict lower solution of PBVP* (4.1) *if 1.* $\alpha'(t) + \lambda\alpha(t) < f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I$.

2. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$.

3. $\alpha(0) \leq \alpha(T)$.

Definition 4.4.12. *A function $\beta \in B$ is called a strict upper solution of PBVP* (4.1) *if 1.* $\beta'(t) + \lambda\beta(t) > f(t,\ \beta(t),\ \beta(t-r)),\ t \in I$.

2. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0]$.

3. $\beta(0) \geq \beta(T)$.

Definition 4.4.13. *A function $\alpha \in W$ is called a strict lower solution of PBVP (4.6) if 1.* $\alpha'(t) + \lambda\alpha(t) < f(t,\ \alpha(t),\ \alpha(\theta(t))),\ t \in I.$

2. $\alpha(0) \leq \alpha(T).$

Definition 4.4.14. *A function $\beta \in W$ is called a strict upper solution of PBVP (4.6) if 1.* $\beta'(t) + \lambda\beta(t) > f(t,\ \beta(t),\ \beta(\theta(t))),\ t \in I.$

2. $\beta(0) \geq \beta(T).$

We now have results on existence of solution between lower and upper solution and between strict lower and strict upper solution for PBVPs (4.1) and (4.6).

Theorem 4.4.5. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta \in X$ be lower and upper solutions of PBVP (4.1) such that $\alpha(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$. Then the PBVP (4.1) has atleast one solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are lower and upper solutions of PBVP (4.1), we have

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I.$
2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \beta(t),\ \beta(t-r)),\ t \in I.$
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$
5. $\alpha(0) \leq \alpha(T).$
6. $\beta(0) \geq \beta(T).$

As $\alpha,\ \beta \in X$, we have $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and

$\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$.

Let $p : I \times \mathbb{R} \to \mathbb{R}$ be defined by $p(t,\ x) = \max\{\alpha(t),\ \min\{x,\ \beta(t)\}\}$,

$\forall\ (t,\ x) \in I \times \mathbb{R}$. Therefore $\alpha(t) \leq p(t,\ x) \leq \beta(t),\ \forall\ (t,\ x) \in I \times \mathbb{R}$.

Also $p : I \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded.

Let $L : X \to Y \times \mathbb{R}$ be defined by $L[y(t)] = (u(t),\ y(0))$, where

$$u(t) = \begin{cases} 0, & -r \le t \le 0, \\ y(t) - y(0) + \lambda \int\limits_0^t y(s)ds, & 0 \le t \le T. \end{cases}$$

By Lemma 3.1.1, $L : X \to Y \times \mathbb{R}$ is linear, bijective and continuous.

Also $L^{-1} : Y \times \mathbb{R} \to X$ exists and is given by

$$L^{-1}(u(t),\ \gamma) = \begin{cases} \gamma, & -r \le t \le 0, \\ \gamma e^{-\lambda t} + \int\limits_0^t e^{-\lambda(t-s)} u'(s)ds, & 0 \le t \le T. \end{cases}$$

Further $L^{-1} : Y \times \mathbb{R} \to X$ is bounded and hence continuous.

Let $N : X \to Y \times \mathbb{R}$ be defined as $N[y(t)] = (v(t),\ p(T,\ y(T)))$,

$$\text{where } v(t) = \begin{cases} 0, & -r \le t \le 0, \\ \int\limits_0^t F^\star(s,\ y(s),\ y(s-r))ds, & 0 \le t \le T. \end{cases}$$

By Lemma 4.1.4, $N : X \to Y \times \mathbb{R}$ is compact and continuous.

Let $F^\star : I \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F^\star(t,\ x,\ y) = \begin{cases} f(t,\ \alpha(t),\ \alpha(t-r)), & x \le \alpha(t),\ y \le \alpha(t-r), \\ f(t,\ x,\ \alpha(t-r)), & \alpha(t) \le x \le \beta(t),\ y \le \alpha(t-r), \\ f(t,\ \beta(t),\ \alpha(t-r)), & \beta(t) \le x,\ y \le \alpha(t-r), \\ f(t,\ \alpha(t),\ y), & x \le \alpha(t),\ \alpha(t-r) \le y \le \beta(t-r), \\ f(t,\ x,\ y), & \alpha(t) \le x \le \beta(t),\ \alpha(t-r) \le y \le \beta(t-r), \\ f(t,\ \beta(t),\ y), & \beta(t) \le x,\ \alpha(t-r) \le y \le \beta(t-r), \\ f(t,\ \alpha(t),\ \beta(t-r)), & x \le \alpha(t),\ \beta(t-r) \le y, \\ f(t,\ x,\ \beta(t-r)), & \alpha(t) \le x \le \beta(t),\ \beta(t-r) \le y, \\ f(t,\ \beta(t),\ \beta(t-r)), & \beta(t) \le x,\ \beta(t-r) \le y. \end{cases}$$

Then, $F^\star : I \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and bounded.

Let $S = L^{-1}N : X \to X$. Then

$$S[y(t)] = \begin{cases} p(T,\ y(T)), & -r \le t \le 0, \\ p(T,\ y(T))e^{-\lambda t} + \int\limits_0^t e^{-\lambda(t-s)} F^\star(s,\ y(s),\ y(s-r))ds, & 0 \le t \le T. \end{cases}$$

A fixed point of $S : X \to X$ is a solution of

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = F^*(t,\ y(t),\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = p(T,\ y(T)). \end{aligned}\right\} \tag{4.18}$$

As $L^{-1} : Y \times \mathbb{R} \to X$ is continuous linear operator and $N : X \to Y \times \mathbb{R}$ is compact, we have $S : X \to X$ is a compact operator. As composition of continuous operators is continuous, therefore $S : X \to X$ is continuous. Further p and $F^\star$ are bounded implies that $S(X)$ is a bounded subset of $(X,\ ||.||_\infty)$. Therefore by Schauder's fixed point theorem, $S : X \to X$ has a fixed point $y \in X$. Thus y is a solution of the BVP (4.18).

We show that $\alpha(t) \le y(t) \le \beta(t),\ \forall\ t \in [-r,\ T]$.

Since $\alpha(0) \le \alpha(T) \le p(T,\ y(T)) \le \beta(T) \le \beta(0)$, we have

$\alpha(0) \le p(T,\ y(T)) \le \beta(0)$. This implies $\alpha(0) \le y(0) \le \beta(0)$.

Therefore $\alpha(t) \le y(t) \le \beta(t),\ \forall\ t \in [-r,\ 0]$.

So it is enough to show that $\alpha(t) \le y(t) \le \beta(t),\ \forall\ t \in (0,\ T]$.

Let $\epsilon > 0$ be arbitrary.

Let α_ϵ and β_ϵ be as defined in the proof of theorem 3.4.3.

Since $\alpha(0) \le y(0) \le \beta(0)$, we have $\alpha_\epsilon(0) < \alpha(0) \le y(0) \le \beta(0) < \beta_\epsilon(0)$.

So we get $\alpha_\epsilon(0) < y(0) < \beta_\epsilon(0)$.

Therefore we prove that $\alpha_\epsilon(t) < y(t) < \beta_\epsilon(t),\ \forall\ t \in (0,\ T]$.

Let if possible $\exists\ t_1\ \in (0,\ T]$ such that $\alpha_\epsilon(t) < y(t) < \beta_\epsilon(t),\ \forall\ t \in (0,\ t_1)$ and $y(t_1) = \beta_\epsilon(t_1)$. Therefore $y(t_1) > \beta(t_1)$ and $\beta'_\epsilon(t_1) \le y'(t_1)$.

CASE-1: Let $0 < t_1 \leq r$.

So we have $-r < t_1 - r \leq 0$.

This implies $\alpha(t_1 - r) \leq y(t_1 - r) \leq \beta(t_1 - r)$. Therefore

$$\begin{aligned} y'(t_1) &= F^\star(t_1,\ y(t_1),\ y(t_1 - r)) - \lambda y(t_1) \\ &= f(t_1,\ \beta(t_1),\ y(t_1 - r)) - \lambda y(t_1) \\ &\leq f(t_1,\ \beta(t_1),\ \beta(t_1 - r)) - \lambda\beta(t_1) \\ &\leq \beta'(t_1) < \beta'_\epsilon(t_1), \text{ which is a contradiction.} \end{aligned}$$

CASE-2: Let $r < t_1 \leq T$.

So we have $0 < t_1 - r \leq t_1$.

This implies $\alpha_\epsilon(t_1 - r) < y(t_1 - r) < \beta_\epsilon(t_1 - r)$.

CASE-2.1: Let $y(t_1 - r) < \alpha(t_1 - r)$. Then

$$\begin{aligned} y'(t_1) &= F^\star(t_1,\ y(t_1),\ y(t_1 - r)) - \lambda y(t_1) \\ &= f(t_1,\ \beta(t_1),\ \alpha(t_1 - r)) - \lambda y(t_1) \\ &\leq f(t_1,\ \beta(t_1),\ \beta(t_1 - r)) - \lambda\beta(t_1) \\ &\leq \beta'(t_1) < \beta'_\epsilon(t_1), \text{ which is a contradiction.} \end{aligned}$$

CASE-2.2: Let $\alpha(t_1 - r) \leq y(t_1 - r) \leq \beta(t_1 - r)$. Then

$$\begin{aligned} y'(t_1) &= F^\star(t_1,\ y(t_1),\ y(t_1 - r)) - \lambda y(t_1) \\ &= f(t_1,\ \beta(t_1),\ y(t_1 - r)) - \lambda y(t_1) \\ &\leq f(t_1,\ \beta(t_1),\ \beta(t_1 - r)) - \lambda\beta(t_1) \\ &\leq \beta'(t_1) < \beta'_\epsilon(t_1), \text{ which is a contradiction.} \end{aligned}$$

CASE-2.3: Let $y(t_1 - r) > \beta(t_1 - r)$. Then

$$\begin{aligned} y'(t_1) &= F^\star(t_1,\ y(t_1),\ y(t_1 - r)) - \lambda y(t_1) \\ &= f(t_1,\ \beta(t_1),\ \beta(t_1 - r)) - \lambda y(t_1) \\ &\leq f(t_1,\ \beta(t_1),\ \beta(t_1 - r)) - \lambda \beta(t_1) \\ &\leq \beta'(t_1) < \beta'_\epsilon(t_1), \text{ which is a contradiction.} \end{aligned}$$

Hence from the above discussed cases, we conclude that
$\alpha_\epsilon(t) < y(t) < \beta_\epsilon(t),\ \forall\ t \in I$ and $\forall\ \epsilon > 0$. Therefore by taking $\epsilon \to 0$, we get
$\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in I$. This gives $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$.
Thus $F^\star(t,\ y(t),\ y(t - r)) = f(t,\ y(t),\ y(t - r)),\ \forall\ t \in I$ and
$p(T,\ y(T)) = y(T)$.
Hence y is a solution of the PBVP (4.1) such that $\alpha(t) \leq y(t) \leq \beta(t)$,
$\forall\ t \in [-r,\ T]$. □

Theorem 4.4.6. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta \in W$ be lower and upper solutions of PBVP (4.6) such that $\alpha(t) \leq \beta(t),\ \forall\ t \in I$. Then the PBVP (4.6) has atleast one solution $y \in W$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in I$.*

Proof. Similar to the proof of the Theorem 4.4.5. □

Theorem 4.4.7. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta \in X$ be strict lower and upper solutions of PBVP (4.1) such that $\alpha(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$. Then the PBVP (4.1) has atleast one solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$ and $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are strict lower and upper solutions of PBVP (4.1), we have 1. $\alpha'(t) + \lambda\alpha(t) < f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I.$

2. $\beta'(t) + \lambda\beta(t) > f(t,\ \beta(t),\ \beta(t-r)),\ t \in I.$
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$
4. $\beta(t) \geq \beta(0),\ t \in [-r, 0].$
5. $\alpha(t) \leq \beta(t),\ t \in [-r, T].$
6. $\alpha(0) \leq \alpha(T).$
7. $\beta(0) \geq \beta(T).$

As $\alpha,\ \beta \in X$, we have $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$.

Let $p,\ F^\star,\ L,\ N$ and S be defined as in the proof of the Theorem 4.4.5. Therefore by Schauder's fixed point theorem, $S : X \to X$ has a fixed point $y \in X$. This will imply that y is a solution of the BVP (4.18).

Arguing in the similar manner as in the proof of the Theorem 4.4.5, we get $\alpha(0) \leq y(0) \leq \beta(0)$. Therefore $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$.

Next we show that $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.

Let if possible $\exists\ t_1\ \in (0,\ T]$ such that $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ t_1)$ and $y(t_1) = \beta(t_1)$. This implies that $y(t_1 - h) < \beta(t_1 - h)$ for sufficiently small $h > 0$. So we have $\frac{\beta(t_1)-\beta(t_1-h)}{h} < \frac{y(t_1)-y(t_1-h)}{h}$. Therefore $\beta'(t_1) \leq y'(t_1)$.

CASE-1: Let $0 < t_1 \leq r$. So we have $-r < t_1 - r \leq 0$.

This implies $\alpha(t_1 - r) \leq y(t_1 - r) \leq \beta(t_1 - r)$. Therefore

$$\begin{aligned} y'(t_1) &= F^\star(t_1,\ y(t_1),\ y(t_1 - r)) - \lambda y(t_1) \\ &= f(t_1,\ \beta(t_1),\ y(t_1 - r)) - \lambda y(t_1) \\ &\leq f(t_1,\ \beta(t_1),\ \beta(t_1 - r)) - \lambda\beta(t_1) \\ &< \beta'(t_1) \text{ which is a contradiction.} \end{aligned}$$

CASE-2: Let $r < t_1 \leq T$. So we have $0 < t_1 - r \leq t_1$.

This implies $\alpha(t_1 - r) < y(t_1 - r) < \beta(t_1 - r)$. Therefore

$$\begin{aligned} y'(t_1) &= F^\star(t_1,\ y(t_1),\ y(t_1 - r)) - \lambda y(t_1) \\ &= f(t_1,\ \beta(t_1),\ y(t_1 - r)) - \lambda y(t_1) \\ &\leq f(t_1,\ \beta(t_1),\ \beta(t_1 - r)) - \lambda \beta(t_1) \\ &< \beta'(t_1), \text{ which is a contradiction.} \end{aligned}$$

Thus from CASE-1 & CASE-2, we have $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.

Therefore $F^\star(t,\ y(t),\ y(t - r)) = f(t,\ y(t),\ y(t - r)),\ \forall\, t \in I$ and

$p(T,\ y(T)) = y(T)$.

Hence y is a solution of the PBVP (4.1) such that $\alpha(t) \leq y(t) \leq \beta(t)$,

$\forall\, t \in [-r,\ 0]$ and $\alpha(t) < y(t) < \beta(t),\ \forall\, t \in (0,\ T]$. □

Theorem 4.4.8. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta \in W$ be strict lower and upper solutions of PBVP (4.6) such that $\alpha(t) \leq \beta(t),\ \forall\ t \in I$. Then the PBVP (4.1) has a solution $y \in W$ such that $\alpha(0) \leq y(0) \leq \beta(0)$ and $\alpha(t) < y(t) < \beta(t)$, $\forall\, t \in (0,\ T]$.*

Proof. Similar to the proof of the Theorem 4.4.7. □

4.5 Method of Weakly Coupled Lower and Upper Solutions

In this section, we define weakly coupled lower and upper solutions for APBVPs, LBVPs and NLBVPs of general first order DECD and first order DEVDA. Again we use Schauder's fixed point theorem to establish few results on existence of solution between weakly coupled lower and upper solutions by assuming f to be non-decreasing in second and third argument. We further define strict weakly coupled lower and upper solutions for APBVPs, LBVPs and NLBVPs to obtain results on existence of solution between the two.

Definition 4.5.1. *A pair of functions $\alpha,\ \beta \in B$ is called weakly coupled lower and upper solutions of APBVP* (4.2) *if*

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I.$
2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \beta(t),\ \beta(t-r)),\ t \in I.$
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T].$
6. $\alpha(0) \leq -\beta(T).$
7. $\beta(0) \geq -\alpha(T).$

Definition 4.5.2. *A pair of functions $\alpha,\ \beta \in W$ is called weakly coupled lower and upper solutions of APBVP* (4.7) *if*

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t),\ \alpha(\theta(t))),\ t \in I.$
2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \beta(t),\ \beta(\theta(t))),\ t \in I.$
3. $\alpha(t) \leq \beta(t),\ t \in I.$
4. $\alpha(0) \leq -\beta(T).$
5. $\beta(0) \geq -\alpha(T).$

Definition 4.5.3. *A pair of functions $\alpha,\ \beta \in B$ is called weakly coupled lower and upper solutions of LBVP* (4.3) *if*

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I.$
2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \beta(t),\ \beta(t-r)),\ t \in I.$
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T].$
6. $\max\{a\alpha(0) + b\alpha(T),\ a\alpha(0) + b\beta(T)\} \leq 0$
   $\leq \min\{a\beta(0) + b\alpha(T),\ a\beta(0) + b\beta(T)\}.$

Definition 4.5.4. *A pair of functions $\alpha,\ \beta \in W$ is called weakly coupled lower and upper solutions of LBVP* (4.8) *if*

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t),\ \alpha(\theta(t))),\ t \in I.$
2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \beta(t),\ \beta(\theta(t))),\ t \in I.$
3. $\alpha(t) \leq \beta(t),\ t \in I.$
4. $\max\{a\alpha(0) + b\alpha(T),\ a\alpha(0) + b\beta(T)\} \leq 0$
   $\leq \min\{a\beta(0) + b\alpha(T),\ a\beta(0) + b\beta(T)\}.$

Definition 4.5.5. *A pair of functions $\alpha,\ \beta \in B$ is called weakly coupled lower and upper solutions of NLBVP* (4.4) *if*

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I.$
2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \beta(t),\ \beta(t-r)),\ t \in I.$
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T].$
6. $\max\{g(\alpha(0),\ \alpha(T)),\ g(\alpha(0),\ \beta(T))\} \leq 0 \leq \min\{g(\beta(0),\ \alpha(T)),\ g(\beta(0),\ \beta(T))\}.$

Definition 4.5.6. *A pair of functions $\alpha,\ \beta \in W$ is called weakly coupled lower and upper solutions of NLBVP* (4.4) *if*

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t),\ \alpha(\theta(t))),\ t \in I.$

2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \beta(t),\ \beta(\theta(t))),\ t \in I.$

3. $\alpha(t) \leq \beta(t),\ t \in I.$

4. $\max\{g(\alpha(0),\ \alpha(T)),\ g(\alpha(0),\ \beta(T))\} \leq 0 \leq \min\{g(\beta(0),\ \alpha(T)),\ g(\beta(0),\ \beta(T))\}.$

Definition 4.5.7. *A pair of weakly coupled lower and upper solutions $\alpha,\ \beta \in B$ of APBVP* (4.2)*, is called strict weakly coupled lower and upper solutions of APBVP* (4.2) *if 1.* $\alpha'(t) + \lambda\alpha(t) < f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I.$

2. $\beta'(t) + \lambda\beta(t) > f(t,\ \beta(t),\ \beta(t-r)),\ t \in I.$

Similarly, we may define strict weakly coupled lower and upper solutions for LBVP (4.3) *and NLBVP* (4.4)*.*

Definition 4.5.8. *A pair of weakly coupled lower and upper solutions $\alpha,\ \beta \in W$ of APBVP* (4.7)*, is called strict weakly coupled lower and upper solutions of APBVP* (4.7) *if 1.* $\alpha'(t) + \lambda\alpha(t) < f(t,\ \alpha(t),\ \alpha(\theta(t))),\ t \in I.$

2. $\beta'(t) + \lambda\beta(t) > f(t,\ \beta(t),\ \beta(\theta(t))),\ t \in I.$

Similarly, we may define strict weakly coupled lower and upper solutions for LBVP (4.8) *and NLBVP* (4.9)*.*

Theorem 4.5.1. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta \in X$ be weakly coupled lower and upper solutions of APBVP* (4.2)*. Then the APBVP* (4.2) *has a solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are weakly coupled lower and upper solutions of APBVP (4.2), we have 1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I.$

2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \beta(t),\ \beta(t-r)),\ t \in I.$
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T].$
6. $\alpha(0) \leq -\beta(T).$
7. $\beta(0) \geq -\alpha(T).$

As $\alpha,\ \beta \in X$, we have $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and

$\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$.

Let p, $F^\star$ and L be defined as in the proof of the Theorem 4.4.5.

Let $N : X \to Y \times \mathbb{R}$ be defined as $N[y(t)] = (v(t),\ -p(T,\ y(T)))$,

$$\text{where } v(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ \int\limits_0^t F^\star(s,\ y(s),\ y(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

Then $N : X \to Y \times \mathbb{R}$ is compact and continuous by Lemma 4.1.2 and Lemma 4.1.4.

Let $S = L^{-1}N : X \to X$. Then

$$S[y(t)] = \begin{cases} -p(T,\ y(T)), & -r \leq t \leq 0, \\ -p(T,\ y(T))e^{-\lambda t} + \int\limits_0^t e^{-\lambda(t-s)}F^\star(s,\ y(s),\ y(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

A fixed point of $S : X \to X$ is a solution of

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= F^*(t,\ y(t),\ y(t-r)),\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= -p(T,\ y(T)). \end{aligned}\right\} \tag{4.19}$$

By Schauder's fixed point theorem, $S : X \to X$ has a fixed point $y \in X$. Therefore y is a solution of (4.19).

Since $-\beta(0) \leq \alpha(T) \leq p(T,\ y(T)) \leq \beta(T) \leq -\alpha(0)$, we have
$-\beta(0) \leq p(T,\ y(T)) \leq -\alpha(0)$. This implies $\alpha(0) \leq -p(T,\ y(T)) \leq \beta(0)$.
Therefore $\alpha(0) \leq y(0) \leq \beta(0)$. Thus $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$.
The remaining proof is similar to the proof of the Theorem 4.4.5. □

Theorem 4.5.2. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta \in W$ be weakly coupled lower and upper solutions of APBVP (4.7). Then the APBVP (4.7) has atleast one solution $y \in W$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in I$.*

Proof. Similar to the proof of theorem 4.5.1. □

Theorem 4.5.3. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta \in X$ be weakly coupled lower and upper solutions of LBVP (4.3). Then the LBVP (4.3) has atleast one solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are weakly coupled lower and upper solutions of LBVP (4.3), we have 1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I$.

2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \beta(t),\ \beta(t-r)),\ t \in I$.
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$.
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0]$.
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T]$.
6. $\max\{a\alpha(0) + b\alpha(T),\ a\alpha(0) + b\beta(T)\}$
$\leq 0 \leq \min\{a\beta(0) + b\alpha(T),\ a\beta(0) + b\beta(T)\}$.

As $\alpha,\ \beta \in X$, we get $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$.
Let p, $F^\star$ and L be defined as in the proof of the Theorem 4.4.5.
Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined as, $g(x, y) = ax + by,\ \forall\ (x,\ y) \in \mathbb{R}^2$.
Therefore $g : \mathbb{R}^2 \to \mathbb{R}$ is continuous and $g(y(0),\ y(T)) = ay(0) + by(T)$.

Let $g^\star : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$\begin{aligned} g^\star(x,\ y) &= p(0,\ x) - g[p(0,\ x),\ p(T, y)],\ \forall\ (x,\ y) \in \mathbb{R}^2. \\ &= p(0,\ x) - ap(0,\ x) - bp(T,\ y),\ \forall\ (x,\ y) \in \mathbb{R}^2. \\ &= (1-a)p(0,\ x) - bp(T,\ y),\ \forall\ (x,\ y) \in \mathbb{R}^2. \end{aligned}$$

Since $\alpha(0) \leq p(0,\ x) \leq \beta(0),\ \forall\ x \in \mathbb{R}$, we have

$|p(0,\ x)| \leq K_1 = \max\{|\alpha(0)| + 1,\ |\beta(0)|\},\ \forall\ x \in \mathbb{R}.$

Also $\alpha(T) \leq p(T,\ y) \leq \beta(T),\ \forall\ y \in \mathbb{R}$ implies that

$|p(T,\ y)| \leq K_2 = \max\{|\alpha(T)| + 1,\ |\beta(T)|\},\ \forall\ y \in \mathbb{R}.$

$$\begin{aligned} \text{Therefore } |g^\star(x,\ y)| &\leq |1-a||p(0,\ x)| + |b||p(T,\ y)|,\ \forall\ (x,\ y) \in \mathbb{R}^2. \\ &\leq |1-a|K_1 + |b|K_2,\ \forall\ (x,\ y) \in \mathbb{R}^2. \end{aligned}$$

Thus $g^\star : \mathbb{R}^2 \to \mathbb{R}$ is bounded. Also $g^\star : \mathbb{R}^2 \to \mathbb{R}$ is continuous, as p and g are continuous.

Let $N : X \to Y \times \mathbb{R}$ be defined as $N[y(t)] = [v(t),\ g^\star(y(0),\ y(T))]$,

$$\text{where } v(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ \int\limits_0^t F^\star(s,\ y(s),\ y(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

By Lemma 4.1.5, N is compact and continuous.

Let $S = L^{-1}N : X \to X$. Then

$$S[y(t)] = \begin{cases} g^\star(y(0),\ y(T)), & -r \leq t \leq 0, \\ g^\star(y(0),\ y(T))e^{-\lambda t} + \int\limits_0^t e^{-\lambda(t-s)}F^\star(s,\ y(s),\ y(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

A fixed point of $S : X \to X$ is a solution of

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = F^*(t,\ y(t),\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = g^\star(y(0),\ y(T)). \end{aligned}\right\} \tag{4.20}$$

Further $S : X \to X$ is compact and continuous. Therefore by Schauder's fixed point theorem, $S : X \to X$ has a fixed point $y \in X$. Thus y is a solution of the BVP (4.20). Arguing in the similar manner as in the proof of the Theorem 3.5.2, we get $\alpha(0) \leq y(0) \leq \beta(0)$.

The remaining part of the proof is similar to the proof of the Theorem 4.4.5. □

Theorem 4.5.4. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy*
$f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ *such that* $u_1 \leq u_2$ *and* $v_1 \leq v_2$. *Let* $\alpha,\ \beta \in W$ *be weakly coupled lower and upper solutions of LBVP* (4.8). *Then the LBVP* (4.8) *has a solution* $y \in W$ *such that* $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in I$.

Proof. Similar to the proof of the Theorem 4.5.3. □

Theorem 4.5.5. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy*
$f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ *such that* $u_1 \leq u_2$ *and* $v_1 \leq v_2$. *Let* $g : \mathbb{R}^2 \to \mathbb{R}$ *be continuous function which satisfies,*
(1) $g(t,\ x) \leq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ *such that* $x \leq y$ *or*
(2) $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ *such that* $x \leq y$.
Let $\alpha,\ \beta \in X$ *be weakly coupled lower and upper solutions of NLBVP* (4.4).
Then the NLBVP (4.4) *has a solution* $y \in X$ *such that*
$\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$.

Proof. Without the loss of generality, let $g(t,\ x) \leq g(t,\ y)$,
$\forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$.

Since $\alpha,\ \beta \in X$ are weakly coupled lower and upper solutions of NLBVP (4.4),

we have 1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I.$

2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \beta(t),\ \beta(t-r)),\ t \in I.$

3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$

4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$

5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T].$

6. $\max\{g(\alpha(0),\ \alpha(T)),\ g(\alpha(0),\ \beta(T))\}$
$\leq 0 \leq \min\{g(\beta(0),\ \alpha(T)),\ g(\beta(0),\ \beta(T))\}.$

As $\alpha,\ \beta \in X$, we get $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$.

As $g(t,\ x) \leq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$, we have

$\max\{g(\alpha(0),\ \alpha(T)),\ g(\alpha(0),\ \beta(T))\} = g(\alpha(0),\ \beta(T))$ and

$\min\{g(\beta(0),\ \alpha(T)),\ g(\beta(0),\ \beta(T))\} = g(\beta(0),\ \alpha(T))$.

This implies $g(\alpha(0),\ \beta(T)) \leq 0 \leq g(\beta(0),\ \alpha(T))$.

Let p, $F^\star$ and L be defined as in theorem 4.4.5.

Let $g^\star : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$g^\star(x,\ y) = p(0,\ x) - g[p(0,\ x),\ p(T,\ y)],\ \forall\ (x,\ y) \in \mathbb{R}^2$.

Then $g^\star : \mathbb{R}^2 \to \mathbb{R}$ is continuous and bounded.

Let $N : X \to Y \times \mathbb{R}$ be defined as $N[y(t)] = [v(t),\ g^\star(y(0),\ y(T))]$,

where $v(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ \int\limits_0^t F^\star(s,\ y(s),\ y(s-r))ds, & 0 \leq t \leq T. \end{cases}$

By Lemma 4.1.5, N is compact and continuous.

Let $S = L^{-1}N : X \to X$. Then

$$S[y(t)] = \begin{cases} g^\star(y(0),\ y(T)), & -r \leq t \leq 0, \\ g^\star(y(0),\ y(T))e^{-\lambda t} + \int\limits_0^t e^{-\lambda(t-s)}F^\star(s,\ y(s),\ y(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

A fixed point of $S : X \to X$ is a solution of

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = F^*(t,\ y(t),\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = g^\star(y(0),\ y(T)). \end{aligned}\right\} \tag{4.21}$$

By Schauder's fixed point theorem, $S : X \to X$ has a fixed point $y \in X$. Therefore y is a solution of the BVP (4.21). Arguing in the similar manner as in the proof of the Theorem 3.5.3, we get $\alpha(0) \leq y(0) \leq \beta(0)$.
The remaining part of the proof is similar to the proof of the Theorem 4.4.5. □

Theorem 4.5.6. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy*
$f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ *such that* $u_1 \leq u_2$ *and* $v_1 \leq v_2$. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous function which satisfies,*
(1) $g(t,\ x) \leq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$ or
(2) $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$.
Let $\alpha,\ \beta \in W$ be weakly coupled lower and upper solutions of NLBVP (4.9). *Then the NLBVP* (4.9) *has a solution $y \in W$ such that*
$\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in I$.

Proof. Similar to the proof of the Theorem 4.5.5. □

We now prove results on existence of solution between strict lower and strict upper solution for APBVP, LBVP and NLBVP of general first order DECD and first order DEVDA.

Theorem 4.5.7. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy*
$f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ *such that*
$u_1 \leq u_2$ *and* $v_1 \leq v_2$. *Let $\alpha,\ \beta \in X$ be strict weakly coupled lower and upper solutions of APBVP* (4.2). *Then the APBVP* (4.2) *has atleast one solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$ and*
$\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.

Proof. Since $\alpha,\ \beta \in X$ are strict weakly coupled lower and upper solutions of APBVP (4.2), we have

1. $\alpha'(t) + \lambda\alpha(t) < f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I.$
2. $\beta'(t) + \lambda\beta(t) > f(t,\ \beta(t),\ \beta(t-r)),\ t \in I.$
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T].$
6. $\alpha(0) \leq -\beta(T).$
7. $\beta(0) \geq -\alpha(T).$

As $\alpha,\ \beta \in X$, we have $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$.

Let $p,\ F^\star,\ L,\ N$ and S be defined as in the proof of the Theorem 4.5.1. Therefore by Schauder's fixed point theorem, $S : X \to X$ has a fixed point $y \in X$. This will imply that y is a solution of the BVP (4.19).

Proceding as in the proof of the Theorem 4.5.1, we obtain $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$.

The remaining part of the proof is similar to the proof of the Theorem 4.4.7. □

Theorem 4.5.8. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta \in W$ be strict weakly coupled lower and upper solutions of APBVP (4.7). Then the APBVP (4.7) has a solution $y \in W$ such that $\alpha(0) \leq y(0) \leq \beta(0)$ and $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.*

Proof. Similar to the proof of the Theorem 4.5.7. □

Theorem 4.5.9. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta \in X$ be strict weakly coupled lower and upper solutions of LBVP (4.3). Then the LBVP (4.3) has a solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$ and $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are strict weakly coupled lower and upper solutions of LBVP (4.3), we have 1. $\alpha'(t) + \lambda\alpha(t) < f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I.$

2. $\beta'(t) + \lambda\beta(t) > f(t, \beta(t), \beta(t-r)),\ t \in I.$

3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$

4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$

5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T].$

6. $\max\{a\alpha(0) + b\alpha(T),\ a\alpha(0) + b\beta(T)\}$
$\leq 0 \leq \min\{a\beta(0) + b\alpha(T),\ a\beta(0) + b\beta(T)\}.$

As $\alpha,\ \beta \in X$, we get $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$. Let $p,\ F^\star,\ L,\ N$ and S be defined as in the proof of the Theorem 4.5.3. Therefore by Schauder's fixed point theorem, $S : X \to X$ has a fixed point $y \in X$. This will imply that y is a solution of the BVP (4.20). Proceding as in the proof of the Theorem 4.5.3, we obtain $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$. The remaining part of the proof is similar to the proof of the Theorem 4.4.7. □

Theorem 4.5.10. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta \in W$ be strict weakly coupled lower and upper solutions of LBVP (4.8). Then the LBVP (4.8) has a solution $y \in X$ such that $\alpha(0) \leq y(0) \leq \beta(0)$ and $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.*

Proof. Similar to the proof of the Theorem 4.5.9. □

Theorem 4.5.11. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous function which satisfies (1) $g(t,\ x) \leq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$ or (2) $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$. Let $\alpha,\ \beta \in X$ be strict weakly coupled lower and upper solutions of NLBVP (4.4). Then the NLBVP (4.4) has a solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$ and $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.*

Proof. Without the loss of generality, let $g(t,\ x) \leq g(t,\ y)$,

$\forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$.

Since $\alpha,\ \beta \in X$ are strict weakly coupled lower and upper solutions of

NLBVP (4.4), we have

1. $\alpha'(t) + \lambda\alpha(t) < f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I$.

2. $\beta'(t) + \lambda\beta(t) > f(t,\ \beta(t),\ \beta(t-r)),\ t \in I$.

3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$.

4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0]$.

5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T]$.

6. $\max\{g(\alpha(0),\ \alpha(T)),\ g(\alpha(0),\ \beta(T))\} \leq 0 \leq \min\{g(\beta(0),\ \alpha(T)),\ g(\beta(0),\ \beta(T))\}$.

As $\alpha,\ \beta \in X$, we have $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0)$,

$\forall\ t \in [-r,\ 0]$.

Let $p,\ F^\star,\ L,\ N$ and S be defined as in the proof of the Theorem 4.5.5.

Therefore by Schauder's fixed point theorem, $S : X \to X$ has a fixed point

$y \in X$. This will imply that y is a solution of the BVP (4.21).

Proceding as in the proof of the Theorem 4.5.5, we obtain

$\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$.

The remaining proof is similar to the proof of theorem 4.4.7. □

Theorem 4.5.12. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy*

$f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ *such that*

$u_1 \leq u_2$ *and* $v_1 \leq v_2$. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous function which satisfies*

(1) $g(t,\ x) \leq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$ or

(2) $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$.

Let $\alpha,\ \beta \in W$ be strict weakly coupled lower and upper solutions of NLBVP

(4.9). *Then the NLBVP* (4.9) *has a solution $y \in W$ such that*

$\alpha(0) \leq y(0) \leq \beta(0)$ *and* $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.

Proof. Similar to the proof of the Theorem 4.5.11. □

4.6 Method of Coupled Lower and Upper Solutions

In this section, we define coupled lower and upper solutions for PBVPs, APBVPs, LBVPs and NLBVPs to establish some results on existence of solution between the two. Further definitions of strict coupled lower and upper solutions for PBVPs, APBVPs, LBVPs and NLBVPs are given to results establish existence of solution between them. Again we make use of Schauder's fixed point theorem to establish our results by assuming f to be non-increasing in second and third arguments.

Definition 4.6.1. *A pair of functions $\alpha,\ \beta \in B$ is called coupled lower and upper solutions of PBVP* (4.1) *if*

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \beta(t),\ \beta(t-r)),\ t \in I.$
2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I.$
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T].$
6. $\alpha(0) \leq \alpha(T).$
7. $\beta(0) \geq \beta(T).$

Definition 4.6.2. *A pair of functions $\alpha,\ \beta \in W$ is called coupled lower and upper solutions of PBVP* (4.6) *if*

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \beta(t),\ \beta(\theta(t))),\ t \in I.$
2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \alpha(t),\ \alpha(\theta(t))),\ t \in I.$
3. $\alpha(t) \leq \beta(t),\ t \in I.$
4. $\alpha(0) \leq \alpha(T).$
5. $\beta(0) \geq \beta(T).$

Definition 4.6.3. *A pair of functions $\alpha,\ \beta \in B$ is called coupled lower and upper solutions of APBVP* (4.2) *if*

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \beta(t),\ \beta(t-r)),\ t \in I.$
2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I.$
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T].$
6. $\alpha(0) \leq -\beta(T).$
7. $\beta(0) \geq -\alpha(T).$

Definition 4.6.4. *A pair of functions $\alpha,\ \beta \in W$ is called coupled lower and upper solutions of APBVP* (4.7) *if*

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \beta(t),\ \beta(\theta(t))),\ t \in I.$
2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \alpha(t),\ \alpha(\theta(t))),\ t \in I.$
3. $\alpha(t) \leq \beta(t),\ t \in I.$
4. $\alpha(0) \leq -\beta(T).$
5. $\beta(0) \geq -\alpha(T).$

Definition 4.6.5. *A pair of functions $\alpha,\ \beta \in B$ is called coupled lower and upper solutions of LBVP* (4.3) *if*

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \beta(t),\ \beta(t-r)),\ t \in I.$
2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I.$
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T].$
6. $\max\{a\alpha(0) + b\alpha(T),\ a\alpha(0) + b\beta(T)\} \leq 0$
   $\leq \min\{a\beta(0) + b\alpha(T),\ a\beta(0) + b\beta(T)\}.$

Definition 4.6.6. *A pair of functions* $\alpha,\ \beta \in W$ *is called coupled lower and upper solutions of LBVP* (4.8) *if*

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \beta(t),\ \beta(\theta(t))),\ t \in I.$
2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \alpha(t),\ \alpha(\theta(t))),\ t \in I.$
3. $\alpha(t) \leq \beta(t),\ t \in I.$
4. $\max\{a\alpha(0) + b\alpha(T),\ a\alpha(0) + b\beta(T)\} \leq 0$
   $\leq \min\{a\beta(0) + b\alpha(T),\ a\beta(0) + b\beta(T)\}.$

Definition 4.6.7. *A pair of functions* $\alpha,\ \beta \in B$ *is called coupled lower and upper solutions of NLBVP* (4.4) *if*

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \beta(t),\ \beta(t-r)),\ t \in I.$
2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I.$
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T].$
6. $\max\{g(\alpha(0),\ \alpha(T)),\ g(\alpha(0),\ \beta(T))\} \leq 0 \leq \min\{g(\beta(0),\ \alpha(T)),\ g(\beta(0),\ \beta(T))\}.$

Definition 4.6.8. *A pair of functions* $\alpha,\ \beta \in W$ *is called coupled lower and upper solutions of NLBVP* (4.9) *if*

1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \beta(t),\ \beta(\theta(t))),\ t \in I.$
2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \alpha(t),\ \alpha(\theta(t))),\ t \in I.$
3. $\alpha(t) \leq \beta(t),\ t \in I.$
4. $\max\{g(\alpha(0),\ \alpha(T)),\ g(\alpha(0),\ \beta(T))\} \leq 0 \leq \min\{g(\beta(0),\ \alpha(T)),\ g(\beta(0),\ \beta(T))\}.$

Definition 4.6.9. *A pair of coupled lower and upper solutions* $\alpha,\ \beta \in B$ *of the PBVP* (4.1)*, is called strict coupled lower and upper solutions of PBVP* (4.1) *if* 1. $\alpha'(t) + \lambda\alpha(t) < f(t,\ \beta(t),\ \beta(t-r)),\ t \in I.$

2. $\beta'(t) + \lambda\beta(t) > f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I.$

Similarly, we may define strict coupled lower and upper solutions for APBVP (4.2)*, LBVP* (4.3) *and NLBVP* (4.4)*.*

Definition 4.6.10. *A pair of coupled lower and upper solutions* $\alpha,\ \beta \in W$ *of the PBVP* (4.6)*, is called strict coupled lower and upper solutions of PBVP* (4.6) *if 1.* $\alpha'(t) + \lambda\alpha(t) < f(t,\ \beta(t),\ \beta(\theta(t))),\ t \in I.$

2. $\beta'(t) + \lambda\beta(t) > f(t,\ \alpha(t),\ \alpha(\theta(t))),\ t \in I.$

Similarly, we may define strict coupled lower and upper solutions for APBVP (4.7)*, LBVP* (4.8) *and NLBVP* (4.9)*.*

We now prove theorems on existence of solution between coupled lower and upper solutions for PBVP, APBVP, LBVP and NLBVP of general first order DECD and first order DEVDA.

Theorem 4.6.1. *Let* $f : I \times \mathbb{R}^2 \to \mathbb{R}$ *be continuous and satisfy* $f(t,\ u_1,\ v_1) \geq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ *such that* $u_1 \leq u_2$ *and* $v_1 \leq v_2$*. Let* $\alpha,\ \beta \in X$ *be coupled lower and upper solutions of PBVP* (4.1)*. Then the PBVP* (4.1) *has a solution* $y \in X$ *such that* $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T].$

Proof. Since $\alpha,\ \beta \in X$ are coupled lower and upper solutions of PBVP (4.1), we have 1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \beta(t),\ \beta(t-r)),\ t \in I.$

2. $\beta'(t) + \lambda\beta(t) \geq f(t, \alpha(t), \alpha(t-r)),\ t \in I.$
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0].$
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0].$
5. $\alpha(0) \leq \alpha(T).$
6. $\beta(0) \geq \beta(T).$

As $\alpha,\ \beta \in X$, we have $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),$ $\forall\ t \in [-r,\ 0].$

Let $p,\ F^\star,\ L,\ N$ and S be defined as in the proof of the Theorem 4.4.5.

Therefore by Schauder's fixed point theorem, $S : X \to X$ has a fixed point $y \in X$.

This will imply that y is a solution of the BVP (4.18).

Proceding as in the proof of theorem 4.4.5, we get $\alpha(0) \leq y(0) \leq \beta(0)$.

This implies $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\, t \in [-r,\ 0]$.

We prove that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\, t \in (0,\ T]$.

Let $\epsilon > 0$ be arbitrary.

Define α_ϵ and β_ϵ as in the proof of theorem 3.4.3.

Since $\alpha(0) \leq y(0) \leq \beta(0)$, we have $\alpha_\epsilon(0) < \alpha(0) \leq y(0) \leq \beta(0) < \beta_\epsilon(0)$.

Therefore $\alpha_\epsilon(0) < y(0) < \beta_\epsilon(0)$.

So we prove that $\alpha_\epsilon(t) < y(t) < \beta_\epsilon(t),\ \forall\, t \in (0,\ T]$.

Let if possible $\exists\ t_1\ \in (0,\ T]$ such that $\alpha_\epsilon(t) < y(t) < \beta_\epsilon(t),\ \forall\, t \in (0,\ t_1)$ and $y(t_1) = \beta_\epsilon(t_1)$. Therefore $y(t_1) > \beta(t_1)$ and $\beta'_\epsilon(t_1) \leq y'(t_1)$.

CASE-1: Let $0 < t_1 \leq r$.

So we have $-r < t_1 - r \leq 0$.

This implies $\alpha(t_1 - r) \leq y(t_1 - r) \leq \beta(t_1 - r)$. Therefore

$$\begin{aligned} y'(t_1) &= F^\star(t_1,\ y(t_1),\ y(t_1 - r)) - \lambda y(t_1) \\ &= f(t_1,\ \beta(t_1),\ y(t_1 - r)) - \lambda y(t_1) \\ &\leq f(t_1,\ \alpha(t_1),\ \alpha(t_1 - r)) - \lambda \beta(t_1) \\ &\leq \beta'(t_1) < \beta'_\epsilon(t_1), \text{ which is a contradiction.} \end{aligned}$$

CASE-2: Let $r < t_1 \leq T$.

So we have $0 < t_1 - r \leq t_1$.

This implies $\alpha_\epsilon(t_1 - r) < y(t_1 - r) < \beta_\epsilon(t_1 - r)$.

CASE-2.1: Let $y(t_1 - r) < \alpha(t_1 - r)$. Then

$$\begin{aligned} y'(t_1) &= F^\star(t_1,\ \beta(t_1),\ y(t_1 - r)) - \lambda y(t_1) \\ &= f(t_1,\ \beta(t_1),\ \alpha(t_1 - r)) - \lambda y(t_1) \\ &\leq f(t_1,\ \beta(t_1),\ \alpha(t_1 - r)) - \lambda \beta(t_1) \\ &\leq \beta'(t_1) < \beta'_\epsilon(t_1), \text{ which is a contradiction.} \end{aligned}$$

CASE-2.2: Let $\alpha(t_1 - r) \leq y(t_1 - r) \leq \beta(t_1 - r)$. Then

$$\begin{aligned} y'(t_1) &= F^\star(t_1,\ y(t_1),\ y(t_1 - r)) - \lambda y(t_1) \\ &= f(t_1,\ \beta(t_1),\ y(t_1 - r)) - \lambda y(t_1) \\ &\leq f(t_1,\ \beta(t_1),\ \alpha(t_1 - r)) - \lambda\beta(t_1) \\ &\leq \beta'(t_1) < \beta'_\epsilon(t_1), \text{ which is a contradiction.} \end{aligned}$$

CASE-2.3: Let $y(t_1 - r) > \beta(t_1 - r)$. Then

$$\begin{aligned} y'(t_1) &= F^\star(t_1,\ y(t_1),\ y(t_1 - r)) - \lambda y(t_1) \\ &= f(t_1,\ \beta(t_1),\ \beta(t_1 - r)) - \lambda y(t_1) \\ &\leq f(t_1,\ \alpha(t_1),\ \alpha(t_1 - r)) - \lambda\beta(t_1) \\ &\leq \beta'(t_1) < \beta'_\epsilon(t_1), \text{ which is a contradiction.} \end{aligned}$$

From the above discussed cases we conclude that $\alpha_\epsilon(t) < y(t) < \beta_\epsilon(t),\ \forall\, t \in I$ and $\forall\, \epsilon > 0$. Thus by taking $\epsilon \to 0$, we get $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\, t \in I$. This implies $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\, t \in [-r,\ T]$. So we have $F^\star(t,\ y(t),\ y(t - r)) = f(t,\ y(t),\ y(t - r)),\ \forall\, t \in I$ and $p(T,\ y(T)) = y(T)$. Hence y is a solution of the PBVP (4.1) such that $\alpha(t) \leq y(t) \leq \beta(t)$, $\forall\, t \in [-r,\ T]$. □

Theorem 4.6.2. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \geq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta \in W$ be coupled lower and upper solutions of PBVP (4.6). Then the PBVP (4.6) has a solution $y \in W$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\, t \in I$.*

Proof. Similar to the proof of the Theorem 4.6.1. □

Theorem 4.6.3. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \geq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta \in X$ be coupled lower and upper solutions of APBVP (4.2). Then the APBVP (4.2) has a solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are coupled lower and upper solutions of APBVP (4.2), we have 1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \beta(t),\ \beta(t-r)),\ t \in I$.

2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I$.
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$.
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0]$.
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T]$.
6. $\alpha(0) \leq -\beta(T)$.
7. $\beta(0) \geq -\alpha(T)$.

As $\alpha,\ \beta \in X$, we have $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and
$\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$.
Let $p,\ F^\star,\ L,\ N$ and S be defined as in the proof of the Theorem 4.5.1.
By Schauder's fixed point theorem, $S : X \to X$ has a fixed point $y \in X$.
Therefore y is a solution of the BVP (4.19). Proceding as in the proof of theorem 4.5.1, we obtain $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$.
The remaining proof is similar to the proof of the Theorem 4.6.1. □

Theorem 4.6.4. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \geq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta \in W$ be coupled lower and upper solutions of APBVP (4.7). Then the APBVP (4.7) has a solution $y \in W$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in I$.*

Proof. Similar to the proof of the Theorem 4.6.3. □

Theorem 4.6.5. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \geq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta \in X$ be coupled lower and upper solutions of LBVP (4.3). Then the LBVP (4.3) has a solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are coupled lower and upper solutions of LBVP (4.3), we have 1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \beta(t),\ \beta(t-r)),\ t \in I$.

2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I$.
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$.
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0]$.
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T]$.
6. $\max\{a\alpha(0) + b\alpha(T), a\alpha(0) + b\beta(T)\}$
   $\leq 0 \leq \min\{a\beta(0) + b\alpha(T), a\beta(0) + b\beta(T)\}$.

As $\alpha,\ \beta \in X$, we have $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$.

Let $p,\ F^\star,\ L,\ N$ and S be defined as in theorem 4.5.3.

Therefore by Schauder's fixed point theorem, $S : X \to X$ has a fixed point $y \in X$. This will imply that y is a solution of the BVP (4.20). Proceding as in proof of theorem 3.5.2, we obtain $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$.

The remaining proof is similar to the proof of the Theorem 4.6.1. □

Theorem 4.6.6. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \geq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta \in W$ be coupled lower and upper solutions of LBVP (4.8). Then the LBVP (4.8) has a solution $y \in W$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$.*

Proof. Similar to the proof of the Theorem 4.6.5. □

Theorem 4.6.7. *Let* $f : I \times \mathbb{R}^2 \to \mathbb{R}$ *be continuous and satisfy* $f(t,\ u_1,\ v_1) \geq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ *such that* $u_1 \leq u_2$ *and* $v_1 \leq v_2$. *Let* $g : \mathbb{R}^2 \to \mathbb{R}$ *be continuous function which satisfies*
(1) $g(t,\ x) \leq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ *such that* $x \leq y$ *or*
(2) $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ *such that* $x \leq y$.
Let $\alpha,\ \beta \in X$ *be coupled lower and upper solutions of NLBVP* (4.4). *Then the NLBVP* (4.4) *has a solution* $y \in X$ *such that* $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$.

Proof. Without the loss of generality, let $g(t,\ x) \leq g(t,\ y)$,
$\forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$.
Since $\alpha,\ \beta \in X$ are coupled lower and upper solutions of NLBVP (4.4),
we have 1. $\alpha'(t) + \lambda\alpha(t) \leq f(t,\ \beta(t),\ \beta(t-r)),\ t \in I$,

2. $\beta'(t) + \lambda\beta(t) \geq f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I$,
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$,
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0]$,
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T]$,
6. $\max\{g(\alpha(0),\ \alpha(T)),\ g(\alpha(0),\ \beta(T))\}$
$\leq 0 \leq \min\{g(\beta(0),\ \alpha(T)),\ g(\beta(0),\ \beta(T))\}$.

As $\alpha,\ \beta \in X$, we have $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and
$\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$.
Let $p,\ F^\star,\ L,\ N$ and S be defined as in theorem 4.5.5.
Then by Schauder's fixed point theorem, $S : X \to X$ has a fixed point $y \in X$.
This will imply that y is a solution of the BVP (4.21). Proceding as in proof of the Theorem 3.5.3, we obtain $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$.
The remaining proof is similar to the proof of the Theorem 4.6.1. □

Theorem 4.6.8. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \geq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous function which satisfies (1) $g(t,\ x) \leq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$ or (2) $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$. Let $\alpha,\ \beta \in W$ be coupled lower and upper solutions of NLBVP (4.9). Then the NLBVP (4.9) has a solution $y \in W$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in I$.*

Proof. Similar to the proof of the Theorem 4.6.7. □

Next, we establish results on existence of solution between strict coupled lower and upper solution for PBVP, APBVP, LBVP and NLBVP of general first order DECD and first order DEVDA.

Theorem 4.6.9. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \geq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta \in X$ be strict coupled lower and upper solutions of PBVP (4.1). Then the PBVP (4.1) has a solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$ and $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are strict coupled lower and upper solutions of PBVP (4.1), we have 1. $\alpha'(t) + \lambda\alpha(t) < f(t,\ \beta(t),\ \beta(t-r)),\ t \in I$.

2. $\beta'(t) + \lambda\beta(t) > f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I$.
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$.
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0]$.
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T]$.
6. $\alpha(0) \leq \alpha(T)$.
7. $\beta(0) \geq \beta(T)$.

As $\alpha,\ \beta \in X$, we have $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$.

Let $p,\ F^\star,\ L,\ N$ and S be defined as in the proof of the Theorem 4.4.5.

Therefore by Schauder's fixed point theorem, $S : X \to X$ has a fixed point $y \in X$.

This will imply that y is a solution of the BVP (4.18). Proceding as in the proof of theorem 4.6.1, we get $\alpha(0) \leq y(0) \leq \beta(0)$.

Therefore $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$.

Next we show that $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.

Let if possible $\exists\ t_1\ \in (0,\ T]$ such that $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ t_1)$ and $y(t_1) = \beta(t_1)$. This implies that $\beta'(t_1) \leq y'(t_1)$.

CASE-1: Let $0 < t_1 \leq r$.

So we get $-r < t_1 - r \leq 0$.

This implies $\alpha(t_1 - r) \leq y(t_1 - r) \leq \beta(t_1 - r)$. Therefore

$$\begin{aligned} y'(t_1) &= F^{\star}(t_1,\ y(t_1),\ y(t_1 - r)) - \lambda y(t_1) \\ &= f(t_1,\ \beta(t_1),\ y(t_1 - r)) - \lambda y(t_1) \\ &\leq f(t_1,\ \alpha(t_1),\ \alpha(t_1 - r)) - \lambda\beta(t_1) \\ &< \beta'(t_1) \text{ which is a contradiction.} \end{aligned}$$

CASE-2: Let $r < t_1 \leq T$.

So we get $0 < t_1 - r \leq t_1$.

This implies $\alpha(t_1 - r) < y(t_1 - r) < \beta(t_1 - r)$. Therefore

$$\begin{aligned} y'(t_1) &= F^{\star}(t_1,\ y(t_1),\ y(t_1 - r)) - \lambda y(t_1) \\ &= f(t_1,\ \beta(t_1),\ y(t_1 - r)) - \lambda y(t_1) \\ &\leq f(t_1,\ \alpha(t_1),\ \alpha(t_1 - r)) - \lambda\beta(t_1) \\ &< \beta'(t_1), \text{ which is a contradiction.} \end{aligned}$$

From CASE-1 & CASE-2, we have $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.

Thus $F^{\star}(t,\ y(t),\ y(t - r)) = f(t,\ y(t),\ y(t - r)),\ \forall\ t \in I$ and

$p(T,\ y(T)) = y(T)$.

Hence y is a solution of the PBVP (4.1) such that $\alpha(t) \leq y(t) \leq \beta(t)$, $\forall\, t \in [-r,\ 0]$ and $\alpha(t) < y(t) < \beta(t),\ \forall\, t \in (0,\ T]$. □

Theorem 4.6.10. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \geq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta \in W$ be strict coupled lower and upper solutions of PBVP (4.6). Then the PBVP (4.6) has a solution $y \in W$ such that $\alpha(0) \leq y(0) \leq \beta(0)$ and $\alpha(t) < y(t) < \beta(t),\ \forall\, t \in (0,\ T]$.*

Proof. Similar to the proof of the Theorem 4.6.9. □

Theorem 4.6.11. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \geq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta \in X$ be strict coupled lower and upper solutions of APBVP (4.2). Then the APBVP (4.2) has a solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\, t \in [-r,\ 0]$ and $\alpha(t) < y(t) < \beta(t),\ \forall\, t \in (0,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are strict coupled lower and upper solutions of APBVP (4.2), we have 1. $\alpha'(t) + \lambda\alpha(t) < f(t,\ \beta(t),\ \beta(t-r)),\ t \in I$.

2. $\beta'(t) + \lambda\beta(t) > f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I$,
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$,
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0]$,
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T]$,
6. $\alpha(0) \leq -\beta(T)$,
7. $\beta(0) \geq -\alpha(T)$.

As $\alpha,\ \beta \in X$, we have $\alpha(t) = \alpha(0),\ \forall\, t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\, t \in [-r,\ 0]$.

Let $p,\ F^\star,\ L,\ N$ and S be defined as in theorem 4.5.1.

Therefore by Schauder's fixed point theorem, $S : X \to X$ has a fixed point $y \in X$. This will imply that y is a solution of the BVP (4.19). Proceding as in proof of the Theorem 3.5.1, we obtain $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\, t \in [-r, 0]$.

The remaining proof is similar to the proof of the Theorem 4.6.9. □

Theorem 4.6.12. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \geq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta \in W$ be strict coupled lower and upper solutions of APBVP (4.7). Then the APBVP (4.7) has a solution $y \in W$ such that $\alpha(0) \leq y(0) \leq \beta(0)$ and $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.*

Proof. Similar to the proof of the Theorem 4.6.11. □

Theorem 4.6.13. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \geq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta \in X$ be strict coupled lower and upper solutions of LBVP (4.8). Then the LBVP (4.8) has a solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$ and $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are strict coupled lower and upper solutions of LBVP (4.8), we have 1. $\alpha'(t) + \lambda\alpha(t) < f(t,\ \beta(t),\ \beta(t-r)),\ t \in I$.

2. $\beta'(t) + \lambda\beta(t) > f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I$.
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$.
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0]$.
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T]$.
6. $\max\{a\alpha(0) + b\alpha(T),\ a\alpha(0) + b\beta(T)\}$
$\leq 0 \leq \min\{a\beta(0) + b\alpha(T),\ a\beta(0) + b\beta(T)\}$.

As $\alpha,\ \beta \in X$, we have $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$.

Let $p,\ F^\star,\ L,\ N$ and S be defined as in theorem 4.5.3. Then by Schauder's fixed point theorem, $S : X \to X$ has a fixed point $y \in X$. This implies that y is a solution of the BVP (4.20). Proceding as in proof of theorem 3.5.2, we obtain $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$.

The remaining proof is similar to the proof of the Theorem 4.6.9. □

Theorem 4.6.14. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \geq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta \in W$ be strict coupled lower and upper solutions of LBVP (4.8). Then the LBVP (4.8) has a solution $y \in W$ such that $\alpha(0) \leq y(0) \leq \beta(0)$ and $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.*

Proof. Similar to the proof of the Theorem 4.6.13. □

Theorem 4.6.15. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \geq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous function which satisfies*
(1) $g(t,\ x) \leq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$ or
(2) $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t, y) \in \mathbb{R}^2$ such that $x \leq y$.
Let $\alpha,\ \beta \in X$ be strict coupled lower and upper solutions of NLBVP (4.4). Then the NLBVP (4.4) has a solution $y \in X$ such that
$\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$ and $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.

Proof. Without the loss of generality, let $g(t,\ x) \leq g(t,\ y)$,
$\forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$.
Since $\alpha,\ \beta \in X$ are strict coupled lower and upper solutions of NLBVP (4.4), we have 1. $\alpha'(t) + \lambda\alpha(t) < f(t,\ \beta(t),\ \beta(t-r)),\ t \in I$.

2. $\beta'(t) + \lambda\beta(t) > f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I$.
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,\ 0]$.
4. $\beta(t) \geq \beta(0),\ t \in [-r,\ 0]$.
5. $\alpha(t) \leq \beta(t),\ t \in [-r,\ T]$.
6. $\max\{g(\alpha(0),\ \alpha(T)),\ g(\alpha(0),\ \beta(T))\}$
$\leq 0 \leq \min\{g(\beta(0),\ \alpha(T)),\ g(\beta(0),\ \beta(T))\}$.

As $\alpha,\ \beta \in X$, we have $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and
$\beta(t) = \beta(0),\ \forall\ t \in [-r,\ 0]$.
Let $p,\ F^\star,\ L,\ N$ and S be defined as in theorem 4.5.5. Then by Schauder's fixed point theorem, $S : X \to X$ has a fixed point $y \in X$. This will imply that

y is a solution of the BVP (4.21). Proceding as in proof of theorem 3.5.3, we obtain $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$.

The remaining proof is similar to the proof of theorem 4.6.9. □

Theorem 4.6.16. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy*

$f(t,\ u_1,\ v_1) \geq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous function which satisfies

(1) $g(t,\ x) \leq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$ or

(2) $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t, y) \in \mathbb{R}^2$ such that $x \leq y$.

Let $\alpha,\ \beta \in W$ be strict coupled lower and upper solutions of NLBVP (4.9). Then the NLBVP (4.9) has a solution $y \in W$ such that $\alpha(0) \leq y(0) \leq \beta(0)$ and $\alpha(t) < y(t) < \beta(t),\ \forall\ t \in (0,\ T]$.

Proof. Similar to the proof of the Theorem 4.6.15. □

4.7 Existence Theorem using Fixed Point Theorem in Partially Ordered Metric Space

In this section, we obtain a result on existence of solution for PBVP of first order DEVDA (4.6) using fixed point theorem on partially ordered metric space.

