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ON A CLASS OF LINEAR HYPERBOLIC DIFFERENTIAL EQUATIONS

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In the present paper the authors first present the usual convergent process of successive approximations associated to the Banach contraction principle for a class of linear hyperbolic partial differential equations. Then the authors obtain as a corollary the Riemann transition function for the telegraph equation, which had been obtained by Titchmarch and Copson by different arguments. Examples on the use of the transition function are given. As another application the authors derive a Wendroff-type inequality.

1. INTRODUCTION

Most dynamical systems are often mathematically modelled in the form of differential equations[2,5]. The existence of closed-form solutions, or at least the qualitative properties of solutions of these equations, is an important consideration in the understanding of the behaviour of the systems. A great variety of methods is available today for handling the theory of existence of solutions. Considering the integral equation equivalent to the linear differential equation

$$\dot{x}(t) - A(t)\,x = 0 \qquad \ldots(1.1)$$

($A(t)$ is an $n \times n$ matrix). Conti[2] proves an existence theorem for (1.1) using the Banach fixed-point principle[6], which leads to computing the solution by iteration process. This generates an evolution matrix which exhibits interesting properties. Employing the same technique, Conti[2] computes closed-form solutions for the linear affine differential equation

$$\dot{x}(t) - A(t)\,x = f(t),$$

which reduces to (1.1) when $f \equiv 0$.

In this paper, the authors establish two-dimensional analogues of the existing results cited above. Apart from these generalizations, it has been possible to illustrate by an example the construction of the Riemann function[3] for a class of LHPDE. A generalized integral inequality of Wendrofftype[1] has also been established.

2. NOTATIONS AND PRELIMINARIES

Let R denote the real line and let

$$J = [a, b],$$
$$K = [c, d],$$

where a, b, c and d are finite.

For any rectangle $J \times K$ in R^2, we define the following classes of real-valued functions :

(i) $C(J \times K)$ = the space of continuous functions

$u : J \times K \to R$,

with norm

$$\|u\| = \sup \{|u(x, y)| : (x, y) \in J \times K\}$$

(ii) $C^{(q)}(J \times K)$ = the space of functions

$u : J \times K \to R$,

having continuous qth order partial derivatives.

$(C^{(0)}(J \times K) = C(J \times K))$.

For any fixed $(x_0, y_0) \in J \times K$, we shall assume

$$D[x, y] = [x_0, x] \times [y_0, y],\ (x, y) \in J \times K,\ x \geqslant x_0,\ y \geqslant y_0.$$

Consider the characteristic initial value problem for the LHPDE

$$\frac{\partial^2 u}{\partial x\, \partial y} = f(x, y)\, u(x, y) \qquad \ldots(2.1)$$

defined on $J \times K$, where

$$u(x, y_0) = \sigma(x),\ x \in J$$

$$u(x_0, y) = \tau(y),\ y \in K$$

$u(x_0, y_0) = \sigma(x_0) = \tau(y_0)$ are prescribed.

Equation (2.1) can be reformulated in terms of the linear Volterra integral equation

$$u(x, y) = g(x, y) + \int_{D[x,y]} f(s, t)\, u(s, t)\, ds\, dt \qquad \ldots(2.2)$$

$$x \geqslant x_0,\ y \geqslant y_0,\ x_0, x \in J,\ y_0, y \in K,$$

where the function $g(x, y)$, uniquely determined by the prescribed initial values, is given by $g(x, y) = \sigma(x) + \tau(y) - \sigma(x_0)$. It is to be noted here, that a solution $u(x, y)$ of (2.1) will satisfy (2.2), then $u(x, y) \in C^{(2)}(J \times K)$ (provided $g(x, y) \in C^{(2)}(J \times K)$) and by differentiation we have (2.1).

3 Main Results

At the outset, we prove the existence of solutions of (2.2) in terms of f and g in a closed form. The technique involves the use of the Banach fixed-point principle, which leads to constructing the solution by iteration.