Theorem 4.7.1. *Let $\lambda > 0,\ \mu_1 > 0,\ \mu_2 > 0$ and $\mu = \max\{\mu_1,\ \mu_2\}$ be such that $\frac{2\mu}{\lambda} < 1$. Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy*

$0 \leq f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \leq \mu_1(x_1 - x_2) + \mu_2(y_1 - y_2)$,

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$. If PBVP (4.6) has lower solution or an upper solution then the PBVP (4.6) has unique solution in $(Z,\ ||.||)$.

Proof. Let $J : Z \to Z$ be defined by

$J[y(t)] = \int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} f(s,\ y(s),\ y(\theta(s)))ds\ +\ \int_0^t e^{-\lambda(t-s)} f(s,\ y(s),\ y(\theta(s)))ds.$

A fixed point of J is a solution of the PBVP (4.6).

Let $x,\ y \in Z$ be such that $x \geq y$. Then

$$
\begin{aligned}
0 &\leq J[x(t)] - J[y(t)] \\
&= \int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} f(s,\ x(s),\ x(\theta(s)))ds\ +\ \int_0^t e^{-\lambda(t-s)} f(s,\ x(s),\ x(\theta(s)))ds \\
&\quad - \int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} f(s,\ y(s),\ y(\theta(s)))ds\ -\ \int_0^t e^{-\lambda(t-s)} f(s,\ y(s),\ y(\theta(s)))ds \\
&= \int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} [f(s,\ x(s),\ x(\theta(s))) - f(s,\ y(s),\ y(\theta(s)))]ds \\
&\quad + \int_0^t e^{-\lambda(t-s)} [f(s,\ x(s),\ x(\theta(s))) - f(s,\ y(s),\ y(\theta(s)))]ds \\
&= \int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} [f(s,\ x(s),\ x(\theta(s))) - f(s,\ y(s),\ y(\theta(s)))]ds \\
&\quad + \int_0^t e^{-\lambda(t-s)} [f(s,\ x(s),\ x(\theta(s))) - f(s,\ y(s),\ y(\theta(s)))]ds \\
&\leq \int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} \{\mu_1[x(s) - y(s)] + \mu_2[x(\theta(s)) - y(\theta(s))]\}ds \\
&\quad + \int_0^t e^{-\lambda(t-s)} \{\mu_1[x(s) - y(s)] + \mu_2[x(\theta(s)) - y(\theta(s))]\}ds \\
&\leq 2\mu||x-y||\{\int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} ds\ +\ \int_0^t e^{-\lambda(t-s)} ds\} \leq \frac{2\mu}{\lambda}||x-y||.
\end{aligned}
$$

So we have $0 \leq J[x(t)] - J[y(t)] \leq \frac{2\mu}{\lambda}||x-y||,\ \forall\ t \in I.$

Thus $||Jx - Jy|| \leq \frac{2\mu}{\lambda}||x-y||,\ \forall\ x,\ y \in Z$ such that $x \geq y$ and $\frac{2\mu}{\lambda} < 1.$

Also $J : Z \to Z$ is an order preserving operator.

Let $u \in Z$ be a lower solution of the PBVP (4.6).

Then $u'(t) + \lambda u(t) \leq f(t,\ u(t),\ u(\theta(t)))$, $t \in I$ and $u(0) \leq u(T)$.

This implies $\dfrac{d}{dt}[e^{\lambda t}u(t)] \leq e^{\lambda t} f(t,\ u(t),\ u(\theta(t)))$. (4.22)

Integrating the inequality (4.22) from 0 to t, we get

$$e^{\lambda t}u(t) - u(0) \leq \int_0^t e^{\lambda s} f(t,\ u(s),\ u(\theta(s)))ds.$$

So we have $u(t) \leq u(0)e^{-\lambda t} + \displaystyle\int_0^t e^{-\lambda(t-s)} f(t, u(s), u(\theta(s)))ds.$ (4.23)

Putting $t = T$ in (4.23) we get

$$u(T) \leq u(0)e^{-\lambda T} + \int_0^T e^{-\lambda(T-s)} f(t,\ u(s),\ u(\theta(s)))ds.$$

Therefore $u(0) \leq u(T) \leq u(0)e^{-\lambda T} + \displaystyle\int_0^T e^{-\lambda(T-s)} f(t,\ u(s),\ u(\theta(s)))ds.$

Thus $u(0) \leq \dfrac{1}{1 - e^{-\lambda T}} \displaystyle\int_0^T e^{-\lambda(T-s)} f(s, u(s), u(\theta(s)))ds.$ (4.24)

Substituting (4.24) in (4.23) we get

$$\begin{aligned} u(t) &\leq \int_0^T \frac{e^{-\lambda(T-s+t)}}{1 - e^{-\lambda T}} f(s,\ u(s),\ u(\theta(s)))ds \\ &\quad + \int_0^t e^{-\lambda(t-s)} f(s,\ u(s),\ u(\theta(s)))ds \\ &= J[u(t)]. \end{aligned}$$

So we have $u \leq Ju$.

Therefore by using fixed point theorem in partially ordered metric space,

$J : Z \to Z$ has a unique fixed point $y \in Z$.

Thus y is a unique solution of the PBVP (4.6). □

4.8 Existence Theorem using Banach Fixed Point Theorem

In this section, we obtain establish a result on existence of solution for PBVP of first order DEVDA (4.6) using Banach fixed point theorem.

Theorem 4.8.1. *Let* $\lambda > 0,\ \mu_1 > 0,\ \mu_2 > 0$ *and* $\mu = \max\{\mu_1,\ \mu_2\}$ *be such that* $\frac{2\mu}{\lambda} < 1$. *Let* $f : I \times \mathbb{R}^2 \to \mathbb{R}$ *be continuous and satisfy*

$|f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2)| \leq \mu_1|x_1 - x_2| + \mu_2|y_1 - y_2|,$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$. *Then the PBVP* (4.6) *has unique solution in* $(Z,\ ||.||)$.

Proof. Let $J : Z \to Z$ be defined by

$J[y(t)] = \int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} f(s,\ y(s),\ y(\theta(s)))ds\ +\ \int_0^t e^{-\lambda(t-s)} f(s,\ y(s),\ y(\theta(s)))ds.$

A fixed point of J is a solution of the PBVP (4.6).

Let $x,\ y \in Z$. Then

$$\begin{aligned} |J[x(t)] - J[y(t)]| = |&\int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} f(s,\ x(s),\ x(\theta(s)))ds \\ &+ \int_0^t e^{-\lambda(t-s)} f(s,\ x(s),\ x(\theta(s)))ds \\ &- \int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} f(s,\ y(s),\ y(\theta(s)))ds \\ &- \int_0^t e^{-\lambda(t-s)} f(s,\ y(s),\ y(\theta(s)))ds|. \end{aligned}$$

So we have

$$
\begin{aligned}
|J[x(t)] - J[y(t)]| &\leq \int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} |f(s,\ x(s),\ x(\theta(s))) - f(s,\ y(s),\ y(\theta(s)))| ds \\
&\quad + \int_0^t e^{-\lambda(t-s)} |f(s,\ x(s),\ x(\theta(s))) - f(s,\ y(s),\ y(\theta(s)))| ds \\
&\leq \int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} \{\mu_1 |x(s) - y(s)| + \mu_2 |x(\theta(s)) - y(\theta(s))|\} ds \\
&\quad + \int_0^t e^{-\lambda(t-s)} \{\mu_1 |x(s) - y(s)| + \mu_2 |x(\theta(s)) - y(\theta(s))|\} ds \\
&\leq 2\mu ||x - y|| \{ \int_0^T \frac{e^{-\lambda(T-s+t)}}{1-e^{-\lambda T}} ds \ + \ \int_0^t e^{-\lambda(t-s)} ds \} \\
&\leq \frac{2\mu}{\lambda} ||x - y||.
\end{aligned}
$$

Therefore $|J[x(t)] - J[y(t)| \leq \frac{2\mu}{\lambda} ||x - y||,\ \forall\ t \in I.$

Thus $||Jx - Jy|| \leq \frac{2\mu}{\lambda} ||x - y||,\ \forall\ x,\ y \in Z$ and $\frac{2\mu}{\lambda} < 1.$

So $J : Z \to Z$ is a contraction mapping.

Therefore by Banach fixed point theorem $J : Z \to Z$ has a unique fixed point $y \in Z$.

Thus y is a unique solution of the PBVP (4.6). □

Chapter 5

MONOTONE ITERATIVE METHOD AND QUASILINEARIZATION METHOD FOR FIRST ORDER DIFFERENTIAL EQUATIONS WITH CONSTANT DELAY

Some content of this chapter is published [see P3].

Some content of this chapter is presented at the Conference [see C2 and C4].

5.1 Introduction

This chapter deals with developing the Monotone Iterative Method and Quasilinearization method to establish the existence of solutions for BVPs discussed in Chapter 3. We make use of results on existence of solution between lower and upper solution for PBVP of first order DECD for our purpose. For LBVP and NLBVP of first order DECD we use theorems on existence of solution between weakly coupled lower and upper solution for establishing existence of solution using monotone iterative method and quasilinearization method. We assume $b < 0$ for LBVP and g to be a non-increasing function in second argument in the case of NLBVP.

5.2 Monotone Iterative Method

We now establish the existence of solution between lower and upper solutions using monotone iterative method for PBVP of first order DECD (3.1).

Theorem 5.2.1. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(t,\ x_1) - f(t,\ x_2) \geq \mu(x_1 - x_2),\ \forall\ (t,\ x_1),\ (t,\ x_2) \in I \times \mathbb{R}^2$ such that $x_2 \leq x_1$ and for some constant $\mu > 0$. Let $\alpha_0,\ \beta_0 \in X$ be lower and upper solutions of the PBVP (3.1) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$. Then there are sequences of monotonic functions $\{\alpha_n\}$ and $\{\beta_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim_{n\to\infty} \alpha_n$ and $\lim_{n\to\infty} \beta_n$ are solutions of the PBVP (3.1).*

Proof. Since, $\alpha_0,\ \beta_0 \in X$ are lower and upper solutions of PBVP (3.1), we have 1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t-r)),\ t \in I.$

2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$

3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0].$

4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0].$

5. $\alpha_0(0) \leq \alpha_0(T).$

6. $\beta_0(0) \geq \beta_0(T).$

As $\alpha_0,\ \beta_0 \in X$, we get $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and $\beta_0(t) = \beta_0(0),$
$\forall\ t \in [-r,\ 0].$

Consider

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ \alpha_0(t-r)) + \mu[y(t-r) - \alpha_0(t-r)],\ t \in I,\\ &y(t) = y(0),\ t \in [-r,\ 0],\\ &y(0) = y(T). \end{aligned}\right\} \qquad (5.1)$$

Let $g_{\alpha_0}(t,\ u) = f(t,\ \alpha_0(t-r)) + \mu[u - \alpha_0(t-r)],\ \forall\ (t,\ u) \in I \times \mathbb{R}.$

Therefore $g_{\alpha_0}(t,\ y(t-r)) = f(t,\ \alpha_0(t-r)) + \mu[y(t-r) - \alpha_0(t-r)],\ \forall\ t \in I.$

Let $(t,\ x_1),\ (t,\ x_2) \in I \times \mathbb{R}$ such that $x_1 \leq x_2$.

This implies $x_1 - \alpha_0(t-r) \leq x_2 - \alpha_0(t-r)$.

This gives $\mu[x_1 - \alpha_0(t-r)] \leq \mu[x_2 - \alpha_0(t-r)]$.

Therefore $f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu[x_1 - \alpha_0(t-r)]$

$$\leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu[x_2 - \alpha_0(t-r)].$$

Thus $g_{\alpha_0}(t,\ x_1) \leq g_{\alpha_0}(t,\ x_2),\ \forall\ (t,\ x_1),\ (t,\ x_2) \in I \times \mathbb{R}$ such that $x_1 \leq x_2$.

$$\begin{aligned} \text{Also } \alpha_0'(t) + \lambda\alpha_0(t) &\leq f(t,\ \alpha_0(t-r))\\ &= f(t,\ \alpha_0(t-r)) + \mu[\alpha_0(t-r) - \alpha_0(t-r)]\\ &= g_{\alpha_0}(t,\ \alpha_0(t-r)). \end{aligned}$$

$$\begin{aligned} \text{Further } \beta_0'(t) + \lambda\beta_0(t) &\geq f(t,\ \beta_0(t-r))\\ &\geq f(t,\ \alpha_0(t-r)) + \mu[\beta_0(t-r) - \alpha_0(t-r)]. \end{aligned}$$

Therefore $\beta_0'(t) + \lambda\beta_0(t) \geq g_{\alpha_0}(t,\ \beta_0(t-r))$.

We already know that

1. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
2. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
3. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
4. $\alpha_0(0) \leq \alpha_0(T)$.
5. $\beta_0(0) \geq \beta_0(T)$.

This implies α_0 and β_0 are lower and upper solutions of the PBVP (5.1) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\, t \in [-r,\ T]$.

Therefore the PBVP (5.1) has a solution $\alpha_1 \in X$ such that $\alpha_0 \leq \alpha_1 \leq \beta_0$ on $[-r,\ T]$. (Using Theorem 3.4.3)

So we have

$$\left.\begin{aligned} &\alpha_1'(t) + \lambda\alpha_1(t) = f(t,\ \alpha_0(t-r)) + \mu[\alpha_1(t-r) - \alpha_0(t-r)],\ t \in I, \\ &\alpha_1(t) = \alpha_1(0),\ t \in [-r,\ 0], \\ &\alpha_1(0) = \alpha_1(T). \end{aligned}\right\}$$

This gives $\alpha_1'(t) + \lambda\alpha_1(t) \leq f(t,\ \alpha_1(t-r)),\ t \in I$.

Also 1. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t-r)),\ t \in I$.

2. $\alpha_1(t) = \alpha_1(0),\ t \in [-r,\ 0]$.
3. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
4. $\alpha_1(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
5. $\alpha_1(0) \leq \alpha_1(T)$.
6. $\beta_0(0) \geq \beta_0(T)$.

Therefore α_1 and $\beta_0 \in X$ are lower and upper solutions of the PBVP (3.1) such that $\alpha_1(t) \leq \beta_0(t),\ \forall\, t \in [-r,\ T]$.

Now Consider

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ \alpha_1(t-r)) + \mu[y(t-r) - \alpha_1(t-r)],\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = y(T). \end{aligned}\right\} \tag{5.2}$$

Let $g_{\alpha_1}(t,\ u) = f(t,\ \alpha_1(t-r)) + \mu[u - \alpha_1(t-r)],\ \forall\ (t,\ u) \in I \times \mathbb{R}$.

Therefore $g_{\alpha_1}(t,\ y(t-r)) = f(t,\ \alpha_1(t-r)) + \mu[y(t-r) - \alpha_1(t-r)],\ \forall\ t \in I$.

So we have $g_{\alpha_1}(t,\ x_1) \leq g_{\alpha_1}(t,\ x_2),\ \forall\ (t,\ x_1),\ (t,\ x_2) \in I \times \mathbb{R}$ such that $x_1 \leq x_2$.

Also $\alpha_1'(t) + \lambda\alpha_1(t) \leq f(t,\ \alpha_1(t-r))$

$$\begin{aligned} &= f(t,\ \alpha_1(t-r)) + \mu[\alpha_1(t-r) - \alpha_1(t-r)] \\ &= g_{\alpha_1}(t,\ \alpha_1(t-r)). \end{aligned}$$

Further $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t-r))$

$$\begin{aligned} &\geq f(t,\ \alpha_1(t-r)) + \mu[\beta_0(t-r) - \alpha_1(t-r)] \\ &= g_{\alpha_1}(t,\ \beta_0(t-r)). \end{aligned}$$

We already know that

1. $\alpha_1(t) = \alpha_1(0),\ t \in [-r,\ 0]$.
2. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
3. $\alpha_1(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
4. $\alpha_1(0) = \alpha_1(T)$.
5. $\beta_0(0) \geq \beta_0(T)$.

This implies α_1 and β_0 are lower and upper solutions of the PBVP (5.2) such that $\alpha_1(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Therefore the PBVP (5.2) has a solution $\alpha_2 \in X$ such that $\alpha_1 \leq \alpha_2 \leq \beta_0$ on $[-r,\ T]$. (Using Theorem 3.4.3)

This gives

$$\left.\begin{aligned} &\alpha_2'(t) + \lambda\alpha_2(t) = f(t,\ \alpha_1(t-r)) + \mu[\alpha_2(t-r) - \alpha_1(t-r)],\ t \in I, \\ &\alpha_2(t) = \alpha_2(0),\ t \in [-r,\ 0], \\ &\alpha_2(0) = \alpha_2(T). \end{aligned}\right\}$$

So we have $\alpha_2'(t) + \lambda\alpha_2(t) \leq f(t,\ \alpha_2(t-r)),\ t \in I.$

Also 1. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t-r)),\ t \in I.$

2. $\alpha_2(t) = \alpha_2(0),\ t \in [-r,\ 0].$

3. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0].$

4. $\alpha_2(t) \leq \beta_0(t),\ t \in [-r,\ T].$

5. $\alpha_2(0) = \alpha_2(T).$

6. $\beta_0(0) \geq \beta_0(T).$

Therefore α_2 and $\beta_0 \in X$ are lower and upper solutions of the PBVP (3.1) such that $\alpha_2(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Continuing in this manner, let $\alpha_n \in X$ be a solution of

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ \alpha_{n-1}(t-r)) \\ &\qquad\qquad + \mu[y(t-r) - \alpha_{n-1}(t-r)],\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = y(T),\ \text{such that } \alpha_{n-1} \leq \alpha_n \leq \beta_0 \text{ on } [-r,\ T]. \end{aligned}\right\} \tag{5.3}$$

Next consider

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ \alpha_n(t-r)) + \mu[y(t-r) - \alpha_n(t-r)],\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = y(T). \end{aligned}\right\} \tag{5.4}$$

Let $g_{\alpha_n}(t,\ u) = f(t,\ \alpha_n(t-r)) + \mu[u - \alpha_n(t-r)],\ \forall\ (t,\ u) \in I \times \mathbb{R}.$

Therefore $g_{\alpha_n}(t,\ y(t-r)) = f(t,\ \alpha_n(t-r)) + \mu[y(t-r) - \alpha_n(t-r)],\ \forall\ t \in I.$

Also $g_{\alpha_n}(t,\ x_1) \leq g_{\alpha_n}(t,\ x_2),\ \forall\ (t,\ x_1),\ (t,\ x_2) \in I \times \mathbb{R}^2$ such that $x_1 \leq x_2$.

Further α_n and β_0 are lower and upper solutions of the PBVP (5.4) such that $\alpha_n(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Therefore the PBVP (5.4) has a solution $\alpha_{n+1} \in X$ such that $\alpha_n \leq \alpha_{n+1} \leq \beta_0$ on $[-r,\ T]$. (Using Theorem 3.4.3)

This gives

$$\left.\begin{aligned} &\alpha'_{n+1}(t) + \lambda\alpha_{n+1}(t) = f(t,\ \alpha_n(t-r)) + \mu[\alpha_{n+1}(t-r) - \alpha_n(t-r)],\ t \in I, \\ &\alpha_{n+1}(t) = \alpha_{n+1}(0),\ t \in [-r,\ 0], \\ &\alpha_{n+1}(0) = \alpha_{n+1}(T). \end{aligned}\right\}$$

Thus we get a monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} &\alpha'_{n+1}(t) + \lambda\alpha_{n+1}(t) = f(t,\ \alpha_n(t-r)) \\ &\qquad\qquad + \mu[\alpha_{n+1}(t-r) - \alpha_n(t-r)],\ t \in I, \\ &\alpha_{n+1}(t) = \alpha_{n+1}(0),\ t \in [-r,\ 0], \\ &\alpha_{n+1}(0) = \alpha_{n+1}(T). \end{aligned}\right\} \tag{5.5}$$

Now for each fixed $t \in [-r,\ T]$, $\{\alpha_n(t)\}$ is a monotonic increasing sequence in $\mathbb{R}$ which is bounded above by $\beta_0(t)$. Therefore by Monotone Convergence Theorem for sequences in $\mathbb{R}$, $\{\alpha_n(t)\}$ is convergent in $\mathbb{R}$, for each $t \in [-r,\ T]$. This implies $\{\alpha_n\}$ is pointwise convergent on $[-r,\ T]$.

Let $\lim\limits_{n\to\infty} \alpha_n(t) = \alpha(t),\ \forall\ t \in [-r,\ T]$.

Since $\alpha'_{n+1}(t) + \lambda\alpha_{n+1}(t) = f(t,\ \alpha_n(t-r)) + \mu_2[\alpha_{n+1}(t-r) - \alpha_n(t-r)]$,

$$\text{we have } \alpha_{n+1}(t)e^{\lambda t} - \alpha_{n+1}(0) = \int_0^t e^{\lambda s} f(s,\ \alpha_n(s-r))ds + \int_0^t e^{\lambda s}\mu[\alpha_{n+1}(s-r) - \alpha_n(s-r)]ds.$$

$$\text{This implies } \alpha_{n+1}(t) = \alpha_{n+1}(0)e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} f(s,\ \alpha_n(s-r))ds + \int_0^t e^{-\lambda(t-s)} \mu[\alpha_{n+1}(s-r) - \alpha_n(s-r)]ds. \tag{5.6}$$

Taking limit as $n \to \infty$ on both sides of equation (5.6) we get

$$\lim_{n\to\infty} \alpha_{n+1}(t) = \lim_{n\to\infty} \alpha_{n+1}(0)e^{-\lambda t} + \lim_{n\to\infty} \int_0^t e^{-\lambda(t-s)} f(s,\ \alpha_n(s-r))ds + \lim_{n\to\infty} \int_0^t e^{-\lambda(t-s)} \mu[\alpha_{n+1}(s-r) - \alpha_n(s-r)]ds. \tag{5.7}$$

Now $\{\alpha_n\}$ is uniformly bounded on $[-r,\ T]$, as $\alpha_0(t) \le \alpha_n(t) \le \beta_0(t)$, $\forall\ t \in [-r,\ T]$. Therefore $\exists\ M > 0$ such that $|\alpha_n(t)| \le M,\ \forall\ t \in [-r,\ T]$ and $\forall\ n \in \mathbb{N}$. Since $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous, $f : I \times [-M,\ M] \to \mathbb{R}$ is continuous. Therefore $f : I \times [-M,\ M] \to \mathbb{R}$ is bounded, as $I \times [-M,\ M]$ is compact. So there is $K > 0$ such that $|f(t,\ \alpha_n(t-r))| \le K,\ \forall\ t \in I$ and $\forall\ n \in \mathbb{N}$.

From equation (5.5), we have

$$\begin{aligned} |\alpha'_{n+1}(t)| &\le |f(t,\ \alpha_n(t-r))| + |\lambda||\alpha_{n+1}(t)| + \mu[|\alpha_{n+1}(t-r)| + |\alpha_n(t-r)|] \\ &\le K + 2\mu M + |\lambda|M = C(\text{say}). \end{aligned}$$

This implies $|\alpha'_n(t)| \le C,\ \forall\ n \in \mathbb{N}$ and $\forall\ t \in I$.

Thus $\{\alpha'_n\}$ is uniformly bounded on I.

We now show that $\{\alpha_n\}$ is equicontinuous on I.

Let $\epsilon > 0$ be arbitrary. Choose $\delta = \min\{\frac{\epsilon}{C}, \frac{T}{2}\}$.

Let $t_1,\ t_2 \in I$ be such that $|t_1 - t_2| < \delta$.

By Lagrange's Mean Value Theorem, $\exists\, \xi$ between t_1 and t_2 such that

$|\alpha_n(t_1) - \alpha_n(t_2)| = |\alpha_n'(\xi)||t_1 - t_2|$.

This implies $|\alpha_n(t_1) - \alpha_n(t_2)| \leq C|t_1 - t_2| < C\delta \leq \epsilon,\ \forall\ t_1,\ t_2 \in I$ such that $|t_1 - t_2| < \delta$. Therefore $\{\alpha_n\}$ is equicontinuous on I.

Using Arzela-Ascoli's lemma we may conclude that $\{\alpha_n\}$ has a subsequence $\{\alpha_{n_k}\}$ converging uniformly on I. So $\alpha_{n_k} \to \alpha$ uniformly on I. Since each α_{n_k} is continuous on I, α is continuous on I. Also $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\alpha(0) = \alpha(T)$. So we conclude that α is continuous on $[-r,\ T]$. Since $\{\alpha_n\}$ is increasing and $\lim\limits_{n\to\infty} \alpha_n(t) = \alpha(t),\ \forall\ t \in I$, by Dini's theorem for uniform convergence of sequence of functions, $\alpha_n \to \alpha$ uniformly on I.

From equation (5.7), we get

$$\lim_{n\to\infty} \alpha_{n+1}(t) = \lim_{n\to\infty} \alpha_{n+1}(0)e^{-\lambda t} + \int_0^t \lim_{n\to\infty} e^{-\lambda(t-s)} f(s,\ \alpha_n(s-r))ds$$
$$+ \int_0^t \lim_{n\to\infty} e^{-\lambda(t-s)} \mu[\alpha_{n+1}(s-r) - \alpha_n(s-r)]ds.$$

So we have $\alpha(t) = \alpha(0)e^{-\lambda t} + \int\limits_0^t e^{-\lambda(t-s)} f(s,\ \alpha(s-r))ds,\ t \in I$.

Also $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\alpha(0) = \alpha(T)$.

Taking limit as $n \to \infty$ on both sides of equation (5.5), we get

$$\lim_{n\to\infty} \alpha_{n+1}'(t) = -\lambda \lim_{n\to\infty} \alpha_{n+1}(t) + \lim_{n\to\infty} f(t,\ \alpha_n(t-r))$$
$$+ \lim_{n\to\infty} \mu_2[\alpha_{n+1}(t-r) - \alpha_n(t-r)].$$

Therefore $\lim\limits_{n\to\infty} \alpha_{n+1}'(t) = -\lambda\alpha(t) + f(t,\ \alpha(t-r)) = \alpha'(t),\ \forall\ t \in I$.

Since the left hand side of the above limit is the uniform limit of sequence of functions, we have $\{\alpha_n'\}$ converging uniformly to α' on I.

As $\alpha_n \to \alpha$ pointwise on $[-r,\ T]$, $\alpha_n \to \alpha$ uniformly on I,

$\alpha_n(t) = \alpha_n(0),\ \forall\ t \in [-r,\ 0]$ and $\forall\ n \in \mathbb{N}$ and $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$,

we have $\alpha_n \to \alpha$ uniformly on $[-r,\ T]$. This implies α is continuous on $[-r,\ T]$.

Also α is differentiable on I and α' is continuous on I. Thus $\alpha \in X$.

As $\alpha_n \to \alpha$ uniformly on $[-r,\ T]$ and $\alpha_n' \to \alpha'$ uniformly on I, we have $\alpha_n \to \alpha$ on $(X,\ ||.||_\infty)$ and α is a solution of the PBVP (3.1).

Next consider

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ \beta_0(t-r)) + \mu[y(t-r) - \beta_0(t-r)],\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = y(T). \end{aligned}\right\} \tag{5.8}$$

Let $h_{\beta_0}(t,\ u) = f(t,\ \beta_0(t-r)) + \mu[u - \beta_0(t-r)],\ \forall\ (t,\ u) \in I \times \mathbb{R}$.

Therefore $h_{\beta_0}(t,\ y(t-r)) = f(t,\ \beta_0(t-r)) + \mu[y(t-r) - \beta_0(t-r)],\ \forall\ t \in I$.

So we have $h_{\beta_0}(t,\ x_1) \leq h_{\beta_0}(t,\ x_2),\ \forall\ (t,\ x_1),\ (t,\ x_2) \in I \times \mathbb{R}^2$ such that $x_1 \leq x_2$.

$$\begin{aligned} \text{Also } \alpha_0'(t) + \lambda\alpha_0(t) &\leq f(t,\ \alpha_0(t-r)) \\ &\leq f(t,\ \beta_0(t-r)) + \mu[\alpha_0(t-r) - \beta_0(t-r)] \\ &= h_{\beta_0}(t,\ \alpha_0(t-r)). \end{aligned}$$

$$\begin{aligned} \text{Further } \beta_0'(t) + \lambda\beta_0(t) &\geq f(t,\ \beta_0(t-r)) \\ &= f(t,\ \beta_0(t-r)) + \mu[\beta_0(t-r) - \beta_0(t-r)] \\ &= h_{\beta_0}(t,\ \beta_0(t-r)). \end{aligned}$$

We already know that

1. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
2. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
3. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
4. $\alpha_0(0) \leq \alpha_0(T)$.
5. $\beta_0(0) \geq \beta_0(T)$.

This implies α_0 and β_0 are lower and upper solutions of the PBVP (5.8) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Therefore the PBVP (5.8) has a solution $\beta_1 \in X$ such that $\alpha_0 \leq \beta_1 \leq \beta_0$ on $[-r,\ T]$. (Using Theorem 3.4.3)

So we have

$$\left.\begin{aligned} &\beta_1'(t) + \lambda\beta_1(t) = f(t,,\ \beta_0(t-r)) + \mu[\beta_1(t-r) - \beta_0(t-r)],\ t \in I, \\ &\beta_1(t) = \beta_1(0),\ t \in [-r,\ 0], \\ &\beta_1(0) = \beta_1(T). \end{aligned}\right\}$$

This gives $\beta_1'(t) + \lambda\beta_1(t) \geq f(t,\ \beta_1(t-r)),\ t \in I$.

Also 1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t-r)),\ t \in I$.

2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
3. $\beta_1(t) = \beta_1(0),\ t \in [-r,\ 0]$.
4. $\alpha_0(t) \leq \beta_1(t),\ t \in [-r,\ T]$.
5. $\alpha_0(0) \leq \alpha_0(T)$.
6. $\beta_1(0) \geq \beta_1(T)$.

Therefore α_0 and $\beta_1 \in X$ are lower and upper solutions of the PBVP (3.1) such that $\alpha_1(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Continuing in this manner, we get a monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\alpha_0 \leq \beta_n,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} &\beta_{n+1}'(t) + \lambda\beta_{n+1}(t) = f(t,\ \beta_n(t-r)) + \mu[\beta_{n+1}(t-r) - \beta_n(t-r)],\ t \in I, \\ &\beta_{n+1}(t) = \beta_{n+1}(0),\ t \in [-r,\ 0], \\ &\beta_{n+1}(0) = \beta_{n+1}(T). \end{aligned}\right\}$$

Further $\{\beta_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty}\beta_n$ is a solution of the PBVP (3.1). □

Next, we have established result on existence of solution using monotone iterative method for the LBVP (3.3) of first order DECD.

Theorem 5.2.2. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(t,\ x_1) - f(t,\ x_2) \geq \mu(x_1 - x_2),\ \forall\ (t,\ x_1),\ (t,\ x_2) \in I \times \mathbb{R}$ such that $x_2 \leq x_1$ and for some constants $\mu > 0$. Let $\alpha_0,\ \beta_0 \in X$ be weakly coupled lower and upper solutions of LBVP given by (3.3). If $b < 0$, then there are sequences of monotonic functions $\{\alpha_n\}$ and $\{\beta_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim_{n\to\infty} \alpha_n$ and $\lim_{n\to\infty} \beta_n$ are solutions of the LBVP (3.3).*

Proof. Since, $\alpha_0,\ \beta_0 \in X$ are weakly coupled lower and upper solutions of LBVP (3.3), we have 1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t-r)),\ t \in I.$

2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t-r)),\ t \in I.$

3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0].$

4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0].$

5. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T].$

6. $\max\{a\alpha_0(0) + b\alpha_0(T),\ a\alpha_0(0) + b\beta_0(T)\}$
$\leq 0 \leq \min\{a\beta_0(0) + b\alpha_0(T),\ a\beta_0(0) + b\beta_0(T)\}.$

As $b < 0$, we have

$\max\{a\alpha_0(0) + b\alpha_0(T),\ a\alpha_0(0) + b\beta_0(T)\} = a\alpha_0(0) + b\alpha_0(T)$ and

$\min\{a\beta_0(0) + b\alpha_0(T),\ a\beta_0(0) + b\beta_0(T)\} = a\beta_0(0) + b\beta_0(T).$

This implies $a\alpha_0(0) + b\alpha_0(T) \leq 0 \leq a\beta_0(0) + b\beta_0(T)$.

Also, as $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and $\beta_0(t) = \beta_0(0),$ $\forall\ t \in [-r,\ 0]$.

Now, by starting with the LBVP

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ \alpha_0(t-r)) + \mu[y(t-r) - \alpha_0(t-r)],\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &ay(0) + by(T) = 0, \end{aligned}\right\} \tag{5.9}$$

and proceeding similarly along the proof of the Theorem 5.2.1, we obtain monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0,$ $\forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} &\alpha'_{n+1}(t) + \lambda\alpha_{n+1}(t) = f(t,\ \alpha_n(t-r)) + \mu[\alpha_{n+1}(t-r) - \alpha_n(t-r)],\ t \in I,\\ &\alpha_{n+1}(t) = \alpha_{n+1}(0),\ t \in [-r,\ 0],\\ &a\alpha_{n+1}(0) + b\alpha_{n+1}(T) = 0. \end{aligned}\right\}$$

Further $\{\alpha_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty} \alpha_n$ is a solution of LBVP (3.3). Next, by considering the LBVP

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ \beta_0(t-r)) + \mu[y(t-r) - \beta_0(t-r)],\ t \in I,\\ &y(t) = y(0),\ t \in [-r,\ 0],\\ &ay(0) + by(T) = 0, \end{aligned}\right\} \tag{5.10}$$

and by proceeding similarly along the proof of the Theorem 5.2.1, we obtain monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that
$\alpha_0 \leq \beta_n,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} &\beta'_{n+1}(t) + \lambda\beta_{n+1}(t) = f(t,\ \beta_n(t-r)) + \mu[\beta_{n+1}(t-r) - \beta_n(t-r)],\ t \in I,\\ &\beta_{n+1}(t) = \beta_{n+1}(0),\ t \in [-r,\ 0],\\ &a\beta_{n+1}(0) + b\beta_{n+1}(T) = 0. \end{aligned}\right\}$$

Further $\{\beta_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty} \beta_n$ is a solution of the LBVP (3.3). □

Now we prove a result on existence of solution for the NLBVP (3.4) of first order DECD by using monotone iterative method.

Theorem 5.2.3. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(t,\ x_1) - f(t,\ x_2) \geq \mu(x_1 - x_2),\ \forall\ (t,\ x_1),\ (t,\ x_2) \in I \times \mathbb{R}$ such that $x_2 \leq x_1$ and for some constant $\mu > 0$. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous function such that $g(x,\ u) \geq g(x,\ v),\ \forall\ (x,\ u),\ (x,\ v) \in \mathbb{R}^2$ such that $u \leq v$. Assume that $\alpha_0,\ \beta_0 \in X$ be weakly coupled lower and upper solutions of NLBVP (3.4). Then there are sequences of monotonic functions $\{\alpha_n\}$ and $\{\beta_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim_{n\to\infty} \alpha_n$ and $\lim_{n\to\infty} \beta_n$ are solutions to the NLBVP (3.4).*

Proof. Since, $\alpha_0,\ \beta_0 \in X$ are weakly coupled lower and upper solutions of NLBVP (3.4), we have

1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t-r)),\ t \in I.$
2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t-r)),\ t \in I.$
3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0].$
4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0].$
5. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T].$
6. $\max\{g(\alpha_0(0),\ \alpha_0(T)),\ g(\alpha_0(0),\ \beta_0(T))\}$
   $\leq 0 \leq \min\{g(\beta_0(0),\ \alpha_0(T)),\ g(\beta_0(0),\ \beta_0(T))\}.$

As $g(x,\ u) \geq g(x,\ v),\ \forall\ (x,\ u),\ (x,\ v) \in \mathbb{R}^2$ such that $u \leq v$, we have

$\max\{g(\alpha_0(0),\ \alpha_0(T)),\ g(\alpha_0(0),\ \beta_0(T))\} = g(\alpha_0(0),\ \alpha_0(T))$ and

$\min\{g(\beta_0(0),\ \alpha_0(T)),\ g(\beta_0(0),\ \beta_0(T))\} = g(\beta_0(0),\ \beta_0(T)).$

This implies $g(\alpha_0(0),\ \alpha_0(T)) \leq 0 \leq g(\beta_0(0),\ \beta_0(T))$.

Also, as $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and $\beta_0(t) = \beta_0(0)$, $\forall\ t \in [-r,\ 0]$.

Now, by starting with the NLBVP

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ \alpha_0(t-r)) + \mu[y(t-r) - \alpha_0(t-r)],\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &g(y(0),\ y(T)) = 0, \end{aligned}\right\} \tag{5.11}$$

and proceeding similarly along the proof of the Theorem 5.2.1, we obtain

monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that
$\alpha_n \leq \beta_0,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} &\alpha'_{n+1}(t) + \lambda\alpha_{n+1}(t) = f(t,\ \alpha_n(t-r)) + \mu[\alpha_{n+1}(t-r) - \alpha_n(t-r)],\ t \in I, \\ &\alpha_{n+1}(t) = \alpha_{n+1}(0),\ t \in [-r,\ 0], \\ &g(\alpha_{n+1}(0),\ \alpha_{n+1}(T)) = 0. \end{aligned}\right\}$$

Further $\{\alpha_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim\limits_{n\to\infty} \alpha_n$ is a solution of NLBVP (3.4). Next, by considering the NLBVP

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ \beta_0(t-r)) + \mu[y(t-r) - \beta_0(t-r)],\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &g(y(0),\ y(T)) = 0, \end{aligned}\right\} \tag{5.12}$$

and by proceeding similarly along the proof of the Theorem 5.2.1, we obtain monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that
$\alpha_0 \leq \beta_n,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} &\beta'_{n+1}(t) + \lambda\beta_{n+1}(t) = f(t,\ \beta_n(t-r)) + \mu[\beta_{n+1}(t-r) - \beta_n(t-r)],\ t \in I, \\ &\beta_{n+1}(t) = \beta_{n+1}(0),\ t \in [-r,\ 0], \\ &g(\beta_{n+1}(0),\ \beta_{n+1}(T)) = 0. \end{aligned}\right\}$$

Further $\{\beta_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim\limits_{n\to\infty} \beta_n$ is a solution of the NLBVP (3.4). □

5.3 Quasilinearization Method

In this section, we develope quasilinearization method to prove existence of solution for PBVP, LBVP and NLBVP of first order DECD. We start with PBVP(3.1).

Theorem 5.3.1. *Let $f \in C^2[I \times \mathbb{R},\ \mathbb{R}]$ and $\alpha_0,\ \beta_0 \in X$ be lower and upper solutions of PBVP (3.1) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$. Assume that $D_2 f(t,\ x) > 0$ and $D_{2,\,2} f(t,\ x) \geq 0,\ \forall\ (t,\ x) \in I \times \mathbb{R}$. Then there is a monotonic increasing sequence of functions $\{\alpha_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim_{n\to\infty} \alpha_n$ is a solution of the PBVP (3.1).*

Proof. It is given that $D_{2,\,2} f(t,\ x) \geq 0,\ \forall\ (t,\ x) \in I \times \mathbb{R}$.

Therefore $f(t,\ x_1) - f(t,\ x_2) \geq D_2 f(t,\ x_2)(x_1 - x_2),\ \forall\ (t,\ x_1),\ (t,\ x_2) \in I \times \mathbb{R}$ such that $x_1 \geq x_2$.

Since $\alpha_0,\ \beta_0 \in X$ are lower and upper solutions of PBVP (3.1), we have

1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t-r)),\ t \in I$.
2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t-r)),\ t \in I$.
3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0]$.
4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0]$.
5. $\alpha_0(0) \leq \alpha_0(T)$.
6. $\beta_0(0) \geq \beta_0(T)$.

Now, as $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and $\beta_0(t) = \beta_0(0)$, $\forall\ t \in [-r,\ 0]$.

By starting with the PBVP

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \alpha_0(t-r)) \\ &\quad + D_2 f(t,\ \alpha_0(t-r))[y(t-r) - \alpha_0(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T), \end{aligned}\right\} \tag{5.13}$$

and proceeding similarly along the proof of the Theorem 5.2.1, we obtain monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0$, $\forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} &\alpha'_{n+1}(t) + \lambda\alpha_{n+1}(t) = f(t,\ \alpha_n(t-r)) \\ &\qquad\qquad + D_2 f(t,\ \alpha_n(t-r))[\alpha_{n+1}(t-r) - \alpha_n(t-r)],\ t \in I, \\ &\alpha_{n+1}(t) = \alpha_{n+1}(0),\ t \in [-r,\ 0], \\ &\alpha_{n+1}(0) = \alpha_{n+1}(T). \end{aligned}\right\}$$

Further $\{\alpha_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty} \alpha_n$ is a solution of the PBVP (3.1). □

Next we discuss the PBVP(3.1) with different conditions than above, leading to existence of a monotonic decreasing sequence of upper solutions converging to a solution of the problem.

Theorem 5.3.2. *Let $f \in C^2[I \times \mathbb{R},\ \mathbb{R}]$ and $\alpha_0,\ \beta_0 \in X$ be lower and upper solutions of PBVP (3.1) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$. Assume that $D_2 f(t,\ x) > 0$ and $D_{2,\,2} f(t,\ x) \leq 0,\ \forall\ (t,\ x) \in I \times \mathbb{R}$. Then there is a monotonic decreasing sequence of functions $\{\beta_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim_{n\to\infty} \beta_n$ is a solution of the PBVP (3.1).*

Proof. It is given that $D_{2,\,2} f(t,\ x) \leq 0,\ \forall\ (t,\ x) \in I \times \mathbb{R}$.

Therefore $f(t,\ x_1) - f(t,\ x_2) \geq D_2 f(t,\ x_1)(x_1 - x_2),\ \forall\ (t,\ x_1),\ (t,\ x_2) \in I \times \mathbb{R}$ such that $x_1 \geq x_2$.

Since $\alpha_0,\ \beta_0 \in X$ are lower and upper solutions of the PBVP (3.1), we have

1. $\alpha'_0(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t-r)),\ t \in I$.
2. $\beta'_0(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t-r)),\ t \in I$.
3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0]$.
4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0]$.
5. $\alpha_0(0) \leq \alpha_0(T)$.
6. $\beta_0(0) \geq \beta_0(T)$.

As $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0)\ \forall\ t \in [-r,\ 0]$ and $\beta_0(t) = \beta_0(0)$, $\forall\ t \in [-r,\ 0]$.

By starting with the PBVP

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t\ \beta_0(t-r)) \\ &\quad + D_2 f(t,\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T), \end{aligned}\right\} \tag{5.14}$$

and proceeding similarly along the proof of the Theorem 5.2.1, we obtain monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\alpha_0 \leq \beta_n,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} \beta'_{n+1}(t) + \lambda \beta_{n+1}(t) &= f(t,\ \beta_n(t-r)) \\ &\quad + D_2 f(t,\ \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)],\ t \in I, \\ \beta_{n+1}(t) &= \beta_{n+1}(0),\ t \in [-r,\ 0], \\ \beta_{n+1}(0) &= \beta_{n+1}(T), \end{aligned}\right\}$$

Further $\{\beta_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim\limits_{n\to\infty} \beta_n$ is a solution of the PBVP (3.1). □

We now prove results on quasilinearization method for LBVP of first order DECD (3.3).

Theorem 5.3.3. *Let $f \in C^2[I \times \mathbb{R},\ \mathbb{R}]$ and $\alpha_0,\ \beta_0 \in X$ be weakly coupled lower and upper solutions of LBVP* (3.3)*. Assume that $D_2 f(t,\ x) > 0$ and $D_{2,\,2} f(t,\ x) \geq 0,\ \forall\ (t,\ x) \in I \times \mathbb{R}$. If $b < 0$, then there is a monotonic increasing sequence of functions $\{\alpha_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim\limits_{n\to\infty} \alpha_n$ is a solution of the LBVP* (3.3)*.*

Proof. It is given that $D_{2,\,2} f(t,\ x) \geq 0,\ \forall\ (t,\ x) \in I \times \mathbb{R}$.

Therefore $f(t,\ x_1) - f(t,\ x_2) \geq D_2 f(t,\ x_2)(x_1 - x_2),\ \forall\ (t,\ x_1),\ (t,\ x_2) \in I \times \mathbb{R}$ such that $x_1 \geq x_2$.

Since $\alpha_0,\ \beta_0 \in X$ are weakly coupled lower and upper solutions of LBVP (3.3), we have 1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t-r)),\ t \in I.$

2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t-r)),\ t \in I.$

3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0].$

4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0].$

5. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T].$

6. $\max\{a\alpha_0(0) + b\alpha_0(T),\ a\alpha_0(0) + b\beta_0(T)\}$
$\leq 0 \leq \min\{a\beta_0(0) + b\alpha_0(T),\ a\beta_0(0) + b\beta_0(T)\}.$

As $b < 0$, we have

$\max\{a\alpha_0(0) + b\alpha_0(T),\ a\alpha_0(0) + b\beta_0(T)\} = a\alpha_0(0) + b\alpha_0(T)$ and

$\min\{a\beta_0(0) + b\alpha_0(T),\ a\beta_0(0) + b\beta_0(T)\} = a\beta_0(0) + b\beta_0(T).$

This implies $a\alpha_0(0) + b\alpha_0(T) \leq 0 \leq a\beta_0(0) + b\beta_0(T)$.

Since $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and $\beta_0(t) = \beta_0(0)$, $\forall\ t \in [-r,\ 0]$.

Now, starting with the LBVP

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \alpha_0(t-r)) \\ &\quad + D_2 f(t,\ \alpha_0(t-r))[y(t-r) - \alpha_0(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ ay(0) + by(T) &= 0, \end{aligned}\right\} \tag{5.15}$$

and proceeding similarly along the proof of the Theorem 5.2.1, we obtain monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} \alpha_{n+1}'(t) + \lambda\alpha_{n+1}(t) &= f(t,\ \alpha_n(t-r)) \\ &\quad + D_2 f(t,\ \alpha_n(t-r))[\alpha_{n+1}(t-r) - \alpha_n(t-r)],\ t \in I, \\ \alpha_{n+1}(t) &= \alpha_{n+1}(0),\ t \in [-r,\ 0], \\ a\alpha_{n+1}(0) + b\alpha_{n+1}(T) &= 0. \end{aligned}\right\}$$

Further $\{\alpha_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty}\alpha_n$ is a solution of the LBVP (3.3). □

Theorem 5.3.4. *Let $f \in C^2[I \times \mathbb{R},\ \mathbb{R}]$ and $\alpha_0,\ \beta_0 \in X$ be weakly coupled lower and upper solutions of LBVP (3.3). Assume that $D_2 f(t,\ x) > 0$ and $D_{2,\,2} f(t,\ x) \leq 0,\ \forall\ (t,\ x) \in I \times \mathbb{R}$ and. If $b < 0$, then there is a monotonic increasing sequence of functions $\{\beta_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim_{n\to\infty}\beta_n$ is a solution of the LBVP (3.3).*

Proof. It is given that $D_{2,\,2} f(t,\ x) \leq 0,\ \forall\ (t,\ x) \in I \times \mathbb{R}$.

Therefore $f(t,\ x_1) - f(t,\ x_2) \geq D_2 f(t,\ x_1)(x_1 - x_2),\ \forall\ (t,\ x_1),\ (t,\ x_2) \in I \times \mathbb{R}$ such that $x_1 \geq x_2$.

Since $\alpha_0,\ \beta_0 \in X$ are weakly coupled lower and upper solutions of LBVP (3.3), we have 1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t-r)),\ t \in I$.

2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t-r)),\ t \in I$.

3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0]$.

4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0]$.

5. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.

6. $\max\{a\alpha_0(0) + b\alpha_0(T),\ a\alpha_0(0) + b\beta_0(T)\}$
$\leq 0 \leq \min\{a\beta_0(0) + b\alpha_0(T),\ a\beta_0(0) + b\beta_0(T)\}$.

As $b < 0$, we have

$\max\{a\alpha_0(0) + b\alpha_0(T),\ a\alpha_0(0) + b\beta_0(T)\} = a\alpha_0(0) + b\alpha_0(T)$ and

$\min\{a\beta_0(0) + b\alpha_0(T),\ a\beta_0(0) + b\beta_0(T)\} = a\beta_0(0) + b\beta_0(T)$.

This implies $a\alpha_0(0) + b\alpha_0(T) \leq 0 \leq a\beta_0(0) + b\beta_0(T)$.

Also, as $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and

$\beta_0(t) = \beta_0(0),\ \forall\ t \in [-r,\ 0]$.

By starting with the LBVP

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \beta_0(t-r)) \\ &\quad + D_2 f(t,\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ ay(0) + by(T) &= 0, \end{aligned}\right\} \tag{5.16}$$

and proceeding similarly along the proof of the Theorem 5.2.1, we obtain monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\alpha_0 \leq \beta_n,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} \beta'_{n+1}(t) + \lambda\beta_{n+1}(t) &= f(t,\ \beta_n(t-r)) \\ &\quad + D_2 f(t,\ \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)],\ t \in I, \\ \beta_{n+1}(t) &= \beta_{n+1}(0),\ t \in [-r,\ 0], \\ a\beta_{n+1}(0) + b\beta_{n+1}(T) &= 0. \end{aligned}\right\}$$

Further $\{\beta_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim\limits_{n\to\infty}\beta_n$ is a solution of the LBVP (3.3). □

Further, we prove similar results on quasilinearization method for NLBVP of first order DECD (3.4).

Theorem 5.3.5. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous function satisfying $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$. Let $f \in C^2[I \times \mathbb{R},\ \mathbb{R}]$ and $\alpha_0,\ \beta_0 \in X$ be weakly coupled lower and upper solutions of NLBVP (3.4). Assume that $D_2 f(t,\ x) > 0$ and $D_{2,\,2} f(t,\ x) \geq 0,\ \forall\ (t,\ x) \in I \times \mathbb{R}$. Then there is a monotonic increasing sequence of functions (α_n) converging in $(X,\ ||.||_\infty)$ such that $\lim\limits_{n\to\infty}\alpha_n$ is a solution of the NLBVP (3.4).*

Proof. It is given that $D_{2,\,2} f(t,\ x) \geq 0,\ \forall\ (t,\ x) \in I \times \mathbb{R}$.

Therefore $f(t,\ x_1) - f(t,\ x_2) \geq D_2 f(t,\ x_2)(x_1 - x_2),\ \forall\ (t,\ x_1),\ (t,\ x_2) \in I \times \mathbb{R}$ such that $x_1 \geq x_2$.

Since $\alpha_0,\ \beta_0 \in X$ are weakly coupled lower and upper solutions of NLBVP (3.4), we have 1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t-r)),\ t \in I.$

2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t-r)),\ t \in I.$
3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0].$
4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0].$
5. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T].$
6. $\max\{g(\alpha_0(0),\ \alpha_0(T)),\ g(\alpha_0(0),\ \beta_0(T))\}$
$\leq 0 \leq \min\{g(\beta_0(0),\ \alpha_0(T)),\ g(\beta_0(0),\ \beta_0(T))\}.$

As $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$, we have $\max\{g(\alpha_0(0),\ \alpha_0(T)),\ g(\alpha_0(0),\ \beta_0(T))\} = g(\alpha_0(0),\ \alpha_0(T))$ and $\min\{g(\beta_0(0),\ \alpha_0(T)),\ g(\beta_0(0),\ \beta_0(T))\} = g(\beta_0(0),\ \beta_0(T))$.
This implies $g(\alpha_0(0),\ \alpha_0(T)) \leq 0 \leq g(\beta_0(0),\ \beta_0(T))$.
Also, as $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and $\beta_0(t) = \beta_0(0)$, $\forall\ t \in [-r,\ 0]$.
Now, starting with the NLBVP

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \alpha_0(t-r)) \\ &\quad + D_2 f(t,\ \alpha_0(t-r))[y(t-r) - \alpha_0(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ g(y(0),\ y(T)) &= 0, \end{aligned}\right\} \tag{5.17}$$

and proceeding similarly along the proof of the Theorem 5.2.1, we obtain monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} \alpha_{n+1}'(t) + \lambda\alpha_{n+1}(t) &= f(t,\ \alpha_n(t-r)) \\ &\quad + D_2 f(t,\ \alpha_n(t-r))[\alpha_{n+1}(t-r) - \alpha_n(t-r)],\ t \in I, \\ \alpha_{n+1}(t) &= \alpha_{n+1}(0),\ t \in [-r,\ 0], \\ g(\alpha_{n+1}(0),\ \alpha_{n+1}(T)) &= 0. \end{aligned}\right\}$$

Further $\{\alpha_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty}\alpha_n$ is a solution of the NLBVP (3.4). □

Theorem 5.3.6. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous function satisfying $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in I \times \mathbb{R}$ such that $x \geq y$. Let $f \in C^2[I \times \mathbb{R},\ \mathbb{R}]$ and $\alpha_0,\ \beta_0 \in X$ be weakly coupled lower and upper solutions of NLBVP (3.4). Assume that $D_2 f(t,\ x) > 0$ and $D_{2,\,2} f(t,\ x) \leq 0$, $\forall\ (t,\ x) \in I \times \mathbb{R}$. Then there is a monotonic decreasing sequence of functions $\{\beta_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim_{n\to\infty}\beta_n$ is a solution of the NLBVP (3.4).*

Proof. It is given that $D_{2,\,2} f(t,\ x) \leq 0,\ \forall\ (t,\ x) \in I \times \mathbb{R}^2$.

Therefore $f(t,\ x_1) - f(t,\ x_2) \geq D_2 f(t,\ x_1)(x_1 - x_2)\ \forall\ (t,\ x_1),\ (t,\ x_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$.

Since $\alpha_0,\ \beta_0 \in X$ are weakly coupled lower and upper solutions of NLBVP (3.4), we have 1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t-r)),\ t \in I$.

2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t-r)),\ t \in I$.
3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0]$.
4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0]$.
5. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
6. $\max\{g(\alpha_0(0),\ \alpha_0(T)),\ g(\alpha_0(0),\ \beta_0(T))\}$
   $\leq 0 \leq \min\{g(\beta_0(0),\ \alpha_0(T)),\ g(\beta_0(0),\ \beta_0(T))\}$.

As $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$, we have $\max\{g(\alpha_0(0),\ \alpha_0(T)),\ g(\alpha_0(0),\ \beta_0(T))\} = g(\alpha_0(0),\ \alpha_0(T))$ and $\min\{g(\beta_0(0),\ \alpha_0(T)),\ g(\beta_0(0),\ \beta_0(T))\} = g(\beta_0(0),\ \beta_0(T))$.

This implies $g(\alpha_0(0),\ \alpha_0(T)) \leq 0 \leq g(\beta_0(0),\ \beta_0(T))$.

Also, as $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and $\beta_0(t) = \beta_0(0)$, $\forall\ t \in [-r,\ 0]$.

By starting with the NLBVP

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \beta_0(t-r)) \\ &\quad + D_2 f(t,\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ g(y(0),\ y(T)) &= 0. \end{aligned}\right\} \tag{5.18}$$

and proceeding similarly along the proof of the Theorem 5.2.1, we obtain monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\alpha_0 \leq \beta_n,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} \beta'_{n+1}(t) + \lambda \beta_{n+1}(t) &= f(t,\ \beta_n(t-r)) \\ &\quad + D_2 f(t,\ \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)],\ t \in I, \\ \beta_{n+1}(t) &= \beta_{n+1}(0),\ t \in [-r,\ 0], \\ g(\beta_{n+1}(0),\ \beta_{n+1}(T)) &= 0. \end{aligned}\right\}$$

Further $\{\beta_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty} \beta_n$ is a solution of the NLBVP (3.4). □

Chapter 6

MONOTONE ITERATIVE METHOD AND QUASILINEARIZATION METHOD FOR FIRST ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

6.1 Introduction

This chapter deals with establishing results using Monotone Iterative Method and Quasilinearization Method for BVPs discussed in Chapter 4. We make use of existence of solution between lower and upper solutions for PBVPs of general first order DECD and first order DEVDA to prove our results. Further, we prove results on existence of solution for LBVP and NLBVP of general first order DECD and first order DEVDA by Monotone Iterative Method and Quasilinearization Method, using results on existence of solution between weakly coupled lower and upper solutions, which we proved in Chapter 4. We assume $b < 0$ for LBVP and in the case of NLBVP, we assume g to be a non-increasing function in second argument.

6.2 Monotone Iterative Method

We now establish results on existence of solution, by using monotone iterative method, for PBVP of general first order DECD (4.1) and for PBVP of first order DEVDA (4.6).

Theorem 6.2.1. *Let* $f : I \times \mathbb{R}^2 \to \mathbb{R}$ *be continuous and satisfy*
$f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq \mu_1(x_1 - x_2) + \mu_2(y_1 - y_2)$,
$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ *such that* $x_2 \leq x_1$, $y_2 \leq y_1$ *and for some constants* $\mu_1 > 0$, $\mu_2 > 0$. *Let* α_0, $\beta_0 \in X$ *be lower and upper solutions of PBVP* (4.1) *such that* $\alpha_0(t) \leq \beta_0(t)$, $\forall\ t \in [-r,\ T]$. *Then there are sequences of monotonic functions* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *converging in* $(X,\ ||.||_\infty)$ *such that* $\lim_{n\to\infty} \alpha_n$ *and* $\lim_{n\to\infty} \beta_n$ *are solutions of the PBVP* (4.1).

Proof. Since, $\alpha_0,\ \beta_0 \in X$ are lower and upper solutions of PBVP (4.1), we have 1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$

2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$

3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0].$

4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0].$

5. $\alpha_0(0) \leq \alpha_0(T).$

6. $\beta_0(0) \geq \beta_0(T).$

As $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and $\beta_0(t) = \beta_0(0),$ $\forall\ t \in [-r,\ 0]$.

Consider

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1[y(t) - \alpha_0(t)] \\ &\quad + \mu_2[y(t-r) - \alpha_0(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T). \end{aligned}\right\} \tag{6.1}$$

Let $g_{\alpha_0}(t,\ u,\ v) = f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1[u - \alpha_0(t)]$
$$+ \mu_2[v - \alpha_0(t-r)],\ \forall\ (t,\ u,\ v) \in I \times \mathbb{R}^2.$$

Therefore $g_{\alpha_0}(t,\ y(t),\ y(t-r)) = f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1[y(t) - \alpha_0(t)]$
$$+ \mu_2[y(t-r) - \alpha_0(t-r)],\ \forall\ t \in I.$$

Let $(t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

This implies $x_1 - \alpha_0(t) \leq x_2 - \alpha_0(t)$ and $y_1 - \alpha_0(t-r) \leq y_2 - \alpha_0(t-r)$.

This gives $\mu_1[x_1 - \alpha_0(t)] \leq \mu_1[x_2 - \alpha_0(t)]$ and

$$\mu_2[y_1 - \alpha_0(t-r)] \leq \mu_2[y_2 - \alpha_0(t-r)].$$

So we get $\mu_1[x_1 - \alpha_0(t)] + \mu_2[y_1 - \alpha_0(t-r)]$

$$\leq \mu_1[x_2 - \alpha_0(t)] + \mu_2[y_2 - \alpha_0(t-r)].$$

Therefore $f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1[x_1 - \alpha_0(t)] + \mu_2[y_1 - \alpha_0(t-r)]$

$$\leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1[x_2 - \alpha_0(t)] + \mu_2[y_2 - \alpha_0(t-r)].$$

Thus $g_{\alpha_0}(t,\ x_1,\ y_1) \leq g_{\alpha_0}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

$$\begin{aligned}
\text{Also } \alpha_0'(t) + \lambda\alpha_0(t) &\leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1[\alpha_0(t) - \alpha_0(t)] \\
&\quad + \mu_2[\alpha_0(t-r) - \alpha_0(t-r)] \\
&= g_{\alpha_0}(t,\ \alpha_0(t),\ \alpha_0(t-r)).
\end{aligned}$$

$$\begin{aligned}
\text{Further } \beta_0'(t) + \lambda\beta_0(t) &\geq f(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\geq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1[\beta_0(t) - \alpha_0(t)] \\
&\quad + \mu_2[\beta_0(t-r) - \alpha_0(t-r)] \\
&= g_{\alpha_0}(t,\ \beta_0(t),\ \beta_0(t-r)).
\end{aligned}$$

We already know that

1. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
2. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
3. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
4. $\alpha_0(0) \leq \alpha_0(T)$.
5. $\beta_0(0) \geq \beta_0(T)$.

This implies, α_0 and β_0 are lower and upper solutions of the PBVP (6.1) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Therefore the PBVP (6.1) has a solution $\alpha_1 \in X$ such that $\alpha_0 \leq \alpha_1 \leq \beta_0$ on $[-r,\ T]$. (Using Theorem 4.4.5)

So we have

$$\left.\begin{aligned}
&\alpha_1'(t) + \lambda\alpha_1(t) = f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1[\alpha_1(t) - \alpha_0(t)] \\
&\qquad\qquad + \mu_2[\alpha_1(t-r) - \alpha_0(t-r)],\ t \in I, \\
&\alpha_1(t) = \alpha_1(0),\ t \in [-r,\ 0], \\
&\alpha_1(0) = \alpha_1(T).
\end{aligned}\right\}$$

This gives $\alpha_1'(t) + \lambda\alpha_1(t) \leq f(t,\ \alpha_1(t),\ \alpha_1(t-r)),\ t \in I.$

Also 1. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$

2. $\alpha_1(t) = \alpha_1(0),\ t \in [-r,\ 0].$

3. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0].$

4. $\alpha_1(t) \leq \beta_0(t),\ t \in [-r,\ T].$

5. $\alpha_1(0) \leq \alpha_1(T).$

6. $\beta_0(0) \geq \beta_0(T).$

Therefore α_1 and $\beta_0 \in X$ are lower and upper solutions of the PBVP (4.1) such that $\alpha_1(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Now Consider

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \alpha_1(t),\ \alpha_1(t-r)) + \mu_1[y(t) - \alpha_1(t)] \\ &\quad + \mu_2[y(t-r) - \alpha_1(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T). \end{aligned}\right\} \tag{6.2}$$

Let $g_{\alpha_1}(t,\ u,\ v) = f(t,\ \alpha_1(t),\ \alpha_1(t-r)) + \mu_1[u - \alpha_1(t)]$
$+ \mu_2[v - \alpha_1(t-r)],\ \forall\ (t,\ u,\ v) \in I \times \mathbb{R}^2.$

Therefore $g_{\alpha_1}(t,\ y(t),\ y(t-r)) = f(t,\ \alpha_1(t),\ \alpha_1(t-r)) + \mu_1[y(t) - \alpha_1(t)]$
$+ \mu_2[y(t-r) - \alpha_1(t-r)],\ \forall\ t \in I.$

So we have $g_{\alpha_1}(t,\ x_1,\ y_1) \leq g_{\alpha_1}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

$$\begin{aligned} \text{Also } \alpha_1'(t) + \lambda\alpha_1(t) &\leq f(t,\ \alpha_1(t),\ \alpha_1(t-r)) \\ &= f(t,\ \alpha_1(t),\ \alpha_1(t-r)) + \mu_1[\alpha_1(t) - \alpha_1(t)] \\ &\quad + \mu_2[\alpha_1(t-r) - \alpha_1(t-r)] \\ &= g_{\alpha_1}(t,\ \alpha_1(t),\ \alpha_1(t-r)). \end{aligned}$$

Further $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r))$

$$\geq f(t,\ \alpha_1(t),\ \alpha_1(t-r)) + \mu_1[\beta_0(t) - \alpha_1(t)]$$
$$+ \mu_2[\beta_0(t-r) - \alpha_1(t-r)]$$
$$= g_{\alpha_1}(t,\ \beta_0(t),\ \beta_0(t-r)).$$

We already know that

1. $\alpha_1(t) = \alpha_1(0),\ t \in [-r,\ 0]$.
2. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
3. $\alpha_1(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
4. $\alpha_1(0) = \alpha_1(T)$.
5. $\beta_0(0) \geq \beta_0(T)$.

This implies α_1 and β_0 are lower and upper solutions of the PBVP (6.2) such that $\alpha_1(t) \leq \beta_0(t),\ \forall\, t \in [-r,\ T]$.

Therefore the PBVP (6.2) has a solution $\alpha_2 \in X$ such that $\alpha_1 \leq \alpha_2 \leq \beta_0$ on $[-r,\ T]$. (Using Theorem 4.4.5)

This gives

$$\left.\begin{aligned} &\alpha_2'(t) + \lambda\alpha_2(t) = f(t,\ \alpha_1(t),\ \alpha_1(t-r)) + \mu_1[\alpha_2(t) - \alpha_1(t)] \\ &\qquad\qquad + \mu_2[\alpha_2(t-r) - \alpha_1(t-r)],\ t \in I, \\ &\alpha_2(t) = \alpha_2(0),\ t \in [-r,\ 0], \\ &\alpha_2(0) = \alpha_2(T). \end{aligned}\right\}$$

So we have $\alpha_2'(t) + \lambda\alpha_2(t) \leq f(t,\ \alpha_2(t),\ \alpha_2(t-r)),\ t \in I$.

Also 1. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I$.

2. $\alpha_2(t) = \alpha_2(0),\ t \in [-r,\ 0]$.
3. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
4. $\alpha_2(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
5. $\alpha_2(0) = \alpha_2(T)$.
6. $\beta_0(0) \geq \beta_0(T)$.

Therefore α_2 and $\beta_0 \in X$ are lower and upper solutions of the PBVP (4.1) such that $\alpha_2(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Continuing in this manner, let $\alpha_n \in X$ be a solution of

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \alpha_{n-1}(t),\ \alpha_{n-1}(t-r)) + \mu_1[y(t) - \alpha_{n-1}(t)] \\ &\quad + \mu_2[y(t-r) - \alpha_{n-1}(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T). \end{aligned}\right\} \tag{6.3}$$

Next consider

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \alpha_n(t),\ \alpha_n(t-r)) + \mu_1[y(t) - \alpha_n(t)] \\ &\quad + \mu_2[y(t-r) - \alpha_n(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T). \end{aligned}\right\} \tag{6.4}$$

Let $g_{\alpha_n}(t,\ u,\ v) = f(t,\ \alpha_n(t),\ \alpha_n(t-r)) + \mu_1[u - \alpha_n(t)]$
$+ \mu_2[v - \alpha_n(t-r)],\ \forall\ (t,\ u,\ v) \in I \times \mathbb{R}^2$.

Therefore $g_{\alpha_n}(t,\ y(t),\ y(t-r)) = f(t,\ \alpha_n(t),\ \alpha_n(t-r)) + \mu_1[y(t) - \alpha_n(t)]$
$+ \mu_2[y(t-r) - \alpha_n(t-r)],\ \forall\ t \in I$.

Also $g_{\alpha_n}(t,\ x_1,\ y_1) \leq g_{\alpha_n}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$. Further α_n and β_0 are lower and upper solutions of (6.4) such that $\alpha_n(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Therefore the PBVP (6.4) has a solution $\alpha_{n+1} \in X$ such that $\alpha_n \leq \alpha_{n+1} \leq \beta_0$ on $[-r,\ T]$. (Using Theorem 4.4.5)

This gives

$$\left.\begin{aligned} \alpha'_{n+1}(t) + \lambda\alpha_{n+1}(t) &= f(t,\ \alpha_n(t),\ \alpha_n(t-r)) + \mu_1[\alpha_{n+1}(t) - \alpha_n(t)] \\ &\quad + \mu_2[\alpha_{n+1}(t-r) - \alpha_n(t-r)],\ t \in I, \\ \alpha_{n+1}(t) &= \alpha_{n+1}(0),\ t \in [-r,\ 0], \\ \alpha_{n+1}(0) &= \alpha_{n+1}(T). \end{aligned}\right\}$$

Thus we get a monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} \alpha'_{n+1}(t) + \lambda\alpha_{n+1}(t) &= f(t,\ \alpha_n(t),\ \alpha_n(t-r)) + \mu_1[\alpha_{n+1}(t) - \alpha_n(t)] \\ &\quad + \mu_2[\alpha_{n+1}(t-r) - \alpha_n(t-r)],\ t \in I, \\ \alpha_{n+1}(t) &= \alpha_{n+1}(0),\ t \in [-r,\ 0], \\ \alpha_{n+1}(0) &= \alpha_{n+1}(T). \end{aligned}\right\} \tag{6.5}$$

Now for each fixed $t \in [-r,\ T]$, $\{\alpha_n(t)\}$ is a monotonic increasing sequence in $\mathbb{R}$ which is bounded above by $\beta_0(t)$. Therefore by Monotone Convergence Theorem for sequences in $\mathbb{R}$, $\{\alpha_n(t)\}$ is convergent in $\mathbb{R}$, for each $t \in [-r,\ T]$. This implies $\{\alpha_n\}$ is pointwise convergent on $[-r,\ T]$.
Let $\lim\limits_{n\to\infty} \alpha_n(t) = \alpha(t),\ \forall\ t \in [-r,\ T]$.

Since $\alpha'_{n+1}(t) + \lambda\alpha_{n+1}(t) = f(t,\ \alpha_n(t),\ \alpha_n(t-r)) + \mu_1[\alpha_{n+1}(t) - \alpha_n(t)]$
$$+ \mu_2[\alpha_{n+1}(t-r) - \alpha_n(t-r)],$$

we have $\alpha_{n+1}(t)e^{\lambda t} - \alpha_{n+1}(0) = \int\limits_0^t e^{\lambda s} f(s,\ \alpha_n(s),\ \alpha_n(s-r))ds$
$$+ \int\limits_0^t e^{\lambda s}\mu_1[\alpha_{n+1}(s) - \alpha_n(s)]ds$$
$$+ \int\limits_0^t e^{\lambda s}\mu_2[\alpha_{n+1}(s-r) - \alpha_n(s-r)]ds.$$

This implies

$$\alpha_{n+1}(t) = \alpha_{n+1}(0)e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} f(s,\ \alpha_n(s),\ \alpha_n(s-r))ds \tag{6.6}$$
$$+ \int_0^t e^{-\lambda(t-s)} \mu_1[\alpha_{n+1}(s) - \alpha_n(s)]ds$$
$$+ \int_0^t e^{-\lambda(t-s)} \mu_2[\alpha_{n+1}(s-r) - \alpha_n(s-r)]ds.$$

Taking limit as $n \to \infty$ on both sides of equation (6.6) we get

$$\lim_{n\to\infty} \alpha_{n+1}(t) = \lim_{n\to\infty} \alpha_{n+1}(0)e^{-\lambda t} \tag{6.7}$$
$$+ \lim_{n\to\infty} \int_0^t e^{-\lambda(t-s)} f(s,\ \alpha_n(s),\ \alpha_n(s-r))ds$$
$$+ \lim_{n\to\infty} \int_0^t e^{-\lambda(t-s)} \mu_1[\alpha_{n+1}(s) - \alpha_n(s)]ds$$
$$+ \lim_{n\to\infty} \int_0^t e^{-\lambda(t-s)} \mu_2[\alpha_{n+1}(s-r) - \alpha_n(s-r)]ds.$$

Now $\{\alpha_n\}$ is uniformly bounded on $[-r,\ T]$, as $\alpha_0(t) \le \alpha_n(t) \le \beta_0(t)$,

$\forall\ t \in [-r,\ T]$. Therefore $\exists\ M > 0$ such that $|\alpha_n(t)| \le M,\ \forall\ t \in [-r,\ T]$ and

$\forall\ n \in \mathbb{N}$. Since $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is continuous,

$f : I \times [-M,\ M] \times [-M,\ M] \to \mathbb{R}$ is continuous.

Therefore $f : I \times [-M,\ M] \times [-M,\ M] \to \mathbb{R}$ is bounded, as

$I \times [-M,\ M] \times [-M,\ M]$ is compact. So there is $K > 0$ such that

$|f(t,\ \alpha_n(t),\ \alpha_n(t-r))| \le K,\ \forall\ t \in I$ and $\forall\ n \in \mathbb{N}$.

From equation (6.5), we have

$$|\alpha'_{n+1}(t)| \le |f(t,\ \alpha_n(t),\ \alpha_n(t-r))| + \mu_1[|\alpha_{n+1}(t)| + |\alpha_n(t)|]$$
$$+ \mu_2[|\alpha_{n+1}(t-r)| + |\alpha_n(t-r)|] + |\lambda||\alpha_{n+1}(t)|$$
$$\le K + 2\mu_1 M + 2\mu_2 M + |\lambda|M = C(\text{say}).$$

This implies $|\alpha_n'(t)| \leq C,\ \forall\ n \in \mathbb{N}$ and $\forall\ t \in I$.

Thus $\{\alpha_n'\}$ is uniformly bounded on I.

We now show that $\{\alpha_n\}$ is equicontinuous on I.

Let $\epsilon > 0$ be arbitrary. Choose $\delta = \min\{\frac{\epsilon}{C}, \frac{T}{2}\}$.

Let $t_1,\ t_2 \in I$ be such that $|t_1 - t_2| < \delta$.

By Lagrange's mean value theorem, $\exists\ \xi$ between t_1 and t_2 such that

$|\alpha_n(t_1) - \alpha_n(t_2)| = |\alpha_n'(\xi)||t_1 - t_2|$.

This implies $|\alpha_n(t_1) - \alpha_n(t_2)| \leq C|t_1 - t_2| < C\delta \leq \epsilon,\ \forall\ t_1,\ t_2 \in I$ such that $|t_1 - t_2| < \delta$. Therefore $\{\alpha_n\}$ is equicontinuous on I.

Using Arzela-Ascoli's lemma we may conclude that $\{\alpha_n\}$ has a subsequence $\{\alpha_{n_k}\}$ converging uniformly on I. So $\alpha_{n_k} \to \alpha$ uniformly on I. Since each α_{n_k} is continuous on I, α is continuous on I. Also $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\alpha(0) = \alpha(T)$. So we conclude that α is continuous on $[-r,\ T]$. Since $\{\alpha_n\}$ is increasing and $\lim_{n\to\infty} \alpha_n(t) = \alpha(t),\ \forall\ t \in I$, by Dini's theorem for uniform convergence of sequence of functions, $\alpha_n \to \alpha$ uniformly on I.

From equation (6.7), we get

$$\begin{aligned}\lim_{n\to\infty} \alpha_{n+1}(t) = \lim_{n\to\infty} \alpha_{n+1}(0)e^{-\lambda t} &+ \int_0^t \lim_{n\to\infty} e^{-\lambda(t-s)} f(s,\ \alpha_n(s),\ \alpha_n(s-r))ds \\ &+ \int_0^t \lim_{n\to\infty} e^{-\lambda(t-s)} \mu_1[\alpha_{n+1}(s) - \alpha_n(s)]ds \\ &+ \int_0^t \lim_{n\to\infty} e^{-\lambda(t-s)} \mu_2[\alpha_{n+1}(s-r) - \alpha_n(s-r)]ds.\end{aligned}$$

So we have $\alpha(t) = \alpha(0)e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} f(s,\ \alpha(s),\ \alpha(s-r))ds,\ t \in I$.

Also $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$ and $\alpha(0) = \alpha(T)$.

Taking limit as $n \to \infty$ on both sides of equation (6.5), we get

$$\begin{aligned}\lim_{n\to\infty} \alpha_{n+1}'(t) = -\lambda \lim_{n\to\infty} \alpha_{n+1}(t) &+ \lim_{n\to\infty} f(t,\ \alpha_n(t),\ \alpha_n(t-r)) \\ &+ \lim_{n\to\infty} \mu_1[\alpha_{n+1}(t) - \alpha_n(t)] + \lim_{n\to\infty} \mu_2[\alpha_{n+1}(t-r) - \alpha_n(t-r)].\end{aligned}$$

Therefore $\lim_{n\to\infty} \alpha'_{n+1}(t) = -\lambda\alpha(t) + f(t,\ \alpha(t),\ \alpha(t-r)) = \alpha'(t),\ \forall\ t \in I.$

Since the left hand side of the above limit is the uniform limit of sequence of functions, we have $\{\alpha'_n\}$ converging uniformly to α' on I.

As $\alpha_n \to \alpha$ pointwise on $[-r,\ T]$, $\alpha_n \to \alpha$ uniformly on I,

$\alpha_n(t) = \alpha_n(0),\ \forall\ t \in [-r,\ 0]$ and $\forall\ n \in \mathbb{N}$ and $\alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$,

we have $\alpha_n \to \alpha$ uniformly on $[-r,\ T]$. This implies α is continuous on $[-r,\ T]$.

Also α is differentiable on I and α' is continuous on I. Thus $\alpha \in X$.

As $\alpha_n \to \alpha$ uniformly on $[-r,\ T]$ and $\alpha'_n \to \alpha'$ uniformly on I, we have $\alpha_n \to \alpha$ on $(X,\ ||.||_\infty)$ and α is a solution of the PBVP (4.1).

Next consider

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + \mu_1[y(t) - \beta_0(t)] \\ &\quad + \mu_2[y(t-r) - \beta_0(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T). \end{aligned}\right\} \tag{6.8}$$

Let $h_{\beta_0}(t,\ u,\ v) = f(t,\ \beta_0(t),\ \beta_0(t-r)) + \mu_1[u - \beta_0(t)]$

$$+ \mu_2[v - \beta_0(t-r)],\ \forall\ (t,\ u,\ v) \in I \times \mathbb{R}^2.$$

Therefore $h_{\beta_0}(t, y(t), y(t-r)) = f(t,\ \beta_0(t),\ \beta_0(t-r)) + \mu_1[y(t) - \beta_0(t)]$

$$+ \mu_2[y(t-r) - \beta_0(t-r)],\ \forall\ t \in I.$$

So we have $h_{\beta_0}(t,\ x_1,\ y_1) \le h_{\beta_0}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \le x_2$ and $y_1 \le y_2$.

$$\begin{aligned} \text{Also } \alpha'_0(t) + \lambda\alpha_0(t) &\le f(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\le f(t,\ \beta_0(t),\ \beta_0(t-r)) + \mu_1[\alpha_0(t) - \beta_0(t)] \\ &\quad + \mu_2[\alpha_0(t-r) - \beta_0(t-r)] \\ &= h_{\beta_0}(t,\ \alpha_0(t),\ \alpha_0(t-r)). \end{aligned}$$

Further $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r))$

$$= f(t,\ \beta_0(t),\ \beta_0(t-r)) + \mu_1[\beta_0(t) - \beta_0(t)]$$
$$+ \mu_2[\beta_0(t-r) - \beta_0(t-r)]$$
$$= h_{\beta_0}(t,\ \beta_0(t),\ \beta_0(t-r)).$$

We already know that

1. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
2. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
3. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
4. $\alpha_0(0) \leq \alpha_0(T)$.
5. $\beta_0(0) \geq \beta_0(T)$.

This implies α_0 and β_0 are lower and upper solutions of the PBVP (6.8) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Therefore the PBVP (6.8) has a solution $\beta_1 \in X$ such that $\alpha_0 \leq \beta_1 \leq \beta_0$ on $[-r,\ T]$. (Using Theorem 4.4.5)

So we have

$$\left.\begin{aligned} &\beta_1'(t) + \lambda\beta_1(t) = f(t,\ \beta_0(t),\ \beta_0(t-r)) + \mu_1[\beta_1(t) - \beta_0(t)] \\ &\qquad\qquad + \mu_2[\beta_1(t-r) - \beta_0(t-r)],\ t \in I, \\ &\beta_1(t) = \beta_1(0),\ t \in [-r,\ 0], \\ &\beta_1(0) = \beta_1(T). \end{aligned}\right\}$$

This gives $\beta_1'(t) + \lambda\beta_1(t) \geq f(t,\ \beta_1(t),\ \beta_1(t-r)),\ t \in I$.

Also 1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I$.

2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
3. $\beta_1(t) = \beta_1(0),\ t \in [-r,\ 0]$.
4. $\alpha_0(t) \leq \beta_1(t),\ t \in [-r,\ T]$.
5. $\alpha_0(0) \leq \alpha_0(T)$.
6. $\beta_1(0) \geq \beta_1(T)$.

Therefore α_0 and $\beta_1 \in X$ are lower and upper solutions of the PBVP (4.1) such that $\alpha_1(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Continuing in this manner, we get a monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\alpha_0 \leq \beta_n,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} \beta'_{n+1}(t) + \lambda\beta_{n+1}(t) &= f(t,\ \beta_n(t),\ \beta_n(t-r)) + \mu_1[\beta_{n+1}(t) - \beta_n(t)] \\ &\quad + \mu_2[\beta_{n+1}(t-r) - \beta_n(t-r)],\ t \in I, \\ \beta_{n+1}(t) &= \beta_{n+1}(0),\ t \in [-r,\ 0], \\ \beta_{n+1}(0) &= \beta_{n+1}(T). \end{aligned}\right\}$$

Further $\{\beta_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim\limits_{n\to\infty}\beta_n$ is a solution of the PBVP (4.1). □

Next we have the following result for PBVP (4.6).

Theorem 6.2.2. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy*

$f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq \mu_1(x_1 - x_2) + \mu_2(y_1 - y_2)$,

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ *such that* $x_2 \leq x_1,\ y_2 \leq y_1$ *and for some constants* $\mu_1 > 0,\ \mu_2 > 0$. *Let* $\alpha_0,\ \beta_0 \in W$ *be lower and upper solutions of PBVP* (4.6) *such that* $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in I$. *Then there are sequences of monotonic functions* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *converging in* $(W,\ ||.||_0)$ *such that* $\lim\limits_{n\to\infty}\alpha_n$ *and* $\lim\limits_{n\to\infty}\beta_n$ *are solutions of the PBVP* (4.6).

Proof. Similar to the proof of the Theorem 6.2.1. □

Next, we establish results on existence of solution, using monotone iterative method, for LBVP of general first order DECD (4.3) and for LBVP of first order DEVDA (4.8).

Theorem 6.2.3. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy*

$f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq \mu_1(x_1 - x_2) + \mu_2(y_1 - y_2)$,

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ *such that* $x_2 \leq x_1,\ y_2 \leq y_1$ *and for some constants* $\mu_1 > 0,\ \mu_2 > 0$. *Let* $\alpha_0,\ \beta_0 \in X$ *be weakly coupled lower and upper solutions of LBVP* (4.3). *If* $b < 0$, *then there are sequences of monotonic functions* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *converging in* $(X,\ ||.||_\infty)$ *such that* $\lim_{n\to\infty} \alpha_n$ *and* $\lim_{n\to\infty} \beta_n$ *are solutions of the LBVP* (4.3).

Proof. Since, $\alpha_0,\ \beta_0 \in X$ are weakly coupled lower and upper solutions of LBVP (4.3), we have 1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I$.

2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I$.

3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0]$.

4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0]$.

5. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.

6. $\max\{a\alpha_0(0) + b\alpha_0(T),\ a\alpha_0(0) + b\beta_0(T)\}$
$\leq 0 \leq \min\{a\beta_0(0) + b\alpha_0(T),\ a\beta_0(0) + b\beta_0(T)\}$.

As $b < 0$, we have

$\max\{a\alpha_0(0) + b\alpha_0(T),\ a\alpha_0(0) + b\beta_0(T)\} = a\alpha_0(0) + b\alpha_0(T)$ and

$\min\{a\beta_0(0) + b\alpha_0(T),\ a\beta_0(0) + b\beta_0(T)\} = a\beta_0(0) + b\beta_0(T)$.

This implies $a\alpha_0(0) + b\alpha_0(T) \leq 0 \leq a\beta_0(0) + b\beta_0(T)$.

Also, as $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and $\beta_0(t) = \beta_0(0)$, $\forall\ t \in [-r,\ 0]$.

Now, by starting with the LBVP

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1[y(t) - \alpha_0(t)] \\ &\quad + \mu_2[y(t-r) - \alpha_0(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ ay(0) + by(T) &= 0, \end{aligned}\right\} \tag{6.9}$$

and proceeding similarly along the proof of the Theorem 6.2.1, we obtain mono-

tonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} \alpha'_{n+1}(t) + \lambda\alpha_{n+1}(t) &= f(t,\ \alpha_n(t),\ \alpha_n(t-r)) + \mu_1[\alpha_{n+1}(t) - \alpha_n(t)] \\ &\quad + \mu_2[\alpha_{n+1}(t-r) - \alpha_n(t-r)],\ t \in I, \\ \alpha_{n+1}(t) &= \alpha_{n+1}(0),\ t \in [-r,\ 0], \\ a\alpha_{n+1}(0) + b\alpha_{n+1}(T) &= 0. \end{aligned}\right\}$$

Further $\{\alpha_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty} \alpha_n$ is a solution of the LBVP (4.3).

Next, by considering the LBVP

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + \mu_1[y(t) - \beta_0(t)] \\ &\quad + \mu_2[y(t-r) - \beta_0(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ ay(0) + by(T) &= 0, \end{aligned}\right\} \tag{6.10}$$

and proceeding similarly along the proof of the Theorem 6.2.1, we obtain a monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\alpha_0 \leq \beta_n$, $\forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} \beta'_{n+1}(t) + \lambda\beta_{n+1}(t) &= f(t,\ \beta_n(t),\ \beta_n(t-r)) + \mu_1[\beta_{n+1}(t) - \beta_n(t)] \\ &\quad + \mu_2[\beta_{n+1}(t-r) - \beta_n(t-r)],\ t \in I, \\ \beta_{n+1}(t) &= \beta_{n+1}(0),\ t \in [-r,\ 0], \\ a\beta_{n+1}(0) + b\beta_{n+1}(T) &= 0. \end{aligned}\right\}$$

Further $\{\beta_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty} \beta_n$ is a solution of the LBVP (4.3). □

Following is the corresponding result for LBVP (4.8).

Theorem 6.2.4. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy*
$f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq \mu_1(x_1 - x_2) + \mu_2(y_1 - y_2)$,
$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ *such that* $x_2 \leq x_1,\ y_2 \leq y_1$ *and for some constants* $\mu_1 > 0,\ \mu_2 > 0$. *Let* $\alpha_0,\ \beta_0 \in W$ *be weakly coupled lower and upper solutions of LBVP* (4.8). *If* $b < 0$, *then there are sequences of monotonic functions* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *converging in* $(W,\ ||.||_0)$ *such that* $\lim_{n\to\infty} \alpha_n$ *and* $\lim_{n\to\infty} \beta_n$ *are solutions of the LBVP* (4.8).

Proof. Similar to the proof of the Theorem 6.2.3. □

We now establish results on existence of solution for NLBVP of general first order DECD and for NLBVP of first order DEVDA by making use of monotone iterative method.

Theorem 6.2.5. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy*
$f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq \mu_1(x_1 - x_2) + \mu_2(y_1 - y_2)$,
$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ *such that* $x_2 \leq x_1,\ y_2 \leq y_1$ *and for some constants* $\mu_1 > 0,\ \mu_2 > 0$. *Let* $g : \mathbb{R}^2 \to \mathbb{R}$ *be continuous function such that* $g(x,\ u) \geq g(x,\ v),\ \forall\ (x,\ u),\ (x,\ v) \in \mathbb{R}^2$ *such that* $u \leq v$. *Assume that* $\alpha_0,\ \beta_0 \in X$ *be weakly coupled lower and upper solutions of NLBVP* (4.4). *Then there are sequences of monotonic functions* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *converging in* $(X,\ ||.||_\infty)$ *such that* $\lim_{n\to\infty} \alpha_n$ *and* $\lim_{n\to\infty} \beta_n$ *are solutions of the NLBVP* (4.4).

Proof. Since, $\alpha_0,\ \beta_0 \in X$ are weakly coupled lower and upper solutions of NLBVP (4.4), we have 1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I$.

2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I$.
3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0]$.
4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0]$.
5. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
6. $\max\{g(\alpha_0(0),\ \alpha_0(T)),\ g(\alpha_0(0),\ \beta_0(T))\}$
   $\leq 0 \leq \min\{g(\beta_0(0),\ \alpha_0(T)),\ g(\beta_0(0),\ \beta_0(T))\}$.

As $g(x,\ u) \geq g(x,\ v),\ \forall\ (x,\ u),\ (x,\ v) \in \mathbb{R}^2$ such that $u \leq v$, we have

$\max\{g(\alpha_0(0),\ \alpha_0(T)),\ g(\alpha_0(0),\ \beta_0(T))\} = g(\alpha_0(0),\ \alpha_0(T))$ and

$\min\{g(\beta_0(0),\ \alpha_0(T)),\ g(\beta_0(0),\ \beta_0(T))\} = g(\beta_0(0),\ \beta_0(T))$.

This implies $g(\alpha_0(0),\ \alpha_0(T)) \leq 0 \leq g(\beta_0(0),\ \beta_0(T))$.

Also, as $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and $\beta_0(t) = \beta_0(0)$, $\forall\ t \in [-r,\ 0]$.

Now, by starting with the NLBVP

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1[y(t) - \alpha_0(t)] \\ &\quad + \mu_2[y(t-r) - \alpha_0(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ g(y(0),\ y(T)) &= 0, \end{aligned}\right\} \tag{6.11}$$

and proceeding similarly along the proof of the Theorem 6.2.1, we obtain monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} \alpha'_{n+1}(t) + \lambda\alpha_{n+1}(t) &= f(t,\ \alpha_n(t),\ \alpha_n(t-r)) + \mu_1[\alpha_{n+1}(t) - \alpha_n(t)] \\ &\quad + \mu_2[\alpha_{n+1}(t-r) - \alpha_n(t-r)],\ t \in I, \\ \alpha_{n+1}(t) &= \alpha_{n+1}(0),\ t \in [-r,\ 0], \\ g(\alpha_{n+1}(0),\ \alpha_{n+1}(T)) &= 0. \end{aligned}\right\}$$

Further $\{\alpha_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim\limits_{n\to\infty} \alpha_n$ is a solution of the NLBVP (4.4).

Next, by considering the NLBVP

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + \mu_1[y(t) - \beta_0(t)] \\ &\quad + \mu_2[y(t-r) - \beta_0(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ g(y(0),\ y(T)) &= 0, \end{aligned}\right\} \tag{6.12}$$

and proceeding similarly along the proof of the Theorem 6.2.1, we obtain a monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\alpha_0 \leq \beta_n$, $\forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} \beta'_{n+1}(t) + \lambda\beta_{n+1}(t) &= f(t,\ \beta_n(t),\ \beta_n(t-r)) + \mu_1[\beta_{n+1}(t) - \beta_n(t)] \\ &\quad + \mu_2[\beta_{n+1}(t-r) - \beta_n(t-r)],\ t \in I, \\ \beta_{n+1}(t) &= \beta_{n+1}(0),\ t \in [-r,\ 0], \\ g(\beta_{n+1}(0),\ \beta_{n+1}(T)) &= 0. \end{aligned}\right\}$$

Further $\{\beta_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim\limits_{n\to\infty}\beta_n$ is a solution of the NLBVP (4.4). □

Next we have a similar result for NLBVP (4.9).

Theorem 6.2.6. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy*
$f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq \mu_1(x_1 - x_2) + \mu_2(y_1 - y_2)$,
$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ *such that* $x_2 \leq x_1,\ y_2 \leq y_1$ *and for some constants* $\mu_1 > 0,\ \mu_2 > 0$. *Let* $g : \mathbb{R}^2 \to \mathbb{R}$ *be continuous function such that* $g(x,\ u) \geq g(x,\ v),\ \forall\ (x,\ u),\ (x,\ v) \in \mathbb{R}^2$ *such that* $u \leq v$. *Assume that* $\alpha_0,\ \beta_0 \in W$ *be weakly coupled lower and upper solutions of NLBVP* (4.9). *Then there are sequences of monotonic functions* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *converging in* $(W,\ ||.||_0)$ *such that* $\lim\limits_{n\to\infty}\alpha_n$ *and* $\lim\limits_{n\to\infty}\beta_n$ *are solutions of the NLBVP* (4.9).

Proof. Similar to the proof of the Theorem 6.2.5. □

6.3 Quasilinearization Method

In this section, we establish results on existence of solution using quasilinearization method for PBVP, LBVP and NLBVP of general first order DECD and first order DEVDA. Here we assume certain conditions on first and second order derivatives of f to obtain our results.

Theorem 6.3.1. *Let $f \in C^2[I \times \mathbb{R}^2, \mathbb{R}]$ and $\alpha_0,\ \beta_0 \in X$ be lower and upper solutions of PBVP (4.1) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$. Assume that $D_i f(t,\ x,\ y) > 0$ and $D_{i,\,j} f(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$. Then there is a monotonic increasing sequence of functions $\{\alpha_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim\limits_{n\to\infty} \alpha_n$ is a solution of the PBVP (4.1).*

Proof. It is given that $D_{i,\,j} f(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and
$\forall\ i,\ j = 2,\ 3$. Therefore
$f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq D_2 f(t,\ x_2,\ y_2)(x_1 - x_2) + D_3 f(t,\ x_2,\ y_2)(y_1 - y_2)$,
$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.
Since $\alpha_0,\ \beta_0 \in X$ are lower and upper solutions of PBVP (4.1), we have

1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I$.
2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I$.
3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0]$.
4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0]$.
5. $\alpha_0(0) \leq \alpha_0(T)$.
6. $\beta_0(0) \geq \beta_0(T)$.

Now, as $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and $\beta_0(t) = \beta_0(0)$,
$\forall\ t \in [-r, 0]$.
Consider

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t) - \alpha_0(t)] \\ &\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r) - \alpha_0(t-r)], \\ &\quad t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T). \end{aligned}\right\} \tag{6.13}$$

Let $g_{\alpha_0}(t,\ u,\ v) = f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[u - \alpha_0(t)]$
$$+ D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[v - \alpha_0(t-r)],$$
$$\forall\ (t,\ u,\ v) \in I \times \mathbb{R}^2.$$

This implies

$$g_{\alpha_0}(t,\ y(t),\ y(t-r)) = f(t,\ \alpha_0(t),\ \alpha_0(t-r))$$
$$+ D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t) - \alpha_0(t)]$$
$$+ D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r) - \alpha_0(t-r)],\ \forall\ t \in I.$$

So we have $g_{\alpha_0}(t,\ x_1,\ y_1) \leq g_{\alpha_0}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

Also $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r))$
$$= f(t,\ \alpha_0(t),\ \alpha_0(t-r))$$
$$+ D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_0(t) - \alpha_0(t)]$$
$$+ D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_0(t-r) - \alpha_0(t-r)]$$
$$= g_{\alpha_0}(t,\ \alpha_0(t),\ \alpha_0(t-r)).$$

Further $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r))$
$$\geq f(t,\ \alpha_0(t),\ \alpha_0(t-r))$$
$$+ D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\beta_0(t) - \alpha_0(t)]$$
$$+ D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\beta_0(t-r) - \alpha_0(t-r)]$$
$$= g_{\alpha_0}(t,\ \beta_0(t),\ \beta_0(t-r)).$$

We already know that

1. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
3. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
4. $\alpha_0(0) \leq \alpha_0(T)$.
5. $\beta_0(0) \geq \beta_0(T)$.

Therefore α_0 and β_0 are lower and upper solutions of the PBVP (6.13) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\, t \in [-r,\ T]$.

Thus the PBVP (6.13) has a solution $\alpha_1 \in X$ such that $\alpha_0 \leq \alpha_1 \leq \beta_0$ on $[-r,\ T]$. (Using Theorem 4.4.5)

So we have

$$\left.\begin{aligned} \alpha_1'(t) + \lambda\alpha_1(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_1(t) - \alpha_0(t)] \\ &\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_1(t-r) - \alpha_0(t-r)],\ t \in I, \\ \alpha_1(t) &= \alpha_1(0),\ t \in [-r,\ 0], \\ \alpha_1(0) &= \alpha_1(T). \end{aligned}\right\}$$

This gives $\alpha_1'(t) + \lambda\alpha_1(t) \leq f(t,\ \alpha_1(t),\ \alpha_1(t-r)),\ t \in I$.

Also 1. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I$.

2. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
3. $\alpha_1(t) = \alpha_1(0),\ t \in [-r,\ 0]$.
4. $\alpha_1(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
5. $\alpha_1(0) \leq \alpha_1(T)$.
6. $\beta_0(0) \geq \beta_0(T)$.

Therefore α_1 and $\beta_0 \in X$ are lower and upper solutions of the PBVP (4.1) such that $\alpha_1(t) \leq \beta_0(t),\ \forall\, t \in [-r,\ T]$.

Continuing in this manner, we get a monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0,\ \forall\, n \in \mathbb{N}$ on $[-r, T]$ and

$$\left.\begin{aligned}
\alpha'_{n+1}(t) + \lambda\alpha_{n+1}(t) &= f(t,\ \alpha_n(t),\ \alpha_n(t-r)) \\
&\quad + D_2 f(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t) - \alpha_n(t)] \\
&\quad + D_3 f(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t-r) - \alpha_n(t-r)], \\
&\quad t \in I, \\
\alpha_{n+1}(t) &= \alpha_{n+1}(0),\ t \in [-r,\ 0], \\
\alpha_{n+1}(0) &= \alpha_{n+1}(T).
\end{aligned}\right\}$$

Further $\{\alpha_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty} \alpha_n$ is a solution of the PBVP (4.1). □

Next we reverse the condition on second order derivatives of f to get monotonic decreasing sequence of upper solutions converging to solution of the PBVP (4.1).

Theorem 6.3.2. *Let $f \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ and $\alpha_0,\ \beta_0 \in X$ be lower and upper solutions of PBVP (4.1) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$. Assume that $D_i f(t,\ x,\ y) > 0$ and $D_{i,\,j} f(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$. Then there is a monotonic decreasing sequence of functions $\{\beta_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim_{n\to\infty} \beta_n$ is a solution of the PBVP (4.1).*

Proof. Since $\alpha_0,\ \beta_0 \in X$ are lower and upper solutions PBVP (4.1), we have

1. $\alpha'_0(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$
2. $\beta'_0(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$
3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0].$
4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0].$
5. $\alpha_0(0) \leq \alpha_0(T).$
6. $\beta_0(0) \geq \beta_0(T).$

Now, as $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0)\ \forall\ t \in [-r,\ 0]$ and

$\beta_0(t) = \beta_0(0),\ \forall\ t \in [-r,\ 0]$.

Also, it is given that $D_{i,\,j}f(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and

$\forall\ i,\ j = 2,\ 3.$ Therefore

$f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq D_2 f(t,\ x_1,\ y_1)(x_1 - x_2) + D_3 f(t,\ x_1,\ y_1)(y_1 - y_2),$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Consider

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &\quad + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[y(t) - \beta_0(t)] \\ &\quad + D_3 f(t, \beta_0(t), \beta_0(t-r))[y(t-r) - \beta_0(t-r)], \\ &\quad t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T). \end{aligned}\right\} \tag{6.14}$$

Let

$$\begin{aligned} h_{\beta_0}(t,\ u,\ v) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[u - \beta_0(t)] \\ &\quad + D_3 f(t,\ \beta_0(t),\ \beta_0(t-r))[v - \beta_0(t-r)],\ \forall\ (t,\ u,\ v) \in I \times \mathbb{R}^2. \end{aligned}$$

Then

$$\begin{aligned} h_{\beta_0}(t,\ y(t),\ y(t-r)) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &\quad + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[y(t) - \beta_0(t)] \\ &\quad + D_3 f(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)], \\ &\quad \forall\ t \in I. \end{aligned}$$

So we have $h_{\beta_0}(t,\ x_1,\ y_1) \leq h_{\beta_0}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

$$\begin{aligned}\text{Also } \alpha_0'(t) + \lambda\alpha_0(t) &\leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\leq f(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &\quad + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[\alpha_0(t) - \beta_0(t)] \\ &\quad + D_3 f(t,\ \beta_0(t),\ \beta_0(t-r))[\alpha_0(t-r) - \beta_0(t-r)] \\ &= h_{\beta_0}(t,\ \alpha_0(t),\ \alpha_0(t-r)).\end{aligned}$$

$$\begin{aligned}\text{Further } \beta_0'(t) + \lambda\beta_0(t) &\geq f(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &= f(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &\quad + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_0(t) - \beta_0(t)] \\ &\quad + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_0(t-r) - \beta_0(t-r)] \\ &= h_{\beta_0}(t,\ \beta_0(t),\ \beta_0(t-r)).\end{aligned}$$

We already know that

1. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
3. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
4. $\alpha_0(0) \leq \alpha_0(T)$.
5. $\beta_0(0) \geq \beta_0(T)$.

Therefore α_0 and β_0 are lower and upper solutions of the PBVP (6.14) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Thus the PBVP (6.14) has a solution $\beta_1 \in X$ such that $\alpha_0 \leq \beta_1 \leq \beta_0$ on $[-r,\ T]$. (Using Theorem 4.4.5)

So we have

$$\left.\begin{aligned}
\beta_1'(t) + \lambda\beta_1(t) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_1(t) - \beta_0(t)] \\
&\quad + D_3 f(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_1(t-r) - \beta_0(t-r)],\ t \in I, \\
\beta_1(t) &= \beta_1(0),\ t \in [-r,\ 0], \\
\beta_1(0) &= \beta_1(T).
\end{aligned}\right\}$$

This implies $\beta_1'(t) + \lambda\beta_1(t) \geq f(t,\ \beta_1(t),\ \beta_1(t-r)),\ t \in I.$

Also 1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$

2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0].$
3. $\beta_1(t) = \beta_1(0),\ t \in [-r,\ 0].$
4. $\alpha_0(t) \leq \beta_1(t),\ t \in [-r,\ T].$
5. $\alpha_0(0) \leq \alpha_0(T).$
6. $\beta_1(0) \geq \beta_1(T).$

Therefore α_0 and $\beta_1 \in X$ are lower and upper solutions of the PBVP (4.1) such that $\alpha_0(t) \leq \beta_1(t),\ \forall\ t \in [-r,\ T]$.

Continuing in this manner, we get a monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\alpha_0 \leq \beta_n,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned}
\beta_{n+1}'(t) + \lambda\beta_{n+1}(t) &= f(t,\ \beta_n(t),\ \beta_n(t-r)) \\
&\quad + D_2 f(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t) - \beta_n(t)] \\
&\quad + D_3 f(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)], \\
&\quad\ t \in I, \\
\beta_{n+1}(t) &= \beta_{n+1}(0),\ t \in [-r,\ 0], \\
\beta_{n+1}(0) &= \beta_{n+1}(T).
\end{aligned}\right\}$$

Further $\{\beta_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty}\beta_n$ is a solution of the PBVP (4.1). □

Following result establishes the existence of solution of the PBVP(4.6).

Theorem 6.3.3. *Let $f \in C^2[I \times \mathbb{R}^2, \mathbb{R}]$ and $\alpha_0,\ \beta_0 \in W$ be lower and upper solutions of PBVP (4.6) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in I$. Assume that $D_i f(t,\ x,\ y) > 0$ and $D_{i,\,j} f(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$. Then there is a monotonic increasing sequence of functions $\{\alpha_n\}$ converging in $(W,\ ||.||_0)$ such that $\lim_{n\to\infty} \alpha_n$ is a solution of the PBVP (4.6).*

Proof. Similar to the proof of the Theorem 6.3.1. □

Again as in case of PBVP (4.1), the change of inequality in the condition on second order derivatives of f results in monotonic decreasing sequence of upper solutions converging to a solution of the PBVP (4.6).

Theorem 6.3.4. *Let $f \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ and $\alpha_0,\ \beta_0 \in W$ be lower and upper solutions of PBVP (4.6) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in I$. Assume that $D_i f(t,\ x,\ y) > 0$ and $D_{i,\,j} f(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$. Then there is a monotonic decreasing sequence of functions $\{\beta_n\}$ converging in $(W,\ ||.||_0)$ such that $\lim_{n\to\infty} \beta_n$ is a solution of the PBVP (4.6).*

Proof. Similar to the proof of the Theorem 6.3.2 □

Next, we prove existence results for LBVP of general first order DECD (4.3) and for LBVP of first order DEVDA (4.8) by using quasilinearization method.

Theorem 6.3.5. *Let $f \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ and $\alpha_0,\ \beta_0 \in X$ be weakly coupled lower and upper solutions of LBVP (4.3). Assume that $D_i f(t,\ x,\ y) > 0$ and $D_{i,\,j} f(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$. If $b < 0$, then there is a monotonic increasing sequence of functions $\{\alpha_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim_{n\to\infty} \alpha_n$ is a solution of the LBVP (4.3).*

Proof. It is given that $D_{i,\,j}f(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and

$\forall\ i,\ j = 2,\ 3.$ Therefore

$f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq D_2 f(t,\ x_2,\ y_2)(x_1 - x_2) + D_3 f(t,\ x_2,\ y_2)(y_1 - y_2),$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Since $\alpha_0,\ \beta_0 \in X$ are weakly coupled lower and upper solutions of LBVP (4.3),

we have 1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$

2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$

3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0].$

4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0].$

5. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T].$

6. $\max\{a\alpha_0(0) + b\alpha_0(T),\ a\alpha_0(0) + b\beta_0(T)\}$
$\leq 0 \leq \min\{a\beta_0(0) + b\alpha_0(T),\ a\beta_0(0) + b\beta_0(T)\}.$

As $b < 0$, we have

$\max\{a\alpha_0(0) + b\alpha_0(T),\ a\alpha_0(0) + b\beta_0(T)\} = a\alpha_0(0) + b\alpha_0(T)$ and

$\min\{a\beta_0(0) + b\alpha_0(T),\ a\beta_0(0) + b\beta_0(T)\} = a\beta_0(0) + b\beta_0(T).$

This implies $a\alpha_0(0) + b\alpha_0(T) \leq 0 \leq a\beta_0(0) + b\beta_0(T)$.

Since $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and

$\beta_0(t) = \beta_0(0),\ \forall\ t \in [-r,\ 0].$

Now, starting with the LBVP

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t) - \alpha_0(t)] \\ &\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r) - \alpha_0(t-r)], \\ &\quad t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ ay(0) + by(T) &= 0, \end{aligned}\right\} \tag{6.15}$$

and proceeding similarly along the proof of the Theorem 6.3.1, we get a monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0,\ \forall\ n \in \mathbb{N}$

on $[-r,\ T]$ and

$$\left.\begin{aligned}
&\alpha'_{n+1}(t) + \lambda\alpha_{n+1}(t) = f(t,\ \alpha_n(t),\ \alpha_n(t-r)) \\
&\qquad\qquad + D_2 f(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t) - \alpha_n(t)] \\
&\qquad\qquad + D_3 f(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t-r) - \alpha_n(t-r)], \\
&\qquad\qquad t \in I, \\
&\alpha_{n+1}(t) = \alpha_{n+1}(0),\ t \in [-r,\ 0], \\
&a\alpha_{n+1}(0) + b\alpha_{n+1}(T) = 0.
\end{aligned}\right\}$$

Further $\{\alpha_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim\limits_{n\to\infty} \alpha_n$ is a solution of the LBVP (4.3). □

Theorem 6.3.6. *Let $f \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ and $\alpha_0,\ \beta_0 \in X$ be weakly coupled lower and upper solutions of LBVP* (4.3). *Assume that $D_i f(t,\ x,\ y) > 0$ and $D_{i,\,j} f(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$. If $b < 0$, then there is a monotonic increasing sequence of functions $\{\beta_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim\limits_{n\to\infty} \beta_n$ is a solution of the LBVP* (4.3).

Proof. It is given that $D_{i,\,j} f(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and
$\forall\ i,\ j = 2,\ 3$. Therefore
$f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq D_2 f(t,\ x_1,\ y_1)(x_1 - x_2) + D_3 f(t, x_1, y_1)(y_1 - y_2)$,
$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.
Since $\alpha_0,\ \beta_0 \in X$ are weakly coupled lower and upper solutions of LBVP (4.3),
we have 1. $\alpha'_0(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I$.

2. $\beta'_0(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I$.
3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0]$.
4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0]$.
5. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
6. $\max\{a\alpha_0(0) + b\alpha_0(T),\ a\alpha_0(0) + b\beta_0(T)\}$
   $\leq 0 \leq \min\{a\beta_0(0) + b\alpha_0(T),\ a\beta_0(0) + b\beta_0(T)\}$.

As $b < 0$, we have

$\max\{a\alpha_0(0) + b\alpha_0(T),\ a\alpha_0(0) + b\beta_0(T)\} = a\alpha_0(0) + b\alpha_0(T)$ and

$\min\{a\beta_0(0) + b\alpha_0(T),\ a\beta_0(0) + b\beta_0(T)\} = a\beta_0(0) + b\beta_0(T)$.

This implies $a\alpha_0(0) + b\alpha_0(T) \leq 0 \leq a\beta_0(0) + b\beta_0(T)$.

Also, as $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and

$\beta_0(t) = \beta_0(0),\ \forall\ t \in [-r,\ 0]$.

Now, starting with the LBVP

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &\quad + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[y(t) - \beta_0(t)] \\ &\quad + D_3 f(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)], \\ &\quad t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ ay(0) + by(T) &= 0, \end{aligned}\right\} \tag{6.16}$$

and proceeding similarly along the proof of the Theorem 6.3.2, we get a monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\alpha_0 \leq \beta_n,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} \beta'_{n+1}(t) + \lambda\beta_{n+1}(t) &= f(t,\ \beta_n(t),\ \beta_n(t-r)) \\ &\quad + D_2 f(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t) - \beta_n(t)] \\ &\quad + D_3 f(t, \beta_n(t), \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)], \\ &\quad t \in I, \\ \beta_{n+1}(t) &= \beta_{n+1}(0),\ t \in [-r,\ 0], \\ a\beta_{n+1}(0) + b\beta_{n+1}(T) &= 0. \end{aligned}\right\}$$

Further $\{\beta_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim\limits_{n\to\infty} \beta_n$ is a solution of the LBVP (4.3). $\square$

Theorem 6.3.7. *Let $f \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ and $\alpha_0,\ \beta_0 \in W$ be weakly coupled lower and upper solutions of LBVP* (4.8)*. Assume that $D_i f(t,\ x,\ y) > 0$ and $D_{i,\,j} f(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$. If $b < 0$, then there is a monotonic increasing sequence of functions $\{\alpha_n\}$ converging in $(W,\ ||.||_0)$ such that $\lim_{n\to\infty} \alpha_n$ is a solution of the LBVP* (4.8)*.*

Proof. Similar to the proof of the Theorem 6.3.5. □

Theorem 6.3.8. *Let $f \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ and $\alpha_0,\ \beta_0 \in W$ be weakly coupled lower and upper solutions of LBVP* (4.8)*. Assume that $D_i f(t,\ x,\ y) > 0$ and $D_{i,\,j} f(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$. If $b < 0$, then there is a monotonic increasing sequence of functions $\{\beta_n\}$ converging in $(W,\ ||.||_0)$ such that $\lim_{n\to\infty} \beta_n$ is a solution of the LBVP* (4.8)*.*

Proof. Similar to the proof of the Theorem 6.3.6. □

Next we use quasilinearization method to obtain the existence results for the NLBVP (4.4) for general first order DECD.

Theorem 6.3.9. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous functions satisfying $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$. Let $f \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ and $\alpha_0,\ \beta_0 \in X$ be weakly coupled lower and upper solutions of NLBVP* (4.4)*. Assume that $D_i f(t,\ x,\ y) > 0$ and $D_{i,\,j} f(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$. Then there is a monotonic increasing sequence of functions $\{\alpha_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim_{n\to\infty} \alpha_n$ is a solution of the NLBVP* (4.4)*.*

Proof. It is given that $D_{i,\,j} f(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$. Therefore

$f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq D_2 f(t,\ x_2,\ y_2)(x_1 - x_2) + D_3 f(t,\ x_2,\ y_2)(y_1 - y_2)$, $\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Since $\alpha_0,\ \beta_0 \in X$ are weakly coupled lower and upper solutions of NLBVP (4.4), we have 1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$

2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$

3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0].$

4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0].$

5. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T].$

6. $\max\{g(\alpha_0(0),\ \alpha_0(T)),\ g(\alpha_0(0),\ \beta_0(T))\}$
$\leq 0 \leq \min\{g(\beta_0(0),\ \alpha_0(T)),\ g(\beta_0(0),\ \beta_0(T))\}.$

As $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$, we have $\max\{g(\alpha_0(0),\ \alpha_0(T)),\ g(\alpha_0(0),\ \beta_0(T))\} = g(\alpha_0(0),\ \alpha_0(T))$ and $\min\{g(\beta_0(0),\ \alpha_0(T)),\ g(\beta_0(0),\ \beta_0(T))\} = g(\beta_0(0),\ \beta_0(T))$.

This implies $g(\alpha_0(0),\ \alpha_0(T)) \leq 0 \leq g(\beta_0(0),\ \beta_0(T))$.

Also, as $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and $\beta_0(t) = \beta_0(0),\ \forall\ t \in [-r,\ 0]$.

Now, starting with the NLBVP

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t) - \alpha_0(t)] \\ &\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r) - \alpha_0(t-r)], \\ &\quad t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ g(y(0),\ y(T)) &= 0, \end{aligned}\right\} \tag{6.17}$$

and proceeding similarly along the proof of the Theorem 6.3.1, we get a monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned}
&\alpha'_{n+1}(t) + \lambda\alpha_{n+1}(t) = f(t,\ \alpha_n(t),\ \alpha_n(t-r)) \\
&\qquad\qquad + D_2 f(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t) - \alpha_n(t)] \\
&\qquad\qquad + D_3 f(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t-r) - \alpha_n(t-r)], \\
&\qquad\qquad t \in I, \\
&\alpha_{n+1}(t) = \alpha_{n+1}(0),\ t \in [-r,\ 0], \\
&g(\alpha_{n+1}(0),\ \alpha_{n+1}(T)) = 0.
\end{aligned}\right\}$$

Further $\{\alpha_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty} \alpha_n$ is a solution of the NLBVP (4.4). □

Theorem 6.3.10. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous functions such that $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in I \times \mathbb{R}$ such that $x \geq y$. Let $f \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ and $\alpha_0,\ \beta_0 \in X$ be weakly coupled lower and upper solutions of NLBVP (4.4). Assume that $D_i f(t,\ x,\ y) > 0$ and $D_{i,\,j} f(t,\ x,\ y) \leq 0$, $\forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$. Then there is a monotonic increasing sequence of functions $\{\beta_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim_{n\to\infty} \beta_n$ is a solution of the NLBVP (4.4).*

Proof. Since $\alpha_0,\ \beta_0 \in X$ are weakly coupled lower and upper solutions of NLBVP (4.4), we have 1. $\alpha'_0(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I$.

2. $\beta'_0(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I$.
3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0]$.
4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0]$.
5. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
6. $\max\{g(\alpha_0(0),\ \alpha_0(T)),\ g(\alpha_0(0),\ \beta_0(T))\}$
   $\leq 0 \leq \min\{g(\beta_0(0),\ \alpha_0(T)),\ g(\beta_0(0),\ \beta_0(T))\}$.

As $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$, we have $\max\{g(\alpha_0(0),\ \alpha_0(T)),\ g(\alpha_0(0),\ \beta_0(T))\} = g(\alpha_0(0),\ \alpha_0(T))$ and $\min\{g(\beta_0(0),\ \alpha_0(T)),\ g(\beta_0(0),\ \beta_0(T))\} = g(\beta_0(0),\ \beta_0(T))$.

This implies $g(\alpha_0(0),\ \alpha_0(T)) \leq 0 \leq g(\beta_0(0),\ \beta_0(T))$.

Also, as $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and

$\beta_0(t) = \beta_0(0),\ \forall\ t \in [-r,\ 0]$.

It is given that $D_{i,\,j}f(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and

$\forall\ i,\ j = 2,\ 3$. Therefore

$f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq D_2f(t,\ x_1,\ y_1)(x_1 - x_2) + D_3f(t,\ x_1,\ y_1)(y_1 - y_2)$,

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Now, starting with the NLBVP

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &\quad + D_2f(t,\ \beta_0(t),\ \beta_0(t-r))[y(t) - \beta_0(t)] \\ &\quad + D_3f(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)], \\ &\quad t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ g(y(0),\ y(T)) &= 0, \end{aligned}\right\} \tag{6.18}$$

and proceeding similarly along the proof of the Theorem 6.3.2, we get a monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\alpha_0 \leq \beta_n,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} \beta'_{n+1}(t) + \lambda\beta_{n+1}(t) &= f(t,\ \beta_n(t),\ \beta_n(t-r)) \\ &\quad + D_2f(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t) - \beta_n(t)] \\ &\quad + D_3f(t, \beta_n(t), \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)], \\ &\quad t \in I, \\ \beta_{n+1}(t) &= \beta_{n+1}(0),\ t \in [-r,\ 0], \\ g(\beta_{n+1}(0),\ \beta_{n+1}(T)) &= 0. \end{aligned}\right\}$$

Further $\{\beta_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim\limits_{n\to\infty}\beta_n$ is a solution of the NLBVP (4.3). $\square$

Following are the existence results for the NLBVP (4.9) for first order DEVDA obtained by using quasilinearisation method.

Theorem 6.3.11. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous functions satisfying $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$. Let $f \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ and $\alpha_0,\ \beta_0 \in W$ be weakly coupled lower and upper solutions of NLBVP (4.9). Assume that $D_i f(t,\ x,\ y) > 0$ and $D_{i,\,j} f(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$. Then there is a monotonic increasing sequence of functions $\{\alpha_n\}$ converging in $(W,\ ||.||_0)$ such that $\lim_{n\to\infty} \alpha_n$ is a solution of the NLBVP (4.9).*

Proof. Similar to the proof of the Theorem 6.3.9. □

Theorem 6.3.12. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous functions satisfying $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in I \times \mathbb{R}$ such that $x \geq y$. Let $f \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ and $\alpha_0,\ \beta_0 \in W$ be weakly coupled lower and upper solutions of NLBVP (4.9). Assume that $D_i f(t,\ x,\ y) > 0$ and $D_{i,\,j} f(t,\ x,\ y) \leq 0$, $\forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$. Then there is a monotonic increasing sequence of functions $\{\beta_n\}$ converging in $(W,\ ||.||_0)$ such that $\lim_{n\to\infty} \beta_n$ is a solution of the NLBVP (4.9).*

Proof. Similar to the proof of the Theorem 6.3.10. □

Chapter 7

EXTENDED AND GENERALIZED QUASILINEARIZATION METHOD FOR FIRST ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

Some content of this chapter is published [see P4].

Some content of this chapter is presented at the Conference [see C5, C7 and C8].

7.1 Introduction

In this chapter, we discuss the extended quasilinearization method for boundary value problems of general first order ordinary differential equation with constant delay of the type
$y'(t) + \lambda y(t) = f(t,\ y(t),\ y(t-r)) + h(t,\ y(t),\ y(t-r))$ and first order ordinary differential equation with variable deviating argument of the type $y'(t) + \lambda y(t) = f(t,\ y(t),\ y(\theta(t))) + h(t,\ y(t),\ y(\theta(t)))$, where there are certain conditions on first and second order partial derivatives with respect to the second and third arguments of f and h.

We also discuss the generalized quasilinearization method for boundary value problems of general first order ordinary differential equation with constant delay of the type $y'(t) + \lambda y(t) = f(t,\ y(t),\ y(t-r))$ and first order ordinary differential equation with variable deviating argument of the type $y'(t) + \lambda y(t) = f(t,\ y(t),\ y(\theta(t)))$, by forcing certain conditions on first and second order partial derivatives with respect to the second and third argument of f and $f + \phi$, for some function ϕ.

Further we introduce extended-generalized quaslinearization method for boundary value problems of general first order ordinary differential equation with constant delay of the type
$y'(t) + \lambda y(t) = f(t,\ y(t),\ y(t-r)) + h(t,\ y(t),\ y(t-r))$ and first order ordinary differential equation with variable deviating argument of the type $y'(t) + \lambda y(t) = f(t,\ y(t),\ y(\theta(t))) + h(t,\ y(t),\ y(\theta(t)))$, where there are certain conditions on first and second order partial derivatives with respect to the second and third arguments of $f,\ h,\ f + \phi$ and $h + \psi$ for some functions ϕ and ψ.

7.2 Extended Quasilinearization Method

Let $T > 0$, $\lambda \geq 0$, $I = [0,\ T]$, $0 < r \leq T$, $\theta : I \to I$ be continuous function and $f,\ h : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous functions. Then we investigate the following BVPs for general first order DECD and first order DEVDA for existence of solution by using extended quasilinearization method. The PBVP of general first order DECD is given by

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(t-r)) + h(t,\ y(t),\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &y(0) = y(T). \end{aligned}\right\} \tag{7.1}$$

The LBVP of general first order DECD is given by

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(t-r)) + h(t,\ y(t),\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &ay(0) + by(T) = 0. \end{aligned}\right\} \tag{7.2}$$

In (7.2), we assume that $b < 0$.

The NLBVP of general first order DECD is given by

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(t-r)) + h(t,\ y(t),\ y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r,\ 0], \\ &g(y(0),\ y(T)) = 0. \end{aligned}\right\} \tag{7.3}$$

In (7.3), we assume that $g : \mathbb{R}^2 \to \mathbb{R}$ is continuous and non-increasing in the second argument.

The PBVP of first order DEVDA is given by

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(\theta(t))) + h(t,\ y(t),\ y(\theta(t))),\ t \in I, \\ &y(0) = y(T). \end{aligned}\right\} \tag{7.4}$$

The LBVP of first order DEVDA is given by

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(\theta(t))) + h(t,\ y(t),\ y(\theta(t))),\ t \in I, \\ &ay(0) + by(T) = 0. \end{aligned}\right\} \tag{7.5}$$

In (7.5), we assume that $b < 0$.

The NLBVP of first order DEVDA is given by

$$\left.\begin{aligned} &y'(t) + \lambda y(t) = f(t,\ y(t),\ y(\theta(t))) + h(t,\ y(t),\ y(\theta(t))),\ t \in I, \\ &g(y(0),\ y(T)) = 0. \end{aligned}\right\} \tag{7.6}$$

In (7.6), we assume that $g : \mathbb{R}^2 \to \mathbb{R}$ is continuous and non-increasing in the second argument.