Theorem 3.1—(Existence and Uniqueness)—If $f(x, y) \in C(J \times K)$ and $g(x, y) \in C(J \times K)$, then $u(x, y)$ defined by

$$u(x, y) = \lim_k \Big[g(x, y) + \int_{D[x,y]} f(s_1, t_1)\, g(s_1, t_1)\, ds_1 dt_1 + \dots$$

$$= \int_{D[x,y]} \dots \int_{D[s_{k-1},t_{k-1}]} f(s_1, t_1) \dots f(s_k, t_k)\, g(s_k, t_k)\, ds_k\, dt_k$$

$$\dots ds_1\, dt_1\Big] \qquad \dots(3.1)$$

$k = 1, 2, \dots, (x, y) \in J \times K$, for any fixed $(x_0, y_0) \in J \times K$, is the only $u(x, y) \in C(J \times K)$ which satisfies (2.2).

Proof : The space $C(J \times K)$ of continuous functions $u : J \times K \to R$, with norm

$$\|u\| = \sup \{ \mid u(x, y) \mid : (x,y) \in J \times K\},$$

is complete.

Let us also consider the space $C_\mu(J \times K)$, $\mu \geqslant 0$, of continuous functions $u : J \times K \to R$, with norm

$$\|u\|_\mu = \sup \{ \mid u(x, y) \mid \exp(-\mu \int_{D[x,y]} \mid f(s, t) \mid ds\, dt)\},$$

$$(x, y) \in J \times K$$

so that $C_0(J \times K) = C(J \times K)$. It is easily verified that the norms $\|u\|_\mu$ are all equivalent for $\mu \geqslant 0$, so that $C_\mu(J \times K)$ is also complete.

Let us now consider the mapping T of $C_\mu(J \times K)$ into itself defined by

$$(Tu)(x, y) = g(x, y) + \int\int_{D[x,y]} f(s, t)\, u(s, t)\, ds\, dt.$$

We therefore have, for $u_1, u_2 \in C_\mu(J \times K)$

$$\mid Tu_1 - Tu_2 \mid \leqslant \int_{D[x,y]} \mid f(s, t) \| u_1(s, t) - u_2(s, t) \mid ds\, dt$$

$$\leqslant \int_{D[x,y]} \mid f(s, t) \| u_1(s, t) - u_2(s, t) \mid$$

$$\exp(-\mu \int_{D[s,t]} \mid f(p, q) \mid dp\, dq) \exp(\mu \int_{D[s,t]} \mid f(p,q) \mid dp\, dq)\, ds\, dt$$

$$\leqslant \|u_1 - u_2\|_\mu \int_{D[x,y]} \mid f(s, t) \mid \exp(\mu \int_{D[s,t]} \mid f(p, q) \mid$$

$$dp\, dq)\, ds\, dt. \qquad \dots(3.2)$$

If for the integral (3.2), we let

$$\int_{D[s,t]} \mid f(p, q) \mid dp\, dq = \Phi(s, t)$$

then $|f(s,t)| = \Phi_{st}(s,t)$, and a simple computation yields

$$\int_{D[x,y]} \mu\,|f(s,t) - \exp(\mu \int_{D[s,t]} |f(p,q)|\,dp\,dq)\,ds\,dt$$

$$\leqslant \exp(\mu \int_{D[x,y]} |f(s,t)|\,ds dt) - 1. \qquad \ldots(3.3)$$

From (3.2) and (3.3) it is clear that for $u_1, u_2 \in C_\mu(J \times K)$

$$|Tu_1 - Tu_2| \leqslant \|u_1 - u_2\|_\mu \mu^{-1} \exp(\mu \int_{D[x,y]} |f(s,t)|\,ds\,dt).$$

Hence

$$|Tu_1 - Tu_2| \exp(-\mu \int_{D[x,y]} |f(s,t)|\,ds\,dt) \leqslant \mu^{-1} \|u_1 - u_2\|_\mu$$

i. e.

$$\|Tu_1 - Tu_2\|_\mu \leqslant \mu^{-1} \|u_1 - u_2\| \mu;\ x_0 \leqslant s \leqslant x,\ y_0 \leqslant t \leqslant y,$$

for $\mu > 0$, so that T is a contraction mapping for $\mu > 1$. By virtue of Banach theorem, T has a unique $u(x, y)$ which satisfies (2.2) and this is represented as the limit in $C\,\mu$ $(J \times K)$ of the sequence (u_k) defined by

$$u^{(1)}(x,y) = g(x,y)$$

$$u^{(k)}(x,y) = g(x,y) + \int_{D[x,y]} f(s,t)\,u^{(k-1)}(s,t)\,ds\,dt$$

$$k = (2, 3, \ldots).$$

This will give (3.1) for $(x, y) \in J \times K$.