We have the following result for PBVP(7.1).

Theorem 7.2.1. *Let $\alpha_0,\ \beta_0 \in X$ such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$ and $\Omega = \{(t,\ u,\ v) : \alpha_0(t) \leq u \leq \beta_0(t)$ and $\alpha_0(t-r) \leq v \leq \beta_0(t-r),\ t \in I\}$. Let $f,\ h \in C^2[\Omega,\ \mathbb{R}]$ be such that $D_i f > 0$, $D_i h > 0$, $D_{i,\,j} f \geq 0$ and $D_{i,\,j} h \leq 0$ on Ω and $\forall\ i,\ j = 2,\ 3$. Assume that α_0 and β_0 are lower and upper solutions of PBVP (7.1). Then there are monotonic sequence of functions $\{\alpha_n\}$ and $\{\beta_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim_{n\to\infty} \alpha_n$ and $\lim_{n\to\infty} \beta_n$ are solutions of the PBVP (7.1).*

Proof. It is given that $D_{i,\,j} f \geq 0$ and $D_{i,\,j} h \leq 0$ on Ω and $\forall\ i,\ j = 2,\ 3$.

Therefore $f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq D_2 f(t,\ x_2,\ y_2)(x_1 - x_2)$

$$+ D_3 f(t,\ x_2,\ y_2)(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2,\ y_1 \geq y_2$ and

$h(t,\ x_1,\ y_1) - h(t,\ x_2,\ y_2) \geq D_2 h(t,\ x_1,\ y_1)(x_1 - x_2)$

$$+ D_3 h(t,\ x_1,\ y_1)(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2,\ y_1 \geq y_2$.

Let $\mu_1 = \min_{(t,\,u,\,v)\in\Omega} D_2 h(t,\ u,\ v)$ and $\mu_2 = \min_{(t,\,u,\,v)\in\Omega} D_3 h(t,\ u,\ v)$. Then $\mu_1 > 0$ and $\mu_2 > 0$.

Thus we have, $h(t,\ x_1,\ y_1) - h(t,\ x_2,\ y_2) \geq \mu_1(x_1 - x_2) + \mu_2(y_1 - y_2)$,

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2,\ y_1 \geq y_2$.

Since $\alpha_0,\ \beta_0 \in X$ are lower and upper solutions of PBVP (7.1), we have

1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$
2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$
3. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0].$
4. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0].$
5. $\alpha_0(0) \leq \alpha_0(T).$
6. $\beta_0(0) \geq \beta_0(T).$

Consider

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t) - \alpha_0(t)] \\ &\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r) - \alpha_0(t-r)] \\ &\quad + \mu_1[y(t) - \alpha_0(t)] + \mu_2[y(t-r) - \alpha_0(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T). \end{aligned}\right\} \tag{7.7}$$

Let $F_{\alpha_0}(t,\ u,\ v) = f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r))$

$$\begin{aligned} &+ [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][u - \alpha_0(t)] \\ &+ [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][v - \alpha_0(t-r)], \\ &\forall\ (t,\ u,\ v) \in \Omega. \end{aligned}$$

This implies

$$\begin{aligned} F_{\alpha_0}(t,\ y(t),\ y(t-r)) = f(t,\ &\alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &+ [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][y(t) - \alpha_0(t)] \\ &+ [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][y(t-r) - \alpha_0(t-r)], \\ &\forall\ t \in I. \end{aligned}$$

Let $(t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

Then $x_1 - \alpha_0(t) \leq x_2 - \alpha_0(t)$ and $y_1 - \alpha_0(t-r) \leq y_2 - \alpha_0(t-r)$.

This implies $[D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][x_1 - \alpha_0(t)]$

$$\leq [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][x_2 - \alpha_0(t)]$$

and $[D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][y_1 - \alpha_0(t-r)]$

$$\leq [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][y_2 - \alpha_0(t-r)].$$

So we have

$$[D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][x_1 - \alpha_0(t)]$$
$$+ [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][y_1 - \alpha_0(t-r)]$$
$$\leq [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][x_2 - \alpha_0(t)]$$
$$+ [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][y_2 - \alpha_0(t-r)].$$

Thus $F_{\alpha_0}(t,\ x_1,\ y_1) \leq F_{\alpha_0}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

Also

$$\begin{aligned}
\alpha_0'(t) + \lambda\alpha_0(t) &\leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad + [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][\alpha_0(t) - \alpha_0(t)] \\
&\quad + [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][\alpha_0(t-r) - \alpha_0(t-r)] \\
&= F_{\alpha_0}(t,\ \alpha_0(t),\ \alpha_0(t-r)).
\end{aligned}$$

Further

$$\begin{aligned}
\beta_0'(t) + \lambda\beta_0(t) &\geq f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\geq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad + [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][\beta_0(t) - \alpha_0(t)] \\
&\quad + [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][\beta_0(t-r) - \alpha_0(t-r)] \\
&= F_{\alpha_0}(t,\ \beta_0(t),\ \beta_0(t-r)).
\end{aligned}$$

We already know that

1. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
3. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
4. $\alpha_0(0) \leq \alpha_0(T)$.
5. $\beta_0(0) \geq \beta_0(T)$.

Therefore α_0 and β_0 are lower and upper solutions of the PBVP (7.7) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Thus the PBVP (7.7) has a solution $\alpha_1 \in X$ such that $\alpha_0 \leq \alpha_1 \leq \beta_0$ on $[-r,\ T]$. (Using Theorem 4.4.5)

Therefore

$$\left.\begin{aligned} \alpha_1'(t) + \lambda\alpha_1(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_1(t) - \alpha_0(t)] \\ &\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_1(t-r) - \alpha_0(t-r)] \\ &\quad + \mu_1[\alpha_1(t) - \alpha_0(t)] + \mu_2[\alpha_1(t-r) - \alpha_0(t-r)],\ t \in I, \\ \alpha_1(t) &= \alpha_1(0),\ t \in [-r,\ 0], \\ \alpha_1(0) &= \alpha_1(T). \end{aligned}\right\}$$

This implies

$\alpha_1'(t) + \lambda\alpha_1(t) \leq f(t,\ \alpha_1(t),\ \alpha_1(t-r)) + h(t,\ \alpha_1(t),\ \alpha_1(t-r)),\ t \in I$.

Also

1. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I$.
2. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
3. $\alpha_1(t) = \alpha_1(0),\ t \in [-r,\ 0]$.
4. $\alpha_1(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
5. $\alpha_1(0) \leq \alpha_1(T)$.
6. $\beta_0(0) \geq \beta_0(T)$.

Therefore α_1 and $\beta_0 \in X$ are lower and upper solutions of the PBVP (7.1) such that $\alpha_1(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Continuing in this manner, we get a monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned}
\alpha'_{n+1}(t) + \lambda\alpha_{n+1}(t) &= f(t,\ \alpha_n(t),\ \alpha_n(t-r)) + h(t,\ \alpha_n(t),\ \alpha_n(t-r)) \\
&\quad + D_2 f(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t) - \alpha_n(t)] \\
&\quad + D_3 f(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t-r) - \alpha_n(t-r)] \\
&\quad + \mu_1[\alpha_{n+1}(t) - \alpha_n(t)] + \mu_2[\alpha_{n+1}(t-r) - \alpha_n(t-r)], \\
&\quad t \in I, \\
\alpha_{n+1}(t) &= \alpha_{n+1}(0),\ t \in [-r,\ 0], \\
\alpha_{n+1}(0) &= \alpha_{n+1}(T).
\end{aligned}\right\}$$

Further $\lim\limits_{n\to\infty} \alpha_n$ exists in $(X,\ ||.||_\infty)$ and $\lim\limits_{n\to\infty} \alpha_n$ is a solution of the PBVP (7.1).

Let $\theta_1 = \min\limits_{(t,\ u,\ v)\in\Omega} D_2 f(t,\ u,\ v)$ and $\theta_2 = \min\limits_{(t,\ u,\ v)\in\Omega} D_3 f(t,\ u,\ v)$. Then $\theta_1 > 0$ and $\theta_2 > 0$.

Thus we have $f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq \theta_1(x_1 - x_2) + \theta_2(y_1 - y_2)$,

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Now consider

$$\left.\begin{aligned}
y'(t) + \lambda y(t) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad + D_2 h(t,\ \beta_0(t),\ \beta_0(t-r))[y(t) - \beta_0(t)] \\
&\quad + D_3 h(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)] \\
&\quad + \theta_1[y(t) - \beta_0(t)] + \theta_2[y(t-r) - \beta_0(t-r)], \\
&\quad t \in I, \\
y(t) &= y(0),\ t \in [-r,\ 0], \\
y(0) &= y(T).
\end{aligned}\right\} \tag{7.8}$$

$$\begin{aligned}\text{Let } G_{\beta_0}(t,\ u,\ v) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &\quad + [\theta_1 + D_2h(t,\ \beta_0(t),\ \beta_0(t-r))][u - \beta_0(t)] \\ &\quad + [\theta_2 + D_3h(t,\ \beta_0(t),\ \beta_0(t-r))][v - \beta_0(t-r)], \\ &\quad \forall\ (t,\ u,\ v) \in \Omega.\end{aligned}$$

This implies

$$\begin{aligned}G_{\beta_0}(t,\ y(t),\ y(t-r)) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &\quad + [\theta_1 + D_2h(t,\ \beta_0(t),\ \beta_0(t-r))][y(t) - \beta_0(t)] \\ &\quad + [\theta_2 + D_3h(t,\ \beta_0(t),\ \beta_0(t-r))][y(t-r) - \beta_0(t-r)], \\ &\quad \forall\ t \in I.\end{aligned}$$

So we have $G_{\beta_0}(t,\ x_1,\ y_1) \leq G_{\beta_0}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

Also

$$\begin{aligned}\alpha_0'(t) + \lambda\alpha_0(t) &\leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\leq f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t-r),\ \beta_0(t)) \\ &\quad + [\theta_1 + D_2h(t,\ \beta_0(t),\ \beta_0(t-r))][\alpha_0(t) - \beta_0(t)] \\ &\quad + [\theta_2 + D_3h(t,\ \beta_0(t),\ \beta_0(t-r))][\alpha_0(t-r) - \beta_0(t-r)] \\ &= G_{\beta_0}(t,\ \alpha_0(t),\ \alpha_0(t-r)).\end{aligned}$$

Further

$$\begin{aligned}\beta_0'(t) + \lambda\beta_0(t) &\geq f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t-r),\ \beta_0(t)) \\ &\quad + [\theta_1 + D_2h(t,\ \beta_0(t),\ \beta_0(t-r))][\beta_0(t) - \beta_0(t)] \\ &\quad + [\theta_2 + D_3h(t,\ \beta_0(t),\ \beta_0(t-r))][\beta_0(t-r) - \beta_0(t-r)] \\ &= G_{\beta_0}(t,\ \beta_0(t),\ \beta_0(t-r)).\end{aligned}$$

We already know that

1. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
3. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
4. $\alpha_0(0) \leq \alpha_0(T)$.
5. $\beta_0(0) \geq \beta_0(T)$.

Therefore α_0 and β_0 are lower and upper solutions of the PBVP (7.8) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\, t \in [-r,\ T]$.

Thus the PBVP (7.8) has a solution $\beta_1 \in X$ such that $\alpha_0 \leq \beta_1 \leq \beta_0$ on $[-r,\ T]$. (Using Theorem 4.4.5)

Therefore

$$\left.\begin{aligned} \beta_1'(t) + \lambda\beta_1(t) = {} & f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t-r),\ \beta_0(t)) \\ & + D_2h(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_1(t) - \beta_0(t)] \\ & + D_3h(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_1(t-r) - \beta_0(t-r)] \\ & + \theta_1[\beta_1(t) - \beta_0(t)] + \theta_2[\beta_1(t-r) - \beta_0(t-r)], \\ & t \in I, \\ \beta_1(t) = \beta_1(0),\ & t \in [-r,\ 0], \\ \beta_1(0) = \beta_1(T). & \end{aligned}\right\}$$

This implies

$\beta_1'(t) + \lambda\beta_1(t) \geq f(t,\ \beta_1(t),\ \beta_1(t-r)) + h(t,\ \beta_1(t),\ \beta_1(t-r)),\ t \in I$.

Also

1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I$.
2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
3. $\beta_1(t) = \beta_1(0),\ t \in [-r,\ 0]$.
4. $\alpha_0(t) \leq \beta_1(t),\ t \in [-r,\ T]$.
5. $\alpha_0(0) \leq \alpha_0(T)$.
6. $\beta_1(0) \geq \beta_1(T)$.

Therefore α_0 and $\beta_1 \in X$ are lower and upper solutions of the PBVP (7.1) such that $\alpha_0(t) \leq \beta_1(t),\ \forall\, t \in [-r,\ T]$.

Continuing in this manner, we get a monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\alpha_0 \leq \beta_n,\ \forall\, n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} \beta'_{n+1}(t) + \lambda\beta_{n+1}(t) &= f(t,\ \beta_n(t),\ \beta_n(t-r)) + h(t,\ \beta_n(t-r),\ \beta_n(t)) \\ &\quad + D_2h(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t) - \beta_n(t)] \\ &\quad + D_3h(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)] \\ &\quad + \theta_1[\beta_{n+1}(t) - \beta_n(t)] + \theta_2[\beta_{n+1}(t-r) - \beta_n(t-r)], \\ &\quad t \in I, \\ \beta_{n+1}(t) &= \beta_{n+1}(0),\ t \in [-r,\ 0], \\ \beta_{n+1}(0) &= \beta_{n+1}(T). \end{aligned}\right\}$$

Further $\lim_{n\to\infty}\beta_n$ exists in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty}\beta_n$ is a solution of the PBVP (7.1). □

Next we obtain similar result for PBVP(7.4).

Theorem 7.2.2. *Let $\alpha_0,\ \beta_0 \in W$ such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in I$ and $\Omega = \{(t,\ u,\ v) : \alpha_0(t) \leq u \leq \beta_0(t)$ and $\alpha_0(\theta(t)) \leq v \leq \beta_0(\theta(t)),\ t \in I\}$. Let $f,\ h \in C^2[\Omega,\ \mathbb{R}]$ be such that $D_if > 0$, $D_ih > 0$, $D_{i,\,j}f \geq 0$ and $D_{i,\,j}h \leq 0$ on Ω and $\forall\ i,\ j = 2,\ 3$. Assume that α_0 and β_0 are lower and upper solutions of PBVP (7.4). Then there are monotonic sequence of functions $\{\alpha_n\}$ and $\{\beta_n\}$ converging in $(W,\ ||.||_0)$ such that $\lim_{n\to\infty}\alpha_n$ and $\lim_{n\to\infty}\beta_n$ are solutions of the PBVP (7.4).*

Proof. Similar to the proof of of Theorem 7.2.1. □

We now prove results on existence of solution for LBVP of general first order DECD(7.2) and for LBVP of first order DEVDA(7.5) by using extended quasilinearization method.

Theorem 7.2.3. *Let* $\alpha_0,\ \beta_0 \in X$ *such that* $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$ *and* $\Omega = \{(t,\ u,\ v) : \alpha_0(t) \leq u \leq \beta_0(t)$ *and* $\alpha_0(t-r) \leq v \leq \beta_0(t-r),\ t \in I\}$. *Let* $f,\ h \in C^2[\Omega,\ \mathbb{R}]$ *be such that* $D_i f > 0,\ D_i h > 0,\ D_{i,\,j} f \geq 0$ *and* $D_{i,\,j} h \leq 0$ *on* $\Omega,\ \forall\ i,\ j = 2,\ 3$. *Assume that* α_0 *and* β_0 *are weakly coupled lower and upper solutions of LBVP* (7.2). *If* $b < 0$, *then there are monotonic sequence of functions* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *converging in* $(X,\ ||.||_\infty)$ *such that* $\lim\limits_{n\to\infty} \alpha_n$ *and* $\lim\limits_{n\to\infty} \beta_n$ *are solutions of the LBVP* (7.2).

Proof. Since, $\alpha_0,\ \beta_0 \in X$ are weakly coupled lower and upper solutions of LBVP (7.2), we have

1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$
2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$
3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0].$
4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0].$
5. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T].$
6. $\max\{a\alpha_0(0) + b\alpha_0(T), a\alpha_0(0) + b\beta_0(T)\}$
   $\leq 0 \leq \min\{a\beta_0(0) + b\alpha_0(T), a\beta_0(0) + b\beta_0(T)\}.$

As $b < 0$, we have

$\max\{a\alpha_0(0) + b\alpha_0(T),\ a\alpha_0(0) + b\beta_0(T)\} = a\alpha_0(0) + b\alpha_0(T)$ and

$\min\{a\beta_0(0) + b\alpha_0(T),\ a\beta_0(0) + b\beta_0(T)\} = a\beta_0(0) + b\beta_0(T).$

This implies $a\alpha_0(0) + b\alpha_0(T) \leq 0 \leq a\beta_0(0) + b\beta_0(T)$.

Also, as $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and

$\beta_0(t) = \beta_0(0),\ \forall\ t \in [-r,\ 0]$.

It is given that $D_{i,\,j} f \geq 0$ and $D_{i,\,j} h \leq 0$ on $\Omega,\ \forall\ i,\ j = 2,\ 3$. Therefore

$$f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq D_2 f(t,\ x_2,\ y_2)(x_1 - x_2) + D_3 f(t,\ x_2,\ y_2)(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2,\ y_1 \geq y_2$ and

$$h(t,\ x_1,\ y_1) - h(t,\ x_2,\ y_2) \geq D_2h(t,\ x_1,\ y_1)(x_1 - x_2) + D_3h(t, x_1, y_1)(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2,\ y_1 \geq y_2$.

Let $\mu_1 = \min\limits_{(t,\ u,\ v)\in\Omega} D_2h(t,\ u,\ v)$ and $\mu_2 = \min\limits_{(t,\ u,\ v)\in\Omega} D_3h(t,\ u,\ v)$. Then $\mu_1 > 0$ and $\mu_2 > 0$.

Thus, we have $h(t,\ x_1,\ y_1) - h(t,\ x_2,\ y_2) \geq \mu_1(x_1 - x_2) + \mu_2(y_1 - y_2)$,

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2,\ y_1 \geq y_2$.

Now, starting with the LBVP

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\quad + D_2f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t) - \alpha_0(t)] \\ &\quad + D_3f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r) - \alpha_0(t-r)] \\ &\quad + \mu_1[y(t) - \alpha_0(t)] + \mu_2[y(t-r) - \alpha_0(t-r)], \\ &\quad t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ ay(0) + by(T) &= 0, \end{aligned}\right\} \tag{7.9}$$

and proceeding similarly along the proof of the Theorem 7.2.1, we get a monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} \alpha'_{n+1}(t) + \lambda\alpha_{n+1}(t) &= f(t,\ \alpha_n(t),\ \alpha_n(t-r)) + h(t,\ \alpha_n(t),\ \alpha_n(t-r)) \\ &\quad + D_2f(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t) - \alpha_n(t)] \\ &\quad + D_3f(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t-r) - \alpha_n(t-r)] \\ &\quad + \mu_1[\alpha_{n+1}(t) - \alpha_n(t)] + \mu_2[\alpha_{n+1}(t-r) - \alpha_n(t-r)], \\ &\quad t \in I, \\ \alpha_{n+1}(t) &= \alpha_{n+1}(0),\ t \in [-r,\ 0], \\ a\alpha_{n+1}(0) + b\alpha_{n+1}(T) &= 0. \end{aligned}\right\}$$

Further $\lim_{n\to\infty}\alpha_n$ exists in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty}\alpha_n$ is a solution of the LBVP (7.2).

Let $\theta_1 = \min_{(t,\ u,\ v)\in\Omega} D_2f(t,\ u,\ v)$ and $\theta_2 = \min_{(t,\ u,\ v)\in\Omega} D_3f(t,\ u,\ v)$. Then $\theta_1 > 0$ and $\theta_2 > 0$.

Thus we have $f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq \theta_1(x_1 - x_2) + \theta_2(y_1 - y_2)$,

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Next, begining with the LBVP

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &\quad + D_2h(t,\ \beta_0(t),\ \beta_0(t-r))[y(t) - \beta_0(t)] \\ &\quad + D_3h(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)], \\ &\quad + \theta_1[y(t) - \beta_0(t)] + \theta_2[y(t-r) - \beta_0(t-r)] \\ &\quad t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ ay(0) + by(T) &= 0, \end{aligned}\right\} \tag{7.10}$$

and proceeding similarly along the proof of the Theorem 7.2.1, we get a monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\alpha_0 \leq \beta_n,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} \beta'_{n+1}(t) + \lambda\beta_{n+1}(t) &= f(t,\ \beta_n(t),\ \beta_n(t-r)) + h(t,\ \beta_n(t-r),\ \beta_n(t)) \\ &\quad + D_2h(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t) - \beta_n(t)] \\ &\quad + D_3h(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)], \\ &\quad + \theta_1[\beta_{n+1}(t) - \beta_n(t)] + \theta_2[\beta_{n+1}(t-r) - \beta_n(t-r)] \\ &\quad t \in I, \\ \beta_{n+1}(t) &= \beta_{n+1}(0),\ t \in [-r,\ 0], \\ a\beta_{n+1}(0) + b\beta_{n+1}(T) &= 0. \end{aligned}\right\}$$

Further $\lim_{n\to\infty}\beta_n$ exists in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty}\beta_n$ is a solution of the LBVP (7.2). □

Theorem 7.2.4. *Let $\alpha_0,\ \beta_0 \in W$ such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in I$ and $\Omega = \{(t,\ u,\ v) : \alpha_0(t) \leq u \leq \beta_0(t)$ and $\alpha_0(\theta(t)) \leq v \leq \beta_0(\theta(t)),\ t \in I\}$. Let $f,\ h \in C^2[\Omega,\ \mathbb{R}]$ be such that $D_i f > 0,\ D_i h > 0,\ D_{i,\,j} f \geq 0$ and $D_{i,\,j} h \leq 0$ on $\Omega,\ \forall\ i,\ j = 2,\ 3$. Assume that α_0 and β_0 are weakly coupled lower and upper solutions of LBVP (7.5). If $b < 0$, then there are monotonic sequence of functions $\{\alpha_n\}$ and $\{\beta_n\}$ converging in $(W,\ ||.||_0)$ such that $\lim_{n\to\infty}\alpha_n$ and $\lim_{n\to\infty}\beta_n$ are solutions of the LBVP (7.5).*

Proof. Similar to the proof of the Theorem 7.2.3 □

Next, we prove results on existence of solutions for NLBVP of first order DECD (7.3) and NLBVP of first order DEVDA (7.6). We make use of extended quasilinearization method to obtain our results.

Theorem 7.2.5. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$. Let $\alpha_0,\ \beta_0 \in X$ such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$ and $\Omega = \{(t,\ u,\ v) : \alpha_0(t) \leq u \leq \beta_0(t)$ and $\alpha_0(t-r) \leq v \leq \beta_0(t-r),\ t \in I\}$. Assume that $f,\ h \in C^2[\Omega,\ \mathbb{R}]$ be such that $D_i f > 0,\ D_i h > 0,\ D_{i,\,j} f \geq 0$ and $D_{i,\,j} h \leq 0$ on Ω and $\forall\ i,\ j = 2,\ 3$. Further, assume that α_0 and β_0 are weakly coupled lower and upper solutions of NLBVP (7.3). Then there are monotonic sequence of functions $\{\alpha_n\}$ and $\{\beta_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim_{n\to\infty}\alpha_n$ and $\lim_{n\to\infty}\beta_n$ are solutions of the NLBVP (7.3).*

Proof. Since, $\alpha_0,\ \beta_0 \in X$ are weakly coupled lower and upper solutions of NLBVP (7.3), we have

1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$
2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$
3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0].$
4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0].$
5. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T].$
6. $\max\{g(\alpha_0(0),\ \alpha_0(T)),\ g(\alpha_0(0),\ \beta_0(T))\}$
   $\leq 0 \leq \min\{g(\beta_0(0),\ \alpha_0(T)),\ g(\beta_0(0),\ \beta_0(T))\}.$

As $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$, we have

$\max\{g(\alpha_0(0),\ \alpha_0(T)),\ g(\alpha_0(0),\ \beta_0(T))\} = g(\alpha_0(0),\ \alpha_0(T))$ and

$\min\{g(\beta_0(0),\ \alpha_0(T)),\ g(\beta_0(0),\ \beta_0(T))\} = g(\beta_0(0),\ \beta_0(T)).$

This implies $g(\alpha_0(0),\ \alpha_0(T)) \leq 0 \leq g(\beta_0(0),\ \beta_0(T)).$

Also, as $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and

$\beta_0(t) = \beta_0(0),\ \forall\ t \in [-r,\ 0].$

It is given that $D_{i,\,j}f \geq 0$ and $D_{i,\,j}h \leq 0$ on $\Omega,\ \forall\ i,\ j = 2,\ 3.$ Therefore

$$f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq D_2f(t,\ x_2,\ y_2)(x_1 - x_2) + D_3f(t,\ x_2,\ y_2)(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2,\ y_1 \geq y_2$ and

$$h(t,\ x_1,\ y_1) - h(t,\ x_2,\ y_2) \geq D_2h(t,\ x_1,\ y_1)(x_1 - x_2) + D_3h(t, x_1, y_1)(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2,\ y_1 \geq y_2.$

Let $\mu_1 = \min\limits_{(t,\,u,\,v)\in\Omega} D_2h(t,\ u,\ v)$ and $\mu_2 = \min\limits_{(t,\,u,\,v)\in\Omega} D_3h(t,\ u,\ v).$ Then $\mu_1 > 0$ and $\mu_2 > 0.$

Thus, we have $h(t,\ x_1,\ y_1) - h(t,\ x_2,\ y_2) \geq \mu_1(x_1 - x_2) + \mu_2(y_1 - y_2),$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2,\ y_1 \geq y_2.$

Now, starting with the NLBVP

$$\left.\begin{aligned} y'(t)+\lambda y(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t)-\alpha_0(t)] \\ &\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r)-\alpha_0(t-r)] \\ &\quad + \mu_1[y(t)-\alpha_0(t)] + \mu_2[y(t-r)-\alpha_0(t-r)], \\ &\quad t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ g(y(0),\ y(T)) &= 0, \end{aligned}\right\} \tag{7.11}$$

and proceeding similarly along the proof of the Theorem 7.2.1, we get a monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} \alpha'_{n+1}(t)+\lambda\alpha_{n+1}(t) &= f(t,\ \alpha_n(t),\ \alpha_n(t-r)) + h(t,\ \alpha_n(t),\ \alpha_n(t-r)) \\ &\quad + D_2 f(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t)-\alpha_n(t)] \\ &\quad + D_3 f(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t-r)-\alpha_n(t-r)] \\ &\quad + \mu_1[\alpha_{n+1}(t)-\alpha_n(t)] + \mu_2[\alpha_{n+1}(t-r)-\alpha_n(t-r)], \\ &\quad t \in I, \\ \alpha_{n+1}(t) &= \alpha_{n+1}(0),\ t \in [-r,\ 0], \\ g(\alpha_{n+1}(0),\ \alpha_{n+1}(T)) &= 0. \end{aligned}\right\}$$

Further $\lim\limits_{n\to\infty} \alpha_n$ exists in $(X,\ ||.||_\infty)$ and $\lim\limits_{n\to\infty} \alpha_n$ is a solution of the NLBVP (7.3).

Let $\theta_1 = \min\limits_{(t,\ u,\ v)\in\Omega} D_2 f(t,\ u,\ v)$ and $\theta_2 = \min\limits_{(t,\ u,\ v)\in\Omega} D_3 f(t,\ u,\ v)$. Then $\theta_1 > 0$ and $\theta_2 > 0$.

Thus we have $f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq \theta_1(x_1 - x_2) + \theta_2(y_1 - y_2)$,

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Next, begining with the NLBVP

$$\left.\begin{aligned} y'(t)+\lambda y(t) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &\quad + D_2h(t,\ \beta_0(t),\ \beta_0(t-r))[y(t)-\beta_0(t)] \\ &\quad + D_3h(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r)-\beta_0(t-r)] \\ &\quad + \theta_1[y(t)-\beta_0(t)] + \theta_2[y(t-r)-\beta_0(t-r)], \\ &\quad t\in I, \\ y(t) &= y(0),\ t\in[-r,\ 0], \\ g(y(0),\ y(T)) &= 0, \end{aligned}\right\} \tag{7.12}$$

and proceeding similarly along the proof of the Theorem 7.2.1, we get a monotonic decreasing sequence of functions $\{\beta_n\}\subseteq X$ such that $\alpha_0\leq\beta_n,\ \forall\ n\in\mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} \beta'_{n+1}(t)+\lambda\beta_{n+1}(t) &= f(t,\ \beta_n(t),\ \beta_n(t-r)) + h(t,\ \beta_n(t-r),\ \beta_n(t)) \\ &\quad + D_2h(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t)-\beta_n(t)] \\ &\quad + D_3h(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t-r)-\beta_n(t-r)], \\ &\quad + \theta_1[\beta_{n+1}(t)-\beta_n(t)] + \theta_2[\beta_{n+1}(t-r)-\beta_n(t-r)] \\ &\quad t\in I, \\ \beta_{n+1}(t) &= \beta_{n+1}(0),\ t\in[-r,\ 0], \\ g(\beta_{n+1}(0),\ \beta_{n+1}(T)) &= 0. \end{aligned}\right\}$$

Further $\lim\limits_{n\to\infty}\beta_n$ exists in $(X,\ ||.||_\infty)$ and $\lim\limits_{n\to\infty}\beta_n$ is a solution of the NLBVP (7.3). □

Theorem 7.2.6. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $g(t,\ x) \geq g(t,\ y)$, $\forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$. Let $\alpha_0,\ \beta_0 \in W$ such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in I$ and $\Omega = \{(t,\ u,\ v) : \alpha_0(t) \leq u \leq \beta_0(t)$ and $\alpha_0(\theta(t)) \leq v \leq \beta_0(\theta(t)),\ t \in I\}$. Assume that $f,\ h \in C^2[\Omega,\ \mathbb{R}]$ be such that $D_i f > 0$, $D_i h > 0$, $D_{i,\,j} f \geq 0$ and $D_{i,\,j} h \leq 0$ on Ω and $\forall\ i,\ j = 2,\ 3$. Further, assume that α_0 and β_0 are weakly coupled lower and upper solutions of NLBVP* (7.6)*. Then there are monotonic sequence of functions $\{\alpha_n\}$ and $\{\beta_n\}$ converging in $(W,\ ||.||_0)$ such that $\lim_{n\to\infty} \alpha_n$ and $\lim_{n\to\infty} \beta_n$ are solutions of the NLBVP* (7.6).

Proof. Similar to the proof of the Theorem 7.2.5. □

7.3 Generalized Quasilinearization Method

In this section, we establish results on existence of solution for boundary value problems of general first order DECD of the type $y'(t)+\lambda y(t) = f(t,\ y(t),\ y(t-r))$ and first order DEVDA of the type $y'(t) + \lambda y(t) = f(t,\ y(t),\ y(\theta(t)))$, by forcing certain conditions on first and second order partial derivatives with respect to the second and third argument of f and $f + \phi$, for some function ϕ. We shall be using the generalized quasilinearization method to obtain our results.

Theorem 7.3.1. *Let $\alpha_0,\ \beta_0 \in X$ be lower and upper solutions of PBVP* (4.1) *such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$. Let $f,\ \phi \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ such that*
(1) $D_{i,\,j} f(t,\ x,\ y) + D_{i,\,j}\phi(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$.
(2) $D_i f(t,\ x,\ y) + D_i\phi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i = 2,\ 3$.
(3) $\phi(t,\ u_1,\ v_1) \geq \phi(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Then there is monotonic increasing sequence of functions $\{\alpha_n\}$converging in $(X,\ ||.||_\infty)$ such that $\lim_{n\to\infty} \alpha_n$ is a solution of the PBVP (4.1).

Proof. Let $F(t,\ x,\ y) = f(t,\ x,\ y) + \phi(t,\ x, y),\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$.

Therefore $D_{i,\,j}F(t,\ x,\ y) \geq 0$ and $D_iF(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$.

This implies $F(t,\ x_1,\ y_1) - F(t,\ x_2,\ y_2) \geq D_2F(t,\ x_2,\ y_2)(x_1 - x_2)$

$$+ D_3F(t,\ x_2,\ y_2)(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

So we have $f(t,\ x_1,\ y_1) + \phi(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) - \phi(t,\ x_2,\ y_2)$

$$\geq D_2f(t,\ x_2,\ y_2)(x_1 - x_2) + D_2\phi(t,\ x_2,\ y_2)(x_1 - x_2)$$
$$+ D_3f(t,\ x_2,\ y_2)(y_1 - y_2) + D_3\phi(t,\ x_2,\ y_2)(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

This gives

$$f(t,\ x_1,\ y_1) \geq f(t,\ x_2,\ y_2) - \phi(t,\ x_1,\ y_1) + \phi(t,\ x_2,\ y_2)$$
$$+ D_2f(t,\ x_2, y_2)(x_1 - x_2) + D_2\alpha(t,\ x_2,\ y_2)(x_1 - x_2)$$
$$+ D_3f(t,\ x_2,\ y_2)(y_1 - y_2) + D_3\alpha(t,\ x_2,\ y_2)(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Since $\alpha_0,\ \beta_0 \in X$ are lower and upper solutions of PBVP (4.1), we have

1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t - r)),\ t \in I$,
2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t - r)),\ t \in I$,
3. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$,
4. $\beta_0(t) = \beta_0(0),\ t \in [-r, 0]$,
5. $\alpha_0(0) \leq \alpha_0(T)$,
6. $\beta_0(0) \geq \beta_0(T)$.

Consider

$$\left.\begin{aligned} y'(t)+\lambda y(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\quad - \phi(t,\ y(t),\ y(t-r)) \\ &\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t)-\alpha_0(t)] \\ &\quad + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t)-\alpha_0(t)] \\ &\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r)-\alpha_0(t-r)] \\ &\quad + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r)-\alpha_0(t-r)], \\ &\quad t\in I, \\ y(t) &= y(0),\ t\in[-r,\ 0], \\ y(0) &= y(T). \end{aligned}\right\} \tag{7.13}$$

Let $F_{\alpha_0}(t,\ u,\ v) = f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r))$

$$\begin{aligned} &- \phi(t,\ u,\ v) + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[u-\alpha_0(t)] \\ &+ D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[u-\alpha_0(t)] \\ &+ D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[v-\alpha_0(t-r)] \\ &+ D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[v-\alpha_0(t-r)], \\ &\forall\ (t,\ u,\ v)\in I\times\mathbb{R}^2. \end{aligned}$$

This implies

$$\begin{aligned} F_{\alpha_0}(t,\ y(t),\ y(t-r)) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\quad - \phi(t,\ y(t),\ y(t-r)) \\ &\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t)-\alpha_0(t)] \\ &\quad + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t)-\alpha_0(t)] \\ &\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r)-\alpha_0(t-r)] \\ &\quad + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r)-\alpha_0(t-r)], \\ &\quad \forall\ t\in I. \end{aligned}$$

Let $(t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2)\in I\times\mathbb{R}^2$ such that $x_1\le x_2$ and $y_1\le y_2$.

Then $x_1-\alpha_0(t)\le x_2-\alpha_0(t)$ and $y_1-\alpha_0(t-r)\le y_2-\alpha_0(t-r)$.

This gives $[D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))][x_1 - \alpha_0(t)]$

$\leq [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))][x_2 - \alpha_0(t)]$

and $[D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))][y_1 - \alpha_0(t-r)]$

$\leq [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))][y_2 - \alpha_0(t-r)]$.

So we have

$$[D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))][x_1 - \alpha_0(t)]$$
$$+ D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))][y_1 - \alpha_0(t-r)]$$
$$\leq [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))][x_2 - \alpha_0(t)]$$
$$+ [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))][y_2 - \alpha_0(t-r)].$$

Therefore

$$f(t,\ \alpha_0(t),\ \alpha_0(t-r))$$
$$+ [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))][x_1 - \alpha_0(t)]$$
$$+ D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))][y_1 - \alpha_0(t-r)]$$
$$\leq f(t,\ \alpha_0(t),\ \alpha_0(t-r))$$
$$+ [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))][x_2 - \alpha_0(t)]$$
$$+ [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))][y_2 - \alpha_0(t-r)].$$

It is given that $\phi(t,\ x_1,\ y_1) \geq \phi(t,\ x_2,\ y_2)$.

This implies $\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) - \phi(t,\ x_1,\ y_1)$

$\leq \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) - \phi(t,\ x_2,\ y_2)$.

Thus we have

$$f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) - \phi(t,\ x_1,\ y_1)$$
$$+ [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))][x_1 - \alpha_0(t)]$$
$$+ D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))][y_1 - \alpha_0(t-r)]$$
$$\leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) - \phi(t,\ x_2,\ y_2)$$
$$+ [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))][x_2 - \alpha_0(t)]$$
$$+ [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))][y_2 - \alpha_0(t-r)].$$

Therefore $F_{\alpha_0}(t,\ x_1,\ y_1) \leq F_{\alpha_0}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

Also

$$\begin{aligned}
\alpha_0'(t) + \lambda\alpha_0(t) &\leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad - \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_0(t) - \alpha_0(t)] \\
&\quad + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_0(t) - \alpha_0(t)] \\
&\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_0(t-r) - \alpha_0(t-r)] \\
&\quad + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_0(t-r) - \alpha_0(t-r)] \\
&= F_{\alpha_0}(t,\ \alpha_0(t),\ \alpha_0(t-r)).
\end{aligned}$$

Further

$$\begin{aligned}
\beta_0'(t) + \lambda\beta_0(t) &\geq f(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\geq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad - \phi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\beta_0(t) - \alpha_0(t)] \\
&\quad + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[\beta_0(t) - \alpha_0(t)] \\
&\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\beta_0(t-r) - \alpha_0(t-r)] \\
&\quad + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))][\beta_0(t-r) - \alpha_0(t-r)] \\
&= F_{\alpha_0}(t,\ \beta_0(t),\ \beta_0(t-r)).
\end{aligned}$$

We already know that

1. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
3. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
4. $\alpha_0(0) \leq \alpha_0(T)$.
5. $\beta_0(0) \geq \beta_0(T)$.

Therefore α_0 and β_0 are lower and upper solutions of the PBVP (7.13) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\, t \in [-r,\ T]$.

Thus the PBVP (7.13) has a solution $\alpha_1 \in X$ such that $\alpha_0 \leq \alpha_1 \leq \beta_0$ on $[-r,\ T]$. (Using Theorem 4.4.5)

Therefore

$$\left.\begin{aligned}
\alpha_1'(t) + \lambda\alpha_1(t) = {} & f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
& - \phi(t,\ \alpha_1(t),\ \alpha_1(t-r)) \\
& + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_1(t) - \alpha_0(t)] \\
& + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_1(t) - \alpha_0(t)] \\
& + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_1(t-r) - \alpha_0(t-r)] \\
& + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_1(t-r) - \alpha_0(t-r)],\ t \in I, \\
\alpha_1(t) = {} & \alpha_1(0),\ t \in [-r,\ 0], \\
\alpha_1(0) = {} & \alpha_1(T).
\end{aligned}\right\}$$

This implies $\alpha_1'(t) + \lambda\alpha_1(t) \leq f(t,\ \alpha_1(t),\ \alpha_1(t-r)),\ t \in I$.

Also 1. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I$.

2. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
3. $\alpha_1(t) = \alpha_1(0),\ t \in [-r,\ 0]$.
4. $\alpha_1(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
5. $\alpha_1(0) \leq \alpha_1(T)$.
6. $\beta_0(0) \geq \beta_0(T)$.

Therefore α_1 and $\beta_0 \in X$ are lower and upper solutions of the PBVP (4.1) such that $\alpha_1(t) \leq \beta_0(t),\ \forall\, t \in [-r,\ T]$.

Continuing in this manner, we get a monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0,\ \forall\, n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned}
\alpha'_{n+1}(t) + \lambda\alpha_{n+1}(t) &= f(t,\ \alpha_n(t),\ \alpha_n(t-r)) + \phi(t,\ \alpha_n(t),\ \alpha_n(t-r)) \\
&\quad - \phi(t,\ \alpha_{n+1}(t),\ \alpha_{n+1}(t-r)) \\
&\quad + D_2 f(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t) - \alpha_n(t)] \\
&\quad + D_2\phi(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t) - \alpha_n(t)] \\
&\quad + D_3 f(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t-r) - \alpha_n(t-r)] \\
&\quad + D_3\phi(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t-r) - \alpha_n(t-r)], \\
&\quad t \in I, \\
\alpha_{n+1}(t) &= \alpha_{n+1}(0),\ t \in [-r,\ 0], \\
\alpha_{n+1}(0) &= \alpha_{n+1}(T).
\end{aligned}\right\}$$

Further, $\lim_{n\to\infty}\alpha_n$ exists in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty}\alpha_n$ is a solution of the PBVP (4.1). □

Theorem 7.3.2. *Let $\alpha_0,\ \beta_0 \in X$ be lower and upper solutions of PBVP (4.1) such that $\alpha_0(t) \le \beta_0(t),\ \forall\ t \in [-r,\ T]$. Assume that $f,\ \psi \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ such that*

(1) $D_{i,\,j}f(t,\ x,\ y) + D_{i,j}\psi(t,\ x,\ y) \le 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$.

(2) $D_i f(t,\ x,\ y) + D_i\psi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i = 2,\ 3$.

(3) $\psi(t,\ u_1,\ v_1) \ge \psi(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$

such that $u_1 \le u_2$ and $v_1 \le v_2$.

Then there is monotonic decreasing sequence of functions $\{\beta_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim_{n\to\infty}\beta_n$ is a solution of the PBVP (4.1).

Proof. Let $G(t,\ x,\ y) = f(t,\ x,\ y) + \psi(t,\ x,\ y),\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$.

Therefore $D_{i,\,j}G(t,\ x,\ y) \le 0$ and $D_iG(t,\ x,\ y) > 0$,

$\forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$.

This implies $G(t,\ x_1,\ y_1) - G(t,\ x_2,\ y_2) \ge D_2G(t,\ x_1,\ y_1)(x_1 - x_2)$

$$+ D_3G(t,\ x_1,\ y_1)(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \ge x_2$ and $y_1 \ge y_2$.

So we have

$$\begin{aligned} f(t,\ x_1,\ y_1) \geq f(t,\ x_2,\ y_2) &- \psi(t,\ x_1,\ y_1) + \psi(t,\ x_2,\ y_2) \\ &+ D_2 f(t,\ x_1,\ y_1)(x_1 - x_2) + D_2\psi(t,\ x_1,\ y_1)(x_1 - x_2) \\ &+ D_3 f(t,\ x_1,\ y_1)(y_1 - y_2) + D_3\psi(t,\ x_1,\ y_1)(y_1 - y_2), \end{aligned}$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Since $\alpha_0,\ \beta_0 \in X$ are lower and upper solutions of PBVP (4.1), we have

1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I,$
2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I,$
3. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0],$
4. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0],$
5. $\alpha_0(0) \leq \alpha_0(T),$
6. $\beta_0(0) \geq \beta_0(T).$

Consider

$$\left.\begin{aligned} y'(t) + \lambda y(t) = f(t,\ \beta_0(t),\ \beta_0(t-r)) &+ \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &- \psi(t,\ y(t),\ y(t-r)) \\ &+ D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[y(t) - \beta_0(t)] \\ &+ D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[y(t) - \beta_0(t)] \\ &+ D_3 f(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)] \\ &+ D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)], \\ t \in I, & \\ y(t) = y(0),\ t \in [-r,\ 0], & \\ y(0) = y(T). & \end{aligned}\right\} \qquad (7.14)$$

$$\begin{aligned}
\text{Let } G_{\beta_0}(t,\ u,\ v) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad - \psi(t,\ u,\ v) + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[u - \beta_0(t)] \\
&\quad + D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[u - \beta_0(t)] \\
&\quad + D_3 f(t,\ \beta_0(t),\ \beta_0(t-r))[v - \beta_0(t-r)] \\
&\quad + D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[v - \beta_0(t-r)], \\
&\quad \forall\ (t,\ u,\ v) \in I \times \mathbb{R}^2.
\end{aligned}$$

This implies

$$\begin{aligned}
G_{\beta_0}(t,\ y(t),\ y(t-r)) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad - \psi(t,\ y(t),\ y(t-r)) \\
&\quad + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[y(t) - \beta_0(t)] \\
&\quad + D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[y(t) - \beta_0(t)] \\
&\quad + D_3 f(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)] \\
&\quad + D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)], \\
&\quad \forall\ t \in I.
\end{aligned}$$

So we have $G_{\beta_0}(t,\ x_1,\ y_1) \leq G_{\beta_0}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

$$\begin{aligned}
\text{Also } \alpha_0'(t) + \lambda\alpha_0(t) &\leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\leq f(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad - \psi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[\alpha_0(t) - \beta_0(t)] \\
&\quad + D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[\alpha_0(t) - \beta_0(t)] \\
&\quad + D_3 f(t,\ \beta_0(t),\ \beta_0(t-r))[\alpha_0(t-r) - \beta_0(t-r)] \\
&\quad + D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[\alpha_0(t-r) - \beta_0(t-r)] \\
&\leq G_{\beta_0}(t,\ \alpha_0(t),\ \alpha_0(t-r)).
\end{aligned}$$

$$\begin{aligned}
\text{Further } \beta_0'(t) + \lambda\beta_0(t) &\geq f(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&= f(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad - \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_0(t) - \beta_0(t)] \\
&\quad + D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_0(t) - \beta_0(t)] \\
&\quad + D_3 f(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_0(t-r) - \beta_0(t-r)] \\
&\quad + D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))][\beta_0(t-r) - \beta_0(t-r)] \\
&= G_{\beta_0}(t,\ \beta_0(t),\ \beta_0(t-r)).
\end{aligned}$$

We already know that

1. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
3. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
4. $\alpha_0(0) \leq \alpha_0(T)$.
5. $\beta_0(0) \geq \beta_0(T)$.

Therefore α_0 and β_0 are lower and upper solutions of the PBVP (7.14) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\, t \in [-r,\ T]$.

Thus the PBVP (7.14) has a solution $\beta_1 \in X$ such that $\alpha_0 \leq \beta_1 \leq \beta_0$ on $[-r,\ T]$. (Using Theorem 4.4.5)

Therefore

$$\left.\begin{aligned}
\beta_1'(t) + \lambda\beta_1(t) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad - \psi(t,\ \beta_1(t),\ \beta_1(t-r)) \\
&\quad + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_1(t) - \beta_0(t)] \\
&\quad + D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_1(t) - \beta_0(t)] \\
&\quad + D_3 f(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_1(t-r) - \beta_0(t-r)] \\
&\quad + D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_1(t-r) - \beta_0(t-r)],\ t \in I, \\
\beta_1(t) &= \beta_1(0),\ t \in [-r,\ 0], \\
\beta_1(0) &= \beta_1(T).
\end{aligned}\right\}$$

This implies $\beta_1'(t) + \lambda\beta_1(t) \geq f(t,\ \beta_1(t),\ \beta_1(t-r)),\ t \in I.$

Also 1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$

2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0].$
3. $\beta_1(t) = \beta_1(0),\ t \in [-r,\ 0].$
4. $\alpha_0(t) \leq \beta_1(t),\ t \in [-r,\ T].$
5. $\alpha_0(0) \leq \alpha_0(T).$
6. $\beta_1(0) \geq \beta_1(T).$

Therefore α_0 and $\beta_1 \in X$ are lower and upper solutions of the PBVP (4.1) such that $\alpha_0(t) \leq \beta_1(t),\ \forall\ t \in [-r,\ T]$.

Continuing in this manner, we get a monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\alpha_0 \leq \beta_n,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned}
\beta'_{n+1}(t) + \lambda\beta_{n+1}(t) = {} & f(t,\ \beta_n(t),\ \beta_n(t-r)) + \beta(t,\ \beta_n(t),\ \beta_n(t-r)) \\
& - \beta(t,\ \beta_{n+1}(t),\ \beta_{n+1}(t-r)) \\
& + D_2 f(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t) - \beta_n(t)] \\
& + D_2\beta(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t) - \beta_n(t)] \\
& + D_3 f(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)] \\
& + D_3\beta(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)], \\
& t \in I, \\
\beta_{n+1}(t) = {} & \beta_{n+1}(0),\ t \in [-r,\ 0], \\
\beta_{n+1}(0) = {} & \beta_{n+1}(T).
\end{aligned}\right\}$$

Further $\lim_{n\to\infty}\beta_n$ exists in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty}\beta_n$ is a solution of the PBVP (4.1). □

Theorem 7.3.3. *Let $\alpha_0,\ \beta_0 \in W$ be lower and upper solutions of PBVP* (4.6) *such that $\alpha_0(t) \le \beta_0(t),\ \forall\ t \in I$. Let $f,\ \phi \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ such that*

(1) $D_{i,\,j}f(t,\ x,\ y) + D_{i,\,j}\phi(t,\ x,\ y) \ge 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$.

(2) $D_i f(t,\ x,\ y) + D_i\phi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i = 2,\ 3$.

(3) $\phi(t,\ u_1,\ v_1) \ge \phi(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \le u_2$ and $v_1 \le v_2$.

Then there is monotonic increasing sequence of functions $\{\alpha_n\}$converging in $(W,\ ||.||_0)$ such that $\lim_{n\to\infty}\alpha_n$ is a solution of the PBVP (4.6).

Proof. Similar to the proof of the Theorem 7.3.1. □

Theorem 7.3.4. *Let $\alpha_0,\ \beta_0 \in W$ be lower and upper solutions of PBVP* (4.6) *such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in I$. Assume that $f,\ \psi \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ such that*
(1) $D_{i,\,j}f(t,\ x,\ y)+D_{i,j}\psi(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in I\times\mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$.
(2) $D_if(t,\ x,\ y) + D_i\psi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i = 2,\ 3$.
(3) $\psi(t,\ u_1,\ v_1) \geq \psi(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$
such that $u_1 \leq u_2$ and $v_1 \leq v_2$.
Then there is monotonic decreasing sequence of functions $\{\beta_n\}$ converging in $(W,\ ||.||_0)$ such that $\lim\limits_{n\to\infty}\beta_n$ is a solution of the PBVP (4.6).

Proof. Similar to the proof of the Theorem 7.3.2. □

We now prove results on generalized quasilinearization method for the LBVP of general first DECD (4.3) and for the LBVP of first order DEVDA (4.8).

Theorem 7.3.5. *Let $\alpha_0,\ \beta_0 \in X$ be weakly coupled lower and upper solutions of LBVP* (4.3). *Let $f,\ \phi \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ satisfy*
(1) $D_{i,\,j}f(t,\ x,\ y)+D_{i,\,j}\phi(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in I\times\mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$.
(2) $D_if(t,\ x,\ y) + D_i\phi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i = 2,\ 3$.
(3) $\phi(t,\ u_1,\ v_1) \geq \phi(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that
$u_1 \leq u_2$ and $v_1 \leq v_2$.
If $b < 0$, then there is monotonic increasing sequence of functions $\{\alpha_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim\limits_{n\to\infty}\alpha_n$ is a solution of the LBVP (4.3).

Proof. Let $F(t,\ x,\ y) = f(t,\ x,\ y) + \phi(t,\ x,\ y),\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$.
Therefore $D_{i,\,j}F(t,\ x,\ y) \geq 0$ and $D_iF(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$.
This implies $F(t,\ x_1,\ y_1) - F(t,\ x_2,\ y_2) \geq D_2F(t,\ x_2,\ y_2)(x_1 - x_2)$
$+ D_3F(t,\ x_2,\ y_2)(y_1 - y_2)$,
$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.
So we have

$$f(t,\ x_1,\ y_1) \geq f(t,\ x_2,\ y_2) - \phi(t,\ x_1,\ y_1) + \phi(t,\ x_2,\ y_2)$$
$$+ D_2 f(t,\ x_2,\ y_2)(x_1 - x_2) + D_2\phi(t,\ x_2,\ y_2)(x_1 - x_2)$$
$$+ D_3 f(t,\ x_2,\ y_2)(y_1 - y_2) + D_3\phi(t,\ x_2,\ y_2)(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Since, $\alpha_0,\ \beta_0 \in X$ are weakly coupled lower and upper solutions of LBVP (4.3),

we have 1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$

2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$

3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0].$

4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0].$

5. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T].$

6. $\max\{a\alpha_0(0) + b\alpha_0(T),\ a\alpha_0(0) + b\beta_0(T)\}$
$\leq 0 \leq \min\{a\beta_0(0) + b\alpha_0(T),\ a\beta_0(0) + b\beta_0(T)\}.$

As $b < 0$, we have

$\max\{a\alpha_0(0) + b\alpha_0(T),\ a\alpha_0(0) + b\beta_0(T)\} = a\alpha_0(0) + b\alpha_0(T)$ and

$\min\{a\beta_0(0) + b\alpha_0(T),\ a\beta_0(0) + b\beta_0(T)\} = a\beta_0(0) + b\beta_0(T).$

This implies $a\alpha_0(0) + b\alpha_0(T) \leq 0 \leq a\beta_0(0) + b\beta_0(T)$.

Also, as $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and

$\beta_0(t) = \beta_0(0),\ \forall\ t \in [-r,\ 0]$.

Now starting with the PBVP

$$\left.\begin{aligned} y'(t)+\lambda y(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r))+\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\quad -\phi(t,\ y(t),\ y(t-r)) \\ &\quad +D_2f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t)-\alpha_0(t)] \\ &\quad +D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t)-\alpha_0(t)] \\ &\quad +D_3f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r)-\alpha_0(t-r)] \\ &\quad +D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r)-\alpha_0(t-r)], \\ &\quad t\in I, \\ y(t) &= y(0),\ t\in[-r,\ 0], \\ ay(0)+by(T) &= 0, \end{aligned}\right\} \tag{7.15}$$

and proceeding similarly along the proof of the Theorem 7.3.1, we get a monotonic increasing sequence of functions $\{\alpha_n\}\subseteq X$ such that $\alpha_n\leq\beta_0,\ \forall\ n\in\mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} \alpha'_{n+1}(t)+\lambda\alpha_{n+1}(t) &= f(t,\ \alpha_n(t),\ \alpha_n(t-r))+\phi(t,\ \alpha_n(t),\ \alpha_n(t-r)) \\ &\quad -\phi(t,\ \alpha_{n+1}(t),\ \alpha_{n+1}(t-r)) \\ &\quad +D_2f(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t)-\alpha_n(t)] \\ &\quad +D_2\phi(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t)-\alpha_n(t)] \\ &\quad +D_3f(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t-r)-\alpha_n(t-r)] \\ &\quad +D_3\phi(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t-r)-\alpha_n(t-r)], \\ &\quad t\in I, \\ \alpha_{n+1}(t) &= \alpha_{n+1}(0),\ t\in[-r,\ 0], \\ a\alpha_{n+1}(0)+b\alpha_{n+1}(T) &= 0. \end{aligned}\right\}$$

Further $\lim\limits_{n\to\infty}\alpha_n$ exists in $(X,\ ||.||_\infty)$ and $\lim\limits_{n\to\infty}\alpha_n$ is a solution of the LBVP (4.3). □

Theorem 7.3.4. *Let $\alpha_0,\ \beta_0 \in X$ be weakly coupled lower and upper solutions of LBVP (4.3). Let $f,\ \psi \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ satisfy*

(1) $D_{i,\,j}f(t,\ x,\ y)+D_{i,\,j}\psi(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in I\times\mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3.$

(2) $D_i f(t,\ x,\ y) + D_i\psi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i = 2,\ 3.$

(3) $\psi(t,\ u_1,\ v_1) \geq \psi(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that

$u_1 \leq u_2$ *and* $v_1 \leq v_2$.

If $b < 0$, then there is monotonic decreasing sequence of functions $\{\beta_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim\limits_{n\to\infty}\beta_n$ is a solution of the LBVP (4.3).

Proof. Since, $\alpha_0,\ \beta_0 \in X$ are weakly coupled lower and upper solutions of LBVP (4.3), we have

1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$
2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$
3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0].$
4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0].$
5. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T].$
6. $\max\{a\alpha_0(0) + b\alpha_0(T),\ a\alpha_0(0) + b\beta_0(T)\}$
   $\leq 0 \leq \min\{a\beta_0(0) + b\alpha_0(T),\ a\beta_0(0) + b\beta_0(T)\}.$

As $b < 0$, we have

$\max\{a\alpha_0(0) + b\alpha_0(T),\ a\alpha_0(0) + b\beta_0(T)\} = a\alpha_0(0) + b\alpha_0(T)$ and

$\min\{a\beta_0(0) + b\alpha_0(T),\ a\beta_0(0) + b\beta_0(T)\} = a\beta_0(0) + b\beta_0(T).$

This implies $a\alpha_0(0) + b\alpha_0(T) \leq 0 \leq a\beta_0(0) + b\beta_0(T).$

Also, as $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and

$\beta_0(t) = \beta_0(0),\ \forall\ t \in [-r,\ 0].$

Let, $G(t,\ x,\ y) = f(t,\ x,\ y) + \psi(t,\ x,\ y),\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2.$

Therefore $D_{i,\,j}G(t,\ x,\ y) \leq 0$ and $D_iG(t,\ x,\ y) > 0,$

$\forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3.$

This implies $G(t,\ x_1,\ y_1) - G(t,\ x_2,\ y_2) \geq D_2G(t,\ x_1,\ y_1)(x_1 - x_2)$
$$+ D_3G(t,\ x_1,\ y_1)(y_1 - y_2),$$
$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

This gives $f(t,\ x_1,\ y_1) \geq f(t,\ x_2,\ y_2) - \psi(t,\ x_1,\ y_1) + \psi(t,\ x_2,\ y_2)$
$$+ D_2f(t,\ x_1,\ y_1)(x_1 - x_2) + D_2\psi(t,\ x_1,\ y_1)(x_1 - x_2)$$
$$+ D_3f(t,\ x_1,\ y_1)(y_1 - y_2) + D_3\psi(t,\ x_1,\ y_1)(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.
Now starting with the LBVP

$$\left.\begin{aligned}
y'(t) + \lambda y(t) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad - \psi(t,\ y(t),\ y(t-r)) \\
&\quad + D_2f(t,\ \beta_0(t),\ \beta_0(t-r))[y(t) - \beta_0(t)] \\
&\quad + D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[y(t) - \beta_0(t)] \\
&\quad + D_3f(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)] \\
&\quad + D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)], \\
&\quad t \in I, \\
y(t) &= y(0),\ t \in [-r,\ 0], \\
ay(0) &+ by(T) = 0,
\end{aligned}\right\} \tag{7.16}$$

and proceeding similarly along the proof of the Theorem 7.3.2, we get a monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\alpha_0 \leq \beta_n,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned}
\beta'_{n+1}(t) + \lambda\beta_{n+1}(t) = {} & f(t,\ \beta_n(t),\ \beta_n(t-r)) + \psi(t,\ \beta_n(t),\ \beta_n(t-r)) \\
& - \psi(t,\ \beta_{n+1}(t),\ \beta_{n+1}(t-r)) \\
& + D_2 f(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t) - \beta_n(t)] \\
& + D_2\psi(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t) - \beta_n(t)] \\
& + D_3 f(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)] \\
& + D_3\psi(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)], \\
& t \in I, \\
\beta_{n+1}(t) = {} & \beta_{n+1}(0),\ t \in [-r,\ 0], \\
a\beta_{n+1}(0) + {} & b\beta_{n+1}(T) = 0.
\end{aligned}\right\}$$

Further $\lim_{n\to\infty}\beta_n$ exists in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty}\beta_n$ is a solution of the LBVP (4.3). □

Theorem 7.3.7. *Let $\alpha_0,\ \beta_0 \in W$ be weakly coupled lower and upper solutions of LBVP* (4.8). *Let $f,\ \phi \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ satisfy*

(1) $D_{i,\,j}f(t,\ x,\ y) + D_{i,\,j}\phi(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$.

(2) $D_i f(t,\ x,\ y) + D_i\phi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i = 2,\ 3$.

(3) $\phi(t,\ u_1,\ v_1) \geq \phi(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that

$u_1 \leq u_2$ *and* $v_1 \leq v_2$.

If $b < 0$, then there is monotonic increasing sequence of functions $\{\alpha_n\}$ converging in $(W,\ ||.||_0)$ such that $\lim_{n\to\infty}\alpha_n$ is a solution of the LBVP (4.8).

Proof. Similar to the proof of the Theorem 7.3.5. □

Theorem 7.3.8. *Let $\alpha_0,\ \beta_0 \in W$ be weakly coupled lower and upper solutions of LBVP (4.8). Let $f,\ \psi \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ satisfy*

(1) $D_{i,\,j}f(t,\ x,\ y)+D_{i,\,j}\psi(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in I\times\mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3.$

(2) $D_i f(t,\ x,\ y) + D_i\psi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i = 2,\ 3.$

(3) $\psi(t,\ u_1,\ v_1) \geq \psi(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$

such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

If $b < 0$, then there is monotonic decreasing sequence of functions $\{\beta_n\}$ converging in $(W,\ ||.||_0)$ such that $\lim_{n\to\infty}\beta_n$ is a solution of the LBVP (4.8).

Proof. Similar to the proof of the Theorem 7.3.6. □

Next, we prove results on generalized quasilinearization method for NLBVP of general first order DECD (4.4) and for NLBVP of first order DEVDA (4.9).

Theorem 7.3.9. *Let $g \in C[\mathbb{R}^2,\ \mathbb{R}]$ satisfies $g(t,\ x) \geq g(t,\ y)$,*
$\forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$. Let $\alpha_0,\ \beta_0 \in X$ be weakly coupled lower and upper solutions of NLBVP (4.4). Assume that $f,\ \phi \in C^2[I\times\mathbb{R}^2,\ \mathbb{R}]$ satisfy

(1) $D_{i,\,j}f(t,\ x,\ y)+D_{i,\,j}\phi(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in I\times\mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3.$

(2) $D_i f(t,\ x,\ y) + D_i\phi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i = 2,\ 3.$

(3) $\phi(t,\ u_1,\ v_1) \geq \phi(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that

$u_1 \leq u_2$ and $v_1 \leq v_2$.

Then there is monotonic increasing sequence of functions $\{\alpha_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim_{n\to\infty}\alpha_n$ is a solution of the NLBVP (4.4).

Proof. Let $F(t,\ x,\ y) = f(t,\ x,\ y) + \phi(t,\ x,\ y),\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2.$
Therefore $D_{i,\,j}F(t,\ x,\ y) \geq 0$ and $D_iF(t,\ x,\ y) > 0,\ \forall\ (t\ ,x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3.$

This implies $F(t,\ x_1,\ y_1) - F(t,\ x_2,\ y_2) \geq D_2F(t,\ x_2,\ y_2)(x_1 - x_2)$
$$+ D_3F(t,\ x_2,\ y_2)(y_1 - y_2),$$
$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

So we have $f(t,\ x_1,\ y_1) \geq f(t,\ x_2,\ y_2) - \phi(t,\ x_1,\ y_1) + \phi(t,\ x_2,\ y_2)$

$+ D_2 f(t,\ x_2,\ y_2)(x_1 - x_2) + D_2\phi(t,\ x_2,\ y_2)(x_1 - x_2)$

$+ D_3 f(t,\ x_2,\ y_2)(y_1 - y_2) + D_3\phi(t,\ x_2,\ y_2)(y_1 - y_2),$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Since, $\alpha_0,\ \beta_0 \in X$ are weakly coupled lower and upper solutions of NLBVP (4.4), we have 1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$

2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$
3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0].$
4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0].$
5. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T].$
6. $\max\{g(\alpha_0(0),\ \alpha_0(T)),\ g(\alpha_0(0),\ \beta_0(T))\}$
   $\leq 0 \leq \min\{g(\beta_0(0),\ \alpha_0(T)),\ g(\beta_0(0),\ \beta_0(T))\}.$

As $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$, we have

$\max\{g(\alpha_0(0),\ \alpha_0(T)),\ g(\alpha_0(0),\ \beta_0(T))\} = g(\alpha_0(0),\ \alpha_0(T))$ and

$\min\{g(\beta_0(0),\ \alpha_0(T)),\ g(\beta_0(0),\ \beta_0(T))\} = g(\beta_0(0),\ \beta_0(T)).$

This implies $g(\alpha_0(0),\ \alpha_0(T)) \leq 0 \leq g(\beta_0(0),\ \beta_0(T)).$

Also, as $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$

and $\beta_0(t) = \beta_0(0),\ \forall\ t \in [-r,\ 0].$

Now starting with the NLBVP

$$\left.\begin{aligned} y'(t)+\lambda y(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r))+\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\quad -\phi(t,\ y(t),\ y(t-r)) \\ &\quad +D_2f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t)-\alpha_0(t)] \\ &\quad +D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t)-\alpha_0(t)] \\ &\quad +D_3f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r)-\alpha_0(t-r)] \\ &\quad +D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r)-\alpha_0(t-r)], \\ &\quad t\in I, \\ y(t) &= y(0),\ t\in[-r,\ 0], \\ g(y(0),\ y(T)) &= 0, \end{aligned}\right\} \tag{7.17}$$

and proceeding similarly along the proof of the Theorem 7.3.1, we get a monotonic increasing sequence of functions $\{\alpha_n\}\subseteq X$ such that $\alpha_n\leq\beta_0,\ \forall\ n\in\mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} \alpha'_{n+1}(t)+\lambda\alpha_{n+1}(t) &= f(t,\ \alpha_n(t),\ \alpha_n(t-r))+\phi(t,\ \alpha_n(t),\ \alpha_n(t-r)) \\ &\quad -\phi(t,\ \alpha_{n+1}(t),\ \alpha_{n+1}(t-r)) \\ &\quad +D_2f(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t)-\alpha_n(t)] \\ &\quad +D_2\phi(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t)-\alpha_n(t)] \\ &\quad +D_3f(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t-r)-\alpha_n(t-r)] \\ &\quad +D_3\phi(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t-r)-\alpha_n(t-r)], \\ &\quad t\in I, \\ \alpha_{n+1}(t) &= \alpha_{n+1}(0),\ t\in[-r,\ 0], \\ g(\alpha_{n+1}(0),\ \alpha_{n+1}(T)) &= 0. \end{aligned}\right\}$$

Further $\lim\limits_{n\to\infty}\alpha_n$ exists in $(X,\ ||.||_\infty)$ and $\lim\limits_{n\to\infty}\alpha_n$ is a solution of the NLBVP (4.4). □

Theorem 7.3.10. *Let* $g \in C[\mathbb{R}^2,\ \mathbb{R}]$ *satisfies* $g(t,\ x) \geq g(t,\ y)$, $\forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ *such that* $x \leq y$. *Let* $\alpha_0,\ \beta_0 \in X$ *be weakly coupled lower and upper solutions of NLBVP* (4.4). *Assume that* $f,\ \psi \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ *satisfy*

(1) $D_{i,\,j}f(t,\ x,\ y) + D_{i,\,j}\psi(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ *and* $\forall\ i,\ j = 2,\ 3.$

(2) $D_i f(t,\ x,\ y) + D_i\psi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ *and* $\forall\ i = 2,\ 3.$

(3) $\psi(t,\ u_1,\ v_1) \geq \psi(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ *such that* $u_1 \leq u_2$ *and* $v_1 \leq v_2$.

Then there is monotonic decreasing sequence of functions $\{\beta_n\}$ *converging in* $(X,\ ||.||_\infty)$ *such that* $\lim_{n\to\infty} \beta_n$ *is a solution of the NLBVP* (4.4).

Proof. Since, $\alpha_0,\ \beta_0 \in X$ are weakly coupled lower and upper solutions of NLBVP (4.4), we have 1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$

2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$

3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0].$

4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0].$

5. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T].$

6. $\max\{g(\alpha_0(0),\ \alpha_0(T)),\ g(\alpha_0(0),\ \beta_0(T))\}$
$\leq 0 \leq \min\{g(\beta_0(0),\ \alpha_0(T)),\ g(\beta_0(0),\ \beta_0(T))\}.$

As $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$, we have
$\max\{g(\alpha_0(0),\ \alpha_0(T)),\ g(\alpha_0(0),\ \beta_0(T))\} = g(\alpha_0(0),\ \alpha_0(T))$ and
$\min\{g(\beta_0(0),\ \alpha_0(T)),\ g(\beta_0(0),\ \beta_0(T))\} = g(\beta_0(0),\ \beta_0(T)).$
This implies $g(\alpha_0(0),\ \alpha_0(T)) \leq 0 \leq g(\beta_0(0),\ \beta_0(T)).$
Also, as $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and
$\beta_0(t) = \beta_0(0),\ \forall\ t \in [-r,\ 0].$
Let $G(t,\ x,\ y) = f(t,\ x,\ y) + \psi(t,\ x,\ y),\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2.$
Therefore $D_{i,\,j}G(t,\ x,\ y) \leq 0$ and $D_i G(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and
$\forall\ i,\ j = 2,\ 3.$

This implies $G(t,\ x_1,\ y_1) - G(t,\ x_2,\ y_2) \geq D_2G(t,\ x_1,\ y_1)(x_1 - x_2)$

$$+ D_3G(t,\ x_1,\ y_1)(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

So we have

$f(t,\ x_1,\ y_1) \geq f(t,\ x_2,\ y_2) - \psi(t,\ x_1,\ y_1) + \psi(t,\ x_2,\ y_2)$

$+ D_2f(t,\ x_1,\ y_1)(x_1 - x_2) + D_2\psi(t,\ x_1,\ y_1)(x_1 - x_2)$

$+ D_3f(t,\ x_1,\ y_1)(y_1 - y_2) + D_3\psi(t,\ x_1,\ y_1)(y_1 - y_2),$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Now, starting with the NLBVP

$$\left.\begin{aligned} y'(t) + \lambda y(t) &= f(t, \beta_0(t), \beta_0(t-r)) + \beta(t, \beta_0(t), \beta_0(t-r)) \\ &\quad - \beta(t, y(t), y(t-r)) \\ &\quad + D_2f(t, \beta_0(t), \beta_0(t-r))[y(t) - \beta_0(t)] \\ &\quad + D_2\beta(t, \beta_0(t), \beta_0(t-r))[y(t) - \beta_0(t)] \\ &\quad + D_3f(t, \beta_0(t), \beta_0(t-r))[y(t-r) - \beta_0(t-r)] \\ &\quad + D_3\beta(t, \beta_0(t), \beta_0(t-r))[y(t-r) - \beta_0(t-r)], \\ &\quad t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ g(y(0),\ y(T)) &= 0, \end{aligned}\right\} \tag{7.18}$$

and proceeding similarly along the proof of theorem 7.3.2, we get a monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\beta_0 \leq \beta_n,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned}
\beta'_{n+1}(t) + \lambda\beta_{n+1}(t) &= f(t,\ \beta_n(t),\ \beta_n(t-r)) + \psi(t,\ \beta_n(t),\ \beta_n(t-r)) \\
&\quad - \psi(t,\ \beta_{n+1}(t),\ \beta_{n+1}(t-r)) \\
&\quad + D_2 f(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t) - \beta_n(t)] \\
&\quad + D_2\psi(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t) - \beta_n(t)] \\
&\quad + D_3 f(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)] \\
&\quad + D_3\psi(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)], \\
&\quad t \in I, \\
\beta_{n+1}(t) &= \beta_{n+1}(0),\ t \in [-r,\ 0], \\
g(\beta_{n+1}(0),\ \beta_{n+1}(T)) &= 0.
\end{aligned}\right\}$$

Further $\lim_{n\to\infty}\beta_n$ exists in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty}\beta_n$ is a solution of the the NLBVP (4.4). □

Theorem 7.3.11. *Let $g \in C[\mathbb{R}^2,\ \mathbb{R}]$ satisfies $g(t,\ x) \geq g(t,\ y)$, $\forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$. Let $\alpha_0,\ \beta_0 \in W$ be weakly coupled lower and upper solutions of NLBVP (4.9). Assume that $f,\ \phi \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ satisfy*

(1) $D_{i,\,j}f(t,\ x,\ y) + D_{i,\,j}\phi(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$.

(2) $D_i f(t,\ x,\ y) + D_i\phi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i = 2,\ 3$.

(3) $\phi(t,\ u_1,\ v_1) \geq \phi(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

Then there is monotonic increasing sequence of functions $\{\alpha_n\}$ converging in $(W,\ ||.||_0)$ such that $\lim_{n\to\infty}\alpha_n$ is a solution of the NLBVP (4.9).

Proof. Similar to the proof of the Theorem 7.3.9. □

Theorem 7.3.12. *Let* $g \in C[\mathbb{R}^2,\ \mathbb{R}]$ *satisfies* $g(t,\ x) \geq g(t,\ y)$, $\forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ *such that* $x \leq y$. *Let* $\alpha_0,\ \beta_0 \in W$ *be weakly coupled lower and upper solutions of NLBVP* (4.9). *Assume that* $f,\ \psi \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ *satisfy*

(1) $D_{i,\,j}f(t,\ x,\ y) + D_{i,\,j}\psi(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ *and* $\forall\ i,\ j = 2,\ 3.$

(2) $D_i f(t,\ x,\ y) + D_i\psi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ *and* $\forall\ i = 2,\ 3.$

(3) $\psi(t,\ u_1,\ v_1) \geq \psi(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ *such that* $u_1 \leq u_2$ *and* $v_1 \leq v_2$.

Then there is monotonic decreasing sequence of functions $\{\beta_n\}$ *converging in* $(W,\ ||.||_0)$ *such that* $\lim_{n\to\infty} \beta_n$ *is a solution of the NLBVP* (4.9).

Proof. Similar to the proof of the Theorem 7.3.10. □

7.4 Extended-Generalized Quasilinearization Method

We now introduce extended-generalized quaslinearization method for boundary value problems of general first order ordinary differential equation with constant delay of the type $y'(t) + \lambda y(t) = f(t,\ y(t),\ y(t-r)) + h(t,\ y(t),\ y(t-r))$ and first order ordinary differential equation with variable deviating argument of the type $y'(t) + \lambda y(t) = f(t,\ y(t),\ y(\theta(t))) + h(t,\ y(t),\ y(\theta(t)))$, where there are certain conditions on first and second order partial derivatives with respect to the second and third arguments of $f,\ h,\ f + \phi$ and $h + \psi$ for some functions ϕ and ψ.

Theorem 7.4.1. *Let* $\alpha_0,\ \beta_0 \in X$ *such that* $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$ *and* $\Omega = \{(t, u, v) : \alpha_0(t) \leq u \leq \beta_0(t)$ *and* $\alpha_0(t-r) \leq v \leq \beta_0(t-r),$ $t \in I\}$. *Assume that* $f,\ h,\ \phi,\ \psi \in C^2[\Omega,\ \mathbb{R}]$ *satisfying*

(1) $D_i f(t,\ x,\ y) + D_i\phi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in \Omega$ *and* $\forall\ i = 2,\ 3.$

(2) $D_i h(t,\ x,\ y) + D_i\psi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in \Omega$ *and* $\forall\ i = 2,\ 3.$

(3) $D_{i,\,j} f(t,\ x,\ y) + D_{i,\,j}\phi(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in \Omega$ *and* $\forall\ i,\ j = 2,\ 3.$

(4) $D_{i,\,j} h(t,\ x,\ y) + D_{i,\,j}\beta(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in \Omega$ *and* $\forall\ i,\ j = 2,\ 3.$

(5) $\phi(t,\ x_1,\ y_1) + \psi(t,\ x_1,\ y_1) \geq \phi(t,\ x_2,\ y_2) + \psi(t, x_2, y_2),$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ *such that* $x_1 \leq x_2$ *and* $y_1 \leq y_2.$

Further, assume that α_0 *and* β_0 *are lower and upper solutions of PBVP* (7.1). *Then there are monotonic sequence of functions* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *converging in* $(X,\ ||.||_\infty)$ *such that* $\lim_{n\to\infty} \alpha_n$ *and* $\lim_{n\to\infty} \beta_n$ *are solutions of the PBVP* (7.1).

Proof. Let $F(t,\ x,\ y) = f(t,\ x,\ y) + \phi(t,\ x,\ y),\ \forall\ (t,\ x,\ y) \in \Omega$ and

$$H(t,\ x,\ y) = h(t,\ x,\ y) + \psi(t,\ x,\ y),\ \forall\ (t,\ x,\ y) \in \Omega.$$

It is assumed that $D_{i,\,j} f(t,\ x,\ y) + D_{i,\,j}\phi(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i,\ j = 2,\ 3.$ Therefore

$$\begin{aligned} f(t,\ x_1,\ y_1) \geq\ & f(t,\ x_2,\ y_2) - \phi(t,\ x_1,\ y_1) + \phi(t,\ x_2,\ y_2) \\ & + [D_2 f(t,\ x_2,\ y_2) + D_2\phi(t,\ x_2,\ y_2)](x_1 - x_2) \\ & + [D_3 f(t,\ x_2,\ y_2) + D_3\phi(t,\ x_2,\ y_2)](y_1 - y_2), \end{aligned}$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2.$

It is also assumed that $D_{i,\,j} h(t,\ x,\ y) + D_{i,\,j}\psi(t,\ x,\ y) \leq 0,$

$\forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i,\ j = 2,\ 3.$

Therefore

$$\begin{aligned} h(t,\ x_1,\ y_1) \geq\ & h(t,\ x_2,\ y_2) - \psi(t,\ x_1,\ y_1) + \psi(t,\ x_2,\ y_2) \\ & + [D_2 h(t,\ x_1,\ y_1) + D_2\psi(t,\ x_1,\ y_1)](x_1 - x_2) \\ & + [D_3 h(t,\ x_1,\ y_1) + D_3\psi(t,\ x_1,\ y_1)](y_1 - y_2), \end{aligned}$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2.$

Let $\mu_1 = \min_{(t,\ u,\ v)\in\Omega}[D_2h(t,\ u,\ v) + D_2\psi(t,\ u,\ v)]$ and

$\mu_2 = \min_{(t,\ u,\ v)\in\Omega}[D_3h(t,\ u,\ v) + D_3\psi(t,u,v)]$. Then, $\mu_1 > 0$ and $\mu_2 > 0$.

So we have

$$h(t,\ x_1,\ y_1) \geq h(t,\ x_2,\ y_2) - \psi(t,\ x_1,\ y_1) + \psi(t,\ x_2,\ y_2) + \mu_1(x_1 - x_2) + \mu_2(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Thus

$$\begin{aligned} f(t,\ x_1,\ y_1) + h(t,\ x_1,\ y_1) \geq\ & f(t,\ x_2,\ y_2) + h(t,\ x_2,\ y_2) + \phi(t,\ x_2,\ y_2) \\ & + \psi(t,\ x_2,\ y_2) - \phi(t,\ x_1,\ y_1) - \psi(t,\ x_1,\ y_1) \\ & + [D_2f(t,\ x_1,\ y_1) + D_2\phi(t,\ x_1,\ y_1)](x_1 - x_2) \\ & + [D_3f(t,\ x_1,\ y_1) + D_3\phi(t,\ x_1,\ y_1)](y_1 - y_2), \\ & + \mu_1(x_1 - x_2) + \mu_2(y_1 - y_2), \end{aligned}$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Since $\alpha_0,\ \beta_0 \in X$ are lower and upper solutions of PBVP (7.1), we have

1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$
2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$
3. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0].$
4. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0].$
5. $\alpha_0(0) \leq \alpha_0(T).$
6. $\beta_0(0) \geq \beta_0(T).$

Consider

$$\left.\begin{aligned} y'(t)+\lambda y(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\quad + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \psi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\quad - \phi(t,\ y(t),\ y(t-r)) - \psi(t,\ y(t),\ y(t-r)) \\ &\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t)-\alpha_0(t)] \\ &\quad + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t)-\alpha_0(t)] \\ &\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r)-\alpha_0(t-r)] \\ &\quad + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r)-\alpha_0(t-r)] \\ &\quad + \mu_1[y(t)-\alpha_0(t)] + \mu_2[y(t-r)-\alpha_0(t-r)], \\ &\quad t\in I, \\ y(t) &= y(0),\ t\in[-r,\ 0], \\ y(0) &= y(T). \end{aligned}\right\} \tag{7.19}$$

Let $F_{\alpha_0}(t,\ u,\ v) = f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r))$

$$\begin{aligned} &+ \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \psi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &- \phi(t,\ u,\ v) - \psi(t,\ u,\ v) \\ &+ D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[u-\alpha_0(t)] \\ &+ [D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][u-\alpha_0(t)] \\ &+ D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[v-\alpha_0(t-r)] \\ &+ [D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][v-\alpha_0(t-r)], \\ &\forall\ (t,\ u,\ v)\in\Omega. \end{aligned}$$

This implies

$$\begin{aligned}F_{\alpha_0}(t,\ y(t),\ y(t-r)) = {} & f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ & + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \psi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ & - \phi(t,\ y(t),\ x(t-r)) - \psi(t,\ y(t),\ y(t-r)) \\ & + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t)-\alpha_0(t)] \\ & + [D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][y(t)-\alpha_0(t)] \\ & + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r)-\alpha_0(t-r)] \\ & + [D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][y(t-r)-\alpha_0(t-r)], \\ & \forall\ t \in I.\end{aligned}$$

Let $(t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \le x_2$ and $y_1 \le y_2$.

Then $x_1 - \alpha_0(t) \le x_2 - \alpha_0(t)$ and $y_1 - \alpha_0(t-r) \le y_2 - \alpha_0(t-r)$.

This implies

$$\begin{aligned}&[D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][x_1 - \alpha_0(t)] \\ \le\ &[D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][x_2 - \alpha_0(t)] \text{ and} \\ &[D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][y_1 - \alpha_0(t-r)] \\ \le\ &[D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][y_2 - \alpha_0(t-r)].\end{aligned}$$

So we have

$$\begin{aligned}&[D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][x_1 - \alpha_0(t)] \\ &+ [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][y_1 - \alpha_0(t-r)] \\ &\le [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][x_2 - \alpha_0(t)] \\ &+ [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][y_2 - \alpha_0(t-r)].\end{aligned}$$

This gives

$f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r))$

$+ \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \psi(t,\ \alpha_0(t),\ \alpha_0(t-r))$

$+ [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][x_1 - \alpha_0(t)]$

$+ [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][y_1 - \alpha_0(t-r)]$

$\leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r))$

$+ \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \psi(t,\ \alpha_0(t),\ \alpha_0(t-r))$

$+ [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][x_2 - \alpha_0(t)]$

$+ [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][y_2 - \alpha_0(t-r)].$

It is given that $\phi(t,\ x_1,\ y_1) + \psi(t,\ x_1,\ y_1) \geq \phi(t,\ x_2,\ y_2) + \psi(t,\ x_2,\ y_2)$.

This implies

$-\phi(t,\ x_1,\ y_1) - \psi(t,\ x_1,\ y_1) \leq -\phi(t,\ x_2,\ y_2) - \psi(t,\ x_2,\ y_2).$

Therefore

$f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r))$

$+ \psi(t,\ \alpha_0(t),\ \alpha_0(t-r)) - \phi(t,\ x_1,\ y_1) - \psi(t,\ x_1,\ y_1)$

$+ [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][x_1 - \alpha_0(t)]$

$+ [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][y_1 - \alpha_0(t-r)]$

$\leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r))$

$+ \psi(t,\ \alpha_0(t),\ \alpha_0(t-r)) - \phi(t,\ x_2,\ y_2) - \psi(t,\ x_2,\ y_2)$

$+ [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][x_2 - \alpha_0(t)]$

$+ [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][y_2 - \alpha_0(t-r)].$

Thus $F_{\alpha_0}(t,\ x_1,\ y_1) \leq F_{\alpha_0}(t,\ x_2,\ y_2)$,

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

Also

$$\begin{aligned}
\alpha_0'(t) + \lambda\alpha_0(t) &\leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \psi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad - \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) - \psi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_0(t) - \alpha_0(t)] \\
&\quad + [D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][\alpha_0(t) - \alpha_0(t)] \\
&\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_0(t-r) - \alpha_0(t-r)] \\
&\quad + [D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][\alpha_0(t-r) - \alpha_0(t-r)] \\
&= F_{\alpha_0}(t,\ \alpha_0(t),\ \alpha_0(t-r)).
\end{aligned}$$

Further

$$\begin{aligned}
\beta_0'(t) + \lambda\beta_0(t) &\geq f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\geq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \psi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad - \phi(t,\ \beta_0(t),\ \beta_0(t-r)) - \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\beta_0(t) - \alpha_0(t)] \\
&\quad + [D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][\beta_0(t) - \alpha_0(t)] \\
&\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\beta_0(t-r) - \alpha_0(t-r)] \\
&\quad + [D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][\beta_0(t-r) - \alpha_0(t-r)] \\
&= F_{\alpha_0}(t,\ \beta_0(t),\ \beta_0(t-r)).
\end{aligned}$$

We already know that

1. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
3. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
4. $\alpha_0(0) \leq \alpha_0(T)$.
5. $\beta_0(0) \geq \beta_0(T)$.

Therefore α_0 and β_0 are lower and upper solutions of the PBVP (7.19) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\, t \in [-r,\ T]$.

Thus the PBVP (7.19) has a solution $\alpha_1 \in X$ such that $\alpha_0 \leq \alpha_1 \leq \beta_0$ on $[-r,\ T]$. (Using Theorem 4.4.5)

Therefore

$$\left.\begin{aligned}
\alpha_1'(t) + \lambda\alpha_1(t) = {} & f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
& + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \psi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
& - \phi(t,\ \alpha_1(t),\ \alpha_1(t-r)) - \psi(t,\ \alpha_1(t),\ \alpha_1(t-r)) \\
& + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_1(t) - \alpha_0(t)] \\
& + [D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][\alpha_1(t) - \alpha_0(t)] \\
& + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_1(t-r) - \alpha_0(t-r)] \\
& + [D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][\alpha_1(t-r) - \alpha_0(t-r)], \\
& t \in I, \\
\alpha_1(t) = \alpha_1(0),\ & t \in [-r,\ 0], \\
\alpha_1(0) = \alpha_1(T).\ &
\end{aligned}\right\}$$

This implies

$\alpha_1'(t) + \lambda\alpha_1(t) \leq f(t,\ \alpha_1(t),\ \alpha_1(t-r)) + h(t,\ \alpha_1(t),\ \alpha_1(t-r)),\ t \in I$.

Also

1. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$
2. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0].$
3. $\alpha_1(t) = \alpha_1(0),\ t \in [-r,\ 0].$
4. $\alpha_1(t) \leq \beta_0(t),\ t \in [-r,\ T].$
5. $\alpha_1(0) \leq \alpha_1(T).$
6. $\beta_0(0) \geq \beta_0(T).$

Therefore α_1 and $\beta_0 \in X$ are lower and upper solutions of PBVP (7.1) such that $\alpha_1(t) \leq \beta_0(t),\ \forall\, t \in [-r,\ T]$.

Continuing in this manner, we get a monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0,\ \forall\, n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned}
\alpha_{n+1}'(t) + \lambda\alpha_{n+1}(t) &= f(t, \alpha_n(t), \alpha_n(t-r)) + h(t, \alpha_n(t), \alpha_n(t-r)) \\
&\quad + \alpha(t, \alpha_n(t), \alpha_n(t-r)) + \beta(t, \alpha_n(t), \alpha_n(t-r)) \\
&\quad - \alpha(t, \alpha_{n+1}(t), \alpha_{n+1}(t-r)) - \beta(t, \alpha_{n+1}(t), \alpha_{n+1}(t-r)) \\
&\quad + D_2 f(t, \alpha_n(t-r), \alpha_n(t-r))[\alpha_{n+1}(t) - \alpha_n(t)] \\
&\quad + D_2\alpha(t, \alpha_n(t), \alpha_n(t-r))[\alpha_{n+1}(t) - \alpha_n(t)] \\
&\quad + D_3 f(t, \alpha_n(t), \alpha_n(t-r))[\alpha_{n+1}(t-r) - \alpha_n(t-r)] \\
&\quad + D_3\alpha(t, \alpha_n(t), \alpha_n(t-r))[\alpha_{n+1}(t-r) - \alpha_n(t-r)], \\
&\quad + \mu_1[\alpha_{n+1}(t) - \alpha_n(t)] + \mu_2[\alpha_{n+1}(t-r) - \alpha_n(t-r)], \\
&\quad\ t \in I, \\
\alpha_{n+1}(t) &= \alpha_{n+1}(0),\ t \in [-r,\ 0], \\
\alpha_{n+1}(0) &= \alpha_{n+1}(T).
\end{aligned}\right\}$$

Further $\lim_{n\to\infty} \alpha_n$ exists in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty} \alpha_n$ is a solution of the PBVP (7.1).

Let $\theta_1 = \min\limits_{(t,\,u,\,v)\in\Omega}[D_2 f(t,\,u,\,v) + D_2\phi(t,\,u,\,v)]$ and

$\theta_2 = \min\limits_{(t,\,u,\,v)\in\Omega}[D_3 f(t,\,u,\,v) + D_3\phi(t,\,u,\,v)]$. Then $\theta_1 > 0$ and $\theta_2 > 0$.

This implies $f(t,\,x_1,\,y_1) \geq f(t,\,x_2,\,y_2) - \phi(t,\,x_1,\,y_1) - \phi(t,\,x_2,\,y_2)$
$+ \theta_1(x_1 - x_2) + \theta_2(y_1 - y_2)$,

$\forall\ (t,\,x_1,\,y_1),\ (t,\,x_2,\,y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

So we have

$$\begin{aligned} f(t,\,x_1,\,y_1) + h(t,\,x_1,\,y_1) \geq\ & f(t,\,x_2,\,y_2) + h(t,\,x_2,\,y_2) + \phi(t,\,x_2,\,y_2) \\ & + \psi(t,\,x_2,\,y_2) - \phi(t,\,x_1,\,y_1) - \psi(t,\,x_1,\,y_1) \\ & + [\theta_1 + D_2 h(t,\,x_1,\,y_1) + D_2\psi(t,\,x_1,\,y_1)](x_1 - x_2) \\ & + [\theta_2 + D_3 h(t,\,x_1,\,y_1) + D_3\psi(t,\,x_1,\,y_1)](y_1 - y_2), \end{aligned}$$

$\forall\ (t,\,x_1,\,y_1),\ (t,\,x_2,\,y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Now consider

$$\left.\begin{aligned} y'(t) + \lambda y(t) = {} & f(t,\,\beta_0(t),\,\beta_0(t-r)) + h(t,\,\beta_0(t),\,\beta_0(t-r)) \\ & + \phi(t,\,\beta_0(t),\,\beta_0(t-r)) + \psi(t,\,\beta_0(t),\,\beta_0(t-r)) \\ & - \phi(t,\,y(t),\,y(t-r)) - \psi(t,\,y(t),\,y(t-r)) \\ & + D_2 h(t,\,\beta_0(t),\,\beta_0(t-r))[y(t) - \beta_0(t)] \\ & + D_2\psi(t,\,\beta_0(t),\,\beta_0(t-r))[y(t) - \beta_0(t)] \\ & + D_3 h(t,\,\beta_0(t),\,\beta_0(t-r))[y(t-r) - \beta_0(t-r)] \\ & + D_3\psi(t,\,\beta_0(t),\,\beta_0(t-r))[y(t-r) - \beta_0(t-r)] \\ & + \theta_1[y(t) - \beta_0(t)] + \theta_2[y(t-r) - \beta_0(t-r)], \\ & t \in I, \\ y(t) = y(0),\ & t \in [-r,\,0], \\ y(0) = y(T). & \end{aligned}\right\} \qquad (7.20)$$

$$\begin{aligned}\text{Let } G_{\beta_0}(t,\ u,\ v) = {} & f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\ & + \phi(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\ & - \phi(t,\ u,\ v) - \psi(t,\ u,\ v) \\ & + [\theta_1 + D_2h(t,\ \beta_0(t),\ \beta_0(t-r))][u-\beta_0(t)] \\ & + D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[u-\beta_0(t)] \\ & + [\theta_2 + D_3h(t,\ \beta_0(t),\ \beta_0(t-r))][v-\beta_0(t-r)] \\ & + D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[v-\beta_0(t-r)],\ \forall\ (t,\ u,\ v) \in \Omega.\end{aligned}$$

This implies

$$\begin{aligned}G_{\beta_0}(t,\ y(t),\ y(t-r)) = {} & f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\ & + \phi(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\ & - \phi(t,\ y(t),\ y(t-r)) - \psi(t,\ y(t),\ y(t-r)) \\ & + [\theta_1 + D_2h(t,\ \beta_0(t),\ \beta_0(t-r))][y(t)-\beta_0(t)] \\ & + D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[y(t)-\beta_0(t)] \\ & + [\theta_2 + D_3h(t,\ \beta_0(t),\ \beta_0(t-r))][y(t-r)-\beta_0(t-r)] \\ & + D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r)-\beta_0(t-r)], \\ & \forall\ t \in I.\end{aligned}$$

So we have $G_{\beta_0}(t,\ x_1,\ y_1) \leq G_{\beta_0}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

Also

$$\begin{aligned}
\alpha_0'(t) + \lambda\alpha_0(t) &\le f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\le f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad + \phi(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad - \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) - \psi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad + [\theta_1 + D_2h(t,\ \beta_0(t),\ \beta_0(t-r))][\alpha_0(t) - \beta_0(t)] \\
&\quad + D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[\alpha_0(t) - \beta_0(t)] \\
&\quad + [\theta_2 + D_3h(t,\ \beta_0(t),\ \beta_0(t-r))][\alpha_0(t-r) - \beta_0(t-r)] \\
&\quad + D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[\alpha_0(t-r) - \beta_0(t-r)] \\
&= G_{\beta_0}(t,\ \alpha_0(t),\ \alpha_0(t-r)).
\end{aligned}$$

Further

$$\begin{aligned}
\beta_0'(t) + \lambda\beta_0(t) &\ge f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&= f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad + \phi(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad - \phi(t,\ \beta_0(t),\ \beta_0(t-r)) - \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad + [\theta_1 + D_2h(t,\ \beta_0(t),\ \beta_0(t-r))][\beta_0(t) - \beta_0(t)] \\
&\quad + D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_0(t) - \beta_0(t)] \\
&\quad + [\theta_2 + D_3h(t,\ \beta_0(t),\ \beta_0(t-r))][\beta_0(t-r) - \beta_0(t-r)] \\
&\quad + D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_0(t-r) - \beta_0(t-r)] \\
&= G_{\beta_0}(t,\ \beta_0(t),\ \beta_0(t-r)).
\end{aligned}$$

We already know that

1. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
3. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
4. $\alpha_0(0) \leq \alpha_0(T)$.
5. $\beta_0(0) \geq \beta_0(T)$.

Therefore, α_0 and β_0 are lower and upper solutions of the PBVP (7.20) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\, t \in [-r,\ T]$.

Thus the PBVP (7.20) has a solution $\beta_1 \in X$ such that $\alpha_0 \leq \beta_1 \leq \beta_0$ on $[-r,\ T]$. (Using Theorem 4.4.5)

This implies

$$\left.\begin{aligned}
\beta_1'(t) + \lambda\beta_1(t) = {} & f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\
& + \phi(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
& - \phi(t,\ \beta_1(t),\ \beta_1(t-r)) - \psi(t,\ \beta_1(t),\ \beta_1(t-r)) \\
& + [\theta_1 + D_2h(t,\ \beta_0(t),\ \beta_0(t-r))][\beta_1(t) - \beta_0(t)] \\
& + D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_1(t) - \beta_0(t)] \\
& + [\theta_2 + D_3h(t,\ \beta_0(t),\ \beta_0(t-r))][\beta_1(t-r) - \beta_0(t-r)] \\
& + D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_1(t-r) - \beta_0(t-r)], \\
& t \in I, \\
\beta_1(t) = \beta_1(0),\ & t \in [-r,\ 0], \\
\beta_1(0) = \beta_1(T).\ &
\end{aligned}\right\}$$

So we have

$\beta_1'(t) + \lambda\beta_1(t) \geq f(t,\ \beta_1(t),\ \beta_1(t-r)) + h(t,\ \beta_1(t),\ \beta_1(t-r)),\ t \in I$.

Also

1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$
2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0].$
3. $\beta_1(t) = \beta_1(0),\ t \in [-r,\ 0].$
4. $\alpha_0(t) \leq \beta_1(t),\ t \in [-r,\ T].$
5. $\alpha_0(0) \leq \alpha_0(T).$
6. $\beta_1(0) \geq \beta_1(T).$

Therefore α_0 and $\beta_1 \in X$ are lower and upper solutions of the PBVP (7.1) such that $\alpha_0(t) \leq \beta_1(t),\ \forall\, t \in [-r,\ T]$.

Continuing in this manner, we get a monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\alpha_0 \leq \beta_n,\ \forall\, n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned}
\beta_{n+1}'(t) + \lambda\beta_{n+1}(t) &= f(t,\ \beta_n(t),\ \beta_n(t-r)) + h(t,\ \beta_n(t),\ \beta_n(t-r)) \\
&\quad + \phi(t,\ \beta_n(t),\ \beta_n(t-r)) + \psi(t,\ \beta_n(t),\ \beta_n(t-r)) \\
&\quad - \phi(t,\ \beta_{n+1}(t),\ \beta_{n+1}(t-r)) - \psi(t,\ \beta_{n+1}(t),\ \beta_{n+1}(t-r)) \\
&\quad + [\theta_1 + D_2h(t,\ \beta_n(t),\ \beta_n(t-r))][\beta_{n+1}(t) - \beta_n(t)] \\
&\quad + D_2\psi(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t) - \beta_n(t)] \\
&\quad + [\theta_2 + D_3h(t,\ \beta_n(t),\ \beta_n(t-r))][\beta_{n+1}(t-r) - \beta_n(t-r)] \\
&\quad + D_3\psi(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)], \\
&\quad t \in I, \\
\beta_{n+1}(t) &= \beta_{n+1}(0),\ t \in [-r,\ 0], \\
\beta_{n+1}(0) &= \beta_{n+1}(T).
\end{aligned}\right\}$$

Further $\lim_{n\to\infty}\beta_n$ exists in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty}\beta_n$ is a solution of the PBVP (7.1). □

Theorem 7.4.2. *Let $\alpha_0,\ \beta_0 \in W$ such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in I$ and $\Omega = \{(t,u,v) : \alpha_0(t) \leq u \leq \beta_0(t)$ and $\alpha_0(\theta(t)) \leq v \leq \beta_0(\theta(t)),\ t \in I\}$. Assume that $f,\ h,\ \phi,\ \psi \in C^2[\Omega,\ \mathbb{R}]$ satisfying*

(1) $D_i f(t,\ x,\ y) + D_i \phi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in \Omega$ *and* $\forall\ i = 2,\ 3.$

(2) $D_i h(t,\ x,\ y) + D_i \psi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in \Omega$ *and* $\forall\ i = 2,\ 3.$

(3) $D_{i,\,j} f(t,\ x,\ y) + D_{i,\,j} \phi(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in \Omega$ *and* $\forall\ i,\ j = 2,\ 3.$

(4) $D_{i,\,j} h(t,\ x,\ y) + D_{i,\,j} \beta(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in \Omega$ *and* $\forall\ i,\ j = 2,\ 3.$

(5) $\phi(t,\ x_1,\ y_1) + \psi(t,\ x_1,\ y_1) \geq \phi(t,\ x_2,\ y_2) + \psi(t, x_2, y_2),$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ *such that* $x_1 \leq x_2$ *and* $y_1 \leq y_2.$

Further, assume that α_0 *and* β_0 *are lower and upper solutions of PBVP* (7.4). *Then there are monotonic sequence of functions* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *converging in* $(W,\ ||.||_0)$ *such that* $\lim_{n\to\infty} \alpha_n$ *and* $\lim_{n\to\infty} \beta_n$ *are solutions of the PBVP* (7.4).

Proof. Similar to the proof of the Theorem 7.4.1. □

We now introduce extended-generalized quasilinearization method to establish exitence theorems for LBVP of general first order DECD (7.2) and for LBVP of first order DEVDA (7.5).

Theorem 7.4.3. *Let* $\alpha_0,\ \beta_0 \in X$ *such that* $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$ *and* $\Omega = \{(t, u, v) : \alpha_0(t) \leq u \leq \beta_0(t)$ *and* $\alpha_0(t - r) \leq v \leq \beta_0(t - r),\ t \in I\}.$ *Assume that* $f,\ h,\ \phi,\ \psi \in C^2[\Omega,\ \mathbb{R}]$ *satisfying*

(1) $D_i f(t,\ x,\ y) + D_i \phi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in \Omega$ *and* $\forall\ i = 2,\ 3.$

(2) $D_i h(t,\ x,\ y) + D_i \psi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in \Omega$ *and* $\forall\ i = 2,\ 3.$

(3) $D_{i,\,j} f(t,\ x,\ y) + D_{i,\,j} \phi(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in \Omega$ *and* $\forall\ i,\ j = 2,\ 3.$

(4) $D_{i,\,j} h(t,\ x,\ y) + D_{i,\,j} \beta(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in \Omega$ *and* $\forall\ i,\ j = 2,\ 3.$

(5) $\phi(t,\ x_1,\ y_1) + \psi(t,\ x_1,\ y_1) \geq \phi(t,\ x_2,\ y_2) + \psi(t, x_2, y_2),$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ *such that* $x_1 \leq x_2$ *and* $y_1 \leq y_2.$

Further, assume that α_0 *and* β_0 *are weakly coupled lower and upper solutions of LBVP* (7.2). *If* $b < 0$, *then there are monotonic sequence of functions* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *converging in* $(X,\ ||.||_\infty)$ *such that* $\lim_{n\to\infty} \alpha_n$ *and* $\lim_{n\to\infty} \beta_n$ *are solutions of the LBVP* (7.2).

Proof. Let $F(t,\ x,\ y) = f(t,\ x,\ y) + \phi(t,\ x,\ y),\ \forall\ (t,\ x,\ y) \in \Omega$ and

$$H(t,\ x,\ y) = h(t,\ x,\ y) + \psi(t,\ x,\ y),\ \forall\ (t,\ x,\ y) \in \Omega.$$

It is assumed that $D_{i,\,j}f(t,\ x,\ y) + D_{i,\,j}\phi(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i,\ j = 2,\ 3.$ Therefore

$$\begin{aligned} f(t,\ x_1,\ y_1) \geq\ & f(t,\ x_2,\ y_2) - \phi(t,\ x_1,\ y_1) + \phi(t,\ x_2,\ y_2) \\ & + [D_2 f(t,\ x_2,\ y_2) + D_2\phi(t,\ x_2,\ y_2)](x_1 - x_2) \\ & + [D_3 f(t,\ x_2,\ y_2) + D_3\phi(t,\ x_2,\ y_2)](y_1 - y_2), \end{aligned}$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

It is also assumed that $D_{i,\,j}h(t,\ x,\ y) + D_{i,\,j}\psi(t,\ x,\ y) \leq 0,$

$\forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i,\ j = 2,\ 3.$

Therefore

$$\begin{aligned} h(t,\ x_1,\ y_1) \geq\ & h(t,\ x_2,\ y_2) - \psi(t,\ x_1,\ y_1) + \psi(t,\ x_2,\ y_2) \\ & + [D_2 h(t,\ x_1,\ y_1) + D_2\psi(t,\ x_1,\ y_1)](x_1 - x_2) \\ & + [D_3 h(t,\ x_1,\ y_1) + D_3\psi(t,\ x_1,\ y_1)](y_1 - y_2), \end{aligned}$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Let $\mu_1 = \min\limits_{(t,\ u,\ v)\in\Omega}[D_2 h(t,\ u,\ v) + D_2\psi(t,\ u,\ v)]$ and

$\mu_2 = \min\limits_{(t,\ u,\ v)\in\Omega}[D_3 h(t,\ u,\ v) + D_3\psi(t,u,v)].$ Then, $\mu_1 > 0$ and $\mu_2 > 0.$

So we have

$$\begin{aligned} h(t,\ x_1,\ y_1) \geq\ & h(t,\ x_2,\ y_2) - \psi(t,\ x_1,\ y_1) + \psi(t,\ x_2,\ y_2) \\ & + \mu_1(x_1 - x_2) + \mu_2(y_1 - y_2), \end{aligned}$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Thus

$$\begin{aligned} f(t,\ x_1,\ y_1) + h(t,\ x_1,\ y_1) \geq\ & f(t,\ x_2,\ y_2) + h(t,\ x_2,\ y_2) + \phi(t,\ x_2,\ y_2) \\ & + \psi(t,\ x_2,\ y_2) - \phi(t,\ x_1,\ y_1) - \psi(t,\ x_1,\ y_1) \\ & + [D_2 f(t,\ x_1,\ y_1) + D_2\phi(t,\ x_1,\ y_1)](x_1 - x_2) \\ & + [D_3 f(t,\ x_1,\ y_1) + D_3\phi(t,\ x_1,\ y_1)](y_1 - y_2), \\ & + \mu_1(x_1 - x_2) + \mu_2(y_1 - y_2), \end{aligned}$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Since $\alpha_0,\ \beta_0 \in X$ are weakly coupled lower and upper solutions of LBVP (7.2), we have

1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$
2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$
3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0].$
4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0].$
5. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T].$
6. $\max\{a\alpha_0(0) + b\alpha_0(T),\ a\alpha_0(0) + b\beta_0(T)\}$
   $\leq 0 \leq \min\{a\beta_0(0) + b\alpha_0(T),\ a\beta_0(0) + b\beta_0(T)\}.$

As $b < 0$, we have

$\max\{a\alpha_0(0) + b\alpha_0(T),\ a\alpha_0(0) + b\beta_0(T)\} = a\alpha_0(0) + b\alpha_0(T)$ and

$\min\{a\beta_0(0) + b\alpha_0(T),\ a\beta_0(0) + b\beta_0(T)\} = a\beta_0(0) + b\beta_0(T).$

This implies $a\alpha_0(0) + b\alpha_0(T) \leq 0 \leq a\beta_0(0) + b\beta_0(T)$.

Also, as $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and

$\beta_0(t) = \beta_0(0),\ \forall\ t \in [-r,\ 0].$

Now, starting with the LBVP

$$\left.\begin{aligned}
y'(t)+\lambda y(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \psi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad - \phi(t,\ y(t),\ y(t-r)) - \psi(t,\ y(t),\ y(t-r)) \\
&\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t)-\alpha_0(t)] \\
&\quad + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t)-\alpha_0(t)] \\
&\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r)-\alpha_0(t-r)] \\
&\quad + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r)-\alpha_0(t-r)], \\
&\quad + \mu_1[y(t)-\alpha_0(t)] + \mu_2[y(t-r)-\alpha_0(t-r)], \\
&\quad t \in I, \\
y(t) &= y(0),\ t \in [-r,\ 0], \\
ay(0) &+ by(T) = 0,
\end{aligned}\right\} \tag{7.21}$$

and proceeding similarly along the proof of the Theorem 7.4.1, we get a monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned}
\alpha'_{n+1}(t) + \lambda\alpha_{n+1}(t) &= f(t, \alpha_n(t), \alpha_n(t-r)) + h(t, \alpha_n(t), \alpha_n(t-r)) \\
&\quad + \alpha(t, \alpha_n(t), \alpha_n(t-r)) + \beta(t, \alpha_n(t), \alpha_n(t-r)) \\
&\quad - \alpha(t, \alpha_{n+1}(t), \alpha_{n+1}(t-r)) - \beta(t, \alpha_{n+1}(t), \alpha_{n+1}(t-r)) \\
&\quad + D_2 f(t, \alpha_n(t-r), \alpha_n(t-r))[\alpha_{n+1}(t) - \alpha_n(t)] \\
&\quad + D_2\alpha(t, \alpha_n(t), \alpha_n(t-r))[\alpha_{n+1}(t) - \alpha_n(t)] \\
&\quad + D_3 f(t, \alpha_n(t), \alpha_n(t-r))[\alpha_{n+1}(t-r) - \alpha_n(t-r)] \\
&\quad + D_3\alpha(t, \alpha_n(t), \alpha_n(t-r))[\alpha_{n+1}(t-r) - \alpha_n(t-r)] \\
&\quad + \mu_1[\alpha_{n+1}(t) - \alpha_n(t)] + \mu_2[\alpha_{n+1}(t-r) - \alpha_n(t-r)], \\
&\quad t \in I, \\
\alpha_{n+1}(t) &= \alpha_{n+1}(0),\ t \in [-r,\ 0], \\
a\alpha_{n+1}(0) + b\alpha_{n+1}(T) &= 0.
\end{aligned}\right\}$$

Further $\lim_{n\to\infty} \alpha_n$ exists in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty} \alpha_n$ is a solution of the LBVP (7.2).

Let $\theta_1 = \min_{(t,\ u,\ v)\in\Omega} [D_2 f(t,\ u,\ v) + D_2\phi(t,\ u,\ v)]$ and

$$\theta_2 = \min_{(t,\ u,\ v)\in\Omega} [D_3 f(t,\ u,\ v) + D_3\phi(t,\ u,\ v)].$$ Then $\theta_1 > 0$ and $\theta_2 > 0$.

This implies $f(t,\ x_1,\ y_1) \geq f(t,\ x_2,\ y_2) - \phi(t,\ x_1,\ y_1) - \phi(t,\ x_2,\ y_2)$
$$+ \theta_1(x_1 - x_2) + \theta_2(y_1 - y_2),$$
$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

So we have

$$\begin{aligned}
f(t,\ x_1,\ y_1) + h(t,\ x_1,\ y_1) &\geq f(t,\ x_2,\ y_2) + h(t,\ x_2,\ y_2) + \phi(t,\ x_2,\ y_2) \\
&\quad + \psi(t,\ x_2,\ y_2) - \phi(t,\ x_1,\ y_1) - \psi(t,\ x_1,\ y_1) \\
&\quad + [\theta_1 + D_2 h(t,\ x_1,\ y_1) + D_2\psi(t,\ x_1,\ y_1)](x_1 - x_2) \\
&\quad + [\theta_2 + D_3 h(t,\ x_1,\ y_1) + D_3\psi(t,\ x_1,\ y_1)](y_1 - y_2),
\end{aligned}$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Next, begining with the LBVP

$$\left.\begin{aligned}
y'(t)+\lambda y(t) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad + \phi(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad - \phi(t,\ y(t),\ y(t-r)) - \psi(t,\ y(t),\ y(t-r)) \\
&\quad + D_2h(t,\ \beta_0(t),\ \beta_0(t-r))[y(t)-\beta_0(t)] \\
&\quad + D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[y(t)-\beta_0(t)] \\
&\quad + D_3h(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r)-\beta_0(t-r)] \\
&\quad + D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r)-\beta_0(t-r)] \\
&\quad + \theta_1[y(t)-\beta_0(t)] + \theta_2[y(t-r)-\beta_0(t-r)], \\
&\quad t \in I, \\
y(t) &= y(0),\ t \in [-r,\ 0], \\
ay(0)+by(T) &= 0,
\end{aligned}\right\} \tag{7.22}$$

and proceeding similarly along the proof of the Theorem 7.4.1, we get a monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\alpha_0 \leq \beta_n,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned}
\beta'_{n+1}(t) + \lambda\beta_{n+1}(t) &= f(t,\ \beta_n(t),\ \beta_n(t-r)) + h(t,\ \beta_n(t),\ \beta_n(t-r)) \\
&\quad + \phi(t,\ \beta_n(t),\ \beta_n(t-r)) + \psi(t,\ \beta_n(t),\ \beta_n(t-r)) \\
&\quad - \phi(t,\ \beta_{n+1}(t),\ \beta_{n+1}(t-r)) - \psi(t,\ \beta_{n+1}(t),\ \beta_{n+1}(t-r)) \\
&\quad + D_2h(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t) - \beta_n(t)] \\
&\quad + D_2\psi(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t) - \beta_n(t)] \\
&\quad + D_3h(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)] \\
&\quad + D_3\psi(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)] \\
&\quad + \theta_1[\beta_{n+1}(t) - \beta_n(t)] + \theta_2[\beta_{n+1}(t-r) - \beta_n(t-r)], \\
&\quad t \in I, \\
\beta_{n+1}(t) &= \beta_{n+1}(0),\ t \in [-r,\ 0], \\
a\beta_{n+1}(0) + b\beta_{n+1}(T) &= 0.
\end{aligned}\right\}$$

Further $\lim_{n\to\infty}\beta_n$ exists in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty}\beta_n$ is a solution of the LBVP (7.2). □

Theorem 7.4.4. *Let $\alpha_0,\ \beta_0 \in W$ such that $\alpha_0(t) \le \beta_0(t),\ \forall\ t \in I$ and $\Omega = \{(t,u,v) : \alpha_0(t) \le u \le \beta_0(t)$ and $\alpha_0(\theta(t)) \le v \le \beta_0(\theta(t)),\ t \in I\}$. Assume that $f,\ h,\ \phi,\ \psi \in C^2[\Omega,\ \mathbb{R}]$ satisfying*

(1) $D_if(t,\ x,\ y) + D_i\phi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i = 2,\ 3$.

(2) $D_ih(t,\ x,\ y) + D_i\psi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i = 2,\ 3$.

(3) $D_{i,\,j}f(t,\ x,\ y) + D_{i,\,j}\phi(t,\ x,\ y) \ge 0,\ \forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i,\ j = 2,\ 3$.

(4) $D_{i,\,j}h(t,\ x,\ y) + D_{i,\,j}\beta(t,\ x,\ y) \le 0,\ \forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i,\ j = 2,\ 3$.

(5) $\phi(t,\ x_1,\ y_1) + \psi(t,\ x_1,\ y_1) \ge \phi(t,\ x_2,\ y_2) + \psi(t,x_2,y_2)$,

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \le x_2$ and $y_1 \le y_2$.

Further, assume that α_0 and β_0 are weakly coupled lower and upper solutions of LBVP (7.5). If $b < 0$, then there are monotonic sequence of functions $\{\alpha_n\}$ and $\{\beta_n\}$ converging in $(W,\ ||.||_0)$ such that $\lim_{n\to\infty}\alpha_n$ and $\lim_{n\to\infty}\beta_n$ are solutions of the LBVP (7.5).

Proof. Similar to the proof of the Theorem 7.4.3. □

Next, we use extended-generalized quasilinearization method for obtaining results on existence of solution for NLBVP of general first order DECD (7.3) and for NLBVP of first order DEVDA (7.6).

Theorem 7.4.5. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$. Let $\alpha_0,\ \beta_0 \in X$ such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$ and*
$\Omega = \{(t, u, v) : \alpha_0(t) \leq u \leq \beta_0(t)$ *and* $\alpha_0(t-r) \leq v \leq \beta_0(t-r),\ t \in I\}$.
Assume that $f,\ h,\ \phi,\ \psi \in C^2[\Omega,\ \mathbb{R}]$ satisfying
(1) $D_i f(t,\ x,\ y) + D_i\phi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i = 2,\ 3$.
(2) $D_i h(t,\ x,\ y) + D_i\psi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i = 2,\ 3$.
(3) $D_{i,\,j} f(t,\ x,\ y) + D_{i,\,j}\phi(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i,\ j = 2,\ 3$.
(4) $D_{i,\,j} h(t,\ x,\ y) + D_{i,\,j}\beta(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i,\ j = 2,\ 3$.
(5) $\phi(t,\ x_1,\ y_1) + \psi(t,\ x_1,\ y_1) \geq \phi(t,\ x_2,\ y_2) + \psi(t, x_2, y_2)$,
$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.
Further, assume that α_0 and β_0 are weakly coupled lower and upper solutions of NLBVP (7.3). Then there are monotonic sequence of functions $\{\alpha_n\}$ and $\{\beta_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim_{n\to\infty}\alpha_n$ and $\lim_{n\to\infty}\beta_n$ are solutions of the NLBVP (7.3).

Proof. Let $F(t,\ x,\ y) = f(t,\ x,\ y) + \phi(t,\ x,\ y),\ \forall\ (t,\ x,\ y) \in \Omega$ and

$$H(t,\ x,\ y) = h(t,\ x,\ y) + \psi(t,\ x,\ y),\ \forall\ (t,\ x,\ y) \in \Omega.$$

It is assumed that $D_{i,\,j} f(t,\ x,\ y) + D_{i,\,j}\phi(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i,\ j = 2,\ 3$.

Therefore $f(t,\ x_1,\ y_1) \geq f(t,\ x_2,\ y_2) - \phi(t,\ x_1,\ y_1) + \phi(t,\ x_2,\ y_2)$
$$+ [D_2 f(t,\ x_2,\ y_2) + D_2\phi(t,\ x_2,\ y_2)](x_1 - x_2)$$
$$+ [D_3 f(t,\ x_2,\ y_2) + D_3\phi(t,\ x_2,\ y_2)](y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

It is also assumed that $D_{i,\,j}h(t,\ x,\ y) + D_{i,\,j}\psi(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i,\ j = 2,\ 3$. Therefore

$$\begin{aligned} h(t,\ x_1,\ y_1) \geq\ & h(t,\ x_2,\ y_2) - \psi(t,\ x_1,\ y_1) + \psi(t,\ x_2,\ y_2) \\ & + [D_2h(t,\ x_1,\ y_1) + D_2\psi(t,\ x_1,\ y_1)](x_1 - x_2) \\ & + [D_3h(t,\ x_1,\ y_1) + D_3\psi(t,\ x_1,\ y_1)](y_1 - y_2), \end{aligned}$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Let $\mu_1 = \min\limits_{(t,\,u,\,v)\in\Omega}[D_2h(t,\ u,\ v) + D_2\psi(t,\ u,\ v)]$ and

$\mu_2 = \min\limits_{(t,\,u,\,v)\in\Omega}[D_3h(t,\ u,\ v) + D_3\psi(t,u,v)]$. Then, $\mu_1 > 0$ and $\mu_2 > 0$.

So we have

$$\begin{aligned} h(t,\ x_1,\ y_1) \geq\ & h(t,\ x_2,\ y_2) - \psi(t,\ x_1,\ y_1) + \psi(t,\ x_2,\ y_2) \\ & + \mu_1(x_1 - x_2) + \mu_2(y_1 - y_2), \end{aligned}$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Thus

$$\begin{aligned} f(t,\ x_1,\ y_1) + h(t,\ x_1,\ y_1) \geq\ & f(t,\ x_2,\ y_2) + h(t,\ x_2,\ y_2) + \phi(t,\ x_2,\ y_2) \\ & + \psi(t,\ x_2,\ y_2) - \phi(t,\ x_1,\ y_1) - \psi(t,\ x_1,\ y_1) \\ & + [D_2f(t,\ x_1,\ y_1) + D_2\phi(t,\ x_1,\ y_1)](x_1 - x_2) \\ & + [D_3f(t,\ x_1,\ y_1) + D_3\phi(t,\ x_1,\ y_1)](y_1 - y_2), \\ & + \mu_1(x_1 - x_2) + \mu_2(y_1 - y_2), \end{aligned}$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Since $\alpha_0,\ \beta_0 \in X$ are weakly coupled lower and upper solutions of NLBVP (7.3), we have

1. $\alpha_0'(t) + \lambda\alpha_0(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I,$
2. $\beta_0'(t) + \lambda\beta_0(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I,$
3. $\alpha_0(t) \leq \alpha_0(0),\ t \in [-r,\ 0],$
4. $\beta_0(t) \geq \beta_0(0),\ t \in [-r,\ 0],$
5. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T],$
6. $\max\{g(\alpha_0(0),\ \alpha_0(T)),\ g(\alpha_0(0),\ \beta_0(T))\}$
   $\leq 0 \leq \min\{g(\beta_0(0),\ \alpha_0(T)),\ g(\beta_0(0),\ \beta_0(T))\}.$

As $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$, we have

$\max\{g(\alpha_0(0),\ \alpha_0(T)),\ g(\alpha_0(0),\ \beta_0(T))\} = g(\alpha_0(0),\ \alpha_0(T))$ and

$\min\{g(\beta_0(0),\ \alpha_0(T)),\ g(\beta_0(0),\ \beta_0(T))\} = g(\beta_0(0),\ \beta_0(T)).$

This implies $g(\alpha_0(0),\ \alpha_0(T)) \leq 0 \leq g(\beta_0(0),\ \beta_0(T)).$

Also, as $\alpha_0,\ \beta_0 \in X$, we have $\alpha_0(t) = \alpha_0(0),\ \forall\ t \in [-r,\ 0]$ and

$\beta_0(t) = \beta_0(0),\ \forall\ t \in [-r,\ 0].$

Now, starting with the NLBVP

$$\left.\begin{aligned}
y'(t) + \lambda y(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \psi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad - \phi(t,\ y(t),\ y(t-r)) - \psi(t,\ y(t),\ y(t-r)) \\
&\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t) - \alpha_0(t)] \\
&\quad + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t) - \alpha_0(t)] \\
&\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r) - \alpha_0(t-r)] \\
&\quad + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r) - \alpha_0(t-r)], \\
&\quad + \mu_1[y(t) - \alpha_0(t)] + \mu_2[y(t-r) - \alpha_0(t-r)], \\
&\quad t \in I, \\
y(t) &= y(0), t \in [-r,\ 0], \\
g(y(0),\ y(T)) &= 0,
\end{aligned}\right\} \tag{7.23}$$

and proceeding similarly along the proof of the Theorem 7.4.1, we get a monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned}
\alpha'_{n+1}(t) + \lambda\alpha_{n+1}(t) &= f(t, \alpha_n(t), \alpha_n(t-r)) + h(t, \alpha_n(t), \alpha_n(t-r)) \\
&\quad + \alpha(t, \alpha_n(t), \alpha_n(t-r)) + \beta(t, \alpha_n(t), \alpha_n(t-r)) \\
&\quad - \alpha(t, \alpha_{n+1}(t), \alpha_{n+1}(t-r)) - \beta(t, \alpha_{n+1}(t), \alpha_{n+1}(t-r)) \\
&\quad + D_2 f(t, \alpha_n(t-r), \alpha_n(t-r))[\alpha_{n+1}(t) - \alpha_n(t)] \\
&\quad + D_2\alpha(t, \alpha_n(t), \alpha_n(t-r))[\alpha_{n+1}(t) - \alpha_n(t)] \\
&\quad + D_3 f(t, \alpha_n(t), \alpha_n(t-r))[\alpha_{n+1}(t-r) - \alpha_n(t-r)] \\
&\quad + D_3\alpha(t, \alpha_n(t), \alpha_n(t-r))[\alpha_{n+1}(t-r) - \alpha_n(t-r)] \\
&\quad + \mu_1[\alpha_{n+1}(t) - \alpha_n(t)] + \mu_2[\alpha_{n+1}(t-r) - \alpha_n(t-r)], \\
&\quad t \in I, \\
\alpha_{n+1}(t) &= \alpha_{n+1}(0),\ t \in [-r,\ 0], \\
a\alpha_{n+1}(0) + b\alpha_{n+1}(T) &= 0.
\end{aligned}\right\}$$

Further $\lim\limits_{n\to\infty} \alpha_n$ exists in $(X,\ ||.||_\infty)$ and $\lim\limits_{n\to\infty} \alpha_n$ is a solution of the NLBVP (7.3).