In particular, if on $J \times K$, $g(x, y) = c$, a constant, we have

Theorem 3.2—If $f(x, y) \in C(J \times K)$, the solutions of (2.2) are repersented by

$$u(x,y) = T_f(x, y; x_0, y_0)\,c \qquad \ldots(3.4)$$

where

$$T_f(x, y; x_0\ y_0) = \lim_k [1 + \int_{D[x,y]} f(s_1, t_1)\,ds_1\,dt_1 + \ldots$$

$$+ \int_{D[x,y]} \cdots \int_{D[s_{k-1},t_{k-1}]} f(s_1, t_1)\,f(s_k, t_k)\,ds_k\,dt_k \ldots ds_1 dt_1] \qquad \ldots(3.5)$$

$$(x, y), (x_0, y_0) \in J \times K.$$

T_f describes the transition of u from (x_0, y_0) to (x, y) and hence, we define $T_f(x, y; x_0, y_0)$ as the 'transition function' generated by f. The importance of Theorems 3.1 and 3.2 is illustrative in the following examples.

Example 3.1—Consider the "Telegraph equation"

$$u_{xy} + \lambda u = 0, \quad \lambda - \text{constant},$$

satisfying the characteristic initial values

$$u\ (x, y_0) = u\ (x_0, y) = u\ (x_0, y_0) = 1.$$

It is clear that the Riemann function (say v) for the above LHPDE satisfies the self-adjoint equation

$$v_{xy} + \lambda v = 0,$$

with boundary conditions,

$$v(x,y_0) = v\ (x_0,y) = v\ (x_0,y) = 1.$$

Under these conditions, it follows from (2.2) that the Riemann function satisfies the integral equation

$$v\ (x, y) = 1 + \int_{D[x,y]} (-\lambda)\ v\ (s,t)\ ds\ dt.$$

By Theorem 3.2, the solutions are represented by

$$v\ (x, y) = T_{-\lambda}\ (x, y; x_0, y_0),$$

where

$$T_{-\lambda}\ (x, y\ ; x_0, y_0) = \lim_k\ [1 + \int_{D[x,y]} (-\lambda)\ ds_1 dt_1 + \ldots$$

$$+ \int_{D[x,y]} \cdots \int_{D[s_{k-1},t_{k-1}]} (-\lambda)^k\ ds_k\ dt_k\ \ldots\ ds_1\ dt_1]$$

$$= \lim_k\ [1 - \lambda\ (x - x_0)\ (y - y_0) + \ldots$$

$$+ \frac{(-1)^k\ (\lambda)^k\ (x - x_0)^k\ (y - y_0)^k}{(k!)^2}\ ]$$

$$= \lim_k \left[1 - (2\sqrt{\lambda\ (x - x_0)\ (y - y_0}/2)^2 + \ldots\right.$$

$$\left. + \frac{(-1)^k\ (2\sqrt{\lambda\ (x - x_0)\ (y - y_0)}/2)^{2k}}{(k!)^2}\right]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k\ (2\sqrt{\lambda\ (x - x_0)\ (y - y_0)}/2)^{2k}}{(k!)^2}$$

$$= J_0\ (2\sqrt{\lambda\ (x - x_0)\ (y - y_0)}).$$

The above example illustrates a method of obtaining the Riemann function for the class of LHPDE under consideration, however, it is to be noted here, that the iteration process for the Riemann function in general, drew attention away from the evaluation of this function[3].

Example 3.2—As an application of Theorem 3.1, consider the above example under the general characteristic initial values

$$u(x, y_0) = x + y_0$$

$$u(x_0, y) = x_0 + y$$

$$u(x_0, y_0) = x_0 + y_0, \text{ with } \lambda = 1/4.$$

It can be shown that a solution to the above LHPDE also satisfies the integral equation

$$u(x, y) = x + y + \int_{D[x,y]} (-1/4)\, u(s, t)\, ds\, dt.$$

By Theorem 3.1, solutions are represented by formal iterated sum

$$u(x, y) = \lim_k \Big[x + y + \int_{D[x,y]} (-1/4)(s_1 + t_1)\, ds_1\, dt_1 + \int_{D[x,y]} \cdots \int_{D[s_{k-1},t_{k-1}]} (-1/4)^k (s_k + t_k)\, ds_k\, dt_k \ldots ds_1\, dt_1\Big]$$