Let $\theta_1 = \min\limits_{(t,\ u,\ v)\in\Omega} [D_2 f(t,\ u,\ v) + D_2\phi(t,\ u,\ v)]$ and

$$\theta_2 = \min_{(t,\ u,\ v)\in\Omega} [D_3 f(t,\ u,\ v) + D_3\phi(t,\ u,\ v)].\ \text{Then } \theta_1 > 0 \text{ and } \theta_2 > 0.$$

This implies $f(t,\ x_1,\ y_1) \geq f(t,\ x_2,\ y_2) - \phi(t,\ x_1,\ y_1) - \phi(t,\ x_2,\ y_2)$

$$+ \theta_1(x_1 - x_2) + \theta_2(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

So we have

$$\begin{aligned} f(t,\ x_1,\ y_1) + h(t,\ x_1,\ y_1) \geq\ & f(t,\ x_2,\ y_2) + h(t,\ x_2,\ y_2) + \phi(t,\ x_2,\ y_2) \\ & + \psi(t,\ x_2,\ y_2) - \phi(t,\ x_1,\ y_1) - \psi(t,\ x_1,\ y_1) \\ & + [\theta_1 + D_2h(t,\ x_1,\ y_1) + D_2\psi(t,\ x_1,\ y_1)](x_1 - x_2) \\ & + [\theta_2 + D_3h(t,\ x_1,\ y_1) + D_3\psi(t,\ x_1,\ y_1)](y_1 - y_2), \end{aligned}$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Next, begining with the NLBVP

$$\left.\begin{aligned} y'(t) + \lambda y(t) = {} & f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\ & + \phi(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\ & - \phi(t,\ y(t),\ y(t-r)) - \psi(t,\ y(t),\ y(t-r)) \\ & + D_2h(t,\ \beta_0(t),\ \beta_0(t-r))[y(t) - \beta_0(t)] \\ & + D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[y(t) - \beta_0(t)] \\ & + D_3h(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)] \\ & + D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)] \\ & + \theta_1[y(t) - \beta_0(t)] + \theta_2[y(t-r) - \beta_0(t-r)], \\ & t \in I, \\ y(t) = {} & y(0),\ t \in [-r,\ 0], \\ g(y(0),\ & y(T)) = 0, \end{aligned}\right\} \tag{7.24}$$

and proceeding similarly along the proof of the Theorem 7.4.1, we get a monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\alpha_0 \leq \beta_n,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned}
\beta'_{n+1}(t) + \lambda\beta_{n+1}(t) &= f(t,\ \beta_n(t),\ \beta_n(t-r)) + h(t,\ \beta_n(t),\ \beta_n(t-r)) \\
&\quad + \phi(t,\ \beta_n(t),\ \beta_n(t-r)) + \psi(t,\ \beta_n(t),\ \beta_n(t-r)) \\
&\quad - \phi(t,\ \beta_{n+1}(t),\ \beta_{n+1}(t-r)) - \psi(t,\ \beta_{n+1}(t),\ \beta_{n+1}(t-r)) \\
&\quad + D_2h(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t) - \beta_n(t)] \\
&\quad + D_2\psi(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t) - \beta_n(t)] \\
&\quad + D_3h(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)] \\
&\quad + D_3\psi(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)] \\
&\quad + \theta_1[\beta_{n+1}(t) - \beta_n(t)] + \theta_2[\beta_{n+1}(t-r) - \beta_n(t-r)], \\
&\quad t \in I, \\
\beta_{n+1}(t) &= \beta_{n+1}(0),\ t \in [-r,\ 0], \\
g(\beta_{n+1}(0),\ \beta_{n+1}(T)) &= 0.
\end{aligned}\right\}$$

Further $\lim_{n\to\infty}\beta_n$ exists in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty}\beta_n$ is a solution of the LBVP (7.2). □

Theorem 7.4.6. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying $g(t,\ x) \geq g(t,\ y),\ \forall\ (t,\ x),\ (t,\ y) \in \mathbb{R}^2$ such that $x \leq y$. Let $\alpha_0,\ \beta_0 \in W$ such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in I$ and $\Omega = \{(t, u, v) : \alpha_0(t) \leq u \leq \beta_0(t)$ and $\alpha_0(\theta(t)) \leq v \leq \beta_0(\theta(t)),\ t \in I\}$. Assume that $f,\ h,\ \phi,\ \psi \in C^2[\Omega,\ \mathbb{R}]$ satisfying*

(1) $D_if(t,\ x,\ y) + D_i\phi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i = 2,\ 3$.

(2) $D_ih(t,\ x,\ y) + D_i\psi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i = 2,\ 3$.

(3) $D_{i,\,j}f(t,\ x,\ y) + D_{i,\,j}\phi(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i,\ j = 2,\ 3$.

(4) $D_{i,\,j}h(t,\ x,\ y) + D_{i,\,j}\beta(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i,\ j = 2,\ 3$.

(5) $\phi(t,\ x_1,\ y_1) + \psi(t,\ x_1,\ y_1) \geq \phi(t,\ x_2,\ y_2) + \psi(t, x_2, y_2)$,

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

Further, assume that α_0 and β_0 are lower and upper solutions of NLBVP (7.6).

Then there are monotonic sequence of functions $\{\alpha_n\}$ *and* $\{\beta_n\}$ *converging in* $(W,\ ||.||_0)$ *such that* $\lim_{n\to\infty}\alpha_n$ *and* $\lim_{n\to\infty}\beta_n$ *are solutions of the NLBVP* (7.6)

Proof. Similar to the proof of the Theorem 7.4.5. □

Chapter 8

SECOND ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

Some content of this chapter is published [see P5].

8.1 Introduction

In previous chapters we studied BVPs for first order DECD and first order DEVDA. In this chapter, we investigate BVPs of second order DECD and second order DEVDA for existence of solution using Schauder's Fixed Point Theorem and Banach Fixed Point Theorem. We also employ method of lower and upper solutions to prove our results on existence of solution between lower and upper solutions for BVPs of second order DECD and second order DEVDA.

Let $a,\ b,\ \gamma_1,\ \gamma_2 \in \mathbb{R}$, $T > 0$, $\lambda \geq 0$, $I = [0,\ T]$, $0 < r \leq T$ and $\theta : I \to I$ be continuous. Then the BVPs for second order DECD under investigation are

$$\left.\begin{aligned} &y''(t) = f(t,\ y(t),\ y(t-r),\ y'(t)),\ t \in I, \\ &y(t) = a,\ t \in [-r,\ 0], \\ &y(T) + y'(T) = b. \end{aligned}\right\} \tag{8.1}$$

$$\left.\begin{aligned} &y''(t) + \gamma_1 y'(t) + \gamma_2 y(t) = f(t,\ y(t),\ y(t-r),\ y'(t)),\ t \in I, \\ &y(t) = a,\ t \in [-r,\ 0], \\ &y(T) + y'(T) = b. \end{aligned}\right\} \tag{8.2}$$

$$\left.\begin{aligned} &y''(t) = f(t,\ y(t),\ y(t-r),\ y'(t)),\ t \in I, \\ &y(t) = 0,\ t \in [-r,\ 0], \\ &y(T) + y'(T) = b. \end{aligned}\right\} \tag{8.3}$$

$$\left.\begin{aligned} -y''(t) + \lambda y(t) &= f(t,\ y(t),\ y(t-r)),\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T), \\ y'(0) &= y'(T). \end{aligned}\right\} \tag{8.4}$$

$$\left.\begin{aligned} -y''(t) &= f(t,\ y(t),\ y(t-r)),\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T), \\ y'(0) &= y'(T). \end{aligned}\right\} \tag{8.5}$$

$$\left.\begin{aligned} -y''(t) + \lambda y(t) &= f(t,\ y(t),\ y(t-r)),\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T), \\ y'(0) &= 0. \end{aligned}\right\} \tag{8.6}$$

$$\left.\begin{aligned} -y''(t) &= f(t,\ y(t),\ y(t-r)),\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T), \\ y'(0) &= 0. \end{aligned}\right\} \tag{8.7}$$

The BVPs for second order DEVDA under investigation are

$$\left.\begin{aligned} y''(t) &= f(t,\ y(t),\ y(\theta(t)),\ y'(t)),\ t \in I, \\ y(0) &= a, \\ y(T) + y'(T) &= b. \end{aligned}\right\} \tag{8.8}$$

$$\left.\begin{aligned} &y''(t)+\gamma_1 y'(t)+\gamma_2 y(t)=f(t,\ y(t),\ y(\theta(t)),\ y'(t)),\ t\in I,\\ &y(0)=a,\\ &y(T)+y'(T)=b. \end{aligned}\right\} \tag{8.9}$$

$$\left.\begin{aligned} &y''(t)=f(t,\ y(t),\ y(\theta(t)),\ y'(t)),\ t\in I,\\ &y(0)=0,\\ &y(T)+y'(T)=b. \end{aligned}\right\} \tag{8.10}$$

$$\left.\begin{aligned} &-y''(t)+\lambda y(t)=f(t,\ y(t),\ y(\theta(t))),\ t\in I,\\ &y(0)=y(T),\\ &y'(0)=y'(T). \end{aligned}\right\} \tag{8.11}$$

$$\left.\begin{aligned} &-y''(t)=f(t,\ y(t),\ y(\theta(t))),\ t\in I,\\ &y(0)=y(T),\\ &y'(0)=y'(T). \end{aligned}\right\} \tag{8.12}$$

$$\left.\begin{aligned} &-y''(t)+\lambda y(t)=f(t,\ y(t),\ y(\theta(t))),\ t\in I,\\ &y(0)=y(T),\\ &y'(0)=0. \end{aligned}\right\} \tag{8.13}$$

$$\left.\begin{aligned} &-y''(t)=f(t,\ y(t),\ y(\theta(t))),\ t\in I,\\ &y(0)=y(T),\\ &y'(0)=0. \end{aligned}\right\} \tag{8.14}$$

We shall need the following lemmas to prove our results on existence of solution for the above BVPs. We state them without proofs.

Lemma 8.1.1. *Let $L : X \to Y \times \mathbb{R}^2$ be an operator defined by $L[y(t)] = (u(t),\ y(0),\ y(T))$, where*

$$u(t) = \begin{cases} 0; & -r \le t \le 0, \\ -y'(t) + y'(0) + \lambda \int_0^t y(s)ds; & 0 \le t \le T. \end{cases}$$

Then

1. *$L : X \to Y \times \mathbb{R}^2$ is a continuous linear operator.*

2. *$L^{-1} : Y \times \mathbb{R}^2 \to X$ exists and is a continuous linear operator.*

Lemma 8.1.2. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, $g^* : \mathbb{R}^3 \to \mathbb{R}$ be continuous and $h^* : \mathbb{R} \to \mathbb{R}$ be continuous. Let $Q : X \to Y \times \mathbb{R}^2$ be defined by $Q[y(t)] = (v(t),\ g^*(y(0),\ y'(0),\ y'(T)),\ h^*(y(0)))$, where*

$$v(t) = \begin{cases} 0; & -r \le t \le 0 \\ \int_0^t f(s,\ y(s),\ y(s-r))ds; & 0 \le t \le T. \end{cases}$$

Then $Q : X \to Y \times \mathbb{R}^2$ is continuous and compact.

Lemma 8.1.3. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, $g^* : \mathbb{R}^2 \to \mathbb{R}$ be continuous and $h^* : \mathbb{R} \to \mathbb{R}$ be continuous. Let $Q : X \to Y \times \mathbb{R}^2$ be defined by $Q[y(t)] = (v(t),\ g^*(y(0),\ y'(0)),\ h^*(y(0)))$, where*

$$v(t) = \begin{cases} 0; & -r \le t \le 0 \\ \int_0^t f(s,\ y(s),\ y(s-r))ds; & 0 \le t \le T. \end{cases}$$

Then $Q : X \to Y \times \mathbb{R}^2$ is continuous and compact.

Lemma 8.1.4. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, $g^* : \mathbb{R}^3 \to \mathbb{R}$ be continuous and $h^* : \mathbb{R} \to \mathbb{R}$ be continuous. Let $Q : W \to V \times \mathbb{R}^2$ be defined by $Q[y(t)] = (v(t),\ g^*(y(0),\ y'(0),\ y'(T)),\ h^*(y(0)))$, where $v(t) = \int_0^t f(s,\ y(s),\ y(\theta(s)))ds,\ t \in I$. Then $Q : W \to V \times \mathbb{R}^2$ is continuous and compact.*

Lemma 8.1.5. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, $g^* : \mathbb{R}^2 \to \mathbb{R}$ be continuous and $h^* : \mathbb{R} \to \mathbb{R}$ be continuous. Let $Q : W \to V \times \mathbb{R}^2$ be defined by $Q[y(t)] = (v(t),\ g^*(y(0),\ y'(0)),\ h^*(y(0)))$, where $v(t) = \int_0^t f(s,\ y(s),\ y(\theta(s)))ds,\ t \in I$. Then $Q : W \to V \times \mathbb{R}^2$ is continuous and compact.*

8.2 Existence Theorems using Banach Fixed Point Theorem

In this section, we estabish results on existence of solution to some of the above BVPs using Banach fixed point theorem. We first define the solution of the BVP (8.1).

Definition 8.2.1. *A function $y : [-r,\ T] \to \mathbb{R}$ is said to be a solution for BVP of second order DECD* (8.1) *if*

1. $y \in C[[-r,\ T],\ \mathbb{R}]$ and $y \in C^2[I,\ \mathbb{R}]$.

2. $y''(t) = f(t,\ y(t),\ y(t-r),\ y'(t)),\ t \in I$.

3. $y(t) = a,\ t \in [-r,\ 0]$.

4. $y(T) + y'(T) = b$.

Similarly, we can give definitions of solution for other BVPs of second order DECD.

Definition 8.2.2. *A function $y : I \to \mathbb{R}$ is said to be a solution for BVP of second order DEVDA* (8.8) *if*

1. $y \in C^2[I,\ \mathbb{R}]$.

2. $y''(t) = f(t,\ y(t),\ y(\theta(t)),\ y'(t)),\ t \in I$.

3. $y(0) = a$.

4. $y(T) + y'(T) = b$.

Similarly, we can give definitions of solution for other BVPs of second order DEVDA.

We need following lemmas and we state them without proofs.

Lemma 8.2.1. *A function $y : [-r,\ T] \to \mathbb{R}$ is a solution of the BVP* (8.1) *iff y satisfies*

$$y(t) = \begin{cases} a, & -r \le t \le 0, \\ \\ a + \frac{(b-a)t}{1+T} + \int\limits_0^T G(t,\ s)F(s,\ y)ds, & 0 \le t \le T, \end{cases} \tag{8.15}$$

$$\text{where } G(t,s) = \begin{cases} \frac{st}{1+T} - s, & 0 \le s \le t \le T, \\ \frac{st}{1+T} - t, & 0 \le t \le s \le T, \end{cases} \tag{8.16}$$

and $F(t;\ y) = f(t,\ y(t),\ y(t-r),\ y'(t)),\ \forall\ t \in I$.

Lemma 8.2.2. *A function $y : I \to \mathbb{R}$ is a solution of the BVP* (8.8) *iff y*

$$\text{satisfies } y(t) = a + \frac{(b-a)t}{1+T} + \int\limits_0^T G(t,\ s)F(s;\ y)ds,\ t \in I, \tag{8.17}$$

where $G(t,s)$ is given by (8.16) *and* $F(t;\ y) = f(t,\ y(t),\ y(\theta(t)),\ y'(t))$,
$\forall\ t \in I$.

We have the following existence theorem for BVP(8.1).

Theorem 8.2.3. *Let* $\mu_1 > 0,\ \mu_2 > 0,\ \mu_3 > 0,\ \mu = \max\{\mu_1,\ \mu_2,\ \mu_3\}$ *such that* $6\mu T(2T+1) + 9\mu T < 1$. *Let* $f : I \times \mathbb{R}^3 \to \mathbb{R}$ *be continuous and satisfy* $|f(t,\ x_1,\ y_1,\ z_1) - f(t,\ x_2,\ y_2,\ z_2)| \leq \mu_1|x_1 - x_2| + \mu_2|y_1 - y_2| + \mu_3|z_1 - z_2|$, $\forall\ (t,\ x_1,\ y_1,\ z_1),\ (t,\ x_2,\ y_2,\ z_2) \in I \times \mathbb{R}^3$. *Then the BVP* (8.1) *has a unique solution* $y \in E$.

Proof. Let $L_1 = 6\mu T(2T+1)$ and $L_2 = 9\mu T$. Then $L = L_1 + L_2 < 1$.

Let $J : E \to E$ be a mapping defined by

$$J[y(t)] = \begin{cases} a, & -r \leq t \leq 0, \\ \\ a + \frac{(b-a)t}{1+T} + \int\limits_0^T G(t,\ s)F(s;\ y)ds, & 0 \leq t \leq T, \end{cases} \tag{8.18}$$

where $G(t,\ s)$ is given by (8.16) and $F(t;\ y) = f(t,\ y(t),\ y(t-r),\ y'(t))$, $\forall\ t \in I$.

For $0 \leq t \leq T$, we may also express $J[y(t)]$ as

$$J[y(t)] = a + \frac{(b-a)t}{1+T} + \int_0^T [\frac{st}{1+T} - t]F(s;\ y)ds\ +\ \int_0^t (t-s)F(s;\ y)ds.$$

$$\text{Therefore } J'[y(t)] = \frac{(b-a)}{1+T} + \int_0^T [\frac{s}{1+T} - 1]F(s;\ y)ds\ +\ \int_0^t F(s;\ y)ds.$$

We show that $J : E \to E$ is a contraction.

Let $w,\ z \in E$ be arbitrary.

Let $-r \le t \le 0$. Then

$|J[w(t)] - J[z(t)]| = |a - a| = 0 \le L_1 d_\infty(w,\ z)$.

Let $0 \le t \le T$. Then

$$
\begin{aligned}
&|J[w(t)] - J[z(t)]| \\
&\le \int_0^T |G(t,\ s)||f(s,\ w(s),\ w(s-r),\ w'(s)) - f(s,\ z(s),\ z(s-r),\ z'(s))|ds \\
&\le \int_0^T |G(t,\ s)|\{\mu_1|w(s) - z(s)| + \mu_2|w(s-r) - z(s-r)| + \mu_3|w'(s) - z'(s)|\}ds \\
&\le 3\mu d_\infty(w,\ z) \int_0^T |G(t,s)|ds \\
&= 3\mu d_\infty(w,\ z)\{\int_0^t |G(t,s)|ds + \int_t^T |G(t,s)|ds\} \\
&= \frac{3\mu d_\infty(w,\ z)}{T+1}\{\int_0^t (st + sT + s)ds + \int_t^T (st + tT + t)ds\} \\
&\le \frac{3\mu d_\infty(w,\ z)}{T}\{\int_0^T (2T^2 + T)ds + \int_0^T (2T^2 + T)ds\} \\
&= 6\mu d_\infty(w,\ z) \int_0^T (2T + 1)ds \\
&= 6\mu T(2T+1) d_\infty(w,\ z) = L_1 d_\infty(w,\ z).
\end{aligned}
$$

Therefore $\sup_{t\in[-r,\ T]} |J[w(t)] - J[z(t)]| \le L_1 d_\infty(w,\ z)$. (8.19)

For $0 \leq t \leq T$, consider

$$
\begin{aligned}
&|J'[w(t)] - J'[z(t)]| \\
&\leq \int_0^T |\frac{s}{1+T} - 1||f(s,\ w(s),\ w(s-r),\ w'(s)) - f(s,\ z(s),\ z(s-r),\ z'(s))|ds \\
&\quad + \int_0^t |f(s,\ w(s),\ w(s-r),\ w'(s)) - f(s,\ z(s),\ z(s-r),\ z'(s))|ds \\
&\leq \int_0^T (\frac{s}{1+T} + 1)\{\mu_1|w(s) - z(s)| + \mu_2|w(s-r) - z(s-r)| + \mu_3|w'(s) - z'(s)|\}ds \\
&\quad + \int_0^t \{\mu_1|w(s) - z(s)| + \mu_2|w(s-r) - z(s-r)| + \mu_3|w'(s) - z'(s)|\}ds \\
&\leq 3\mu d_\infty(w,\ z) \int_0^T (\frac{s}{T} + 1)ds \ +\ 3\mu d_\infty(w, z) \int_0^T ds \\
&\leq 6\mu d_\infty(w,\ z) \int_0^T ds \ +\ 3\mu d_\infty(w, z) \int_0^T ds \\
&= 9\mu T d_\infty(w,\ z) \\
&= L_2 d_\infty(w,\ z).
\end{aligned}
$$

Therefore $\sup_{t \in [-r,\ T]} |J'[w(t)] - J'[z(t)]| \leq L_2 d_\infty(w,\ z).$ (8.20)

Adding inequalities (8.19) and (8.20), we get $d_\infty(Jw,\ Jz) \leq L d_\infty(w,\ z)$, $\forall\ w,\ z \in E$ and $L < 1$. Therefore $J : E \to E$ is a contraction mapping. Thus J has a unique fixed point $y \in E$. Hence y is a unique solution of the BVP (8.1). □

Next we prove the result for BVP(8.8).

Theorem 8.2.4. *Let* $\mu_1 > 0,\ \mu_2 > 0,\ \mu_3 > 0,\ \mu = \max\{\mu_1,\ \mu_2,\ \mu_3\}$ *such that* $6\mu T(2T+1) + 9\mu T < 1$. *Let* $f : I \times \mathbb{R}^3 \to \mathbb{R}$ *be continuous and satisfy*

$|f(t,\ x_1,\ y_1,\ z_1) - f(t,\ x_2,\ y_2,\ z_2)| \leq \mu_1|x_1 - x_2| + \mu_2|y_1 - y_2| + \mu_3|z_1 - z_2|,$

$\forall\ (t,\ x_1,\ y_1,\ z_1),\ (t,\ x_2,\ y_2,\ z_2) \in I \times \mathbb{R}^3$. *Then the BVP* (8.8) *has a unique solution* $y \in U$.

Proof. Similar to the proof of the Theorem 8.2.3. □

Following corollary gives the existence result for BVP(8.2).

Corollary 8.2.5. *Let* $\mu_1 > 0,\ \mu_2 > 0,\ \mu_3 > 0$ *and*

$\mu = \max\{\mu_1 + |\gamma_1|,\ \mu_2,\ \mu_3 + |\gamma_2|\}$ *such that* $6\mu T(2T+1) + 9\mu T < 1$.

Let $f : I \times \mathbb{R}^3 \to \mathbb{R}$ *be continuous and satisfy*

$|f(t,\ x_1,\ y_1,\ z_1) - f(t,\ x_2,\ y_2,\ z_2)| \leq \mu_1|x_1 - x_2| + \mu_2|y_1 - y_2| + \mu_3|z_1 - z_2|,$

$\forall\ (t,\ x_1,\ y_1,\ z_1),\ (t,\ x_2,\ y_2,\ z_2) \in I \times \mathbb{R}^3$. *Then the BVP* (8.2) *has unique solution* $y \in E$.

Proof. Let $F^*(t,\ u,\ v,\ w) = f(t,\ u,\ v,\ w) - \gamma_1 u - \gamma_2 w,$

$\forall\ (t,\ u,\ v,\ w) \in I \times \mathbb{R}^3.$

Consider the BVP given by

$$\left.\begin{aligned} &y''(t) = F^*(t,\ y(t),\ y(t-r),\ y'(t)),\ t \in I, \\ &y(t) = a,\ t \in [-r,\ 0], \\ &y(T) + y'(T) = b. \end{aligned}\right\} \tag{8.21}$$

Let $(t,\ u_1,\ v_1,\ w_1),\ (t,\ u_2,\ v_2,\ w_2) \in I \times \mathbb{R}^3$ be arbitrary. Then

$|F^*(t,\ u_1,\ v_1,\ w_1) - F^*(t,\ u_2,\ v_2,\ w_2)|$

$= |f(t,\ u_1,\ v_1,\ w_1) - \gamma_1 u_1 - \gamma_2 w_1 - f(t,\ u_2,\ v_2,\ w_2) + \gamma_1 u_2 + \gamma_2 w_2|$

$\leq |f(t,\ u_1,\ v_1,\ w_1) - f(t,\ u_2,\ v_2,\ w_2)| + |\gamma_1||u_1 - u_2| + |\gamma_2||w_1 - w_2|$

$\leq (\mu_1 + |\gamma_1|)|u_1 - u_2| + \mu_2|v_1 - v_2| + (\mu_3 + |\gamma_2|)|w_1 - w_2|.$

Therefore by applying Theorem 8.2.2, we conclude that the BVP (8.21) has unique solution $y \in E$.

So we have

$$\left.\begin{aligned} &y''(t) = f(t,\ y(t),\ y(t-r),\ y'(t)) - \gamma_1 y(t) - \gamma_2 y'(t),\ t \in I, \\ &y(t) = a,\ t \in [-r,\ 0], \\ &y(T) + y'(T) = b. \end{aligned}\right\}$$

Hence y is unique solution of the BVP (8.2). □

Next we obtain the result for BVP(8.9).

Corollary 8.2.6. *Let* $\mu_1 > 0$, $\mu_2 > 0$, $\mu_3 > 0$ *and* $\mu = \max\{\mu_1 + |\gamma_1|,\ \mu_2,\ \mu_3 + |\gamma_2|\}$ *such that* $6\mu T(2T+1) + 9\mu T < 1$. *Let* $f : I \times \mathbb{R}^3 \to \mathbb{R}$ *be continuous and satisfy*

$|f(t,\ x_1,\ y_1,\ z_1) - f(t,\ x_2,\ y_2,\ z_2)| \leq \mu_1|x_1 - x_2| + \mu_2|y_1 - y_2| + \mu_3|z_1 - z_2|$,

$\forall\ (t,\ x_1,\ y_1,\ z_1),\ (t,\ x_2,\ y_2,\ z_2) \in I \times \mathbb{R}^3$. *Then the BVP* (8.9) *has unique solution* $y \in U$.

Proof. Similar to the proof of the Corollary 8.2.5. □

8.3 Existence Theorems using Schauder's Fixed Point Theorem

In this section, we estabish results on existence of solution to the BVP (8.3) using Schauder's fixed point theorem. We break down our main theorem into two lemmas.

Lemma 8.3.1. *Let $f : I \times \mathbb{R}^3 \to \mathbb{R}$ be continuous. Then $N : Y \to Y$*

$$\text{defined by } N[y(t)] = \begin{cases} 0, & -r \le t \le 0, \\ \frac{bt}{1+T} + \int_0^T G(t,\ s)F(s;\ y)ds, & 0 \le t \le T, \end{cases}$$

is continuous, where $G(t,\ s)$ is given by (8.16) *and*

$F(t;\ y) = f(t,\ y(t),\ y(t-r),\ y'(t)),\ \forall\ t \in I.$

Proof. For $0 \le t \le T$, we may express $N[y(t)]$ as

$$N[y(t)] = \frac{bt}{1+T} + \int_0^T [\frac{st}{1+T} - t]F(s;\ y)ds\ +\ \int_0^t (t-s)F(s;\ y)ds.$$

$$\text{Therefore } N'[y(t)] = \frac{b}{1+T} + \int_0^T [\frac{s}{1+T} - 1]F(s;\ y)ds\ +\ \int_0^t F(s;\ y)ds.$$

Let $\{y_n\}$ be a sequence in $(Y,\ ||.||_\infty)$ such that $y_n \to y$ in $(Y,\ ||.||_\infty)$.

Then $||y_n - y||_\infty \to 0$ as $n \to \infty$. Therefore $y_n \to y$ uniformly on $[-r,\ T]$ and $y_n' \to y'$ uniformly on I.

i.e $\sup\limits_{t \in [-r,\ T]} |y_n(t) - y(t)| \to 0$ as $n \to \infty$ and $\sup\limits_{t \in I} |y_n'(t) - y'(t)| \to 0$ as $n \to \infty$.

Let $v_n = Ny_n$ and $v = Ny$. We prove that $v_n \to v$ in $(Y,\ ||.||_\infty)$. We first show that $v_n \to v$ uniformly on $[-r,\ T]$.

Let $\epsilon > 0$ be arbitrary. Since $y_n \to y$ uniformly on $[-r,\ T]$, $y_n \to y$ uniformly on I. Therefore $y_n(t-r) \to y(t-r)$ uniformly on I.

This implies $f(t,\ y_n(t),\ y_n(t-r),\ y_n'(t)) \to f(t,\ y(t),\ y(t-r),\ y'(t))$

uniformly on I as $f : I \times \mathbb{R}^3 \to \mathbb{R}$ is continuous.

Therefore $\exists\ n_0 \in \mathbb{N}$ such that

$|f(t,\ y_n(t),\ y_n(t-r),\ y_n'(t)) \to f(t,\ y(t),\ y(t-r),\ y'(t))| < \frac{\epsilon}{8T^2}$,

$\forall\ n \ge n_0$ and $\forall\ t \in I$.

i.e $|F(t;\ y_n) - F(t;\ y)| < \frac{\epsilon}{8T^2},\ \forall\ n \geq n_0$ and $\forall\ t \in I$.

Let $-r \leq t \leq 0$. Then

$$|v_n(t) - v(t)| = 0 < \epsilon,\ \forall\ n \geq n_0 \text{ and } \forall\ t \in [-r,\ 0]. \tag{8.22}$$

Let $0 \leq t \leq T$. Then

$$|v_n(t) - v(t)| = |\int_0^T G(t,\ s)F(s;\ y_n)ds - \int_0^T G(t,\ s)F(s;\ y)ds|.$$

So we have

$$\begin{aligned}
|v_n(t) - v(t)| &\leq \int_0^T |G(t,\ s)||F(s;\ y_n) - F(s;\ y)|ds \\
&< \frac{\epsilon}{8T^2}\int_0^T |G(t,\ s)|ds \\
&= \frac{\epsilon}{8T^2}\{\int_0^t |G(t,\ s)|ds + \int_t^T |G(t,\ s)|ds\} \\
&= \frac{\epsilon}{8T^2}\{\int_0^t |\frac{st}{1+T} - s|ds + \int_t^T |\frac{st}{1+T} - t|ds\} \\
&\leq \frac{\epsilon}{8T^2}\{\int_0^T (\frac{st}{T} + s)ds + \int_0^T (\frac{st}{T} + t)ds\} \\
&\leq \frac{\epsilon}{8T^2}\{\int_0^T 2Tds + \int_0^T 2Tds\} \\
&= \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

Therefore $|v_n(t) - v(t)| < \epsilon,\ \forall\ n \geq n_0$ and $\forall\ t \in I$. (8.23)

Thus from (8.22) and (8.23), we conclude that, $|v_n(t) - v(t)| < \epsilon,\ \forall\ n \geq n_0$ and $\forall\ t \in [-r,\ T]$. This implies $v_n \to v$ uniformly on $[-r,\ T]$.

We next show that $v_n' \to v'$ uniformly on I.

Let $\epsilon > 0$ be arbitrary.

Since $F(t;\ y_n) \to F(t;\ y)$ uniformly on I, $\exists\ n_0 \in \mathbb{N}$ such that

$|F(t;\ y_n) - F(t;\ y)| < \frac{\epsilon}{8T},\ \forall\ n \geq n_0$ and $\forall\ t \in I$.

Let $0 \leq t \leq T$. Then

$$|v_n'(t) - v'(t)| \leq \int_0^T |\frac{s}{1+T} - 1||F(s;\ y_n) - F(s;\ y)|ds + \int_0^t |F(s;\ y_n) - F(s;\ y)|ds.$$

So we have

$$\begin{aligned}
|v_n'(t) - v'(t)| &\leq \int_0^T |\frac{s}{1+T} - 1||F(s;\ y_n) - F(s;\ y)|ds \\
&\quad + \int_0^t |F(s;\ y_n) - F(s;\ y)|ds \\
&\leq \int_0^T (\frac{s}{T} + 1)|F(s;\ y_n) - F(s;\ y)|ds + \int_0^t |F(s;\ y_n) - F(s;\ y)|ds \\
&\leq 2\int_0^T |F(s;\ y_n) - F(s;\ y)|ds + \int_0^T |F(s;\ y_n) - F(s;\ y)|ds \\
&< \frac{\epsilon}{4T}\int_0^T ds + \frac{\epsilon}{8T}\int_0^T ds \\
&= \frac{3\epsilon}{8} < \epsilon.
\end{aligned}$$

Therefore $|v_n'(t) - v'(t)| < \epsilon,\ \forall\ n \geq n_0$ and $\forall\ t \in I$.

This implies $v_n' \to v'$ uniformly on I.

So we have $v_n \to v$ uniformly on $[-r,\ T]$ and $v_n' \to v'$ uniformly on I.

Therefore $v_n \to v$ on $(Y,\ ||.||_\infty)$. i.e $Ny_n \to Ny$ on $(Y,\ ||.||_\infty)$.

Hence $N : Y \to Y$ is continuous. $\square$

Lemma 8.3.2. *Let $f : I\times\mathbb{R}^3 \to \mathbb{R}$ be continuous and bounded. Let $N : Y \to Y$ be defined by*

$$N[y(t)] = \begin{cases} 0, & -r \le t \le 0, \\ \frac{bt}{1+T} + \int_0^T G(t,\ s)F(s;\ y)ds, & 0 \le t \le T, \end{cases}$$

where $G(t,\ s)$ is given by (8.16) and $F(t;\ y) = f(t,\ y(t),\ y(t-r),\ y'(t))$, $\forall\ t\ \in\ I$. Then $N(Y)$ is bounded subset of $(Y,\ ||.||_\infty)$ and $N : Y \to Y$ is compact.

Proof. Since $f : I \times \mathbb{R}^3 \to \mathbb{R}$ is bounded, $\exists\ M > 0$ such that

$|f(t,\ u,\ v,\ w)| \le M,\ \forall\ (t,\ u,\ v,\ w) \in I \times \mathbb{R}^3$.

For $0 \le t \le T$, we can express $N[y(t)]$ as

$$N[y(t)] = \frac{bt}{1+T} + \int_0^T [\frac{st}{1+T} - t]F(s;\ y)ds\ +\ \int_0^t (t-s)F(s;\ y)ds.$$

$$\text{Therefore } N'[y(t)] = \frac{b}{1+T} + \int_0^T [\frac{s}{1+T} - 1]F(s;\ y)ds\ +\ \int_0^t F(s;\ y)ds.$$

We first show that $N(Y)$ is a bounded subset of $(Y,\ ||.||_\infty)$.

Let $y \in Y$ be arbitrary.

If $-r \le t \le 0$, then $|N[y(t)]| = 0 \le |b| + 4MT^2$.

Let $0 \le t \le T$. Then

$$\begin{aligned} |N[y(t)]| &\le \frac{|b||t|}{1+T} + \int_0^T |\frac{st}{1+T} - t||F(s;\ y)|ds\ +\ \int_0^t |t-s||F(s;\ y)|ds \\ &\le |b| + \int_0^T (\frac{st}{T} + t)|F(s;\ y)|ds\ +\ \int_0^T (t+s)|F(s;\ y)|ds \\ &\le |b| + 2TM\int_0^T ds\ +\ 2TM\int_0^T ds = |b| + 4MT^2. \end{aligned}$$

Therefore $|N[y(t)]| \leq |b| + 4MT^2,\ \forall\ t \in [-r,\ T]$ and

$\sup\limits_{t\in[-r,\ T]} |N[y(t)]| \leq |b| + 4MT^2.$

Now for $0 \leq t \leq T$, we have

$$|N'[y(t)]| \leq \frac{|b|}{1+T} + \int_0^T |\frac{s}{1+T} - 1||F(s;\ y)|ds\ +\ \int_0^t |F(s;\ y)|ds$$
$$\leq \frac{|b|}{T} + \int_0^T (\frac{s}{T} + 1)|F(s;\ y)|ds\ +\ \int_0^T |F(s;\ y)|ds.$$

Therefore

$$|N'[y(t)]| \leq \frac{|b|}{T} + 2M\int_0^T ds\ +\ M\int_0^T ds$$
$$= \frac{|b|}{T} + 3MT,\ \forall\ t \in I.$$

So we have $\sup\limits_{t\in I} |N'[y(t)]| \leq \frac{|b|}{T} + 3MT$. Thus

$$||Ny||_\infty = \sup_{t\in[-r,\ T]} |N[y(t)]| + \sup_{t\in I} |N'[y(t)]|$$
$$\leq |b| + 4MT^2 + \frac{|b|}{T} + 3MT.$$

Hence $||Ny||_\infty \leq |b| + 4MT^2 + \frac{|b|}{T} + 3MT,\ \forall\ y \in Y.$

This shows that $N(Y)$ is bounded subset of $(Y,\ ||.||_\infty)$.

Next we show that $N : Y \to Y$ is a compact operator.

Let $\{y_n\}$ be a bounded sequence in $(Y,\ ||.||_\infty)$ and let $v_n = Ny_n$.

Since $N(Y)$ is a bounded subset of $(Y,\ ||.||_\infty)$, $\{v_n\}$ is a bounded sequence in $(Y,\ ||.||_\infty)$. Therefore $\{v_n\}$ is uniformly bounded on $[-r,\ T]$ and $\{v_n'\}$ is uniformly bounded on I.

Claim 1:- $\{v_n\}$ is equicontinuous on $[-r,\ T]$.

Let $t_0 \in [-r,\ T]$ and $\epsilon > 0$ be arbitrary.

CASE 1.1:- Let $t_0 = -r$. Choose $\delta = \min\{\frac{r}{2},\ \epsilon\}$.

Then, for $-r \leq t < -r + \delta,\ |v_n(t) - v_n(t_0)| = 0 < \epsilon$.

This implies $\{v_n\}$ is equicontinuous at $t_0 = -r$.

CASE 1.2:- Let $t_0 = 0$. Choose $\delta = \min\{\frac{r}{2},\ \frac{T}{|b| + 4MT^2}\epsilon\}$.

Let t be such that $|t| < \delta$. If $t \leq t_0$ then $|v_n(t) - v_n(t_0)| = 0 < \epsilon$.

Let $t \geq t_0$. Then

$$|v_n(t) - v_n(t_0)|$$
$$\leq \frac{|b||t|}{1+T} + \int_0^T |\frac{st}{1+T} - t||F(s;\ y_n)|ds\ +\ \int_0^t |t-s||F(s;\ y_n)|ds.$$

Therefore

$$|v_n(t) - v_n(t_0)|$$
$$\leq \frac{|b||t|}{T} + |t| \int_0^T (\frac{s}{T} + 1)|F(s;\ y_n)|ds\ +\ \int_0^t (t+s)|F(s;\ y_n)|ds$$
$$\leq \frac{|b||t|}{T} + 2M|t| \int_0^T ds\ +\ 2TM \int_0^t ds$$
$$= \frac{|b||t|}{T} + 4MT|t|$$
$$< (\frac{|b| + 4MT^2}{T})\delta$$
$$\leq (\frac{|b| + 4MT^2}{T})(\frac{T}{|b| + 4MT^2})\epsilon = \epsilon.$$

Therefore $|v_n(t) - v_n(t_0)| < \epsilon,\ \forall\ t$ such that $|t| < \delta$.

This implies $\{v_n\}$ is equicontinuous at $t_0 = 0$.

CASE 1.3:- Let $t_0 = T$. Choose $\delta = \min\{\frac{T}{2},\ \frac{T}{|b| + 5MT^2}\epsilon\}$.

Let t be such that $T - \delta < t \leq T$.

Then

$$
\begin{aligned}
|v_n(t) - v_n(t_0)| &= |v_n(T) - v_n(t)| \\
&= |\frac{b(T-t)}{1+T} + \int_0^T (T-t)[\frac{s}{1+T} - 1]F(s;\ y_n)ds \\
&\quad + \int_0^T (T-s)F(s;\ y_n)ds\ -\ \int_0^t (t-s)F(s;\ y_n)ds| \\
&\le \frac{|b|(T-t)}{T} + (T-t)\int_0^T (\frac{s}{T}+1)|F(s;\ y_n)|ds \\
&\quad + |\int_0^T (T-s)F(s;\ y_n)ds\ +\ \int_t^0 (t-s)F(s;\ y_n)ds| \\
&\le \frac{|b|(T-t)}{T} + 2MT(T-t) \\
&\quad + |\int_0^T TF(s;\ y_n)ds\ +\ \int_t^0 tF(s;\ y_n)ds\ -\ \int_t^T sF(s;\ y_n)ds| \\
&\le \frac{|b|(T-t)}{T} + 2MT(T-t) \\
&\quad + |\int_0^T TF(s;\ y_n)ds\ +\ \int_t^0 tF(s;\ y_n)ds|\ +\ \int_t^T s|F(s;\ y_n)|ds \\
&\le \frac{|b|(T-t)}{T} + 3MT(T-t)\ +\ |\int_0^T TF(s;\ y_n)ds \\
&\quad + \int_t^0 tF(s;\ y_n)ds| \\
&= \frac{|b|(T-t)}{T} + 3MT(T-t) \\
&\quad + |\int_0^T TF(s;\ y_n)ds\ -\ \int_0^T tF(s;\ y_n)ds \\
&\quad + \int_0^T tF(s;\ y_n)ds\ +\ \int_t^0 tF(s;\ y_n)ds|.
\end{aligned}
$$

Therefore

$$|v_n(t) - v_n(t_0)| = \frac{|b|(T-t)}{T} + 3MT(T-t) \; + \; |\int_0^T (T-t)F(s;\; y_n)ds$$
$$+ \int_t^T tF(s;\; y_n)ds|$$
$$\leq \frac{|b|(T-t)}{T} + 3MT(T-t) \; + \; M(T-t)\int_0^T ds \; + \; MT\int_t^T ds$$
$$\leq \frac{|b|(T-t)}{T} + 5MT(T-t)$$
$$< (\frac{|b| + 5MT^2}{T})\delta$$
$$\leq (\frac{|b| + 5MT^2}{T})(\frac{T}{|b| + 5MT^2})\epsilon = \epsilon.$$

Thus $|v_n(t) - v_n(t_0)| < \epsilon, \; \forall \; t$ such that $T - \delta < t \leq T$.

This implies $\{v_n\}$ is equicontinuous at $t_0 = T$.

CASE 1.4:- Let $-r < t_0 < 0$. Choose $\delta = \min\{t_0 + \frac{r}{2}, \; -\frac{t_0}{2}\}$.

Then $|v_n(t) - v_n(t_0)| = 0 < \epsilon, \; \forall \; t$ such that $|t - t_0| < \epsilon$. This implies $\{v_n\}$ is equicontinuous at t_0, where $-r < t_0 < 0$.

CASE 1.5:- Let $0 < t_0 < T$. Choose $\delta = \min\{\frac{t_0}{2}, \; \frac{T-t_0}{2}, \; \frac{T}{|b| + 5MT^2}\epsilon\}$.

Let t be such that $|t - t_0| < \delta$. Then

$$|v_n(t) - v_n(t_0)| = |\frac{b(t-t_0)}{1+T} + \int_0^T (t-t_0)[\frac{s}{1+T} - 1]F(s;\; y_n)ds$$
$$+ \int_0^t (t-s)F(s;\; y_n)ds \; - \; \int_0^{t_0} (t_0 - s)F(s;\; y_n)ds$$
$$\leq \frac{|b||t-t_0|}{T} + |t-t_0|\int_0^T (\frac{s}{T} + 1)|F(s;\; y_n)|ds$$
$$+ \; |\int_0^t (t-s)F(s;\; y_n)ds \; + \; \int_{t_0}^0 (t_0 - s)F(s;\; y_n)ds|.$$

So we have

$$
\begin{aligned}
|v_n(t) - v_n(t_0)| &\leq \frac{|b||t-t_0|}{T} + 2MT|t-t_0| \\
&\quad + \Big|\int_0^t tF(s;\, y_n)ds \;+\; \int_{t_0}^0 t_0F(s;\, y_n)ds\Big| \;+\; \Big|\int_{t_0}^t s|F(s;\, y_n)|ds\Big| \\
&\leq \frac{|b||t-t_0|}{T} + 3MT|t-t_0| \\
&\quad + \Big|\int_0^t tF(s;\, y_n)ds \;-\; \int_0^t t_0F(s;\, y_n)ds \\
&\quad + \int_0^t t_0F(s;\, y_n)ds \;+ \int_{t_0}^0 t_0F(s;\, y_n)ds\Big| \\
&= \frac{|b||t-t_0|}{T} + 3MT|t-t_0| \;+\; \Big|\int_0^t (t-t_0)F(s;\, y_n)ds \\
&\quad + \int_{t_0}^t t_0F(s;\, y_n)ds\Big| \\
&\leq \frac{|b||t-t_0|}{T} + 3MT|t-t_0| \;+\; |t-t_0|\int_0^T |F(s;\, y_n)|ds \\
&\quad + \Big|\int_{t_0}^t t_0|F(s;\, y_n)|ds\Big| \\
&\leq \frac{|b||t-t_0|}{T} + 3MT|t-t_0| \;+\; M|t-t_0|\int_0^T ds \;+\; MT\Big|\int_{t_0}^t ds\Big| \\
&\leq \frac{|b||t-t_0|}{T} + 5MT|t-t_0| \\
&< (\frac{|b|+5MT^2}{T})\delta \\
&\leq (\frac{|b|+5MT^2}{T})(\frac{T}{|b|+5MT^2})\epsilon = \epsilon.
\end{aligned}
$$

Therefore $|v_n(t) - v_n(t_0)| < \epsilon,\ \forall\, t$ such that $|t - t_0| < \delta$.

This implies $\{v_n\}$ is equicontinuous at t_0, where $0 < t_0 < T$.

Hence, from CASE 1.1 - CASE 1.5, we may conclude that $\{v_n\}$ is equicontinuous on $[-r,\ T]$ uniformly.

Now using Arzela-Ascoli's lemma, $\{v_n\}$ has a uniformly convergent subsequence $\{v_{(n,\,m)}\}$ converging uniformly to v on $[-r,\ T]$.

i.e $v_{(n,\,m)} \to v$ uniformly $[-r,\ T]$ as $m \to \infty$.

Since each $v_{(n,\,m)}$ is continuous on $[-r,\ T]$, v is continuous on $[-r,\ T]$.

As $\{v_n\}$ and $\{v_n'\}$ are uniformly bounded on $[-r,\ T]$ and I respectively, $\{v_{(n,\,m)}\}$ and $\{v'_{(n,\,m)}\}$ are uniformly bounded on $[-r,\ T]$ and I respectively.

Let $z_m = v'_{(n,\,m)}$.

Claim 2:- $\{z_m\}$ is equicontinuous on I uniformly.

Let $t_0 \in I$ be arbitrary and $\epsilon > 0$ be arbitrary.

CASE 2.1:- Let $t_0 = 0$. Choose $\delta = \min\{\frac{T}{2},\ \frac{\epsilon}{M}\}$.

Let t be such that $0 \le t < \delta$. Then

$$\begin{aligned} |z_m(t) - z_m(t_0)| &= |\int_0^t F(s;\ y_{(n,\,m)})ds| \\ &\le \int_0^t |F(s;\ y_{(n,\,m)})|ds \\ &\le M\int_0^t ds = tM < M\delta \le \epsilon. \end{aligned}$$

Thus $|z_m(t) - z_m(t_0)| < \epsilon,\ \forall\ t$ such that $0 \le t < \delta$. Therefore $\{z_m\}$ is equicontinuous at $t_0 = 0$.

CASE 2.2:- Let $t_0 = T$. Choose $\delta = \min\{\frac{T}{2},\ \frac{\epsilon}{M}\}$.

Let t be such that $T - \delta < t \le T$. Then

$$\begin{aligned} |z_m(t) - z_m(t_0)| &= |z_m(T) - z_m(t)| \\ &= |\int_0^T F(s;\ y_{(n,\,m)})ds\ -\ \int_0^t F(s;\ y_{(n,\,m)})ds| \\ &\le \int_t^T |F(s;\ y_{(n,\,m)})|ds \le M\int_t^T ds = (T-t)M < M\delta \le \epsilon. \end{aligned}$$

Thus $|z_m(t) - z_m(t_0)| < \epsilon,\ \forall\ t$ such that $T - \delta < t \leq T$.

Therefore $\{z_m\}$ is equicontinuous at $t_0 = T$.

CASE 2.3:- Let $0 < t_0 < T$. Choose $\delta = \min\{\frac{t_0}{2},\ \frac{T-t_0}{2},\ \frac{\epsilon}{M}\}$.

Let t be such that $|t - t_0| < \delta$. Then

$$\begin{aligned} |z_m(t) - z_m(t_0)| &= |\int_0^t F(s;\ y_{(n,\,m)})ds\ -\ \int_0^{t_0} F(s;\ y_{(n,\,m)})ds| \\ &\leq \int_{t_0}^t |F(s;\ y_{(n,\,m)})|ds \\ &\leq M\int_{t_0}^t ds = M|t - t_0| < M\delta \leq \epsilon. \end{aligned}$$

Thus $|z_m(t) - z_m(t_0)| < \epsilon,\ \forall\ t$ such that $|t - t_0| < \delta$.

This implies $\{z_m\}$ is equicontinuous at t_0, where $0 < t_0 < T$.

Hence, from CASE 2.1 - CASE 2.3, we conclude that $\{z_m\}$ is equicontinuous on I uniformly.

i.e $\{v'_{(n,\,m)}\}$ is equicontinuous on I uniformly.

Again making use of Arzela-Ascoli's lemma, we may find a subsequence $\{v'_{(n,\,m,\,p)}\}$ of $\{v'_{(n,\,m)}\}$ converging uniformly to u on I.

i.e $v'_{(n,\,m,\,p)} \to u$ uniformly on I as $p \to \infty$.

Since $v_{(n,\,m)} \to v$ uniformly on $[-r,\ T]$ as $m \to \infty$, $v_{(n,\,m)} \to v$ uniformly on I as $m \to \infty$. As $\{v'_{(n,\,m,\,p)}\}$ is a subsequence of $\{v_{(n,\,m)}\}$, we may conclude that $v_{(n,\,m,\,p)} \to v$ uniformly on I as $p \to \infty$. Therefore $v_{(n,\,m,\,p)} \to v$ uniformly on I as $p \to \infty$ and $v'_{(n,m,p)} \to u$ uniformly on I as $p \to \infty$ implies that v is differentiable on I and $v' = u$ on I. Thus $v_{(n,\,m,\,p)} \to v$ uniformly on $[-r,\ T]$ as $p \to \infty$ and $v'_{(n,\,m,\,p)} \to v'$ uniformly on I as $p \to \infty$.

i.e $||v_{(n,\,m,\,p)} - v||_\infty \to 0$ as $p \to \infty$.

Also $\{v_{(n,\,m,\,p)}\} \subseteq Y$ and $(Y,\ ||.||_\infty)$ is a Banach space implies that $v \in Y$.

Therefore $v_{(n,\,m,\,p)} \to v$ in $(Y,\ ||.||_\infty)$. Thus $\{Ny_n\}$ has a subsequence $\{Ny_{(n,\,m,\,p)}\}$

converging in $(Y,\ ||.||_\infty)$. Hence $N : Y \to Y$ is a compact operator. □

We now combine Lemma 8.3.1 and Lemma 8.3.2 to get the following theorem on existence of solution.

Theorem 8.3.3. *Let $f : I \times \mathbb{R}^3 \to \mathbb{R}$ be continuous and bounded. Then the BVP* (8.3) *has atleast one solution.*

Proof. Let $N : Y \to Y$ be defined by

$$N[y(t)] = \begin{cases} 0, & -r \le t \le 0, \\ \frac{bt}{1+T} + \int_0^T G(t,\ s)F(s;\ y)ds, & 0 \le t \le T, \end{cases}$$

where $G(t,\ s)$ is given by (8.16) and
$F(t;\ y) = f(t,\ y(t),\ y(t-r),\ y'(t)),\ \forall\ t \in I$. Since $f : I \times \mathbb{R}^3 \to \mathbb{R}$ is continuous, $N : Y \to Y$ is continuous. (By Lemma 8.3.1)
As $f : I \times \mathbb{R}^3 \to \mathbb{R}$ is continuous and bounded, $N(Y)$ is bounded subset of $(Y,\ ||.||_\infty)$ and $N : Y \to Y$ is compact. (By Lemma 8.3.2)
Therefore by applying Schauder's fixed point theorem, $N : Y \to Y$ has a fixed point $y \in Y$.
Thus y satisfies $y(t) = 0,\ t \in [-r,\ 0]$, $y(T) + y'(T) = b$ and
$y(t) = \frac{bt}{1+T} + \int_0^T [\frac{st}{1+T} - t]F(s;\ y)ds\ +\ \int_0^t (t-s)F(s;\ y)ds,\ t \in I.$
Therefore $y'(t) = \frac{b}{1+T} + \int_0^T [\frac{s}{1+T} - 1]F(s;\ y)ds\ +\ \int_0^t F(s;\ y)ds,\ t \in I.$
Since $f : I \times \mathbb{R}^3 \to \mathbb{R}$ is continuous, y' is differentiable on I and
$y''(t) = f(t,\ y(t),\ y(t-r),\ x'(t)),\ t \in I$. Thus $y \in C[[-r,\ T],\ \mathbb{R}]$ and $y \in C^2[I,\ \mathbb{R}]$. Hence y is a solution of the BVP (8.3). □

Next, we estabish existence of solution to the BVP (8.10) using Schauder's fixed point theorem. Again we may break down our main theorem into two lemmas.

Lemma 8.3.4. *Let $f : I \times \mathbb{R}^3 \to \mathbb{R}$ be continuous. Then $N : V \to V$ defined by $N[y(t)] = \frac{bt}{1+T} + \int_0^T G(t,\ s)F(s;\ y)ds,\ t \in I$ is continuous, where $G(t,\ s)$ is given by* (8.16) *and $F(t;\ y) = f(t,\ y(t),\ y(\theta(t)),\ y'(t)),\ \forall\ t \in I$.*

Proof. Similar to the proof of the Lemma 8.3.1. □

Lemma 8.3.5. *Let $f : I \times \mathbb{R}^3 \to \mathbb{R}$ be continuous. Let $N : V \to V$ defined by $N[y(t)] = \frac{bt}{1+T} + \int_0^T G(t,\ s)F(s;\ y)ds,\ t \in I$, where $G(t,\ s)$ is given by* (8.16) *and $F(t;\ y) = f(t,\ y(t),\ y(\theta(t)),\ y'(t)),\ \forall\ t \in I$. Then $N(V)$ is a bounded subset of $(V,\ ||.||_0)$ and $N : V \to V$ is compact.*

Proof. Similar to the proof of the Lemma 8.3.2. □

Theorem 8.3.6. *Let $f : I \times \mathbb{R}^3 \to \mathbb{R}$ be continuous and bounded. Then the BVP* (8.10) *has atleast one solution.*

Proof. Similar to the proof of the Theorem 8.3.3. □

We shall illustrate Theorem 8.3.3 with the following example.

Example 8.3.1. Consider the following BVP

$$\left.\begin{aligned} y''(t) &= \cos[y(t - \frac{\pi}{2})],\ t \in I, \\ y(t) &= y(0),\ t \in [-\frac{\pi}{2},\ 0], \\ y(\frac{\pi}{2}) &+ y'(\frac{\pi}{2}) = 1. \end{aligned}\right\}$$

Here $r = T = \frac{\pi}{2}$, $I = [0,\ \frac{\pi}{2}]$, $f(t,\ u,\ v,\ w) = \cos v$,

$\forall\ (t,\ u,\ v,\ w) \in I \times \mathbb{R}^3$ and $b = 1$.

Clearly $f : I \times \mathbb{R}^3 \to \mathbb{R}$ is bounded and continuous.

Let $0 \leq t \leq \frac{\pi}{2}$. Then $-\frac{\pi}{2} \leq t - \frac{\pi}{2} \leq 0$.

Therefore $y(t - \frac{\pi}{2}) = 0,\ \forall\ t \in [0,\ \frac{\pi}{2}]$.

This implies $\cos[y(t - \frac{\pi}{2})] = 1,\ \forall\ t \in [0,\ \frac{\pi}{2}]$. So $y''(t) = 1,\ \forall\ t \in [0,\ \frac{\pi}{2}]$.

Thus $y(t) = y'(0)t + \frac{t^2}{2},\ t \in [0,\ \frac{\pi}{2}]$ and $y'(t) = y'(0) + t,\ t \in [0,\ \frac{\pi}{2}]$.

Substituting $t = \frac{\pi}{2}$ in above expressions of $y(t)$ and $y'(t)$ we get
$y'(0) = \frac{8-4\pi-\pi^2}{8+4\pi}$.
Hence $y(t) = (\frac{8-4\pi-\pi^2}{8+4\pi})t + \frac{t^2}{2},\ t \in [0, \frac{\pi}{2}]$ is a solution to the given BVP.

8.4 Method of Lower and Upper Solutions

In this section, we prove existence of solution between lower and upper solutions to BVPs of second order DECD and second order DEVDA. We first give definitions of lower and upper solutions to BVPs of second order DECD and second order DEVDA.

Definition 8.4.1. *A function $\alpha : [-r,\ T] \to \mathbb{R}$ is said to be a lower solution of BVP* (8.4) *if 1.* $\alpha \in C[[-r,\ T],\ \mathbb{R}]$ *and* $\alpha \in C^2[I,\ \mathbb{R}]$.

2. $-\alpha''(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I$.

3. $\alpha(t) = \alpha(0),\ t \in [-r,\ 0]$.

4. $\alpha'(0) \geq \alpha'(T)$.

5. $\alpha(0) = \alpha(T)$.

Similarly we may give definitions of a lower solution for BVPs of second order DECD (8.5), (8.6) *and* (8.7).

Definition 8.4.2. *A function $\beta : [-r,\ T] \to \mathbb{R}$ is said to be an upper solution of BVP* (8.4) *if 1.* $\beta \in C[[-r,\ T],\ \mathbb{R}]$ *and* $\beta \in C^2[I,\ \mathbb{R}]$.

2. $-\beta''(t) + \lambda\beta(t) \geq f(t,\ \beta(t),\ \beta(t-r)),\ t \in I$.

3. $\beta(t) = \beta(0),\ t \in [-r,\ 0]$.

4. $\beta'(0) \leq \beta'(T)$

5. $\beta(0) = \beta(T)$.

Similarly we may give definitions of an upper solution for BVPs of second order DECD (8.5), (8.6) *and* (8.7).

Definition 8.4.3. *A function $\alpha : I \to \mathbb{R}$ is said to be a lower solution of BVP* (8.11) *if 1.* $\alpha \in C^2[I,\ \mathbb{R}]$.

2. $-\alpha''(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t),\ \alpha(\theta(t))),\ t \in I$.

3. $\alpha'(0) \geq \alpha'(T)$.

4. $\alpha(0) = \alpha(T)$.

Similarly we may give definitions of a lower solution for BVPs of second order DEVDA (8.12), (8.13) *and* (8.14).

Definition 8.4.4. *A function $\beta : I \to \mathbb{R}$ is said to be an upper solution of BVP* (8.11) *if 1.* $\beta \in C^2[I,\ \mathbb{R}]$.

2. $-\beta''(t) + \lambda\beta(t) \geq f(t,\ \beta(t),\ \beta(\theta(t))),\ t \in I$.

3. $\beta'(0) \leq \beta'(T)$

4. $\beta(0) = \beta(T)$.

Similarly we may give definitions of an upper solution for BVPs of second order DEVDA (8.12), (8.13) *and* (8.14).

We now establish the results on existence of solution between lower and upper solutions for BVP of second order DECD (8.4) and (8.5).