$$= \lim_k \left[x + y - (1/4)(x - x_0)(y - y_0)\left(\frac{x + x_0}{2\,!\,1\,!} + \frac{y + y_0}{2\,!\,1\,!}\right) + \ldots \frac{(-1)^k}{4^k}(x - x_0)^k (y - y_0)^k \left(\frac{x + kx_0}{(k+1)\,!\,k\,!} + \frac{y + ky_0}{(k+1)\,!\,k}\right)\right]$$

$$= (x + y)\sum_{k=0}^{\infty} \frac{(-1)^k (\sqrt{(x - x_0)(y - y_0)}/2)^{2k}}{(k+1)\,!\,k\,!} + (x_0 + y_0)\sum_{k=1}^{\infty} \frac{(-1)^k (\sqrt{(x - x_0)(y - y_0)}/2^{2k}.k :}{(k+1)!\ k\,!}$$

$$= \frac{(x + y)(x - x_0)(y - y_0)}{2} J_1\left(\sqrt{(x - x_0)(y - y_0)}\right) + (x_0 + y_0)\{J_0(\sqrt{(x - x_0)(y - y_0)}) - 1\} - \frac{2(x_0 + y_0)}{(x - x_0)(y - y_0)}\{J_1(\sqrt{(x - x_0)(y - y_0)}) - 1\}.$$

This series represents a unique solution of the LHPDE.

Corollary 3.1—If $f(x, y) = \lambda$, a constant, we have

$$T_f(x, y; x_0, y_0) = J_0\left(2i\sqrt{\lambda (x - x_0)(y - y_0)}\right)$$

$$(x_0, y_0),\ (x, y) \in J \times K.$$

Corollary 3.2—It is clear that by integration we have

(a) $T_f(x, y; x_0, y_0) = 1 + \int_{D[x,y]} f(s, t)\, T_f(s, t; x_0, y_0)\, ds dt.$

(b) $T_f(x, y; x_0, y) = 1 + \int_{D[x,y]} f(s, t)\ T_f(x, y; s, t)\, ds\, dt$

$$(x_0, y_0), (x, y) \in J \times K.$$

Remark 3.1 : Consider the characteristic initial value problem

$$u_{xy} = h(x, y) + f(x, y)\, u(x, y) \qquad \ldots(3.6)$$

$$h(x, y),\ f(x, y) \in C^{(0)}(J \times K)$$

where

$$u(x, y_0) = \sigma(x)$$

$$u(x_0, y) = \tau(y)$$

$u(x_0, y_0) = \sigma(x_0) = \tau(y_0)$ are prescribed.

As in the case of (2.1), equation (3.6) can be reformulated in terms of the Volterra integral equation

$$u(x, y) = k(x, y) + \int_{D[x,y]} h(s, t)\, ds\, dt + \int_{D[x,y]} f(s, t)\, u(s, t)\, ds\, dt. \qquad \ldots(3.7)$$

$$x \geqslant x_0,\ y \geqslant y_0.$$

where the function $k(x, y)$, uniquely determined by the prescribed initial values, is given by $k(x, y) = \sigma(x) + \tau(y) - \sigma(x_0)$.

By virtue of Theorem (3.1), we know that for every $(x_0, y_0) \in J \times K$, there is a unique $u(x, y)$ satisfying (3.7) and it is represented by (3.1) with

$$g(x, y) = k(x, y) + \int_{D[x,y]} h(s, t)\, ds\, dt,$$

where $g(x, y) \in C^{(2)}(J \times K)$.

Taking into account the function generated by f and g, it can easily be shown by a simple computation that

$$u(x, y) = \lim_{k} \Big[k(x, y) + \int_{D[x,y]} f(s_1, t_1)\, k(s_1 t_1)\, ds_1\, dt_1 + \int_{D[x,y]} \ldots \int_{D[s_{k-1} t_{k-1}]} f(s_1, t_1) \ldots f(s_k, t_k)\, k(s_k, t_k)\, ds_k dt_k \ldots ds_1\, dt_1\Big]$$

$$+ \int_{D[x,y]} T_f(x, y; s, t)\, h(s, t)\, ds\, dt. \qquad \ldots(3.8)$$

The following theorems are now immediate :

Theorem 3.3—If $f(x, y) \in C(J \times K)$, $h(x, y) \in C(J \times K)$ and $k(x, y) \in C^{(2)}(J \times K)$, the solutions of (3.7) are represented by (3.8).

Theorem 3.4—Under the hypotheses of Theorem 3.3, if $k(x, y) = c$, a constant, the solutions of (3.7) are represented by the formula

$$u(x, y) = T_f(x, y; x_0, y_0)\, c + \int_{D[x,y]} T_f(x, y; s, t)\, h(s, t)\, ds\, dt.$$
$$x \geqslant x_0, \quad y \geqslant y_0.$$

4. Integral Inequalities

Wendroff's inequality has been enriched in various directions mainly due to its potential in the study of the qualitative properties of solutions of various PDE and multiple LVIE. In the sequel we obtain some generalised integral inequalities of Wendroff-type.

For $u \in C(J \times K)$ let $u(x_0, y_0) = c$ (a constant) where x_0 and y_0 are the left end points of J and K respectively, define the partial order $\propto$ in $C(J \times K)$ as follows: for $u_1, u_2 \in C(J \times K)$, call $u_1 \propto u_2$ if $u_1(x, y) \leqslant u_2(x, y)$ for all $(x, y) \in J \times K$. In the sequel, we shall require the following lemma :

Lemma 4.1[6] (p 18)—Let X be a complete normed linear space partially ordered by the relation $\propto$ in such a manner that if an increasing sequence $\{x_n\}$ has the limit x_0, then $x_n \propto x_0$ for all n. Let T be an order-preserving contraction on X with unique fixed point f_0. Then

$$f \propto Tf \Rightarrow f \propto f_0.$$

Theorem 4.1—Suppose that u and f are scalar, non-negative functions such that the product fu is $dx\, dy$ – integrable on $J \times K$, and $u \in C(J \times K)$. Then for any positive constant C, the inequality

$$u(x, y) \leqslant c + \int_{D[x,y]} f(s, t)\, u(s, t)\, ds\, ds\, dt \qquad \ldots(4.1)$$
$$x \geqslant x_0, y \geqslant y_0.$$

implies

$$u(x, y) \leqslant c\, T_f(x, y; x_0, y_0) \qquad \ldots(4.2)$$
$$x \geqslant x_0, y \geqslant y_0.$$

Proof : Let $\{u_n\}$ be a sequence in the partially ordered space $C(J \times K)$ such that $u_1 \propto u_2 \propto \ldots$ and $u_n \to u$. Then $u \in C(J \times K)$ and $u(x_0, y_0) = c$, since for each n, $u_n(x_0, y_0) = c$. Furthermore, $u_n(x, y) \leqslant u(x, y)$ for each 'n', $n = 1, 2, \ldots, (x, y) \in J \times K$. Define an operator

$$T : C(J \times K) \to C(J \times K) \text{ by,}$$

$$(Tu)(x, y) = c + \int_{D[x,y]} f(s, t)\, u(s, t)\, ds\, dt.$$

As in Theorem 3.1, it can be shown that T is a contraction. T is also order-preserving i. e. for $u_1, u_2 \in C(J \times K)$ and $u_1 \propto u_2 \Rightarrow Tu_1 \propto Tu_2$. The desired result now follows from Lemma 4.1, completing the proof.

Corollary 4.1—If, under the hypotheses of Theorem 4.1, c is replaced by a function $g(x, y)$ where $g(x, y) \geqslant 0$ is a continuous, monotone, nondecreasing function in $x \in J$ and $y \in K$, the inequality

$$u(x, y) \leqslant g(x, y) + \int_{D[x,y]} f(s, t)\, u(s, t)\, ds\, dt \qquad \ldots(4.3)$$

$$x \geqslant x_0,\ y \geqslant y_0,$$

implies

$$u(x, y) \leqslant g(x, y)\, T_f(x, y; x_0, y_0),$$

$$x \geqslant x_0,\ y \geqslant y_0$$

$$x_0, x \in J,\ y_0, y \in K.$$

Remark 4. 1 : Using the notion of resolvent kernel from the theory of Volterra integral equations, a stronger version of the Wendroff's inequality[1] (p. 154) was obtained by Corduneanu[4].

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