Theorem 8.4.1. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let α, β be respectively lower and upper solutions of BVP* (8.4) *such that $\alpha(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$. Then the BVP* (8.4) *has atleast one solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$.*

Proof. Since α, β are respectively lower and upper solutions of BVP (8.4), we have 1. $-\alpha''(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I$,

2. $-\beta''(t) + \lambda\beta(t) \geq f(t,\ \beta(t),\ \beta(t-r)),\ t \in I$,

3. $\alpha(t) = \alpha(0),\ t \in [-r,\ 0]$,

4. $\beta(t) = \beta(0),\ t \in [-r,\ 0]$,

5. $\alpha'(0) \geq \alpha'(T)$,

6. $\beta'(0) \leq \beta'(T)$,

7. $\alpha(0) = \alpha(T)$,

8. $\beta(0) = \beta(T)$.

Also $\alpha,\ \beta \in X$.

Let $p : I \times \mathbb{R} \to \mathbb{R}$ be defined by

$p(t,\ x) = \max\{\alpha(t),\ \min\{x,\ \beta(t)\}\},\ \forall\ (t,\ x) \in I \times \mathbb{R}$.

Therefore $\alpha(t) \leq p(t,\ x) \leq \beta(t),\ \forall\ (t,\ x) \in I \times \mathbb{R}$.

Further $p : I \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded.

Let $g^* : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$g^*(x,\ y,\ z) = p(0,\ x + y - z),\ \forall\ (x,\ y,\ z) \in \mathbb{R}^3$. Then $g^* : \mathbb{R}^3 \to \mathbb{R}$ is continuous and bounded.

Let $h^* : \mathbb{R} \to \mathbb{R}$ be defined by

$h^*(x) = p(0,\ x),\ \forall\ x \in \mathbb{R}$. Then $h^* : \mathbb{R}^3 \to \mathbb{R}$ is continuous and bounded.

Let $L : X \to Y \times \mathbb{R}^2$ be an operator defined by

$L[y(t)] = (u(t),\ y(0),\ y(T))$, where

$$u(t) = \begin{cases} 0; & -r \leq t \leq 0, \\ -y'(t) + y'(0) + \lambda \int_0^t y(s)ds; & 0 \leq t \leq T. \end{cases}$$

Then, by Lemma 8.1.1, we have

1. $L : X \to Y \times \mathbb{R}^2$ is a continuous linear operator.

2. $L^{-1} : Y \times \mathbb{R}^2 \to X$ exists and it is continuous linear operator.

Let $F^\star : I \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F^\star(t,\ x,\ y) = \begin{cases} f(t,\ \alpha(t),\ \alpha(t-r)), & x \le \alpha(t),\ y \le \alpha(t-r), \\ f(t,\ x,\ \alpha(t-r)), & \alpha(t) \le x \le \beta(t),\ y \le \alpha(t-r), \\ f(t,\ \beta(t),\ \alpha(t-r)), & \beta(t) \le x,\ y \le \alpha(t-r), \\ f(t,\ \alpha(t),\ y), & x \le \alpha(t),\ \alpha(t-r) \le y \le \beta(t-r), \\ f(t,\ x,\ y), & \alpha(t) \le x \le \beta(t),\ \alpha(t-r) \le y \le \beta(t-r), \\ f(t,\ \beta(t),\ y), & \beta(t) \le x,\ \alpha(t-r) \le y \le \beta(t-r), \\ f(t,\ \alpha(t),\ \beta(t-r)), & x \le \alpha(t),\ \beta(t-r) \le y, \\ f(t,\ x,\ \beta(t-r)), & \alpha(t) \le x \le \beta(t),\ \beta(t-r) \le y, \\ f(t,\ \beta(t),\ \beta(t-r)), & \beta(t) \le x,\ \beta(t-r) \le y. \end{cases}$$

Then $F^\star : I \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and bounded.

Let $Q : X \to Y \times \mathbb{R}^2$ be defined by

$Q[y(t)] = (v(t),\ g^*(y(0),\ y'(0),\ y'(T)),\ h^*(y(0)))$, where

$$v(t) = \begin{cases} 0; & -r \le t \le 0 \\ \int_0^t F^*(s,\ y(s),\ y(s-r))ds; & 0 \le t \le T. \end{cases}$$

By Lemma 8.1.2, $Q : X \to Y \times \mathbb{R}^2$ is continuous and compact.

Let $S = L^{-1}Q : X \to X$. Then $S : X \to X$ is continuous and compact.

Since F^*, g^* and h^* are bounded, $Z(X)$ is a bounded subset of $(X,\ ||.||_\infty)$.

Therefore by Schauder's fixed point theorem, S has a fixed point $y \in X$.

This implies $y(t) = y(0),\ \forall\ t \in [-r,\ 0]$.

Also, for $0 \le t \le T$ we have $y(t) = S[y(t)]$.

This implies $L[y(t)] = Q[y(t)]$.

i.e $[-y'(t) + y'(0) + \lambda \int_0^t y(s)ds,\ y(0),\ y(T)]$

$= [\int_0^t F^*(s,\ y(s),\ y(s-r))ds,\ g^*(y(0),\ y'(0),\ y'(T)),\ h^*(y(0))]$.

Therefore

$$\left.\begin{aligned} -y'(t) + y'(0) + \lambda \int_0^t y(s)ds &= \int_0^t F^*(s,\ y(s),\ y(s-r))ds,\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= g^*(y(0),\ y'(0),\ y'(T)), \\ y(T) &= h^*(y(0)). \end{aligned}\right\}$$

Since F^* is continuous on $I \times \mathbb{R}^2$, $\int_0^t F^*(s,\ y(s),\ y(s-r))ds$ is differentiable on I. Therefore $y \in C^2[I,\ \mathbb{R}]$.

Thus

$$\left.\begin{aligned} -y''(t) + \lambda y(t) &= F^*(t,\ y(t),\ y(t-r)),\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= g^*(y(0),\ y'(0),\ y'(T)), \\ y(T) &= h^*(y(0)). \end{aligned}\right\} \tag{8.24}$$

Therefore $y \in X$ is a fixed point of S iff y is a solution of the BVP (8.24).

We now show that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$.

Since $\alpha(T) = \alpha(0) \leq p(0, y(0)) \leq \beta(0) = \beta(T)$, we have

$\alpha(T) \leq h^*(y(0)) \leq \beta(T)$. This implies $\alpha(T) \leq y(T) \leq \beta(T)$.

Also as $\alpha(0) \leq p(0, y(0) + y'(0) - y'(T)) \leq \beta(0)$, we have

$\alpha(0) \leq g^*(y(0), y'(0), y'(T)) \leq \beta(0)$. Therefore $\alpha(0) \leq y(0) \leq \beta(0)$.

Thus $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ 0]$.

Therefore it is enough to show that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in (0,\ T)$.

Let if possible $y - \beta$ has a positive maximum at $t_1\ \in (0,\ T)$.

Then $y(t_1) > \beta(t_1)$, $y'(t_1) = \beta'(t_1)$ and $y''(t_1) < \beta''(t_1)$.

Therefore $-y''(t_1) + \lambda y(t_1) > -\beta''(t_1) + \lambda\beta(t_1)$.

So we have $f(t_1,\ \beta(t_1),\ \beta(t_1 - r)) < F^*(t_1,\ y(t_1),\ y(t_1 - r))$. (8.25)

CASE-1: Let $y(t_1 - r) \leq \alpha(t_1 - r)$. Then

$$\begin{aligned} f(t_1,\ \beta(t_1),\ \beta(t_1 - r)) &< F^*(t_1,\ y(t_1),\ y(t_1 - r)) \\ &= f(t_1,\ \beta(t_1),\ \alpha(t_1 - r)) \\ &\leq f(t_1,\ \beta(t_1),\ \beta(t_1 - r)), \text{ which is a contradiction.} \end{aligned}$$

CASE-2: Let $\alpha(t_1 - r) \leq y(t_1 - r) \leq \beta(t_1 - r)$. Then

$$\begin{aligned} f(t_1,\ \beta(t_1),\ \beta(t_1 - r)) &< F^*(t_1,\ y(t_1),\ y(t_1 - r)) \\ &= f(t_1,\ \beta(t_1),\ y(t_1 - r)) \\ &\leq f(t_1,\ \beta(t_1),\ \beta(t_1 - r)), \text{ which is a contradiction.} \end{aligned}$$

CASE-3: Let $y(t_1 - r) \geq \beta(t_1 - r)$. Then

$$\begin{aligned} f(t_1,\ \beta(t_1),\ \beta(t_1 - r)) &< F^*(t_1,\ y(t_1),\ y(t_1 - r)) \\ &= f(t_1,\ \beta(t_1),\ \beta(t_1 - r)), \text{ which is a contradiction.} \end{aligned}$$

So we conclude that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$.

This implies $\alpha(t - r) \leq y(t - r) \leq \beta(t - r),\ \forall\ t \in I$.

Thus $F^\star(t,\ y(t),\ y(t - r)) = f(t,\ y(t),\ y(t - r)),\ \forall\ t \in I$.

Also $y(T) = h^*(y(0)) = p(0, y(0)) = y(0)$.

Next we show that $p(0,\ y(0) + y'(0) - y'(T)) = y(0) + y'(0) - y'(T)$.

It is enough to show that $\alpha(0) \leq y(0) + y'(0) - y'(T) \leq \beta(0)$.

Let if possible $\alpha(0) > y(0) + y'(0) - y'(T)$.

Then

$$\begin{aligned} y(0) &= p(0,\ y(0) + y'(0) - y'(T)) \\ &= \max\{\alpha(0),\ \min\{y(0) + y'(0) - y'(T),\ \beta(0)\}\} \\ &= \max\{\alpha(0),\ y(0) + y'(0) - y'(T)\} \\ &= \alpha(0). \end{aligned}$$

Therefore $y(T) = y(0) = \alpha(0) = \alpha(T)$.

Since $\alpha(t) \leq y(t),\ \forall\ t \in I$, we have $\alpha'(0) \leq y'(0)$ and $y'(T) \leq \alpha'(T)$.

$$\begin{aligned} \text{Therefore } y(0) + y'(0) - y'(T) &= \alpha(0) + y'(0) - y'(T) \\ &\geq \alpha(0) + \alpha'(0) - \alpha'(T) \\ &\geq \alpha(0),\ \text{which is contradiction.} \end{aligned}$$

Thus $\alpha(0) \leq y(0) + y'(0) - y'(T)$.

Similarly it can be proved that $y(0) + y'(0) - y'(T) \leq \beta(0)$.

This implies $p(0,\ y(0) + y'(0) + y'(T)) = y(0) + y'(0) - y'(T)$.

Hence y is a solution of (8.4) such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$. $\square$

Corollary 8.4.2. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta$ be respectively lower and upper solutions of BVP (8.5) such that $\alpha(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$. Then the BVP (8.5) has atleast one solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are respectively lower and upper solutions of BVP (8.5), we have 1. $-\alpha''(t) \leq f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I,$

2. $-\beta''(t) \geq f(t,\ \beta(t),\ \beta(t-r)),\ t \in I.$
3. $\alpha(t) = \alpha(0),\ t \in [-r,\ 0].$
4. $\beta(t) = \beta(0),\ t \in [-r,\ 0].$
5. $\alpha'(0) \geq \alpha'(T).$
6. $\beta'(0) \leq \beta'(T).$
7. $\alpha(0) = \alpha(T).$
8. $\beta(0) = \beta(T).$

For $\lambda > 0$, define $F(t,\ u,\ v) = f(t,\ u,\ v) + \lambda u,\ \forall\ (t,\ u,\ v) \in I \times \mathbb{R}^2$.

Consider the BVP

$$\left.\begin{aligned} -y''(t) + \lambda y(t) &= F(t,\ y(t),\ y(t-r)),\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T), \\ y'(0) &= y'(T). \end{aligned}\right\} \tag{8.26}$$

Since $-\alpha''(t) \leq f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I$, we have

$-\alpha''(t) + \lambda\alpha(t) \leq F(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I.$

Also $-\beta''(t) \geq f(t,\ \beta(t),\ \beta(t-r)),\ t \in I$ implies

$-\beta''(t) + \lambda\beta(t) \geq F(t,\ \beta(t),\ \beta(t-r)),\ t \in I.$

Therefore α and β are respectively lower and upper solutions of the BVP (8.26) such that $\alpha(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$.

Let $(t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$. Then

$$\begin{aligned} F(t,\ x_1,\ y_1) &= f(t,\ x_1,\ y_1) + \lambda x_1 \\ &\leq f(t,\ x_2,\ y_2) + \lambda x_1 \\ &\leq f(t,\ x_2,\ y_2) + \lambda x_2 \\ &= F(t,\ x_2,\ y_2). \end{aligned}$$

So we have $F(t,\ x_1,\ y_1) \leq F(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

Thus by using Theorem 8.4.1 we conclude that the BVP (8.26) has atleast one solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$. Hence y is a solution of the BVP (8.5) such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$. □

Similarly, we can prove results on existence of solution between lower and upper solutions for BVP of second order DEVDA (8.11) and (8.12).

Theorem 8.4.3. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta$ be respectively lower and upper solutions of BVP (8.11) such that $\alpha(t) \leq \beta(t),\ \forall\ t \in I$. Then the BVP (8.11) has atleast one solution $y \in W$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in I$.*

Proof. Similar to the proof of the Theorem 8.4.1. □

Corollary 8.4.4. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta$ be respectively lower and upper solutions of BVP (8.12) such that $\alpha(t) \leq \beta(t),\ \forall\ t \in I$. Then the BVP (8.12) has atleast one solution $y \in W$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in I$.*

Proof. Similar to the proof of the Corollary 8.4.2. □

Finally we prove results on existence of solution between lower and upper solutions for BVP of second order DECD (8.6) and (8.7).

Theorem 8.4.5. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta$ respectively be lower and upper solutions of BVP (8.6) such that $\alpha(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$. Then the BVP (8.6) has atleast one solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$.*

Proof. Since $\alpha,\ \beta \in X$ are respectively lower and upper solutions of BVP (8.6), we have 1. $-\alpha''(t) + \lambda\alpha(t) \leq f(t,\ \alpha(t),\ \alpha(t-r)),\ t \in I,$

2. $-\beta''(t) + \lambda\beta(t) \geq f(t,\ \beta(t),\ \beta(t-r)),\ t \in I,$
3. $\alpha(t) = \alpha(0),\ t \in [-r,\ 0],$
4. $\beta(t) = \beta(0),\ t \in [-r,\ 0],$
5. $\alpha'(0) \geq 0,$
6. $\beta'(0) \leq 0,$
7. $\alpha(0) = \alpha(T),$
8. $\beta(0) = \beta(T).$

Also $\alpha,\ \beta \in X$.

Let $p : I \times \mathbb{R} \to \mathbb{R}$, $F^\star : I \times \mathbb{R}^2 \to \mathbb{R}$, $h^* : \mathbb{R} \to \mathbb{R}$ and $L : X \to Y \times \mathbb{R}^2$ be defined as in the proof of the Theorem 8.4.1.

Let $g^* : \mathbb{R}^2 \to \mathbb{R}$ be defined by $g^*(x,\ y) = p(0,\ x+y),\ \forall\ (x,\ y) \in \mathbb{R}^2$.

Then $g^* : \mathbb{R}^2 \to \mathbb{R}$ is continuous and bounded.

Let $Q : X \to Y \times \mathbb{R}^2$ be defined by $Q[y(t)] = (v(t),\ g^*(y(0),\ y'(0)),\ h^*(y(0)))$, where

$$v(t) = \begin{cases} 0; & -r \leq t \leq 0 \\ \int_0^t F^*(s,\ y(s),\ y(s-r))ds; & 0 \leq t \leq T. \end{cases}$$

By Lemma 8.1.3, $Q : X \to Y \times \mathbb{R}^2$ is continuous and compact.

Let $S = L^{-1}Q : X \to X$. Then $S : X \to X$ is continuous and compact.

Since F^*, g^* and h^* are bounded, $S(X)$ is a bounded subset of $(X,\ ||.||_\infty)$.

Therefore by Schauder's fixed point theorem, S has a fixed point $y \in X$.

This implies $y(t) = y(0),\ \forall\ t \in [-r,\ 0]$.

Also, for $0 \leq t \leq T$ we have $y(t) = S[y(t)]$.

This implies $L[y(t)] = Q[y(t)]$.

i.e $[-y'(t) + y'(0) + \lambda \int_0^t y(s)ds,\ y(0),\ y(T)]$

$= [\int_0^t F^*(s,\ y(s),\ y(s-r))ds,\ g^*(y(0),\ y'(0)),\ h^*(y(0))].$

Therefore

$$\left.\begin{aligned} -y'(t) + y'(0) + \lambda \int_0^t y(s)ds &= \int_0^t F^*(s,\ y(s),\ y(s-r))ds,\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= g^*(y(0),\ y'(0)), \\ y(T) &= h^*(y(0)). \end{aligned}\right\}$$

Since F^* is continuous on $I \times \mathbb{R}^2$, $\int_0^t F^*(s,\ y(s),\ y(s-r))ds$ is differentiable on I. Therefore $y \in C^2[I,\ \mathbb{R}]$. So we have

$$\left.\begin{aligned} -y''(t) + \lambda y(t) &= F^*(t,\ y(t),\ y(t-r)),\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= g^*(y(0),\ y'(0)), \\ y(T) &= h^*(y(0)). \end{aligned}\right\} \tag{8.27}$$

Therefore $y \in X$ is a fixed point of S iff y is a solution of the BVP (8.27).

Proceding similarly as in the proof of the Theorem 8.4.1, it can be easily shown that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$. This implies

$\alpha(t-r) \leq y(t-r) \leq \beta(t-r),\ \forall\ t \in I.$

Thus $F^\star(t,\ y(t),\ y(t-r)) = f(t,\ y(t),\ y(t-r)),\ \forall\ t \in I.$

Also $y(T) = h^*(y(0)) = p(0, y(0)) = y(0).$

Next we show that $p(0,\ y(0) + y'(0)) = y(0) + y'(0).$

It is enough to show that $\alpha(0) \leq y(0) + y'(0) \leq \beta(0).$

Let if possible $\alpha(0) > y(0) + y'(0).$

Then

$$\begin{aligned} y(0) &= p(0,\ y(0) + y'(0)) \\ &= \max\{\alpha(0),\ \min\{y(0) + y'(0),\ \beta(0)\}\} \\ &= \max\{\alpha(0),\ y(0) + y'(0)\} \\ &= \alpha(0). \end{aligned}$$

Therefore $y(T) = y(0) = \alpha(0) = \alpha(T)$.

Since $\alpha(t) \leq y(t),\ \forall\ t \in I,\ \alpha'(0) \leq y'(0)$ and $y'(T) \leq \alpha'(T)$.

So we have $\begin{aligned} y(0) + y'(0) &= \alpha(0) + y'(0) \\ &\geq \alpha(0) + \alpha'(0) \\ &\geq \alpha(0), \text{ which is contradiction.} \end{aligned}$

Thus $\alpha(0) \leq y(0) + y'(0)$.

Similarly it can be proved that $y(0) + y'(0) \leq \beta(0)$.

This implies $p(0,\ y(0) + y'(0)) = y(0) + y'(0)$. Hence y is a solution of the BVP (8.6) such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$. □

Corollary 8.4.6. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let $\alpha,\ \beta$ be respectively lower and upper solutions of BVP (8.7) such that $\alpha(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$. Then the BVP (8.7) has atleast one solution $y \in X$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in [-r,\ T]$.*

Proof. Similar to the proof of the Corollary 8.4.2. □

Similarly, we can prove results on existence of solution between lower and upper solutions for BVP of second order DEVDA (8.13) and (8.14).

Theorem 8.4.7. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let α, β respectively be lower and upper solutions of BVP (8.13) such that $\alpha(t) \leq \beta(t),\ \forall\ t \in I$. Then the BVP (8.13) has atleast one solution $y \in W$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in I$.*

Proof. Similar to the proof of the Theorem 8.4.5. □

Corollary 8.4.8. *Let $f : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $f(t,\ u_1,\ v_1) \leq f(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$. Let α, β be respectively lower and upper solutions of BVP (8.14) such that $\alpha(t) \leq \beta(t),\ \forall\ t \in I$. Then the BVP (8.14) has atleast one solution $y \in W$ such that $\alpha(t) \leq y(t) \leq \beta(t),\ \forall\ t \in I$.*

Proof. Similar to the proof of the Corollary 8.4.2. □

Chapter 9

MONOTONE ITERATIVE METHOD AND QUASILINEARIZATION METHOD FOR SECOND ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

9.1 Introduction

This chapter deals with developing monotone iterative method and quasilinearization method to obtain existence theorems for BVPs of second order DECD (8.7) and second order DEVDA (8.14), discussed in Chapter 8. We make use of Corollary 8.4.6 on existence of solution between lower and upper solutions for BVP of second order DECD (8.7) and Corollary 8.4.8 on existence of solution between lower and upper solutions for BVP of second order DEVDA (8.14), to prove our results.

9.2 Monotone Iterative Method

We first establish the existence result by using the monotone iterative method for BVP of second order DECD (8.7).

Theorem 9.2.1. *Let* $f : I \times \mathbb{R}^2 \to \mathbb{R}$ *be continuous and satisfy*
$f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq \mu_1(x_1 - x_2) + \mu_2(y_1 - y_2)$,
$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ *such that* $x_2 \leq x_1,\ y_2 \leq y_1$ *and for some constants* $\mu_1 > 0,\ \mu_2 > 0$. *Let* $\alpha_0,\ \beta_0 \in X$ *be lower and upper solutions of BVP* (8.7) *such that* $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$. *Then there are sequences of monotonic functions* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *converging in* $(X,\ ||.||_\infty)$ *such that* $\lim_{n\to\infty} \alpha_n$ *and* $\lim_{n\to\infty} \beta_n$ *are solutions of the BVP* (8.7).

Proof. Since, $\alpha_0,\ \beta_0 \in X$ are lower and upper solutions of BVP (8.7), we have

1. $-\alpha_0''(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$
2. $-\beta_0'(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$
3. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0].$
4. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0].$
5. $\alpha_0(0) = \alpha_0(T).$
6. $\beta_0(0) = \beta_0(T).$
7. $\alpha_0'(0) \geq 0.$
8. $\beta_0'(0) \leq 0.$

Consider

$$\left.\begin{aligned} -y''(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1[y(t) - \alpha_0(t)] \\ &\quad + \mu_2[y(t-r) - \alpha_0(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T), \\ y'(0) &= 0. \end{aligned}\right\} \tag{9.1}$$

Let $g_{\alpha_0}(t,\ u,\ v) = f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1[u - \alpha_0(t)] + \mu_2[v - \alpha_0(t-r)]$, $\forall\ (t,\ u,\ v) \in I \times \mathbb{R}^2$.

Therefore $g_{\alpha_0}(t,\ y(t),\ y(t-r)) = f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1[y(t) - \alpha_0(t)]$
$+ \mu_2[y(t-r) - \alpha_0(t-r)],\ \forall\ t \in I.$

So we have $g_{\alpha_0}(t,\ x_1,\ y_1) \leq g_{\alpha_0}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

$$\begin{aligned} \text{Also } -\alpha_0''(t) &\leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1[\alpha_0(t) - \alpha_0(t)] \\ &\quad + \mu_2[\alpha_0(t-r) - \alpha_0(t-r)] \\ &= g_{\alpha_0}(t,\ \alpha_0(t),\ \alpha_0(t-r)). \end{aligned}$$

$$\begin{aligned}\text{Further } -\beta_0''(t) &\geq f(t,\ \beta_0(t),\ \beta_0(t-r))\\ &\geq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1[\beta_0(t) - \alpha_0(t)]\\ &\quad + \mu_2[\beta_0(t-r) - \alpha_0(t-r)]\\ &= g_{\alpha_0}(t,\ \beta_0(t),\ \beta_0(t-r)).\end{aligned}$$

We already know that

1. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
2. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
3. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
4. $\alpha_0(0) = \alpha_0(T)$.
5. $\beta_0(0) = \beta_0(T)$.
6. $\alpha_0'(0) \geq 0$.
7. $\beta_0'(0) \leq 0$.

This implies α_0 and β_0 are lower and upper solutions of the BVP (9.1) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Therefore the BVP (9.1) has a solution $\alpha_1 \in X$ such that $\alpha_0 \leq \alpha_1 \leq \beta_0$ on $[-r,\ T]$. (Using Corollary 8.4.6)

So we have

$$\left.\begin{aligned}-\alpha_1''(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1[\alpha_1(t) - \alpha_0(t)]\\ &\quad + \mu_2[\alpha_1(t-r) - \alpha_0(t-r)],\ t \in I,\\ \alpha_1(t) &= \alpha_1(0),\ t \in [-r,\ 0],\\ \alpha_1(0) &= \alpha_1(T),\\ \alpha_1'(0) &= 0.\end{aligned}\right\}$$

This gives $-\alpha_1''(t) \leq f(t,\ \alpha_1(t),\ \alpha_1(t-r)),\ t \in I$.

Also 1. $-\beta_0''(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$

2. $\alpha_1(t) = \alpha_1(0),\ t \in [-r,\ 0].$

3. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0].$

4. $\alpha_1(t) \leq \beta_0(t),\ t \in [-r,\ T].$

5. $\alpha_1(0) = \alpha_1(T).$

6. $\beta_0(0) = \beta_0(T).$

7. $\alpha_1'(0) = 0.$

8. $\beta_0'(0) \leq 0.$

Therefore α_1 and $\beta_0 \in X$ are lower and upper solutions of the BVP (8.7) such that $\alpha_1(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Next consider

$$\left.\begin{aligned} -y''(t) &= f(t,\ \alpha_1(t),\ \alpha_1(t-r)) + \mu_1[y(t) - \alpha_1(t)] \\ &\quad + \mu_2[y(t-r) - \alpha_1(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T), \\ y'(0) &= 0. \end{aligned}\right\} \tag{9.2}$$

Let $g_{\alpha_1}(t,\ u,\ v) = f(t,\ \alpha_1(t),\ \alpha_1(t-r)) + \mu_1[u - \alpha_1(t)] + \mu_2[v - \alpha_1(t-r)]$, $\forall\ (t,\ u,\ v) \in I \times \mathbb{R}^2$.

Therefore $g_{\alpha_1}(t,\ y(t),\ y(t-r)) = f(t,\ \alpha_1(t),\ \alpha_1(t-r)) + \mu_1[y(t) - \alpha_1(t)]$
$+ \mu_2[y(t-r) - \alpha_1(t-r)],\ \forall\ t \in I.$

So we have $g_{\alpha_1}(t,\ x_1,\ y_1) \leq g_{\alpha_1}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

$$\begin{aligned} \text{Also } -\alpha_1''(t) &\leq f(t,\ \alpha_1(t),\ \alpha_1(t-r)) \\ &= f(t,\ \alpha_1(t),\ \alpha_1(t-r)) + \mu_1[\alpha_1(t) - \alpha_1(t)] \\ &\quad + \mu_2[\alpha_1(t-r) - \alpha_1(t-r)] \\ &= g_{\alpha_1}(t,\ \alpha_1(t),\ \alpha_1(t-r)). \end{aligned}$$

$$\begin{aligned}\text{Further } -\beta_0''(t) &\geq f(t,\ \beta_0(t),\ \beta_0(t-r))\\ &\geq f(t,\ \alpha_1(t),\ \alpha_1(t-r)) + \mu_1[\beta_0(t) - \alpha_1(t)]\\ &\quad + \mu_2[\beta_0(t-r) - \alpha_1(t-r)]\\ &= g_{\alpha_1}(t,\ \beta_0(t),\ \beta_0(t-r)).\end{aligned}$$

We already know that

1. $\alpha_1(t) = \alpha_1(0),\ t \in [-r,\ 0]$.
2. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
3. $\alpha_1(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
4. $\alpha_1(0) = \alpha_1(T)$.
5. $\beta_0(0) = \beta_0(T)$.
6. $\alpha_1'(0) = 0$.
7. $\beta_0'(0) \leq 0$.

This implies α_1 and β_0 are lower and upper solutions of the BVP (9.2) such that $\alpha_1(t) \leq \beta_0(t),\ \forall\, t \in [-r,\ T]$.

Therefore the BVP (9.2) has a solution $\alpha_2 \in X$ such that $\alpha_1 \leq \alpha_2 \leq \beta_0$ on $[-r,\ T]$. (Using Corollary 8.4.6)

This gives

$$\left.\begin{aligned}-\alpha_2''(t) &= f(t,\ \alpha_1(t),\ \alpha_1(t-r)) + \mu_1[\alpha_2(t) - \alpha_1(t)]\\ &\quad + \mu_2[\alpha_2(t-r) - \alpha_1(t-r)],\ t \in I,\\ \alpha_2(t) &= \alpha_2(0),\ t \in [-r,\ 0],\\ \alpha_2(0) &= \alpha_2(T),\\ \alpha_2'(0) &= 0.\end{aligned}\right\}$$

So we have $-\alpha_2''(t) \leq f(t,\ \alpha_2(t),\ \alpha_2(t-r)),\ t \in I$.

Also 1. $-\beta_0''(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$

2. $\alpha_2(t) = \alpha_2(0),\ t \in [-r,\ 0].$

3. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0].$

4. $\alpha_2(t) \leq \beta_0(t),\ t \in [-r,\ T].$

5. $\alpha_2(0) = \alpha_2(T).$

6. $\beta_0(0) = \beta_0(T).$

7. $\alpha_2'(0) = 0.$

8. $\beta_0'(0) \leq 0.$

Therefore α_2 and $\beta_0 \in X$ are lower and upper solutions of the BVP (8.7) such that $\alpha_2(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Continuing in this manner, let $\alpha_n \in X$ be a solution of

$$\left.\begin{aligned} -y''(t) &= f(t,\ \alpha_{n-1}(t),\ \alpha_{n-1}(t-r)) + \mu_1[y(t) - \alpha_{n-1}(t)] \\ &\quad + \mu_2[y(t-r) - \alpha_{n-1}(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T), \\ y'(0) &= 0, \end{aligned}\right\} \tag{9.3}$$

such that $\alpha_{n-1} \leq \alpha_n \leq \beta_0$ on $[-r,\ T]$.

Next consider

$$\left.\begin{aligned} -y''(t) &= f(t,\ \alpha_n(t),\ \alpha_n(t-r)) + \mu_1[y(t) - \alpha_n(t)] \\ &\quad + \mu_2[y(t-r) - \alpha_n(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T), \\ y'(0) &= 0. \end{aligned}\right\} \tag{9.4}$$

Let $g_{\alpha_n}(t,\ u,\ v) = f(t,\ \alpha_n(t),\ \alpha_n(t-r)) + \mu_1[u - \alpha_n(t)]$

$$+ \mu_2[v - \alpha_n(t-r)],\ \forall\ (t,\ u,\ v) \in I \times \mathbb{R}^2.$$

Therefore $g_{\alpha_n}(t,\ y(t),\ y(t-r)) = f(t,\ \alpha_n(t),\ \alpha_n(t-r)) + \mu_1[y(t) - \alpha_n(t)]$
$+ \mu_2[y(t-r) - \alpha_n(t-r)],\ \forall\ t \in I.$

Also $g_{\alpha_n}(t,\ x_1,\ y_1) \leq g_{\alpha_n}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

Further α_n and β_0 are lower and upper solutions of the BVP (9.4) such that $\alpha_n(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Therefore the BVP (9.4) has a solution $\alpha_{n+1} \in X$ such that $\alpha_n \leq \alpha_{n+1} \leq \beta_0$ on $[-r,\ T]$. (Using Corollary 8.4.6)

This gives

$$\left.\begin{aligned} -\alpha''_{n+1}(t) &= f(t,\ \alpha_n(t),\ \alpha_n(t-r)) + \mu_1[\alpha_{n+1}(t) - \alpha_n(t)] \\ &\quad + \mu_2[\alpha_{n+1}(t-r) - \alpha_n(t-r)],\ t \in I, \\ \alpha_{n+1}(t) &= \alpha_{n+1}(0),\ t \in [-r,\ 0], \\ \alpha_{n+1}(0) &= \alpha_{n+1}(T), \\ \alpha'_{n+1}(0) &= 0. \end{aligned}\right\}$$

Thus we get a monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} -\alpha''_{n+1}(t) &= f(t,\ \alpha_n(t),\ \alpha_n(t-r)) + \mu_1[\alpha_{n+1}(t) - \alpha_n(t)] \\ &\quad + \mu_2[\alpha_{n+1}(t-r) - \alpha_n(t-r)],\ t \in I, \\ \alpha_{n+1}(t) &= \alpha_{n+1}(0),\ t \in [-r,\ 0], \\ \alpha_{n+1}(0) &= \alpha_{n+1}(T), \\ \alpha'_{n+1}(0) &= 0. \end{aligned}\right\} \tag{9.5}$$

Now for each fixed $t \in [-r,\ T]$, $\{\alpha_n(t)\}$ is a monotonic increasing sequence in $\mathbb{R}$ which is bounded above by $\beta_0(t)$. Therefore by Monotone Convergence Theorem for sequences in $\mathbb{R}$, $\{\alpha_n(t)\}$ is convergent in $\mathbb{R}$, for each $t \in [-r,\ T]$.

This implies $\{\alpha_n\}$ is pointwise convergent on $[-r,\ T]$.

Let $\lim_{n\to\infty} \alpha_n(t) = \alpha(t),\ \forall\ t \in [-r,\ T]$.

From (9.5), we get

$$\alpha'_{n+1}(t) = -\int_0^t f(s,\ \alpha_n(s),\ \alpha_n(s-r))ds - \mu_1 \int_0^t [\alpha_{n+1}(s) - \alpha_n(s)]ds \qquad (9.6)$$
$$- \mu_2 \int_0^t [\alpha_{n+1}(s-r) - \alpha_n(s-r)]ds.$$

Integrating (9.6) from 0 to t, we get

$$\alpha_{n+1}(t) = \alpha_{n+1}(0) - \int_0^t (t-s) f(s,\ \alpha_n(s),\ \alpha_n(s-r))ds \qquad (9.7)$$
$$- \mu_1 \int_0^t (t-s)[\alpha_{n+1}(s) - \alpha_n(s)]ds$$
$$- \mu_2 \int_0^t (t-s)[\alpha_{n+1}(s-r) - \alpha_n(s-r)]ds.$$

Now $\{\alpha_n\}$ is uniformly bounded on $[-r,\ T]$ as $\alpha_0(t) \le \alpha_n(t) \le \beta_0(t)$, $\forall\ t \in [-r,\ T]$. Therefore $\exists\ M > 0$ such that $|\alpha_n(t)| \le M,\ \forall\ t \in [-r,\ T]$ and $\forall\ n \in \mathbb{N}$. Since $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $f : I \times [-M,\ M] \times [-M,\ M] \to \mathbb{R}$ is continuous. Therefore $f : I \times [-M,\ M] \times [-M,\ M] \to \mathbb{R}$ is bounded, as $I \times [-M,\ M] \times [-M,\ M]$ is compact. So there is $K > 0$ such that $|f(t,\ \alpha_n(t),\ \alpha_n(t-r))| \le K,\ \forall\ t \in I$ and $\forall\ n \in \mathbb{N}$.

Now from equation (9.6), we get

$$
\begin{aligned}
|\alpha'_{n+1}(t)| &\leq |\int_0^t f(s, \alpha_n(s), \alpha_n(s-r))ds| \\
&\quad + |\mu_1||\int_0^t [\alpha_{n+1}(s) - \alpha_n(s)]ds| \\
&\quad + |\mu_2||\int_0^t [\alpha_{n+1}(s-r) - \alpha_n(s-r)]ds| \\
&\leq \int_0^t |f(s, \alpha_n(s), \alpha_n(s-r))|ds \ + |\mu_1| \int_0^t \{|\alpha_{n+1}(s)| + |\alpha_n(s)|\}ds \\
&\quad + |\mu_2| \int_0^t \{|\alpha_{n+1}(s-r)| + |\alpha_n(s-r)|\}ds \\
&\leq K \int_0^T ds \ + \ 2M|\mu_1| \int_0^T ds \ + 2M|\mu_2| \int_0^T ds \\
&= TK \ + \ 2TM|\mu_1| \ + 2TM|\mu_2| = C \text{ (Say).}
\end{aligned}
$$

Thus $\{\alpha'_n\}$ is uniformly bounded on I.

We now show that $\{\alpha_n\}$ is equicontinuous on I.

Let $\epsilon > 0$ be arbitrary. Choose $\delta = \min\{\frac{\epsilon}{C}, \ \frac{T}{2}\}$.

Let $t_1, \ t_2 \in I$ be such that $|t_1 - t_2| < \delta$.

By Lagrange's Mean Value Theorem, $\exists\, \xi$ between t_1 and t_2 such that

$|\alpha_n(t_1) - \alpha_n(t_2)| = |\alpha'_n(\xi)||t_1 - t_2|$.

This implies $|\alpha_n(t_1) - \alpha_n(t_2)| \leq C|t_1 - t_2| < C\delta \leq \epsilon, \ \forall \ t_1, \ t_2 \in I$ such that $|t_1 - t_2| < \delta$. Therefore $\{\alpha_n\}$ is equicontinuous on I. Using Arzela-Ascoli's lemma we may conclude that $\{\alpha_n\}$ has a subsequence $\{\alpha_{n_k}\}$ converging uniformly on I. So $\alpha_{n_k} \to \alpha$ uniformly on I. Since each α_{n_k} is continuous on I, α is continuous on I. Also $\alpha(t) = \alpha(0), \ \forall \ t \in [-r, \ 0]$ and $\alpha(0) = \alpha(T)$. So we conclude that α is continuous on $[-r, \ T]$. Since $\{\alpha_n\}$ is increasing and $\lim_{n\to\infty} \alpha_n(t) = \alpha(t), \ \forall \ t \in I$, by Dini's Theorem for uniform convergence of se-

quence of functions, $\alpha_n \to \alpha$ uniformly on I.

Taking limit as $n \to \infty$ on both sides of equation (9.7) we get

$\alpha(t) = \alpha(0) - \int_0^t (t-s) f(s,\ \alpha(s),\ \alpha(s-r))ds,\ \forall\ t \in I.$

This implies that α is differentiable on I and

$\alpha'(t) = -\int_0^t f(s,\ \alpha(s),\ \alpha(s-r))ds,\ \forall\ t \in I.$

Since the right hand side of equation (9.6) forms a uniformly convergent sequence of functions, we conclude that $\{\alpha_n'\}$ is uniformly convergent on I.

Now taking limit as $n \to \infty$ on both sides of equation (9.6) we get

$\lim_{n\to\infty} \alpha_{n+1}'(t) = \alpha'(t).$

So we conclude that $\alpha_n' \to \alpha'$ uniformly on I. Also $\alpha_n \to \alpha$ pointwise on $[-r,\ T]$ and $\alpha_n \to \alpha$ uniformly on I. Further $\alpha_n(t) = \alpha_n(0),\ \forall\ t \in [-r,\ 0]$ and $\forall\ n \in \mathbb{N},\ \alpha(t) = \alpha(0),\ \forall\ t \in [-r,\ 0]$.

Therefore $\alpha_n \to \alpha$ uniformly on $[-r,\ T]$.

Thus $\{\alpha_n\}$ converges to α in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty} \alpha_n = \alpha(t)$ is a solution of the BVP (8.7).

Next consider

$$\left.\begin{aligned} -y''(t) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + \mu_1[y(t) - \beta_0(t)] \\ &\quad + \mu_2[y(t-r) - \beta_0(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T), \\ y'(0) &= 0. \end{aligned}\right\} \tag{9.8}$$

Let $h_{\beta_0}(t,\ u,\ v) = f(t,\ \beta_0(t),\ \beta_0(t-r)) + \mu_1[u - \beta_0(t)]$
$+ \mu_2[v - \beta_0(t-r)],\ \forall\ (t,\ u,\ v) \in I \times \mathbb{R}^2.$

Therefore $h_{\beta_0}(t, y(t), y(t-r)) = f(t,\ \beta_0(t),\ \beta_0(t-r)) + \mu_1[y(t) - \beta_0(t)]$
$+ \mu_2[y(t-r) - \beta_0(t-r)],\ \forall\ t \in I.$

So we have $g_{\beta_0}(t,\ x_1,\ y_1) \leq g_{\beta_0}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

$$\begin{aligned}
\text{Also } -\alpha_0''(t) &\leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\leq f(t,\ \beta_0(t),\ \beta_0(t-r)) + \mu_1[\alpha_0(t) - \beta_0(t)] \\
&\quad + \mu_2[\alpha_0(t-r) - \beta_0(t-r)] \\
&= h_{\beta_0}(t,\ \alpha_0(t),\ \alpha_0(t-r)).
\end{aligned}$$

$$\begin{aligned}
\text{Further } -\beta_0''(t) &\geq f(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&= f(t,\ \beta_0(t),\ \beta_0(t-r)) + \mu_1[\beta_0(t) - \beta_0(t)] \\
&\quad + \mu_2[\beta_0(t-r) - \beta_0(t-r)] \\
&= h_{\beta_0}(t,\ \beta_0(t),\ \beta_0(t-r)).
\end{aligned}$$

We already know that

1. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
2. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
3. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
4. $\alpha_0(0) = \alpha_0(T)$.
5. $\beta_0(0) = \beta_0(T)$.
6. $\alpha_0'(0) \geq 0$.
7. $\beta_0'(0) \leq 0$.

This implies that α_0 and β_0 are lower and upper solutions of the BVP (9.8) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Therefore the BVP (9.8) has a solution $\beta_1 \in X$ such that $\alpha_0 \leq \beta_1 \leq \beta_0$ on $[-r,\ T]$. (Using Corollary 8.4.6)

So we have

$$\left.\begin{aligned} -\beta_1''(t) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + \mu_1[\beta_1(t) - \beta_0(t)] \\ &\quad + \mu_2[\beta_1(t-r) - \beta_0(t-r)],\ t \in I, \\ \beta_1(t) &= \beta_1(0),\ t \in [-r,\ 0], \\ \beta_1(0) &= \beta_1(T), \\ \beta_1'(0) &= 0. \end{aligned}\right\}$$

This gives $-\beta_1''(t) \geq f(t,\ \beta_1(t),\ \beta_1(t-r)),\ t \in I.$

Also 1. $-\alpha_0''(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$

2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0].$
3. $\beta_1(t) = \beta_1(0),\ t \in [-r,\ 0].$
4. $\alpha_0(t) \leq \beta_1(t),\ t \in [-r,\ T].$
5. $\alpha_0(0) = \alpha_0(T).$
6. $\beta_1(0) = \beta_1(T).$
7. $\alpha_0'(0) \geq 0.$
8. $\beta_1'(0) = 0.$

Therefore α_0 and $\beta_1 \in X$ are lower and upper solutions of the BVP (8.7) such that $\alpha_1(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Continuing in this manner, we get a monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\alpha_0 \leq \beta_n,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned} -\beta_{n+1}''(t) &= f(t,\ \beta_n(t),\ \beta_n(t-r)) + \mu_1[\beta_{n+1}(t) - \beta_n(t)] \\ &\quad + \mu_2[\beta_{n+1}(t-r) - \beta_n(t-r)],\ t \in I, \\ \beta_{n+1}(t) &= \beta_{n+1}(0),\ t \in [-r,\ 0], \\ \beta_{n+1}(0) &= \beta_{n+1}(T), \\ \beta_{n+1}'(0) &= 0. \end{aligned}\right\}$$

Further, $\{\beta_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty} \beta_n$ is a solution of the BVP (8.7). $\square$

Next, we establish the existence result for BVP of second order DEVDA (8.14) by using the monotone iterative method.

Theorem 9.2.2. *Let* $f : I \times \mathbb{R}^2 \to \mathbb{R}$ *be continuous and satisfy*
$f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq \mu_1(x_1 - x_2) + \mu_2(y_1 - y_2)$,
$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ *such that* $x_2 \leq x_1$, $y_2 \leq y_1$ *and for some constants* $\mu_1 > 0$, $\mu_2 > 0$. *Let* α_0, $\beta_0 \in W$ *be lower and upper solutions of BVP* (8.4) *such that* $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in I$. *Then there are sequences of monotonic functions* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *converging in* $(W,\ ||.||_0)$ *such that* $\lim_{n\to\infty} \alpha_n$ *and* $\lim_{n\to\infty} \beta_n$ *are solutions of the BVP* (8.14).

Proof. Similar to the proof of the Theorem 9.2.1. □

9.3 Quasilinearization Method

In this section, we obtain results on existence of solution for BVP of second order DECD (8.7) and for BVP of second order DEVDA (8.14) by using quasilinearization method.

Theorem 9.3.1. *Let* $f \in C^2[I \times \mathbb{R}^2, \mathbb{R}]$ *and* α_0, $\beta_0 \in X$ *be lower and upper solutions of the BVP* (8.7) *such that* $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$. *Assume that* $D_i f(t,\ x,\ y) > 0$ *and* $D_{i,\,j} f(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ *and* $\forall\ i,\ j = 2,\ 3$. *Then there is a monotonic increasing sequence of functions* $\{\alpha_n\}$ *converging in* $(X,\ ||.||_\infty)$ *such that* $\lim_{n\to\infty} \alpha_n$ *is a solution of the BVP* (8.7).

Proof. It is given that $D_{i,\,j} f(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and
$\forall\ i,\ j = 2,\ 3$.

Therefore $f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq D_2 f(t,\ x_2,\ y_2)(x_1 - x_2)$

$$+ D_3 f(t,\ x_2,\ y_2)(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Since $\alpha_0,\ \beta_0 \in X$ are lower and upper solutions of BVP (8.7), we have

1. $-\alpha_0''(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$
2. $-\beta_0''(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$
3. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0].$
4. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0].$
5. $\alpha_0(0) = \alpha_0(T).$
6. $\beta_0(0) = \beta_0(T).$
7. $\alpha_0'(0) \geq 0.$
8. $\beta_0'(0) \leq 0.$

Consider

$$\left.\begin{aligned} -y''(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t) - \alpha_0(t)] \\ &\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r) - \alpha_0(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T), \\ y'(0) &= 0. \end{aligned}\right\} \tag{9.9}$$

Let

$$g_{\alpha_0}(t,\ u,\ v) = f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[u - \alpha_0(t)]$$

$$+ D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[v - \alpha_0(t-r)],\ \forall\ (t,\ u,\ v) \in I \times \mathbb{R}^2.$$

This implies

$$g_{\alpha_0}(t,\ y(t),\ y(t-r)) = f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t)-\alpha_0(t)] + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r)-\alpha_0(t-r)],\ \forall\ t \in I.$$

So we have $g_{\alpha_0}(t,\ x_1,\ y_1) \leq g_{\alpha_0}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$. Also

$$\begin{aligned} -\alpha_0''(t) &\leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_0(t)-\alpha_0(t)] \\ &\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_0(t-r)-\alpha_0(t-r)] \\ &= g_{\alpha_0}(t,\ \alpha_0(t),\ \alpha_0(t-r)). \end{aligned}$$

Further

$$\begin{aligned} -\beta_0''(t) &\geq f(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &\geq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\beta_0(t)-\alpha_0(t)] \\ &\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\beta_0(t-r)-\alpha_0(t-r)] \\ &= g_{\alpha_0}(t,\ \beta_0(t),\ \beta_0(t-r)). \end{aligned}$$

We already know that

1. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
3. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
4. $\alpha_0(0) = \alpha_0(T)$.
5. $\beta_0(0) = \beta_0(T)$.
4. $\alpha_0'(0) \geq 0$.
5. $\beta_0'(0) \leq 0$.

Therefore α_0 and β_0 are lower and upper solutions of the BVP (9.9) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Thus the BVP (9.9) has a solution $\alpha_1 \in X$ such that

$\alpha_0 \leq \alpha_1 \leq \beta_0$ on $[-r,\ T]$. (Using Corollary 8.4.6)

So we have

$$\left.\begin{aligned} -\alpha_1''(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_1(t) - \alpha_0(t)] \\ &\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_1(t-r) - \alpha_0(t-r)],\ t \in I, \\ \alpha_1(t) &= \alpha_1(0),\ t \in [-r,\ 0], \\ \alpha_1(0) &= \alpha_1(T), \\ \alpha_1'(0) &= 0. \end{aligned}\right\}$$

This gives $-\alpha_1''(t) \leq f(t,\ \alpha_1(t),\ \alpha_1(t-r)),\ t \in I$.

Also 1. $-\beta_0''(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I$.

2. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
3. $\alpha_1(t) = \alpha_1(0),\ t \in [-r,\ 0]$.
4. $\alpha_1(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
5. $\alpha_1(0) = \alpha_1(T)$.
6. $\beta_0(0) = \beta_0(T)$.
7. $\alpha_1'(0) = 0$.
8. $\beta_0'(0) \leq 0$.

Therefore α_1 and $\beta_0 \in X$ are lower and upper solutions of the BVP (8.7) such that $\alpha_1(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Continuing in this manner, we get a monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned}
-\alpha''_{n+1}(t) &= f(t,\ \alpha_n(t),\ \alpha_n(t-r)) \\
&\quad + D_2 f(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t) - \alpha_n(t)] \\
&\quad + D_3 f(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t-r) - \alpha_n(t-r)],\ t \in I, \\
\alpha_{n+1}(t) &= \alpha_{n+1}(0),\ t \in [-r,\ 0], \\
\alpha_{n+1}(0) &= \alpha_{n+1}(T), \\
\alpha'_{n+1}(0) &= 0.
\end{aligned}\right\}$$

Further $\{\alpha_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty} \alpha_n$ is a solution of the BVP (8.7). □

Theorem 9.3.2. *Let $f \in C^2[I \times \mathbb{R}^2, \mathbb{R}]$ and $\alpha_0,\ \beta_0 \in W$ be lower and upper solutions of the BVP* (8.14) *such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$. Assume that $D_i f(t,\ x,\ y) > 0$ and $D_{i,\,j} f(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$. Then there is a monotonic increasing sequence of functions $\{\alpha_n\}$ converging in $(W,\ ||.||_0)$ such that $\lim_{n\to\infty} \alpha_n$ is a solution of the BVP* (8.14).

Proof. Similar to the proof of the Theorem 9.3.1. □

We now obtain similar results by changing the condition on decond order partial deivatives of f.

Theorem 9.3.3. *Let $f \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ and $\alpha_0,\ \beta_0 \in X$ be lower and upper solutions of BVP* (8.7) *such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$. Assume that $D_i f(t,\ x,\ y) > 0$ and $D_{i,\,j} f(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$. Then there is a monotonic decreasing sequence of functions $\{\beta_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim_{n\to\infty} \beta_n$ is a solution of the BVP* (8.7).

Proof. It is given that $D_{i,\,j} f(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and
$\forall\ i,\ j = 2,\ 3$.

Therefore $f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq D_2 f(t,\ x_1,\ y_1)(x_1 - x_2)$

$$+ D_3 f(t,\ x_1,\ y_1)(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Since $\alpha_0,\ \beta_0 \in X$ are lower and upper solutions of BVP (8.7), we have

1. $-\alpha_0''(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$
2. $-\beta_0''(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$
3. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0].$
4. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0].$
5. $\alpha_0(0) = \alpha_0(T).$
6. $\beta_0(0) = \beta_0(T).$
7. $\alpha_0(0) \geq 0.$
8. $\beta_0(0) \leq 0.$

Consider

$$\left.\begin{aligned} -y''(t) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &\quad + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[y(t) - \beta_0(t)] \\ &\quad + D_3 f(t, \beta_0(t), \beta_0(t-r))[y(t-r) - \beta_0(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T), \\ y'(0) &= 0. \end{aligned}\right\} \tag{9.10}$$

Let

$$\begin{aligned} h_{\beta_0}(t,\ u,\ v) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &\quad + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[u - \beta_0(t)] \\ &\quad + D_3 f(t,\ \beta_0(t),\ \beta_0(t-r))[v - \beta_0(t-r)],\ \forall\ (t,\ u,\ v) \in I \times \mathbb{R}^2. \end{aligned}$$

Then

$$\begin{aligned} h_{\beta_0}(t,\ y(t),\ y(t-r)) = f(t,\ \beta_0(t),\ \beta_0(t-r)) \\ + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[y(t) - \beta_0(t)] \\ + D_3 f(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)], \\ \forall\ t \in I. \end{aligned}$$

So we have $h_{\beta_0}(t,\ x_1,\ y_1) \leq h_{\beta_0}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$. Also

$$\begin{aligned} -\alpha_0''(t) &\leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\leq f(t,\ \beta_0(t),\ \beta_0(t-r)) + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[\alpha_0(t) - \beta_0(t)] \\ &\quad + D_3 f(t,\ \beta_0(t),\ \beta_0(t-r))[\alpha_0(t-r) - \beta_0(t-r)] \\ &= h_{\beta_0}(t,\ \alpha_0(t),\ \alpha_0(t-r)). \end{aligned}$$

Further

$$\begin{aligned} -\beta_0''(t) &\geq f(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_0(t) - \beta_0(t)] \\ &\quad + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_0(t-r) - \beta_0(t-r)] \\ &= h_{\beta_0}(t,\ \beta_0(t),\ \beta_0(t-r)). \end{aligned}$$

We already know that

1. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
3. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
4. $\alpha_0(0) = \alpha_0(T)$.
5. $\beta_0(0) = \beta_0(T)$.
6. $\alpha_0'(0) \geq 0$.
7. $\beta_0'(0) \leq 0$.

Therefore α_0 and β_0 are lower and upper solutions of the BVP (9.10) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Thus the BVP (9.10) has a solution $\beta_1 \in X$ such that $\alpha_0 \leq \beta_1 \leq \beta_0$ on $[-r,\ T]$. (Using Corollary 8.4.6)

So we have

$$\left.\begin{aligned} -\beta_1''(t) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_1(t) - \beta_0(t)] \\ &\quad + D_3 f(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_1(t-r) - \beta_0(t-r)],\ t \in I, \\ \beta_1(t) &= \beta_1(0),\ t \in [-r,\ 0], \\ \beta_1(0) &= \beta_1(T), \\ \beta_1'(0) &= 0. \end{aligned}\right\}$$

This implies $-\beta_1''(t) \geq f(t,\ \beta_1(t),\ \beta_1(t-r)),\ t \in I$.

Also 1. $-\alpha_0''(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I$.

2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
3. $\beta_1(t) = \beta_1(0),\ t \in [-r,\ 0]$.
4. $\alpha_0(t) \leq \beta_1(t),\ t \in [-r,\ T]$.
5. $\alpha_0(0) = \alpha_0(T)$.
6. $\beta_1(0) = \beta_1(T)$.
7. $\alpha_0'(0) \geq 0$.
8. $\beta_1'(0) = 0$.

Therefore α_0 and $\beta_1 \in X$ are lower and upper solutions of the BVP (8.7) such that $\alpha_0(t) \leq \beta_1(t),\ \forall\ t \in [-r,\ T]$.

Continuing in this manner, we get a monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\alpha_0 \leq \beta_n,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned}
-\beta''_{n+1}(t) &= f(t,\ \beta_n(t),\ \beta_n(t-r)) \\
&\quad + D_2 f(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t) - \beta_n(t)] \\
&\quad + D_3 f(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)],\ t \in I, \\
\beta_{n+1}(t) &= \beta_{n+1}(0),\ t \in [-r,\ 0], \\
\beta_{n+1}(0) &= \beta_{n+1}(T), \\
\beta'_{n+1}(0) &= 0.
\end{aligned}\right\}$$

Further $\{\beta_n\}$ converges in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty} \beta_n$ is a solution of the BVP (8.7). □

Theorem 9.3.4. *Let $f \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ and $\alpha_0,\ \beta_0 \in W$ be lower and upper solutions of BVP (8.14) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in I$. Assume that $D_i f(t,\ x,\ y) > 0$ and $D_{i,\,j} f(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$. Then there is a monotonic decreasing sequence of functions $\{\beta_n\}$ converging in $(W,\ ||.||_0)$ such that $\lim_{n\to\infty} \beta_n$ is a solution of the BVP (8.14).*

Proof. Similar to the proof of the Theorem 9.3.3. □

Chapter 10

EXTENDED AND GENERALIZED QUASILINEARIZATION METHOD FOR SECOND ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

10.1 Introduction

In this chapter, we shall discuss the BVPs of second order DECD of the type $-y''(t) = f(t,\ y(t),\ y(t-r)) + h(t,\ y(t),\ y(t-r))$ and BVPs of second order DEVDA of the type $-y''(t) = f(t,\ y(t),\ y(\theta(t))) + h(t,\ y(t),\ y(\theta(t)))$, where there are certain conditions on first and second order partial derivatives with respect to the second and third arguments of f and h. We shall establish the existence of solution results by employing the extended quasilinearisation method. We shall also establish results on existence of solution, by using generalized quasilinearization method, for BVPs of second order DECD (8.7) and for BVPs of second order DEVDA (8.14), discussed in Chapter 8, by forcing certain conditions on first and second order partial derivatives with respect to the second and third argument of f and $f+\phi$, for some function ϕ.

Further we introduce extended-generalized quaslinearization Method for BVP of second order DECD of the type
$-y''(t) = f(t,\ y(t),\ y(t-r)) + h(t,\ y(t),\ y(t-r))$ and for BVP of second order DEVDA of the type $-y''(t) = f(t,\ y(t),\ y(\theta(t))) + h(t,\ y(t),\ y(\theta(t)))$, by imposing certain conditions on first and second order partial derivatives with respect to the second and third arguments of $f,\ h,\ f+\phi$ and $h+\psi$ for some functions ϕ and ψ.

We make use of Corollary 8.4.6 and Corollary 8.4.8 to obtain all our results.

10.2 Extended Quasilinearization Method

Let $T > 0$, $\lambda \geq 0$, $I = [0,\ T]$, $0 < r \leq T$, $\theta : I \to I$ be continuous function and $f,\ h : I \times \mathbb{R}^2 \to \mathbb{R}$ be continuous functions. Then we investigate the following BVPs for extended quasilinearization method. The BVP of second order DECD is given by

$$\left.\begin{aligned} -y''(t) &= f(t,\ y(t),\ y(t-r)) + h(t,\ y(t),\ y(t-r)),\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T), \\ y'(0) &= 0. \end{aligned}\right\} \tag{10.1}$$

The BVP of second order DEVDA is given by

$$\left.\begin{aligned} -y''(t) &= f(t,\ y(t),\ y(\theta(t))) + h(t,\ y(t),\ y(\theta(t))),\ t \in I, \\ y(0) &= y(T), \\ y'(0) &= 0. \end{aligned}\right\} \tag{10.2}$$

We prove the following result for the BVP(10.1).

Theorem 10.2.1. *Let $\alpha_0,\ \beta_0 \in X$ such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$ and $\Omega = \{(t,\ u,\ v) : \alpha_0(t) \leq u \leq \beta_0(t)$ and $\alpha_0(t-r) \leq v \leq \beta_0(t-r),\ t \in I\}$. Let $f,\ h \in C^2[\Omega,\ \mathbb{R}]$ be such that $D_i f > 0,\ D_i h > 0,\ D_{i,\,j} f \geq 0$ and $D_{i,\,j} h \leq 0$ on Ω and $\forall\ i,\ j = 2,\ 3$. Assume that α_0 and β_0 are lower and upper solutions of BVP (10.1). Then there are monotonic sequences of functions $\{\alpha_n\}$ and $\{\beta_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim\limits_{n\to\infty} \alpha_n$ and $\lim\limits_{n\to\infty} \beta_n$ are solutions of the BVP (10.1).*

Proof. It is given that $D_{i,\,j} f \geq 0$ and $D_{i,\,j} h \leq 0$ on Ω and $\forall\ i,\ j = 2,\ 3$.

Therefore $f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq D_2 f(t,\ x_2,\ y_2)(x_1 - x_2)$

$$+ D_3 f(t,\ x_2,\ y_2)(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2,\ y_1 \geq y_2$.

Also $h(t,\ x_1,\ y_1) - h(t,\ x_2,\ y_2) \geq D_2 h(t,\ x_1,\ y_1)(x_1 - x_2)$

$$+ D_3 h(t,\ x_1,\ y_1)(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2,\ y_1 \geq y_2$.

Let $\mu_1 = \min\limits_{(t,\,u,\,v)\in\Omega} D_2 h(t,\ u,\ v)$ and $\mu_2 = \min\limits_{(t,\,u,\,v)\in\Omega} D_3 h(t,\ u,\ v)$. Then $\mu_1 > 0$ and $\mu_2 > 0$.

Thus we have, $h(t,\ x_1,\ y_1) - h(t,\ x_2,\ y_2) \geq \mu_1(x_1 - x_2) + \mu_2(y_1 - y_2)$,

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2,\ y_1 \geq y_2$.

Since α_0, β_0 are lower and upper solutions of BVP (10.1), we have

1. $-\alpha_0''(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$
2. $-\beta_0''(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$
3. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0].$
4. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0].$
5. $\alpha_0(0) = \alpha_0(T).$
6. $\beta_0(0) = \beta_0(T).$
7. $\alpha_0'(0) \geq 0.$
8. $\beta_0'(0) \leq 0.$

Consider

$$\left.\begin{aligned} -y''(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\quad + [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][y(t) - \alpha_0(t)] \\ &\quad + [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][y(t-r) - \alpha_0(t-r)], \\ &\quad t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T), \\ y'(0) &= 0. \end{aligned}\right\} \tag{10.3}$$

Let $F_{\alpha_0}(t,\ u,\ v) = f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r))$

$$+ [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][u - \alpha_0(t)]$$

$$+ [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][v - \alpha_0(t-r)],$$

$$\forall\ (t,\ u,\ v) \in \Omega.$$

This implies

$$\begin{aligned} F_{\alpha_0}(t,\ y(t),\ y(t-r)) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\quad + [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][y(t) - \alpha_0(t)] \\ &\quad + [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][y(t-r) - \alpha_0(t-r)], \\ &\quad \forall\ t \in I. \end{aligned}$$

So we have $F_{\alpha_0}(t,\ x_1,\ y_1) \leq F_{\alpha_0}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

$$\begin{aligned} \text{Also } -\alpha_0''(t) &\leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\quad + [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][\alpha_0(t) - \alpha_0(t)] \\ &\quad + [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][\alpha_0(t-r) - \alpha_0(t-r)] \\ &= F_{\alpha_0}(t,\ \alpha_0(t),\ \alpha_0(t-r)). \end{aligned}$$

$$\begin{aligned} \text{Further } -\beta_0''(t) &\geq f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &\geq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\quad + [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][\beta_0(t) - \alpha_0(t)] \\ &\quad + [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][\beta_0(t-r) - \alpha_0(t-r)] \\ &= F_{\alpha_0}(t,\ \beta_0(t),\ \beta_0(t-r)). \end{aligned}$$

We already know that

1. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
3. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
4. $\alpha_0(0) = \alpha_0(T)$.
5. $\beta_0(0) = \beta_0(T)$.
6. $\alpha_0'(0) \geq 0$.

7. $\beta_0'(0) \leq 0$.

Therefore α_0 and β_0 are lower and upper solutions of the BVP (10.3) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Thus the BVP (10.3) has a solution $\alpha_1 \in X$ such that $\alpha_0 \leq \alpha_1 \leq \beta_0$ on $[-r,\ T]$. (Using Corollary 8.4.6)

Therefore

$$\left.\begin{aligned} -\alpha_1''(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\quad + [D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][\alpha_1(t) - \alpha_0(t)] \\ &\quad + [D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][\alpha_1(t-r) - \alpha_0(t-r)],\ t \in I, \\ \alpha_1(t) &= \alpha_1(0),\ t \in [-r,\ 0], \\ \alpha_1(0) &= \alpha_1(T), \\ \alpha_1'(0) &= 0. \end{aligned}\right\}$$

This implies $-\alpha_1''(t) \leq f(t,\ \alpha_1(t),\ \alpha_1(t-r)) + h(t,\ \alpha_1(t),\ \alpha_1(t-r)),\ t \in I$.

Also 1. $-\beta_0''(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I$.

2. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
3. $\alpha_1(t) = \alpha_1(0),\ t \in [-r,\ 0]$.
4. $\alpha_1(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
5. $\alpha_1(0) = \alpha_1(T)$.
6. $\beta_0(0) = \beta_0(T)$.
7. $\alpha_1'(0) = 0$.
8. $\beta_0'(0) \leq 0$.

Therefore α_1 and $\beta_0 \in X$ are lower and upper solutions of the BVP (10.1) such that $\alpha_1(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Continuing in this manner, we get a monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned}
-\alpha''_{n+1}(t) &= f(t,\ \alpha_n(t),\ \alpha_n(t-r)) + h(t, \alpha_n(t), \alpha_n(t-r)) \\
&\quad + [D_2 f(t,\ \alpha_n(t),\ \alpha_n(t-r)) + \mu_1][\alpha_{n+1}(t) - \alpha_n(t)] \\
&\quad + [D_3 f(t,\ \alpha_n(t),\ \alpha_n(t-r)) + \mu_2][\alpha_{n+1}(t-r) - \alpha_n(t-r)],\ t \in I, \\
\alpha_{n+1}(t) &= \alpha_{n+1}(0),\ t \in [-r,\ 0], \\
\alpha_{n+1}(0) &= \alpha_{n+1}(T), \\
\alpha'_{n+1}(0) &= 0.
\end{aligned}\right\}$$

Further $\lim_{n\to\infty} \alpha_n$ exists in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty} \alpha_n$ is a solution of the BVP (10.1).

Let $\theta_1 = \min_{(t,\ u,\ v)\in\Omega} D_2 f(t,\ u,\ v)$ and $\theta_2 = \min_{(t,\ u,\ v)\in\Omega} D_3 f(t,\ u,\ v)$. Then $\theta_1 > 0$ and $\theta_2 > 0$.

Thus we have $f(t,\ x_1,\ y_1) - f(t,\ x_2,\ y_2) \geq \theta_1(x_1 - x_2) + \theta_2(y_1 - y_2)$,

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Now consider

$$\left.\begin{aligned}
-y''(t) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad + [\theta_1 + D_2 h(t,\ \beta_0(t),\ \beta_0(t-r))][y(t) - \beta_0(t)] \\
&\quad + [\theta_2 + D_3 h(t,\ \beta_0(t),\ \beta_0(t-r))][y(t-r) - \beta_0(t-r)], \\
&\quad t \in I, \\
y(t) &= y(0),\ t \in [-r,\ 0], \\
y(0) &= y(T), \\
y'(0) &= 0.
\end{aligned}\right\} \tag{10.4}$$

$$\begin{aligned}
\text{Let } G_{\beta_0}(t,\ u,\ v) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad + [\theta_1 + D_2 h(t,\ \beta_0(t),\ \beta_0(t-r))][u - \beta_0(t)] \\
&\quad + [\theta_2 + D_3 h(t,\ \beta_0(t),\ \beta_0(t-r))][v - \beta_0(t-r)], \\
&\quad \forall\ (t,\ u,\ v) \in \Omega.
\end{aligned}$$

This implies

$$\begin{aligned} G_{\beta_0}(t,\ y(t),\ y(t-r)) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &\quad + [\theta_1 + D_2h(t,\ \beta_0(t),\ \beta_0(t-r))][y(t) - \beta_0(t)] \\ &\quad + [\theta_2 + D_3h(t,\ \beta_0(t),\ \beta_0(t-r))][y(t-r) - \beta_0(t-r)], \\ &\quad \forall\ t \in I. \end{aligned}$$

So we have $G_{\beta_0}(t,\ x_1,\ y_1) \leq G_{\beta_0}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

$$\begin{aligned} \text{Also } -\alpha_0''(t) &\leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\leq f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t-r),\ \beta_0(t)) \\ &\quad + [\theta_1 + D_2h(t,\ \beta_0(t),\ \beta_0(t-r))][\alpha_0(t) - \beta_0(t)] \\ &\quad + [\theta_2 + D_3h(t,\ \beta_0(t),\ \beta_0(t-r))][\alpha_0(t-r) - \beta_0(t-r)] \\ &= G_{\beta_0}(t,\ \alpha_0(t),\ \alpha_0(t-r)). \end{aligned}$$

$$\begin{aligned} \text{Further } -\beta_0''(t) &\geq f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t-r),\ \beta_0(t)) \\ &\quad + [\theta_1 + D_2h(t,\ \beta_0(t),\ \beta_0(t-r))][\beta_0(t) - \beta_0(t)] \\ &\quad + [\theta_2 + D_3h(t,\ \beta_0(t),\ \beta_0(t-r))][\beta_0(t-r) - \beta_0(t-r)] \\ &= G_{\beta_0}(t,\ \beta_0(t),\ \beta_0(t-r)). \end{aligned}$$

We already know that

1. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
3. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
4. $\alpha_0(0) = \alpha_0(T)$.
5. $\beta_0(0) = \beta_0(T)$.

6. $\alpha_0'(0) \geq 0$.

7. $\beta_0'(0) \leq 0$.

Therefore α_0 and β_0 are lower and upper solutions of the BVP (10.413) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\, t \in [-r,\ T]$.

Thus the BVP (10.4) has a solution $\beta_1 \in X$ such that $\alpha_0 \leq \beta_1 \leq \beta_0$ on $[-r,\ T]$. (Using Corollary 8.4.6)

Therefore

$$\left.\begin{aligned}
\beta_1'(t) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t-r),\ \beta_0(t)) \\
&\quad + [\theta_1 + D_2h(t,\ \beta_0(t),\ \beta_0(t-r))][\beta_1(t) - \beta_0(t)] \\
&\quad + [\theta_2 + D_3h(t,\ \beta_0(t),\ \beta_0(t-r))][\beta_1(t-r) - \beta_0(t-r)],\ t \in I, \\
\beta_1(t) &= \beta_1(0),\ t \in [-r,\ 0], \\
\beta_1(0) &= \beta_1(T), \\
\beta_1'(0) &= 0.
\end{aligned}\right\}$$

This implies $-\beta_1''(t) \geq f(t,\ \beta_1(t),\ \beta_1(t-r)) + h(t,\ \beta_1(t),\ \beta_1(t-r)),\ t \in I$.

Also 1. $-\alpha_0''(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I$.

2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.

3. $\beta_1(t) = \beta_1(0),\ t \in [-r,\ 0]$.

4. $\alpha_0(t) \leq \beta_1(t),\ t \in [-r,\ T]$.

5. $\alpha_0(0) = \alpha_0(T)$.

6. $\beta_1(0) = \beta_1(T)$.

7. $\alpha_0'(0) \geq 0$.

8. $\beta_1'(0) = 0$.

Therefore α_0 and $\beta_1 \in X$ are lower and upper solutions of the BVP (10.1) such that $\alpha_0(t) \leq \beta_1(t),\ \forall\, t \in [-r,\ T]$.

Continuing in this manner, we get a monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\alpha_0 \leq \beta_n,\ \forall\, n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned}
-\beta''_{n+1}(t) &= f(t,\ \beta_n(t),\ \beta_n(t-r)) + h(t,\ \beta_n(t-r),\ \beta_n(t)) \\
&\quad + [\theta_1 + D_2h(t,\ \beta_n(t),\ \beta_n(t-r))][\beta_{n+1}(t) - \beta_n(t)] \\
&\quad + [\theta_2 + D_3h(t,\ \beta_n(t),\ \beta_n(t-r))][\beta_{n+1}(t-r) - \beta_n(t-r)], \\
&\quad t \in I, \\
\beta_{n+1}(t) &= \beta_{n+1}(0),\ t \in [-r,\ 0], \\
\beta_{n+1}(0) &= \beta_{n+1}(T), \\
\beta'_{n+1}(0) &= 0.
\end{aligned}\right\}$$

Further $\lim_{n\to\infty}\beta_n$ exists in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty}\beta_n$ is a solution of the BVP (10.1). □

Next we have a similar result for the BVP (10.2).

Theorem 10.2.2. *Let $\alpha_0,\ \beta_0 \in W$ such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in I$ and $\Omega = \{(t,\ u,\ v) : \alpha_0(t) \leq u \leq \beta_0(t)$ and $\alpha_0(\theta(t)) \leq v \leq \beta_0(\theta(t)),\ t \in I\}$. Let $f,\ h \in C^2[\Omega,\ \mathbb{R}]$ be such that $D_if > 0$, $D_ih > 0$, $D_{i,\,j}f \geq 0$ and $D_{i,\,j}h \leq 0$ on Ω and $\forall\ i,\ j = 2,\ 3$. Assume that α_0 and β_0 are lower and upper solutions of BVP (10.2). Then there are monotonic sequences of functions $\{\alpha_n\}$ and $\{\beta_n\}$ converging in $(W,\ ||.||_0)$ such that $\lim_{n\to\infty}\alpha_n$ and $\lim_{n\to\infty}\beta_n$ are solutions of the BVP (10.2).*

Proof. Similar to the proof of the Theorem 10.2.1. □

10.3 Generalized Quasilinearization Method

In this section, we undertake the study of existence of solution for the BVP of second order DECD (8.7) and BVP of second order DEVDA (8.14), by forcing certain conditions on first and second order partial derivatives with respect to the second and third argument of f and $f+\phi$, for some function ϕ. We obtain our results by using generalised quasilinearization method.

Theorem 10.3.1. *Let $\alpha_0,\ \beta_0 \in X$ be lower and upper solutions of BVP (8.7) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$. Let $f,\ \phi \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ such that*

(1) $D_{i,\,j}f(t,\ x,\ y)+D_{i,\,j}\phi(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in I\times\mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$.

(2) $D_if(t,\ x,\ y) + D_i\phi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i = 2,\ 3$.

(3) $\phi(t,\ u_1,\ v_1) \geq \phi(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

Then there is a monotonic increasing sequence of functions $\{\alpha_n\}$converging in $(X,\ ||.||_\infty)$ such that $\lim\limits_{n\to\infty}\alpha_n$ is a solution to the BVP (8.7).

Proof. Let $F(t,\ x,\ y) = f(t,\ x,\ y) + \phi(t,\ x, y),\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$.

Therefore $D_{i,\,j}F(t,\ x,\ y) \geq 0$ and $D_iF(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$. This implies

$F(t,\ x_1,\ y_1)-F(t,\ x_2,\ y_2) \geq D_2F(t,\ x_2,\ y_2)(x_1-x_2)+D_3F(t,\ x_2,\ y_2)(y_1-y_2)$,

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

So we have

$$\begin{aligned} f(t,\ x_1,\ y_1) \geq\ & f(t,\ x_2,\ y_2) - \phi(t,\ x_1,\ y_1) + \phi(t,\ x_2,\ y_2) \\ & + D_2f(t,\ x_2, y_2)(x_1 - x_2) + D_2\alpha(t,\ x_2,\ y_2)(x_1 - x_2) \\ & + D_3f(t,\ x_2,\ y_2)(y_1 - y_2) + D_3\alpha(t,\ x_2,\ y_2)(y_1 - y_2), \end{aligned}$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Since $\alpha_0,\ \beta_0$ are lower and upper solutions of BVP (8.7), we have

1. $-\alpha_0''(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I$,
2. $-\beta_0''(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I$,
3. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$,
4. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$,
5. $\alpha_0(0) = \alpha_0(T)$,
6. $\beta_0(0) = \beta_0(T)$,
7. $\alpha_0'(0) \geq 0$,
8. $\beta_0'(0) \leq 0$.

Consider

$$\left.\begin{aligned}
-y''(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad - \phi(t,\ y(t),\ y(t-r)) \\
&\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t) - \alpha_0(t)] \\
&\quad + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t) - \alpha_0(t)] \\
&\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r) - \alpha_0(t-r)] \\
&\quad + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r) - \alpha_0(t-r)],\ t \in I, \\
y(t) &= y(0),\ t \in [-r,\ 0], \\
y(0) &= y(T), \\
y'(0) &= 0.
\end{aligned}\right\} \tag{10.5}$$

Let $F_{\alpha_0}(t,\ u,\ v) = f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r))$

$$\begin{aligned}
&- \phi(t,\ u,\ v) + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[u - \alpha_0(t)] \\
&+ D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[u - \alpha_0(t)] \\
&+ D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[v - \alpha_0(t-r)] \\
&+ D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[v - \alpha_0(t-r)], \\
&\forall\ (t,\ u,\ v) \in I \times \mathbb{R}^2.
\end{aligned}$$

This implies

$$\begin{aligned}
F_{\alpha_0}(t,\ y(t),\ y(t-r)) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad - \phi(t,\ y(t),\ y(t-r)) \\
&\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t) - \alpha_0(t)] \\
&\quad + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t) - \alpha_0(t)] \\
&\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r) - \alpha_0(t-r)] \\
&\quad + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r) - \alpha_0(t-r)],\ \forall\ t \in I.
\end{aligned}$$

So we have $F_{\alpha_0}(t,\ x_1,\ y_1) \leq F_{\alpha_0}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

$$\begin{aligned}
\text{Also } -\alpha_0''(t) &\leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad - \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_0(t) - \alpha_0(t)] \\
&\quad + D_2 \phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_0(t) - \alpha_0(t)] \\
&\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_0(t-r) - \alpha_0(t-r)] \\
&\quad + D_3 \phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_0(t-r) - \alpha_0(t-r)] \\
&= F_{\alpha_0}(t,\ \alpha_0(t),\ \alpha_0(t-r)).
\end{aligned}$$

$$\begin{aligned}
\text{Further } -\beta_0''(t) &\geq f(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\geq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad - \phi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\beta_0(t) - \alpha_0(t)] \\
&\quad + D_2 \phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[\beta_0(t) - \alpha_0(t)] \\
&\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\beta_0(t-r) - \alpha_0(t-r)] \\
&\quad + D_3 \phi(t,\ \alpha_0(t),\ \alpha_0(t-r))][\beta_0(t-r) - \alpha_0(t-r)] \\
&= F_{\alpha_0}(t,\ \beta_0(t),\ \beta_0(t-r)).
\end{aligned}$$

We already know that

1. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
3. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
4. $\alpha_0(0) = \alpha_0(T)$.
5. $\beta_0(0) = \beta_0(T)$.

6. $\alpha_0'(0) \geq 0$.

7. $\beta_0'(0) \leq 0$.

Therefore α_0 and β_0 are lower and upper solutions of the BVP (10.5) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Thus the BVP (10.5) has a solution $\alpha_1 \in X$ such that $\alpha_0 \leq \alpha_1 \leq \beta_0$ on $[-r,\ T]$. (Using Corollary 8.4.6)

Therefore

$$\left.\begin{aligned}
-\alpha_1''(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad - \phi(t,\ \alpha_1(t),\ \alpha_1(t-r)) \\
&\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_1(t) - \alpha_0(t)] \\
&\quad + D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_1(t) - \alpha_0(t)] \\
&\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_1(t-r) - \alpha_0(t-r)] \\
&\quad + D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_1(t-r) - \alpha_0(t-r)],\ t \in I, \\
\alpha_1(t) &= \alpha_1(0),\ t \in [-r,\ 0], \\
\alpha_1(0) &= \alpha_1(T), \\
\alpha_1'(0) &= 0.
\end{aligned}\right\}$$

This implies $-\alpha_1''(t) \leq f(t,\ \alpha_1(t),\ \alpha_1(t-r)),\ t \in I$.

Also 1. $-\beta_0''(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I$.

2. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.

3. $\alpha_1(t) = \alpha_1(0),\ t \in [-r,\ 0]$.

4. $\alpha_1(t) \leq \beta_0(t),\ t \in [-r,\ T]$.

5. $\alpha_1(0) = \alpha_1(T)$.

6. $\beta_0(0) = \beta_0(T)$.

7. $\alpha_1'(0) = 0$.

8. $\beta_0'(0) \leq 0$.

Therefore α_1 and $\beta_0 \in X$ are lower and upper solutions of the BVP (8.7) such that $\alpha_1(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Continuing in this manner, we get a monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned}
-\alpha''_{n+1}(t) &= f(t,\ \alpha_n(t),\ \alpha_n(t-r)) + \phi(t,\ \alpha_n(t),\ \alpha_n(t-r)) \\
&\quad - \phi(t,\ \alpha_{n+1}(t),\ \alpha_{n+1}(t-r)) \\
&\quad + D_2 f(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t) - \alpha_n(t)] \\
&\quad + D_2\phi(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t) - \alpha_n(t)] \\
&\quad + D_3 f(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t-r) - \alpha_n(t-r)] \\
&\quad + D_3\phi(t,\ \alpha_n(t),\ \alpha_n(t-r))[\alpha_{n+1}(t-r) - \alpha_n(t-r)],\ t \in I, \\
\alpha_{n+1}(t) &= \alpha_{n+1}(0),\ t \in [-r,\ 0], \\
\alpha_{n+1}(0) &= \alpha_{n+1}(T), \\
\alpha'_{n+1}(0) &= 0.
\end{aligned}\right\}$$

Further $\lim\limits_{n\to\infty} \alpha_n$ exists in $(X,\ ||.||_\infty)$ and $\lim\limits_{n\to\infty} \alpha_n$ is a solution of the BVP (8.7). □

Next we have a similar result for the BVP(8.14).

Theorem 10.3.2. *Let $\alpha_0,\ \beta_0 \in W$ be lower and upper solutions of BVP* (8.14) *such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in I$. Let $f,\ \phi \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ such that*

(1) $D_{i,\,j} f(t,\ x,\ y) + D_{i,\,j}\phi(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$.

(2) $D_i f(t,\ x,\ y) + D_i\phi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i = 2,\ 3$.

(3) $\phi(t,\ u_1,\ v_1) \geq \phi(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

Then there is a monotonic increasing sequence of functions $\{\alpha_n\}$converging in $(W,\ ||.||_0)$ such that $\lim\limits_{n\to\infty} \alpha_n$ is a solution to the BVP (8.14).

Proof. Similar to the proof of the Theorem 10.3.1. □

Next we prove results for the BVPs under consideration by reversing the inequality in the first condition. We see that in case of BVP (8.7), we get a monotonic decreasing sequence of functions $\{\beta_n\}$ converging to a

Theorem 10.3.3. *Let $\alpha_0,\ \beta_0 \in X$ be lower and upper solutions of BVP (8.7) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$. Assume that $f,\ \psi \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ such that*

(1) $D_{i,\,j}f(t,\ x,\ y)+D_{i,j}\psi(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in I\times\mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$.

(2) $D_if(t,\ x,\ y) + D_i\psi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i = 2,\ 3$.

(3) $\psi(t,\ u_1,\ v_1) \geq \psi(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$

such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

Then there is a monotonic decreasing sequence of functions $\{\beta_n\}$converging in $(X,\ ||.||_\infty)$ such that $\lim\limits_{n\to\infty}\beta_n$ is a solution of the BVP (8.7).

Proof. Let $G(t,\ x,\ y) = f(t,\ x,\ y) + \psi(t,\ x,\ y),\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$.

Therefore $D_{i,\,j}G(t,\ x,\ y) \leq 0$ and $D_iG(t,\ x,\ y) > 0$,

$\forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$. This implies

$$G(t,\ x_1,\ y_1) - G(t,\ x_2,\ y_2) \geq D_2G(t,\ x_1,\ y_1)(x_1 - x_2) + D_3G(t,\ x_1,\ y_1)(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

So we have

$$f(t,\ x_1,\ y_1) \geq f(t,\ x_2,\ y_2) - \psi(t,\ x_1,\ y_1) + \psi(t,\ x_2,\ y_2) + D_2f(t,\ x_1,\ y_1)(x_1 - x_2) + D_2\psi(t,\ x_1,\ y_1)(x_1 - x_2) + D_3f(t,\ x_1,\ y_1)(y_1 - y_2) + D_3\psi(t,\ x_1,\ y_1)(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Since $\alpha_0,\ \beta_0 \in X$ are lower and upper solutions of the BVP (8.7), we have

1. $-\alpha_0''(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I$.
2. $-\beta_0''(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I$.
3. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.

4. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.

5. $\alpha_0(0) = \alpha_0(T)$.

6. $\beta_0(0) = \beta_0(T)$.

7. $\alpha_0'(0) \geq 0$.

8. $\beta_0'(0) \leq 0$.

Consider

$$\left.\begin{aligned} -y''(t) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &\quad - \psi(t,\ y(t),\ y(t-r)) \\ &\quad + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[y(t) - \beta_0(t)] \\ &\quad + D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[y(t) - \beta_0(t)] \\ &\quad + D_3 f(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)] \\ &\quad + D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)],\ t \in I, \\ y(t) &= y(0),\ t \in [-r,\ 0], \\ y(0) &= y(T), \\ y'(0) &= 0. \end{aligned}\right\} \tag{10.6}$$

Let $G_{\beta_0}(t,\ u,\ v) = f(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r))$

$$\begin{aligned} &- \psi(t,\ u,\ v) + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[u - \beta_0(t)] \\ &+ D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[u - \beta_0(t)] \\ &+ D_3 f(t,\ \beta_0(t),\ \beta_0(t-r))[v - \beta_0(t-r)] \\ &+ D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[v - \beta_0(t-r)], \\ &\forall\ (t,\ u,\ v) \in I \times \mathbb{R}^2. \end{aligned}$$

This implies

$$\begin{aligned} G_{\beta_0}(t,\ y(t),\ y(t-r)) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &\quad - \psi(t,\ y(t),\ y(t-r)) \\ &\quad + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[y(t) - \beta_0(t)] \\ &\quad + D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[y(t) - \beta_0(t)] \\ &\quad + D_3 f(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)] \\ &\quad + D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)],\ \forall\ t \in I. \end{aligned}$$

So we have $G_{\beta_0}(t,\ x_1,\ y_1) \leq G_{\beta_0}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in I \times \mathbb{R}^2$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

$$\begin{aligned} \text{Also } -\alpha_0''(t) &\leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\leq f(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\ &\quad - \psi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\ &\quad + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[\alpha_0(t) - \beta_0(t)] \\ &\quad + D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[\alpha_0(t) - \beta_0(t)] \\ &\quad + D_3 f(t,\ \beta_0(t),\ \beta_0(t-r))[\alpha_0(t-r) - \beta_0(t-r)] \\ &\quad + D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[\alpha_0(t-r) - \beta_0(t-r)] \\ &= G_{\beta_0}(t,\ \alpha_0(t),\ \alpha_0(t-r)). \end{aligned}$$

$$\begin{aligned}
\text{Further } -\beta_0''(t) &\geq f(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&= f(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad - \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_0(t) - \beta_0(t)] \\
&\quad + D_2 \psi(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_0(t) - \beta_0(t)] \\
&\quad + D_3 f(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_0(t-r) - \beta_0(t-r)] \\
&\quad + D_3 \psi(t,\ \beta_0(t),\ \beta_0(t-r))][\beta_0(t-r) - \beta_0(t-r)] \\
&= G_{\beta_0}(t,\ \beta_0(t),\ \beta_0(t-r)).
\end{aligned}$$

We already know that

1. $\alpha_0(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
3. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
4. $\alpha_0(0) = \alpha_0(T)$.
5. $\beta_0(0) = \beta_0(T)$.
6. $\alpha_0'(0) \geq 0$.
7. $\beta_0'(0) \leq 0$.

Therefore α_0 and β_0 are lower and upper solutions of the BVP (10.6) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Thus the BVP (10.6) has a solution $\beta_1 \in X$ such that $\alpha_0 \leq \beta_1 \leq \beta_0$ on $[-r,\ T]$. (Using Corollary 8.4.6)

So we have

$$\left.\begin{aligned}
-\beta_1''(t) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad - \psi(t,\ \beta_1(t),\ \beta_1(t-r)) \\
&\quad + D_2 f(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_1(t) - \beta_0(t)] \\
&\quad + D_2 \psi(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_1(t) - \beta_0(t)] \\
&\quad + D_3 f(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_1(t-r) - \beta_0(t-r)] \\
&\quad + D_3 \psi(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_1(t-r) - \beta_0(t-r)],\ t \in I, \\
\beta_1(t) &= \beta_1(0),\ t \in [-r,\ 0], \\
\beta_1(0) &= \beta_1(T), \\
\beta_1'(0) &= 0.
\end{aligned}\right\}$$

This implies $-\beta_1''(t) \geq f(t,\ \beta_1(t),\ \beta_1(t-r)),\ t \in I.$

Also 1. $-\alpha_0''(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$

2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0].$
3. $\beta_1(t) = \beta_1(0),\ t \in [-r,\ 0].$
4. $\alpha_0(t) \leq \beta_1(t),\ t \in [-r,\ T].$
5. $\alpha_0(0) = \alpha_0(T).$
6. $\beta_1(0) = \beta_1(T).$
7. $\alpha_0'(0) \geq 0.$
8. $\beta_1'(0) = 0.$

Therefore α_0 and $\beta_1 \in X$ are lower and upper solutions of the BVP (8.7) such that $\alpha_0(t) \leq \beta_1(t),\ \forall\, t \in [-r,\ T]$.

Continuing in this manner, we get a monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\alpha_0 \leq \beta_n,\ \forall\, n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned}
-\beta''_{n+1}(t) &= f(t,\ \beta_n(t),\ \beta_n(t-r)) + \beta(t,\ \beta_n(t),\ \beta_n(t-r)) \\
&\quad - \beta(t,\ \beta_{n+1}(t),\ \beta_{n+1}(t-r)) \\
&\quad + D_2 f(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t) - \beta_n(t)] \\
&\quad + D_2\beta(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t) - \beta_n(t)] \\
&\quad + D_3 f(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)] \\
&\quad + D_3\beta(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)],\ t \in I, \\
\beta_{n+1}(t) &= \beta_{n+1}(0),\ t \in [-r,\ 0], \\
\beta_{n+1}(0) &= \beta_{n+1}(T), \\
\beta'_{n+1}(0) &= 0.
\end{aligned}\right\}$$

Further $\lim_{n\to\infty}\beta_n$ exists in $(X,\ ||.||_\infty)$ and $\lim_{n\to\infty}\beta_n$ is a solution of the BVP (8.7). □

Incase of the BVP(8.14), we see that as a result of reversing the inequality in first condition, we get a monotonic decreasing sequence of functions converging to a solution of the BVP.

Theorem 10.3.4. *Let $\alpha_0,\ \beta_0 \in W$ be lower and upper solutions of BVP* (8.14) *such that $\alpha_0(t) \le \beta_0(t),\ \forall\ t \in I$. Assume that $f,\ \psi \in C^2[I \times \mathbb{R}^2,\ \mathbb{R}]$ such that*
(1) $D_{i,\,j}f(t,\ x,\ y) + D_{i,j}\psi(t,\ x,\ y) \le 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i,\ j = 2,\ 3$.
(2) $D_i f(t,\ x,\ y) + D_i\psi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in I \times \mathbb{R}^2$ and $\forall\ i = 2,\ 3$.
(3) $\psi(t,\ u_1,\ v_1) \ge \psi(t,\ u_2,\ v_2),\ \forall\ (t,\ u_1,\ v_1),\ (t,\ u_2,\ v_2) \in I \times \mathbb{R}^2$
such that $u_1 \le u_2$ and $v_1 \le v_2$.
Then there is monotonic decreasing sequence of functions $\{\beta_n\}$converging in $(W,\ ||.||_0)$ such that $\lim_{n\to\infty}\beta_n$ is a solution of the BVP (8.14).

Proof. Similar to the proof of the Theorem 10.3.3. □

10.4 Extended-Generalized Quasilinearization Method

In this section , we make use of extended-generalized quaslinearization method to prove the existence of solution for the BVP of second order DECD (10.1) and BVP of second order DEVDA (8.14). The results are obtained under certain conditions on first and second order partial derivatives with respect to the second and third arguments of $f,\ h,\ f+\phi$ and $h+\psi$ for some functions ϕ and ψ.

Theorem 10.4.1. *Let $\alpha_0,\ \beta_0 \in X$ such that $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$ and $\Omega = \{(t,u,v) : \alpha_0(t) \leq u \leq \beta_0(t)$ and $\alpha_0(t-r) \leq v \leq \beta_0(t-r),\ t \in I\}$. Assume that $f,\ h,\ \phi,\ \psi \in C^2[\Omega,\ \mathbb{R}]$ satisfying*

(1) $D_i f(t,\ x,\ y) + D_i\phi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i = 2,\ 3.$

(2) $D_i h(t,\ x,\ y) + D_i\psi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i = 2,\ 3.$

(3) $D_{i,\,j} f(t,\ x,\ y) + D_{i,\,j}\phi(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i,\ j = 2,\ 3.$

(4) $D_{i,\,j} h(t,\ x,\ y) + D_{i,\,j}\beta(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i,\ j = 2,\ 3.$

(5) $\phi(t,\ x_1,\ y_1) + \psi(t,\ x_1,\ y_1) \geq \phi(t,\ x_2,\ y_2) + \psi(t, x_2, y_2),$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$. Further, assume that α_0 and β_0 are lower and upper solutions of BVP (10.1). Then there are monotonic sequences of functions $\{\alpha_n\}$ and $\{\beta_n\}$ converging in $(X,\ ||.||_\infty)$ such that $\lim\limits_{n\to\infty}\alpha_n$ and $\lim\limits_{n\to\infty}\beta_n$ are solutions to the BVP (10.1).

Proof. Let $F(t,\ x,\ y) = f(t,\ x,\ y) + \phi(t,\ x,\ y),\ \forall\ (t,\ x,\ y) \in \Omega$ and

$$H(t,\ x,\ y) = h(t,\ x,\ y) + \psi(t,\ x,\ y),\ \forall\ (t,\ x,\ y) \in \Omega.$$

It is assumed that $D_{i,\,j} f(t,\ x,\ y) + D_{i,\,j}\phi(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i,\ j = 2,\ 3.$

Therefore $f(t,\ x_1,\ y_1) \geq f(t,\ x_2,\ y_2) - \phi(t,\ x_1,\ y_1) + \phi(t,\ x_2,\ y_2)$

$$+ [D_2 f(t,\ x_2,\ y_2) + D_2\phi(t,\ x_2,\ y_2)](x_1 - x_2)$$

$$+ [D_3 f(t,\ x_2,\ y_2) + D_3\phi(t,\ x_2,\ y_2)](y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

It is also assumed that $D_{i,\,j}h(t,\ x,\ y) + D_{i,\,j}\psi(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in \Omega$ and $\forall\ i,\ j = 2,\ 3$.

So we have $h(t,\ x_1,\ y_1) \geq h(t,\ x_2,\ y_2) - \psi(t,\ x_1,\ y_1) + \psi(t,\ x_2,\ y_2)$

$$+ [D_2 h(t,\ x_1,\ y_1) + D_2\psi(t,\ x_1,\ y_1)](x_1 - x_2)$$

$$+ [D_3 h(t,\ x_1,\ y_1) + D_3\psi(t,\ x_1,\ y_1)](y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Let $\mu_1 = \min\limits_{(t,\,u,\,v)\in\Omega}[D_2 h(t,\ u,\ v) + D_2\psi(t,\ u,\ v)]$ and

$\mu_2 = \min\limits_{(t,\,u,\,v)\in\Omega}[D_3 h(t,\ u,\ v) + D_3\psi(t,u,v)]$. Then $\mu_1 > 0$ and $\mu_2 > 0$.

So we have $h(t,\ x_1,\ y_1) \geq h(t,\ x_2,\ y_2) - \psi(t,\ x_1,\ y_1) + \psi(t,\ x_2,\ y_2)$

$$+ \mu_1(x_1 - x_2) + \mu_2(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Thus

$$f(t,\ x_1,\ y_1) + h(t,\ x_1,\ y_1) \geq f(t,\ x_2,\ y_2) + h(t,\ x_2,\ y_2) + \phi(t,\ x_2,\ y_2)$$

$$+ \psi(t,\ x_2,\ y_2) - \phi(t,\ x_1,\ y_1) - \psi(t,\ x_1,\ y_1)$$

$$+ [D_2 f(t,\ x_1,\ y_1) + D_2\phi(t,\ x_1,\ y_1) + \mu_1](x_1 - x_2)$$

$$+ [D_3 f(t,\ x_1,\ y_1) + D_3\phi(t,\ x_1,\ y_1) + \mu_2](y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Since $\alpha_0,\ \beta_0 \in X$ are lower and upper solutions of BVP (10.1), we have

1. $-\alpha_0''(t) \le f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I.$
2. $-\beta_0''(t) \ge f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I.$
3. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0].$
4. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0].$
5. $\alpha_0(0) = \alpha_0(T).$
6. $\beta_0(0) = \beta_0(T).$
7. $\alpha_0'(0) \ge 0.$
8. $\beta_0'(0) \le 0.$

Consider

$$\left.\begin{aligned}
-y''(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \psi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad - \phi(t,\ y(t),\ y(t-r)) - \psi(t,\ y(t),\ y(t-r)) \\
&\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t) - \alpha_0(t)] \\
&\quad + [D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][y(t) - \alpha_0(t)] \\
&\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r) - \alpha_0(t-r)] \\
&\quad + [D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][y(t-r) - \alpha_0(t-r)], \\
&\quad t \in I, \\
y(t) &= y(0),\ t \in [-r,\ 0], \\
y(0) &= y(T), \\
y'(0) &= 0.
\end{aligned}\right\} \qquad (10.7)$$

$$\begin{aligned}
\text{Let } F_{\alpha_0}(t,\ u,\ v) = {} & f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
& + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \psi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
& - \phi(t,\ u,\ v) - \psi(t,\ u,\ v) \\
& + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[u - \alpha_0(t)] \\
& + [D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][u - \alpha_0(t)] \\
& + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[v - \alpha_0(t-r)] \\
& + [D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][v - \alpha_0(t-r)], \\
& \forall\ (t,\ u,\ v) \in \Omega.
\end{aligned}$$

This implies

$$\begin{aligned}
F_{\alpha_0}(t, y(t), y(t-r)) = {} & f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
& + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \psi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
& - \phi(t,\ y(t),\ y(t-r)) - \psi(t,\ y(t),\ y(t-r)) \\
& + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t) - \alpha_0(t)] \\
& + [D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][y(t) - \alpha_0(t)] \\
& + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[y(t-r) - \alpha_0(t-r)] \\
& + [D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][y(t-r) - \alpha_0(t-r)], \\
& \forall\ t \in I.
\end{aligned}$$

So we have $F_{\alpha_0}(t,\ x_1,\ y_1) \leq F_{\alpha_0}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

$$\begin{aligned}
\text{Also } -\alpha_0''(t) &\le f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \psi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad - \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) - \psi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_0(t) - \alpha_0(t)] \\
&\quad + [D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][\alpha_0(t) - \alpha_0(t)] \\
&\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_0(t-r) - \alpha_0(t-r)] \\
&\quad + [D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][\alpha_0(t-r) - \alpha_0(t-r)] \\
&= F_{\alpha_0}(t,\ \alpha_0(t),\ \alpha_0(t-r)).
\end{aligned}$$

$$\begin{aligned}
\text{Further } -\beta_0'(t) &\ge f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\ge f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \psi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad - \phi(t,\ \beta_0(t),\ \beta_0(t-r)) - \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\beta_0(t) - \alpha_0(t)] \\
&\quad + [D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][\beta_0(t) - \alpha_0(t)] \\
&\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\beta_0(t-r) - \alpha_0(t-r)] \\
&\quad + [D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][\beta_0(t-r) - \alpha_0(t-r)] \\
&= F_{\alpha_0}(t,\ \beta_0(t),\ \beta_0(t-r)).
\end{aligned}$$

We already know that

1. $\alpha_0(t) \le \beta_0(t),\ t \in [-r,\ T]$.
2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
3. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
4. $\alpha_0(0) = \alpha_0(T)$.

5. $\beta_0(0) = \beta_0(T)$.
6. $\alpha_0'(0) \geq 0$.
7. $\beta_0'(0) \leq 0$.

Therefore α_0 and β_0 are lower and upper solutions of the BVP (10.7) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\, t \in [-r,\ T]$.

Thus the BVP (10.7) has a solution $\alpha_1 \in X$ such that $\alpha_0 \leq \alpha_1 \leq \beta_0$ on $[-r,\ T]$. (Using Corollary 8.4.6)

Therefore

$$\left.\begin{aligned}
-\alpha_1''(t) &= f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad + \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \psi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad - \phi(t,\ \alpha_1(t),\ \alpha_1(t-r)) - \psi(t,\ \alpha_1(t),\ \alpha_1(t-r)) \\
&\quad + D_2 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_1(t) - \alpha_0(t)] \\
&\quad + [D_2\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_1][\alpha_1(t) - \alpha_0(t)] \\
&\quad + D_3 f(t,\ \alpha_0(t),\ \alpha_0(t-r))[\alpha_1(t-r) - \alpha_0(t-r)] \\
&\quad + [D_3\phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) + \mu_2][\alpha_1(t-r) - \alpha_0(t-r)],\ t \in I, \\
\alpha_1(t) &= \alpha_1(0),\ t \in [-r,\ 0], \\
\alpha_1(0) &= \alpha_1(T), \\
\alpha_1'(0) &= 0.
\end{aligned}\right\}$$

This implies $-\alpha_1''(t) \leq f(t,\ \alpha_1(t),\ \alpha_1(t-r)) + h(t,\ \alpha_1(t),\ \alpha_1(t-r)),\ t \in I$.

Also

1. $-\beta_0''(t) \geq f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)),\ t \in I$.
2. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
3. $\alpha_1(t) = \alpha_1(0),\ t \in [-r,\ 0]$.
4. $\alpha_1(t) \leq \beta_0(t),\ t \in [-r,\ T]$.
5. $\alpha_1(0) = \alpha_1(T)$.
6. $\beta_0(0) = \beta_0(T)$.

7. $\alpha_1'(0) = 0$.

8. $\beta_0'(0) \leq 0$.

Therefore α_1 and $\beta_0 \in X$ are lower and upper solutions of the BVP (10.1) such that $\alpha_1(t) \leq \beta_0(t),\ \forall\ t \in [-r,\ T]$.

Continuing in this manner, we get a monotonic increasing sequence of functions $\{\alpha_n\} \subseteq X$ such that $\alpha_n \leq \beta_0,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned}
-\alpha_{n+1}''(t) &= f(t, \alpha_n(t), \alpha_n(t-r)) + h(t, \alpha_n(t), \alpha_n(t-r)) \\
&\quad + \alpha(t, \alpha_n(t), \alpha_n(t-r)) + \beta(t, \alpha_n(t), \alpha_n(t-r)) \\
&\quad - \alpha(t, \alpha_{n+1}(t), \alpha_{n+1}(t-r)) - \beta(t, \alpha_{n+1}(t), \alpha_{n+1}(t-r)) \\
&\quad + D_2 f(t, \alpha_n(t-r), \alpha_n(t-r))[\alpha_{n+1}(t) - \alpha_n(t)] \\
&\quad + [D_2\alpha(t, \alpha_n(t), \alpha_n(t-r)) + \mu_1][\alpha_{n+1}(t) - \alpha_n(t)] \\
&\quad + D_3 f(t, \alpha_n(t), \alpha_n(t-r))[\alpha_{n+1}(t-r) - \alpha_n(t-r)] \\
&\quad + [D_3\alpha(t, \alpha_n(t), \alpha_n(t-r)) + \mu_2][\alpha_{n+1}(t-r) - \alpha_n(t-r)], \\
&\quad\ t \in I, \\
\alpha_{n+1}(t) &= \alpha_{n+1}(0),\ t \in [-r,\ 0], \\
\alpha_{n+1}(0) &= \alpha_{n+1}(T), \\
\alpha_{n+1}'(0) &= 0.
\end{aligned}\right\}$$

Further $\lim\limits_{n\to\infty} \alpha_n$ exists in $(X,\ ||.||_\infty)$ and $\lim\limits_{n\to\infty} \alpha_n$ is a solution of the BVP (10.1).

Let $\theta_1 = \min\limits_{(t,\ u,\ v)\in\Omega} [D_2 f(t,\ u,\ v) + D_2\phi(t,\ u,\ v)]$ and

$\theta_2 = \min\limits_{(t,\ u,\ v)\in\Omega} [D_3 f(t,\ u,\ v) + D_3\phi(t,\ u,\ v)]$. Then $\theta_1 > 0$ and $\theta_2 > 0$.

This implies $f(t,\ x_1,\ y_1) \geq f(t,\ x_2,\ y_2) - \phi(t,\ x_1,\ y_1) - \phi(t,\ x_2,\ y_2)$

$$+ \theta_1(x_1 - x_2) + \theta_2(y_1 - y_2),$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

So we have

$$\begin{aligned} f(t,\ x_1,\ y_1) + h(t,\ x_1,\ y_1) \geq f(t,\ x_2,\ y_2) + h(t,\ x_2,\ y_2) + \phi(t,\ x_2,\ y_2) \\ + \psi(t,\ x_2,\ y_2) - \phi(t,\ x_1,\ y_1) - \psi(t,\ x_1,\ y_1) \\ + [\theta_1 + D_2h(t,\ x_1,\ y_1) + D_2\psi(t,\ x_1,\ y_1)](x_1 - x_2) \\ + [\theta_2 + D_3h(t,\ x_1,\ y_1) + D_3\psi(t,\ x_1,\ y_1)](y_1 - y_2), \end{aligned}$$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \geq x_2$ and $y_1 \geq y_2$.

Now consider

$$\left.\begin{aligned} -y''(t) = f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\ + \phi(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\ - \phi(t,\ y(t),\ y(t-r)) - \psi(t,\ y(t),\ y(t-r)) \\ + [\theta_1 + D_2h(t,\ \beta_0(t),\ \beta_0(t-r))][y(t) - \beta_0(t)] \\ + D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[y(t) - \beta_0(t)] \\ + [\theta_2 + D_3h(t,\ \beta_0(t),\ \beta_0(t-r))][y(t-r) - \beta_0(t-r)] \\ + D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)], \\ t \in I, \\ y(t) = y(0),\ t \in [-r,\ 0], \\ y(0) = y(T), \\ y'(0) = 0. \end{aligned}\right\} \quad (10.8)$$

$$\begin{aligned}
\text{Let } G_{\beta_0}(t,\ u,\ v) = {} & f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\
& + \phi(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
& - \phi(t,\ u,\ v) - \psi(t,\ u,\ v) \\
& + [\theta_1 + D_2 h(t,\ \beta_0(t),\ \beta_0(t-r))][u - \beta_0(t)] \\
& + D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[u - \beta_0(t)] \\
& + [\theta_2 + D_3 h(t,\ \beta_0(t),\ \beta_0(t-r))][v - \beta_0(t-r)] \\
& + D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[v - \beta_0(t-r)], \\
& \forall\ (t,\ u,\ v) \in \Omega.
\end{aligned}$$

This implies

$$\begin{aligned}
G_{\beta_0}(t,\ y(t),\ y(t-r)) = {} & f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\
& + \phi(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
& - \phi(t,\ y(t),\ y(t-r)) - \psi(t,\ y(t),\ y(t-r)) \\
& + [\theta_1 + D_2 h(t,\ \beta_0(t),\ \beta_0(t-r))][y(t) - \beta_0(t)] \\
& + D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[y(t) - \beta_0(t)] \\
& + [\theta_2 + D_3 h(t,\ \beta_0(t),\ \beta_0(t-r))][y(t-r) - \beta_0(t-r)] \\
& + D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[y(t-r) - \beta_0(t-r)], \\
& \forall\ t \in I.
\end{aligned}$$

So we have $G_{\beta_0}(t,\ x_1,\ y_1) \leq G_{\beta_0}(t,\ x_2,\ y_2),\ \forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

$$\begin{aligned}
\text{Also } -\alpha_0''(t) &\le f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\le f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad + \phi(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad - \phi(t,\ \alpha_0(t),\ \alpha_0(t-r)) - \psi(t,\ \alpha_0(t),\ \alpha_0(t-r)) \\
&\quad + [\theta_1 + D_2 h(t,\ \beta_0(t),\ \beta_0(t-r))][\alpha_0(t) - \beta_0(t)] \\
&\quad + D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[\alpha_0(t) - \beta_0(t)] \\
&\quad + [\theta_2 + D_3 h(t,\ \beta_0(t),\ \beta_0(t-r))][\alpha_0(t-r) - \beta_0(t-r)] \\
&\quad + D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[\alpha_0(t-r) - \beta_0(t-r)] \\
&= G_{\beta_0}(t,\ \alpha_0(t),\ \alpha_0(t-r)).
\end{aligned}$$

$$\begin{aligned}
\text{Further } -\beta_0''(t) &\ge f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&= f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad + \phi(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad - \phi(t,\ \beta_0(t),\ \beta_0(t-r)) - \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad + [\theta_1 + D_2 h(t,\ \beta_0(t),\ \beta_0(t-r))][\beta_0(t) - \beta_0(t)] \\
&\quad + D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_0(t) - \beta_0(t)] \\
&\quad + [\theta_2 + D_3 h(t,\ \beta_0(t),\ \beta_0(t-r))][\beta_0(t-r) - \beta_0(t-r)] \\
&\quad + D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_0(t-r) - \beta_0(t-r)] \\
&= G_{\beta_0}(t,\ \beta_0(t),\ \beta_0(t-r)).
\end{aligned}$$

We already know that

1. $\alpha_0(t) \le \beta_0(t),\ t \in [-r,\ T]$.
2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
3. $\beta_0(t) = \beta_0(0),\ t \in [-r,\ 0]$.
4. $\alpha_0(0) = \alpha_0(T)$.
5. $\beta_0(0) = \beta_0(T)$.

6. $\alpha_0'(0) \geq 0$.

7. $\beta_0'(0) \leq 0$.

Therefore, α_0 and β_0 are lower and upper solutions of the BVP (10.8) such that $\alpha_0(t) \leq \beta_0(t),\ \forall\, t \in [-r,\ T]$.

Thus the BVP (10.8) has a solution $\beta_1 \in X$ such that $\alpha_0 \leq \beta_1 \leq \beta_0$ on $[-r,\ T]$. (Using Corollary 8.4.6)

This implies

$$\left.\begin{aligned}
-\beta_1''(t) &= f(t,\ \beta_0(t),\ \beta_0(t-r)) + h(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad + \phi(t,\ \beta_0(t),\ \beta_0(t-r)) + \psi(t,\ \beta_0(t),\ \beta_0(t-r)) \\
&\quad - \phi(t,\ \beta_1(t),\ \beta_1(t-r)) - \psi(t,\ \beta_1(t),\ \beta_1(t-r)) \\
&\quad + [\theta_1 + D_2h(t,\ \beta_0(t),\ \beta_0(t-r))][\beta_1(t) - \beta_0(t)] \\
&\quad + D_2\psi(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_1(t) - \beta_0(t)] \\
&\quad + [\theta_2 + D_3h(t,\ \beta_0(t),\ \beta_0(t-r))][\beta_1(t-r) - \beta_0(t-r)] \\
&\quad + D_3\psi(t,\ \beta_0(t),\ \beta_0(t-r))[\beta_1(t-r) - \beta_0(t-r)],\ t \in I, \\
\beta_1(t) &= \beta_1(0),\ t \in [-r,\ 0], \\
\beta_1(0) &= \beta_1(T), \\
\beta_1'(0) &= 0.
\end{aligned}\right\}$$

This implies $-\beta_1''(t) \geq f(t,\ \beta_1(t),\ \beta_1(t-r)) + h(t,\ \beta_1(t),\ \beta_1(t-r)),\ t \in I$.

Also

1. $-\alpha_0''(t) \leq f(t,\ \alpha_0(t),\ \alpha_0(t-r)) + h(t,\ \alpha_0(t),\ \alpha_0(t-r)),\ t \in I$.
2. $\alpha_0(t) = \alpha_0(0),\ t \in [-r,\ 0]$.
3. $\beta_1(t) = \beta_1(0),\ t \in [-r,\ 0]$.
4. $\alpha_0(t) \leq \beta_1(t),\ t \in [-r,\ T]$.
5. $\alpha_0(0) = \alpha_0(T)$.
6. $\beta_1(0) = \beta_1(T)$.
7. $\alpha_0'(0) \geq 0$.

8. $\beta_1'(0) = 0$.

Therefore α_0 and $\beta_1 \in X$ are lower and upper solutions of the BVP (10.1) such that $\alpha_0(t) \leq \beta_1(t),\ \forall\ t \in [-r,\ T]$.

Continuing in this manner, we get a monotonic decreasing sequence of functions $\{\beta_n\} \subseteq X$ such that $\alpha_0 \leq \beta_n,\ \forall\ n \in \mathbb{N}$ on $[-r,\ T]$ and

$$\left.\begin{aligned}
-\beta_{n+1}''(t) &= f(t,\ \beta_n(t),\ \beta_n(t-r)) + h(t,\ \beta_n(t),\ \beta_n(t-r)) \\
&\quad + \phi(t,\ \beta_n(t),\ \beta_n(t-r)) + \psi(t,\ \beta_n(t),\ \beta_n(t-r)) \\
&\quad - \phi(t,\ \beta_{n+1}(t),\ \beta_{n+1}(t-r)) - \psi(t,\ \beta_{n+1}(t),\ \beta_{n+1}(t-r)) \\
&\quad + [\theta_1 + D_2h(t,\ \beta_n(t),\ \beta_n(t-r))][\beta_{n+1}(t) - \beta_n(t)] \\
&\quad + D_2\psi(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t) - \beta_n(t)] \\
&\quad + [\theta_2 + D_3h(t,\ \beta_n(t),\ \beta_n(t-r))][\beta_{n+1}(t-r) - \beta_n(t-r)] \\
&\quad + D_3\psi(t,\ \beta_n(t),\ \beta_n(t-r))[\beta_{n+1}(t-r) - \beta_n(t-r)],\ t \in I, \\
\beta_{n+1}(t) &= \beta_{n+1}(0),\ t \in [-r,\ 0], \\
\beta_{n+1}(0) &= \beta_{n+1}(T), \\
\beta_{n+1}'(0) &= 0.
\end{aligned}\right\}$$

Further, $\lim\limits_{n\to\infty}\beta_n$ exists in $(X,\ ||.||_\infty)$ and $\lim\limits_{n\to\infty}\beta_n$ is a solution of the BVP (10.1). □

Finally we state a similar result for the BVP (10.2).

Theorem 10.4.2. *Let* $\alpha_0,\ \beta_0 \in W$ *such that* $\alpha_0(t) \leq \beta_0(t),\ \forall\ t \in I$ *and* $\Omega = \{(t,u,v) : \alpha_0(t) \leq u \leq \beta_0(t)$ *and* $\alpha_0(\theta(t)) \leq v \leq \beta_0(\theta(t)),\ t \in I\}$. *Assume that* $f,\ h,\ \phi,\ \psi \in C^2[\Omega,\ \mathbb{R}]$ *satisfying*

(1) $D_i f(t,\ x,\ y) + D_i\phi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in \Omega$ *and* $\forall\ i = 2,\ 3.$

(2) $D_i h(t,\ x,\ y) + D_i\psi(t,\ x,\ y) > 0,\ \forall\ (t,\ x,\ y) \in \Omega$ *and* $\forall\ i = 2,\ 3.$

(3) $D_{i,\,j} f(t,\ x,\ y) + D_{i,\,j}\phi(t,\ x,\ y) \geq 0,\ \forall\ (t,\ x,\ y) \in \Omega$ *and* $\forall\ i,\ j = 2,\ 3.$

(4) $D_{i,\,j} h(t,\ x,\ y) + D_{i,\,j}\beta(t,\ x,\ y) \leq 0,\ \forall\ (t,\ x,\ y) \in \Omega$ *and* $\forall\ i,\ j = 2,\ 3.$

(5) $\phi(t,\ x_1,\ y_1) + \psi(t,\ x_1,\ y_1) \geq \phi(t,\ x_2,\ y_2) + \psi(t, x_2, y_2),$

$\forall\ (t,\ x_1,\ y_1),\ (t,\ x_2,\ y_2) \in \Omega$ *such that* $x_1 \leq x_2$ *and* $y_1 \leq y_2$.

Further, assume that α_0 *and* β_0 *are lower and upper solutions of BVP* (10.2). *Then there are monotonic sequences of functions* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *converging in* $(W,\ ||.||_0)$ *such that* $\lim\limits_{n\to\infty} \alpha_n$ *and* $\lim\limits_{n\to\infty} \beta_n$ *are solutions to the BVP* (10.2).

Proof. Similar to the proof of the Theorem 10.4.1. □

Summary

In this thesis, an effort is made to establish some existence of solutions results to BVPs of first and second order differential equations with deviating arguments. The BVPs discussed in this work includes PBVP, APBVP, LBVP, NLBVP of first order differential equations with deviating arguments and various BVPs of second order differential equations with deviating arguments. Leray-Schauder Alternative, Schauder's fixed point theorem, Fixed point theorem on partially ordered metric space and Banach fixed point theorem are used to obtain the results.

In this work, we take f to be continuous and non-decreasing in the second coordinate or second and third co-ordinate, as the case may be, to prove our most of the results. However, while using method of coupled lower and upper solutions, we assume f to be non increasing in second and third arguments.

It is to be noted that in existing literature the function f is taken to be continuous and Lipchitz or continuous and decreasing in second coordinate for establishing existence of a solution for initial value problems as well as boundary value problems.

Some salient features of the thesis are as follows:

1. Some results on existence of solution for BVPs of first order DEDA are obtained.

2. Results on existence of solution between lower and upper solutions for BVPs of first order DEDA are proved.

3. Existence of solution results for BVPs of first order DEDA are proved using MIM, QM, EQM, GQM and EGQM.

4. Existence of solution results for BVPs of second order DEDA are obtained.

5. Results on existence of solution between lower and upper solutions for BVPs of second order DEDA are proved.

6. Existence of solution results for BVPs of second order DEDA are proved by using techniques of MIM, QM, EQM, GQM and EGQM .

Problems for Further Study

We suggest the following topics for further study.

1. Study of Qualitative Properties for BVPs First and Second Order Differential Equations with Deviating Arguments.

2. Study of Hyers - Ulam Stability for BVPs of First and Second Order Differential Equations with Deviating Arguments.

3. To Study BVPs of nth Order Differential Equations with Deviating Arguments.

4. To Study BVPs of Partial Differential Equations with Deviating Arguments.

Papers Published in Journals

(P1) **Some Existence Theorems for Periodic Boundary Value Problem of First Order Delay Differential Equation** in *Mapana Journal of Sciences* (UGC CARE LIST) 2023, Vol. 22, Special Issue 1, 13-31, ISSN 0975-3303—https://doi.org/10.12723/mjs.sp1.2, July 2023.

(P2) **Weakly Coupled Lower and Upper Solutions for anti-Periodic Boundary Value Problem of First Order Delay Differential Equation** in *Indian Journal of Natural Sciences* (Web of Science), Vol.14, Issue 80, October 2023.

(P3) **Monotone iterative method for Boundary Value Problem with Linear Condition of First Order Delay Differential Equation** in *Serdica Mathematical Journal* (UGC CARE LIST), Vol.49, No.4, September 2023.

(P4) **Extended Quasilinearization Method for Periodic Boundary Value Problem of First Order Delay Differential Equations** in *Jnanabha* (UGC CARE LIST), Vol.54, No.2, November 2024.

(P5) **On Boundary Value Problems for Second Order Ordinary Differential Equation with Constant Delay** in *Serdica Mathematical Journal* (UGC CARE LIST), Vol.54, April 2025.

Papers Communicated to Journals

1. Lower and Upper Solution for Periodic Boundary Value Problem of Second Order Ordinary Differential Equation With Constant Delay.

2. Application of Some Fixed Point Theorems to Existence of Solution for Periodic Boundary Value Problem of First Order Ordinary Differential Equation with Deviating Argument.

Papers Presented in Conferences

(C1) "Existence of Solutions to Anti-periodic Boundary Value Problem for First Order Delay Differential Equation" at National Conference on Mathematical Analysis and Applications, organized by Dept. of Mathematics, Shivaji University, Kolhapur, during 21 -22 March 2022.

(C2) "Some Existence of Theorems for Periodic Boundary Value Problem of First Order Delay Differential Equation" at International Conference on Recent Developments on Pure and Applied Mathematics, organized by Dept. of Mathematics, St. Xavier's College(Autonomous) Palayamkottai, Tirunalveli - Tamilnadu held on 19 - 21 July 2022.

(C3) "Monotone Iterative Technique for Periodic Boundary Value Problem of First Order Delay Differential Equation" at International Conference on Mathematical Sciences and Its Applications, organized by School of Mathematical Sciences, SRTMU, Nanded - Maharashtra held on 28 - 30 July 2022.

(C4) "Quasilinearization Technique for Periodic Boundary Value Problem of First Order Delay Differential Equations" at International Conference on Nonlinear Analysis and Applications, organized by Dept. of Mathematics, Assam Don Bosco University, Sonapur - Guwahati - Assam held on 22 - 23 November 2022.

(C5) "Exended Quasilinearization Method for Periodic Boundary Value Problem of First Order Delay Differential Equation" at International Conference onrecent Trends in Applied Mathematics, organized by Dept. of Mathematics, Loyola College, Chennai, held on 24 - 25 February 2023.

(C6) "Weakly Coupled Lower and Upper Solutions for Anti-periodic Boundary Value Problem of First Order Delay Differential Equation" at International Conference on Blooming Fuzzifier Logic, organized by Dept. of Mathematics, Chikkanna Govt. Arts College, Tirupur - Tamil Nadu held on 23 - 24 August 2023.

(C7) "Generalized Quasilinearization Method for Boundary Value Problem with Nonlinear Condition of First Order Delay Differential Equation" at International Conference on Mathematical Sciences and Its Application to Artificial Intelligence, organized by Dept. of Mathematics, SRM University, Sonipat - Haryana, held on 17 - 19 January 2024.

(C8) "Extended Quasilinearization Method for Boundary Value Problem with Nonlinear Condition of First Order Delay Differential Equation" at International Conference on Analysis and Its Application, organized by Dept. of Mathematics, Jamia Millia Islamia, New Delhi, held on 19 - 21 January 2024.

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